Lecture Notes on Topological Insulators

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These are preliminary lecture notes on PDE models of topological insulators following courses at the University of Chicago in Fall 2022 and Fall 2023. The main topics of these lectures are:

Lecture 1. Motivations: Robust asymmetric transport at interfaces separating topological insulators

Lecture 2. Derivation of macroscopic Dirac equations from microscopic tight-binding models

Lecture 3. IQHE and Landau levels in Magnetic Schrödinger equations

Lecture 4. Avron-Seiler-Simon theory of the integer Quantum Hall Effect

Lecture 5. Bulk invariants for two-dimensional regularized Dirac models

Lecture 6. Asymmetric transport in one space dimension and index of Toeplitz operators

Lecture 7. Edge invariant for two-dimensional Dirac models and spectral flow

Lecture 8. Stability of topological indices under perturbations

Lecture 9. Pseudo-differential operators and quantization of Edge Conductivity

Lecture 10. Classification of PDE models by Domain Walls and Bulk-Edge correspondence

Lecture 11. Bulk-difference invariants, Chern numbers and winding numbers

Lecture 12. Survey of topological invariants (cohomology, degree theory, Chern-Weil, Chern numbers)

Lecture 13. Quantitative computations for Dirac models; integral formulation and scattering theory

Lecture 14. Gated Twisted Bilayer Graphene, PDE model and asymmetric transport

A number of appendices review relevant mathematical tools used throughout the lectures:

Appendix A. Linear operators, compact operators, trace-class operators

Appendix B. Fredholm operators and index computation

Appendix C. Pseudo-differential Calculus and Helffer Sjöstrand formula

Appendix D. Semiclassical Calculus

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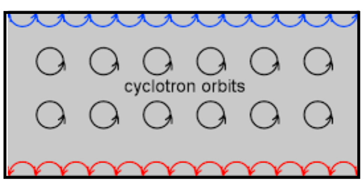
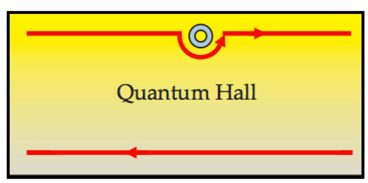
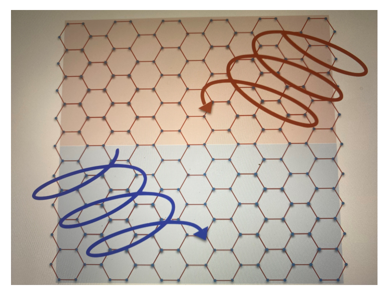
The topics covered in these notes are presented with varying levels of details and mathematical rigor. Lecture 1 introduces several models of topological insulators considered in later lectures and briefly describes the main topic of interest: asymmetric transport. Lecture 2 focuses on the derivation of macroscopic partial differential models from more microscopic descriptions. Lectures 3 to 5 analyze bulk invariants for magnetic Schrödinger and Dirac models of topological insulators. The core of these notes is the material in Lectures 6 to 10. Asymmetric transport is first considered in a one-dimensional setting. Two-dimensional Hamiltonians are then modeled by general classes of pseudo-differential operators, are classified by domain wall extensions, and are assigned several equivalent topological invariants in the form of an edge conductivity and indices of Fredholm operators, all computed by an explicit Fedosov-Hörmander formula, a form of bulk-edge correspondence. Lectures 11 and 12 describe the notion of bulk-different invariants and survey the definition and computation of several invariants including degrees of maps, winding numbers and Chern numbers. Lecture 13 presents a reformulation of interface transport problems as an integral equation. This allows us to perform accurate numerical simulations of interface transport and verify the robustness of the topological invariants. Lecture 14 applies the theory developed in these lectures to the analysis of gated twisted bilayer graphene.

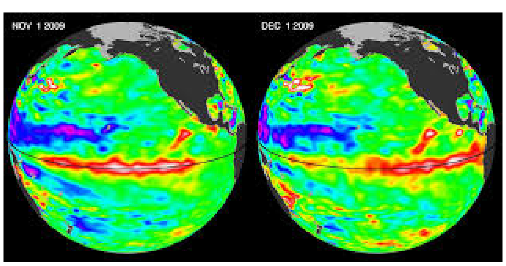
1 Lecture 1.

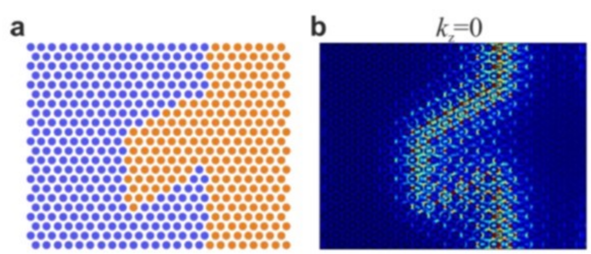
Preliminaries on quantized asymmetric transport. We start with four typical examples of topological insulators displaying asymmetric transport at an interface separating two insulators.











The top-left corner sketches the orbits of a gas of two-dimensional electrons in the presence of a strong magnetic fields. Away from interfaces, these electrons do not undergo any long-distance transport and the material is insulating. At the edge of the domain, which we interpret as the interface between the magnetic phase (in grey) and the vacuum phase (in white all around), however, transport is possible and in a somewhat caricatural semiclassical picture, it is asymmetric (for instance, from right to left at the picture's top). Moreover, in the presence of defect (top central picture), the transport from left to right is barely affected. Experiments displayed this behavior in the early 1980s and this edge behavior was rapidly associated with topological invariants characterizing its quantization [10, 15, 23].

Similar behaviors were then rapidly observed in other areas of science. The top-right corner of the above figure represents a sheet of graphene with electromagnetic forcing that is supposed to generate insulators (unperturbed graphene is a semi-metal) in different (red and blue) topological phases so that asymmetric transport may again be observed at the interface.

The figure at the bottom right corner, taken from [42], displays two photonic crystals engineered in different topological phases (blue and orange), with again asymmetric transport observed along the edge. Note that the transport (from top to bottom) remains strong even in the presence of a very jagged interface. (Hardly any transmission would be observed if signals were injected at the bottom of the sample because of scattering effects.)

Finally, the bottom left picture displays the eastward transport of atmospheric modes (in red), the so-called Kelvin and Yanai modes, along the equator over the pacific ocean. While these modes have been known for quite some time, that their protected transport afforded a topological interpretation was observed only recently [17]. The Coriolis force has opposite signs in the northern and southern hemispheres, where it acts as an energy barrier. This generates asymmetric transport in the vicinity of the equator where the force vanishes.

Our main objective in this course is a mathematical description of the above transport phenomena. In particular, we focus on: asymmetric transport, stability against perturbations, topological invariants, relation of asymmetric transport to bulk properties; computations of indices; quantitative theoretical and computational descriptions of asymmetric transport. We will mostly consider equations in two space dimensions in Euclidean space: no *complicated* geometries, no *somewhat arbitrary* boundary conditions.

The two main mechanisms to generate insulators with non-trivial topologies are: (i) magnetic fields; and (ii) massive terms. The topological invariants characterizing these topologies typically take the form of non-trivial indices of Fredholm operators. The mechanism we consider in this course to generate asymmetric transport is a *domain wall* modeling the transition between two insulators.

Bulk Insulators. Two typical mechanisms creating energy barriers in two-dimensional materials and leading to insulators are:

(i): *Magnetic fields* such as in the following *Magnetic Schrödinger* equation

$$i\partial_t\psi = H\psi, \quad H = (D+A)^2 + V, \quad D = (D_x, D_y) = (\frac{1}{i}\partial_x, \frac{1}{i}\partial y), \quad A = (A_x, A_y) \tag{1.1}$$

where $\nabla \times A = \partial_x A_y - \partial_y A_x = B$ is magnetic field and V is electric potential. (We assume $q = -e = -1$ and $\frac{1}{2m} = 1$ and use $(D+A)^2 = (D+A)\cdot(D+A)$.) In the Landau gauge, $(A_x = 0, A_y = Bx)$ generates a constant magnetic field B. Landau levels for magnetic Schrödinger equations are analyzed in detail in a later lecture. The Laudau levels are discrete. For energies between these Landau levels, the system is gapped, i.e., insulating.

(ii): *Mass terms* as in the following *Dirac operator* for a two-band model

$$H = D_x\sigma_1 + D_y\sigma_2 + m\sigma_3 \tag{1.2}$$

where $m(x, y)$ is the mass term and the Pauli matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.3}$$

We will also consider more general two-band models

$$H = h(x, D) \cdot \sigma, \qquad h = (h_1, h_2, h_3),$$

where $h_j(x, D)$ are (scalar) differential or pseudo-differential operators. The operators act on two-vector valued functions where (heuristically and possibly after a unitary transform), the first component models the upper energy band while the second component describes the lower energy band. Here and below, $h \cdot \sigma$ means $h_1\sigma_1 + h_2\sigma_2 + h_3\sigma_3$ when $h = (h_1, h_2, h_3)$ is a three-vector while it means $h_1\sigma_1 + h_2\sigma_2$ when $h = (h_1, h_2)$ is a two-vector.

We start with the simple Dirac model (1.2) written as

$$H = D \cdot \sigma + m\sigma_3, \qquad D = (D_x, D_y)$$

with m a constant mass term. We observe (Exercise: check) that

$$H^2 = (D^2 + m^2)I_2 = (-\Delta + m^2)I_2.$$

This is an operator with purely absolutely continuous spectrum $[m^2, \infty)$. The spectrum of H is also absolutely continuous and given by $(-\infty, -|m|] \cup [|m|, \infty)$ (Exercise or see [44]). In other words, the mass term m generates a *spectral gap*. No excitation with energy $|E| < |m|$ is allowed to propagate in the bulk.

The operator may be diagonalized in the Fourier domain $H = \mathcal{F}^{-1}\hat{H}(\xi, \zeta)\mathcal{F}$ with

$$\hat{H} = \xi\sigma_1 + \zeta\sigma_2 + m\sigma_3.$$

This 2×2 matrix has two eigenvalues given by

$$E_\pm(\xi, \zeta) = \pm\sqrt{\xi^2 + \zeta^2 + m^2}. \tag{1.4}$$

When $m = 0$, we observe that the two sheets touch at $(\xi, \zeta) = 0$ and form a *Dirac cone*. The operator forms a reasonable model of transport in two-dimensional materials composed of a single sheet of graphene.

Remark 1.1 *Both models break* time-reversal *symmetry.*

For the Schrödinger equation, time reversal symmetry is observed when $H(D) = \mathcal{K}H(D)\mathcal{K}$, where $\mathcal{K}$ is the anti-linear operator of complex conjugation. In the presence of a non-trivial magnetic potential, time-reversal symmetry is not preserved. While 'position' remains invariant by TR, the momentum operator $p \equiv D$ is mapped to $-p = -D = \mathcal{K}D\mathcal{K}$ by time reversion. Thus $\mathcal{K}(D+A)^2\mathcal{K} = (-D + A)^2 = (D - A)^2$ is time-reversal symmetric when A vanishes.

For the Dirac model (with spinful Fermions in which spin is also flipped by time reversion), the Fermionic time reversal (FTR) symmetry holds when $H(D) = \mathcal{T}H(D)\mathcal{T}^{-1}$ where $\mathcal{T}$ is an anti-linear map (i.e., $\mathcal{T}(\alpha\psi) = \bar{\alpha}\mathcal{T}\psi$) such that $\mathcal{T}^2 = -1$. For instance, choosing $\mathcal{T} = i\sigma_2\mathcal{K}$ with $\mathcal{T}^{-1} = -\mathcal{T}$, we verify that for $H = D \cdot \sigma + m\sigma_3$, then $\mathcal{T}^*H\mathcal{T} = D \cdot \sigma - m\sigma_3$ so that H is FTR-symmetric if and only if $m = 0$. (Exercise: check)

The topological indices we consider in this course all vanish for time-reversal symmetric operators in two space dimensions. For Fermionic Time Reversal symmetric operators (only in electronic settings), a $\mathbb{Z}_2$-invariant may be defined but this is not covered in the current set of lecture notes.

As most common materials do not naturally break TR symmetry, at least in the absence of magnetic fields, topological insulators typically need to be engineered and this is no simple task especially if one wants the spectral gap ($2|m|$ above) to be large.

We now move to a description of interface transport at the transition zone separating insulators. The upshot is: Asymmetric transport is *necessary* and in fact *quantized* when the two insulators are in *different topological classes*; this is a physical general principle called a *bulk-edge* correspondence.

Domain Walls. The simplest example of transition between insulators is the model of a Dirac equation with spatially varying mass term. Assume $m = m(y)$ a (reasonable) function bounded away from 0 outside of a compact domain (say, including $y = 0$) and consider

$$H = D \cdot \sigma + m(y)\sigma_3.$$

An equivalent formulation of H (in a different basis) is more convenient for analysis. Define the isometry (on $\mathbb{C}^2$)

$$Q = Q^{-1} = \frac{1}{\sqrt{2}}(\sigma_1 + \sigma_3), \qquad Q(\sigma_1, \sigma_2, \sigma_3)Q^{-1} = (\sigma_3, -\sigma_2, \sigma_1).$$

We still call H the operator QHQ^{-1} given by

$$H = D_x\sigma_3 - D_y\sigma_2 + m(y)\sigma_1 = \begin{pmatrix} D_x & \mathfrak{a} \\ \mathfrak{a}^* & -D_x \end{pmatrix}, \qquad \mathfrak{a} = \partial_y + m(y), \quad \mathfrak{a}^* = -\partial_y + m(y)$$

with $\mathfrak{a}^*$ a formal adjoint to $\mathfrak{a}$. These may be interpreted as the creation/annihilation operators of the quantum harmonic oscillator when $m(y) = y$. Since H is invariant by translation in the x variable, we consider the partial Fourier transform $\hat{H} = \mathcal{F}_{x\to\xi} H \mathcal{F}^{-1}_{\xi\to x}$ with

$$\hat{H} = \hat{H}(\xi) = \begin{pmatrix} \xi & \mathfrak{a} \\ \mathfrak{a}^* & -\xi \end{pmatrix}, \qquad \hat{H}^2(\xi) = \begin{pmatrix} \xi^2 + \mathfrak{a}\mathfrak{a}^* & 0 \\ 0 & \xi^2 + \mathfrak{a}^*\mathfrak{a} \end{pmatrix}.$$

A more explicit diagonalization (spectral decomposition) of $\hat{H}(\xi)$ for each $\xi \in \mathbb{R}$ will be considered in detail later. (Spectrum is exclusively point spectrum when $|m(y)| \to \infty$ as $|y| \to \infty$ and includes continuous spectrum when $|m(y)|$ is bounded.)

Here, we simply identify some of the *point spectrum* by looking for non-trivial kernels of $\hat{H} - E$ for $|E|$ small, which implies non-trivial kernels of $\hat{H}^2 - E^2$. This involves the spectral decomposition of the self-adjoint operators $\mathfrak{a}^*\mathfrak{a}$ and $\mathfrak{a}\mathfrak{a}^*$. It is known that they share the same non-vanishing point spectrum.

Typical example of an edge mode. Looking at the bottom spectrum means here looking at the kernel of $\mathfrak{a}^*\mathfrak{a}$, which means looking at the kernel of $\mathfrak{a}$ (warning, the spectrum of the non-self-adjoint operator $\mathfrak{a}$ (as an unbounded operator on $L^2(\mathbb{R})$ with a appropriate natural domain) is the whole complex plane when $m(y) = y$ while that of $\mathfrak{a}^*$ is empty). At any rate, we easily find that

$$L^2(\mathbb{R}) \ni e^{-M(y)}, \qquad M(y) = \int_0^y m(z)dz$$

is in the $L^2(\mathbb{R})$-kernel of $\mathfrak{a}$ provided that $m(y) \geq m_0 > 0$ for $y > y_0$ and $m(y) \leq -m_0 < 0$ for $y > -y_0$ for some $y_0 \in \mathbb{R}$. Such calculations are best carried out by observing the conjugation

$$\partial_y + m(y) = e^{-M(y)} \partial_y e^{M(y)}.$$

We verify that the kernel of $\mathfrak{a}^*$ is trivial in such circumstances. We thus obtain a non-trivial eigenstate of $\hat{H}(\xi)$ with

$$(\hat{H}(\xi) + \xi) e^{-M(y)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0, \qquad E(\xi) = -\xi.$$

When $m(y)$ is negative for large y and positive for large $-y$, then we find

$$(\hat{H}(\xi) - \xi) e^{M(y)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0, \qquad E(\xi) = \xi,$$

using now that the kernel of $\mathfrak{a}^*$ is non-trivial while that of $\mathfrak{a}$ is.

In the former case, we observe that $\partial_\xi E(\xi) = v_F = -1$, i.e., the group velocity of this mode is constant and negative: we have a non-dispersive (relativistic) mode propagating with Fermi velocity $v_F = -1$. In the latter case, $\partial_\xi E(\xi) = 1$ and the non-dispersive (relativistic) mode with Fermi velocity $v_F = 1$ propagates in the opposite direction.

We can (and will in some cases) verify that the rest of the point spectrum comes in flavors of

$$E_p(\xi) = \pm\sqrt{\varepsilon_n^2 + \xi^2}, \qquad p = (n, \pm)$$

where ε_n^2 is a non-vanishing eigenvalue of $\mathfrak{a}^*\mathfrak{a}$ (as well as $\mathfrak{a}\mathfrak{a}^*$). These modes are symmetrical in terms of leftward/rightward propagation and hence do not contribute to the transport asymmetry.

An index characterizing asymmetric transport. We also note that when $m(y)$ is bounded below by $m_0 > 0$ for $|y|$ large or bounded above by $-m_0 < 0$ for $|y|$ large, then the kernels of both $\mathfrak{a}$ and $\mathfrak{a}^*$ are trivial. Assume m converges to $m_\pm$ as $y \to \pm\infty$. We may thus summarize the above observations as

$$-\mathrm{Index}\,\mathfrak{a} = \frac{1}{2}(\,\mathrm{sign}(m_-) - \,\mathrm{sign}(m_+)) \in \{-1, 0, 1\}.$$

This is the simplest example of a *bulk-edge correspondence*: *asymmetric transport*, characterized here by the index of $\mathfrak{a}$ (which is a Fredholm operator in an appropriate topology and whose index is indeed given by the above formula) may be non-trivial and is related to *bulk properties* of the insulators (modeled by $m_\pm$).

Observable characterizing asymmetric transport. While the above $\mathrm{Index}\,\mathfrak{a}$ is interesting mathematically, we would like to relate it to a physical observable modeling asymmetric transport. This is the role of the following object

$$\sigma_I = \mathrm{Tr}\; i[H, P]\varphi'(H). \tag{1.5}$$

Here, H is a (self-adjoint) Hamiltonian of interest.

We will repeatedly need to construct *switch functions*. We introduce the notation

$$\mathfrak{S}(a, b; c, d), \qquad \mathbb{R} \ni a < b \in \mathbb{R}, \qquad \mathbb{R} \ni c \leq d \in \mathbb{R} \tag{1.6}$$

as the set of (measurable) bounded real-valued functions f on $\mathbb{R}$ such that $f(x) = a$ for $x < c$ and $f(x) = b$ for $x > d$. We denote by $\mathfrak{S}(a, b)$ the union of the above set over $-\infty < c \le d < \infty$. We will denote by *smooth switch functions* the intersection of the above sets with the space of $C_b^\infty(\mathbb{R})$ functions.

Then, $\mathbb{R} \ni t \to \varphi(t) \in [0, 1]$ is a smooth non-decreasing switch function in $\mathfrak{S}(0, 1; t_0, t_1)$. The interval $[t_0, t_1]$ is assumed to belong to *the bulk band gap*. What this means is the following. Let m_0 be the minimum of $|m_-|$ and $|m_+|$ above. No signal with energy $|E| < m_0$ can propagate in the bulk, i.e., away from the interface $y = 0$ since the material is an insulator for both y large and $-y$ large. The only possible propagation for such a signal is then along the interface $y = 0$. We therefore *assume* that the system is prepared so that all modes compatible with H are present with a density $\varphi'(H)$ supported in the interval $(-m_0, m_0)$.

The second object appearing in (1.5) is the (current) observable $i[H, P]$ which is the rate of change in time of the observable P. The latter is multiplication by a function $P(x, y) = P(x)$, which for concreteness, we may take as a smooth switch function in $\mathfrak{S}(0, 1; x_0 - \eta, x_0 + \eta)$ for some $0 < \eta \ll 1$. In other words, multiplication by P projects onto the part of the spatial domain on the *right* of the hyperplane (line) $x = x_0$. Then, $i[H, P]$ is interpreted as the amount of current crossing from the left of x_0 to the right of x_0 per unit time, hence as a physically motivated measure of transport from left to right.

The interpretation as a rate of change is as follows. Let $\psi(t) = e^{-itH}\psi$ be the solution of the Schrödinger equation $i\partial_t\psi(t) = H\psi(t)$ with initial condition $\psi(0) = \psi$. Then $\langle P\rangle(t) = \langle\psi(t)|P|\psi(t)\rangle = \langle\psi|e^{itH}Pe^{-itH}|\psi\rangle$ so that

$$\frac{d}{dt}\langle P\rangle(t) = \langle\psi(t)|i[H, P]|\psi(t)\rangle = \operatorname{Tr} i[H, P]\psi(t)\psi^*(t).$$

In analogy with the (two-dimensional) Hall conductivity in electronics, which we will introduce later, σ_I is often referred to as an *interface conductivity*. We will also refer to σ_I as an *edge current* (observable).

Remark 1.2 A few remarks are in order: (0) $\varphi'(H)$ is constructed spectrally when H is self-adjoint (and thus cannot as such include dissipative effects).

(i) Importantly, $i[H, P]\varphi'(H)$ is assumed to be trace-class (the Banach space of trace-class operators is a subset of the ideal of compact operators). If $\varphi'(H)$ were to have partial support in energies that are allowed to propagate in the bulk, then $i[H, P]\varphi'(H)$ would *not* be trace-class. Showing that $i[H, P]\varphi'(H)$ is indeed trace-class is one of the challenges of the theory.

(ii) The second main objective will be to show that $2\pi\sigma_I \in \mathbb{Z}$ is *quantized*. In other words, *asymmetric transport* is quantized as expected from the bulk-edge correspondence.

(iii) A third main objective is to show that σ_I is immune to *perturbations*, i.e., $\sigma_I[H] = \sigma_I[H+V]$ for V perturbations that model defects and impurities (possibly a lot of randomness) in the material. It is this *robustness* that makes the materials interesting practically and the theoretical descriptions challenging.

(iv) Finally, we would like to devise methods that compute this interface conductivity explicitly in many cases of practical interest. In particular, we wish to relate it to the bulk properties of the two insulators the interface separates. This relation is typically referred to as a bulk-edge correspondence.

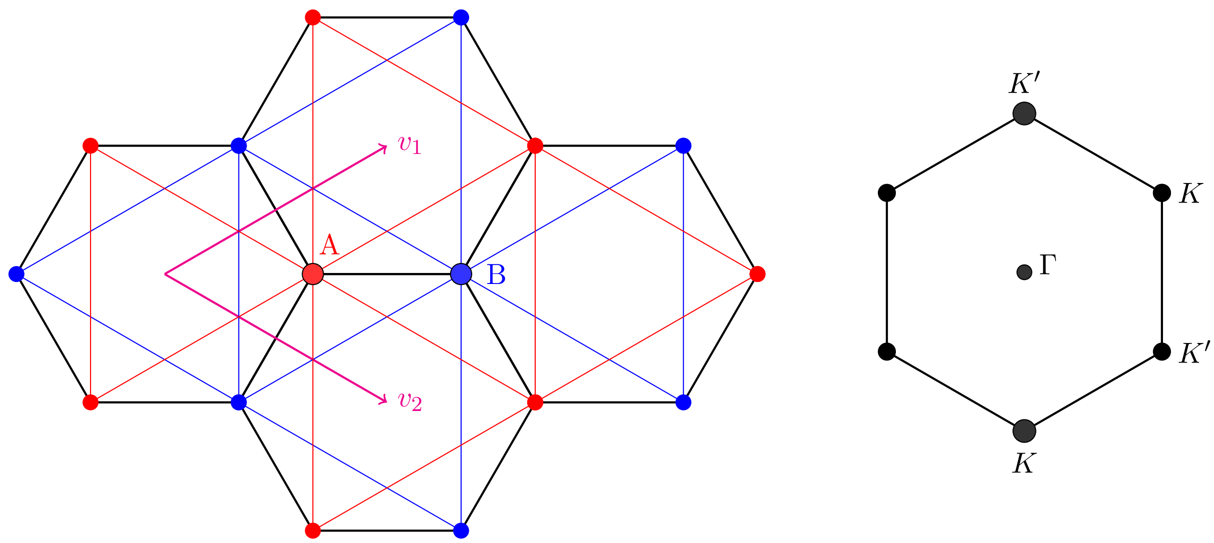


Figure 1: Geometry of graphene bipartite lattice. Right: spatial lattice. Left: dual lattice.

2 Lecture 2.

Derivation of Dirac equations from microscopic Hamiltonians. This lecture briefly presents the derivation of Dirac equations from microscopic models. We present a rigorous derivation from a tight-binding Haldane model and sketch the derivation from a periodic Schrödinger equation referring to the existing literature for details.

Dirac approximation for the Haldane model. We start with a hexagonal bipartite model with a periodic array of sites A and sites B as shown in Figure 1. With v_1 and v_2 describing the lattice as in Figure 1, we have $a_1 = \frac{1}{3}(v_1 + v_2)$ with $a_2 = R_{2\pi/3}a_1$ and $a_3 = R_{-2\pi/3}a_1$ the three vectors linking a site A to its nearest neighbors B. Here R_θ is the anti-clockwise rotation by θ. We then define $b_1 = v_1$ and $b_2 = v_2 - v_1$ with $b_3 = -(b_1 + b_2)$. Thus, the six nearest same-type neighbors are described by shifts by $\pm b_j$.

Note that both the sites A and the sites B occupy the edges of a lattice $\Lambda = \mathbb{Z}v_1 + \mathbb{Z}v_2$, whence the name of a bipartite lattice modeling the honeycomb structure.

Periodic lattice model. A tight-binding model describes interactions among the atoms located at the sites A and B. The interaction strength t_1 models interactions between nearest neighbors. This leads to a Hamiltonian in position space given by

$$H_1 = t_1\big(\sum_{j=1}^{3} \tau_{a_j}\sigma_+ + \tau_{-a_j}\sigma_-\big)$$

where $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$ and the shift operators are defined as $\tau_a f(x) = f(x+a)$.

A term t_2 next models interactions between next-nearest neighbors. This interaction also includes a local magnetic component (Aharonov-Bohm phase) $e^{\pm i\phi}$ leading to the Hamiltonian

$$H_2 = t_2 \sum_{j=1}^{3}(e^{i\phi}\tau_{b_j} + e^{-i\phi}\tau_{-b_j})\frac{I-\sigma_3}{2} + t_2 \sum_{j=1}^{3}(e^{i\phi}\tau_{-b_j} + e^{-i\phi}\tau_{b_j})\frac{I+\sigma_3}{2}.$$

We finally have a local mass term M that we assume is asymmetric at the sites A and B. The unperturbed Haldane model is given by

$$H = H_1 + H_2 + M\sigma_3. \tag{2.1}$$

The above Hamiltonian is naturally defined on (smooth vector-valued functions in) $L^2(\mathbb{R}^2;\mathbb{C}^2)$ and not only on a discrete lattice. Standard lattice models may be obtained by restricting it to distributions supported on the lattice. It is however convenient to look at the Hamiltonian as acting on functions defined on the continuum $\mathbb{R}^2$.

In the Fourier domain, $H = \mathcal{F}^{-1}(\hat{H}_1 + \hat{H}_2 + M\sigma_3)\mathcal{F}$, where

$$\begin{aligned}
\hat{H}_1 &= t_1\big(\sum_{j=1}^{3} e^{ia_j\cdot\xi}\sigma_+ + e^{-ia_j\cdot\xi}\sigma_-\big) = t_1\sum_{j=1}^{3}\big(\cos a_j\cdot\xi\sigma_1 - \sin a_j\cdot\xi\sigma_2\big)\\
\hat{H}_2 &= 2t_2\sum_{j=1}^{3}\cos(\xi\cdot b_j+\phi)\frac{1}{2}(\sigma_3+I) + 2t_2\sum_{j=1}^{3}\cos(-\xi\cdot b_j+\phi)\frac{1}{2}(I-\sigma_3)\\
&= 2t_2\cos\phi\sum_{j=1}^{3}\cos\xi\cdot b_j I + 2t_2\sin\phi\sum_{j=1}^{3}\sin\xi\cdot b_j\sigma_3.
\end{aligned}$$

With this convention, we find that

$$\hat{H} = \sum_{j=0}^{3} h_j\sigma_j$$

with $\sigma_0 = I_2$ the identity matrix and

$$\begin{aligned}
h_0 &= 2t_2\cos\phi\sum_{j=1}^{3}\cos\xi\cdot b_j, \qquad & h_3 &= M + 2t_2\sin\phi\sum_{j=1}^{3}\sin\xi\cdot b_j\\
h_1 &= t_1\sum_{j=1}^{3}\cos a_j\cdot\xi, \qquad & h_2 &= -t_1\sum_{j=1}^{3}\sin a_j\cdot\xi
\end{aligned}$$

The Hamiltonian $\hat{H}(\xi)$ is a 2×2 Hermitian matrix for each $\xi\in\mathbb{R}^2$. We easily verify that its eigenvalues are given by

$$E_\pm(\xi) = h_0 \pm \sqrt{h_1^2+h_2^2+h_3^2}. \tag{2.2}$$

When the mass terms M and t_2 vanish, a number of wavenumbers are special in that the two energy bands meet, i.e., $E_\pm(\xi) = h_0$. This occurs when $h_1(\xi) = h_2(\xi) = 0$. Such solutions come in pairs $\xi = K$ and $\xi = K' = -K$. When K is such a point, then so are its rotations by multiples of $2\pi/3$. This defines 6 solutions with minimal norm that may be seen as the vertices of a hexagon; see right of Fig.1.

More precisely, we find that $h_1 = h_2 = 0$ is equivalent to

$$\sum_{j=1}^{3} e^{ia_j\cdot K} = 0 \tag{2.3}$$

which implies that $a_1\cdot K = \varphi$ a phase and then $a_2\cdot K = \varphi + \frac{2\pi}{3}$ and $a_3\cdot K = \varphi - \frac{2\pi}{3}$. There are six solutions to the equation with two of them with K orthogonal to a_1. With $a_1 = a(1,0)^t$, we find two solutions

$$K = -\frac{1}{a}\frac{4\pi}{3\sqrt{3}}(0,1)^t, \qquad K' = \frac{1}{a}\frac{4\pi}{3\sqrt{3}}(0,1)^t = -K. \tag{2.4}$$

The other solutions are rotations by $2n\pi/3$ of K and K'. Tiling the plane with such hexagons generates an infinite family of vertices such that (2.3) holds. Half of these points are in a dual lattice to the A sites while the other half are in the dual lattice to the B sites. Points on the same dual lattice space are typically identified as the K or K' *valley* Dirac points.

Since our objective is to break the periodic lattice structure when we introduce spatial perturbations, we do not identify the aforementioned lattice points. The objective of the Dirac equation is to model wavepackets formed of wavenumbers that are close to any of the Dirac points. For concreteness, we identify the specific K and K' points given in (2.4). Similar (but different) Dirac equations can be obtained for any of the solutions of (2.3).

We are concerned with wavenumbers of the form $K' + \xi$ with $|\xi| \ll 1$. We first observe that

$$\sum_{j=1}^{3} e^{ia_j\cdot K'} a_j = \frac{3}{2}(a_1 + ia_1^{\perp}) = \frac{3}{2}a(1,i)^t.$$

Defining $h'_j(\xi)$ as $h_j(K'+\xi)$ (and with an abuse of language $h_j(\xi)$ as $h_j(K+\xi)$), we find

$$h'_1/t_1 = \sum_{j=1}^{3} \cos a_j\cdot K' \cos a_j\cdot\xi - \sin a_j\cdot K' \sin a_j\xi = -\sum_{j=1}^{3} \sin a_j\cdot K' a_j\cdot\xi + O(|\xi|^2) = -\frac{3}{2}a\xi_2 + O(|\xi|^2).$$

Similarly,

$$h'_2/t_1 = -\sum_{j=1}^{3} \sin a_j\cdot K' \cos a_j\cdot\xi + \cos a_j\cdot K' \sin a_j\cdot\xi = -\sum_{j=1}^{3} \cos a_j\cdot K' a_j\cdot\xi + O(|\xi|^2) = -\frac{3}{2}a\xi_1 + O(|\xi|^2).$$

Replacing K' by K, we obtain instead

$$h_1/t_1 = \frac{3}{2}a\xi_2 + O(|\xi|^2), \qquad h_2/t_1 = -\frac{3}{2}a\xi_1 + O(|\xi|^2).$$

We also compute

$$\sum_{j=1}^{3} \sin b_j\cdot K' = \frac{3}{2}\sqrt{3} = -\sum_{j=1}^{3} \sin b_j\cdot K.$$

As a consequence,

$$h'_3 = M + 3\sqrt{3}t_2\sin\phi + O(|\xi|), \qquad h_3 = M - 3\sqrt{3}t_2\sin\phi + O(|\xi|).$$

We thus obtain two different mass terms for the valleys K and K'. For specific choices of (M, t_2, ϕ), these two mass terms may in fact have different signs. This is one of the origins of the non-trivial topology associated with the Haldane model.

Note that the errors in the approximation of the mass terms are of order $O(|\xi|)$ while the errors on the terms h_j and h'_j are of order $O(|\xi|^2)$. The derivation of the Dirac equation will thus require us to assume that the mass terms M and t_2 are themselves small. This is one of the necessary conditions for the validity of the Dirac model. To allow for slowly varying spatial variations (at the lattice scale), we will therefore assume that M and t_2 are of the form $\delta M(\delta x)$ and $\delta t_2(\delta x)$ with $0 < \delta \ll 1$ a small parameter. In contrast we will assume that t_1 is of the form $t_1(\delta x)$.

For the rest of the section, we assume a Aharonov-Bohm phase $\phi = \frac{\pi}{2}$ to simplify calculations. Thus, $h_0 = 0$ and the resulting limiting Dirac Hamiltonians are given explicitly by

$$\hat{H}_K = t_1\frac{3}{2}a(\xi_2\sigma_1 - \xi_1\sigma_2) + (M - 3\sqrt{3}t_2)\sigma_3, \quad \hat{H}_{K'} = -t_1\frac{3}{2}a(\xi_2\sigma_1 + \xi_1\sigma_2) + (M + 3\sqrt{3}t_2)\sigma_3.$$

A slowly varying spatial model. The Hamiltonian H is written as a pseudo-differential operator, which is diagonalized in the Fourier domain since its coefficients are invariant with respect to spatial translations. It is then straightforward to include slowly varying spatial variations by defining the Hamiltonian as a pseudo-differential operator in the Weyl calculus $H_\delta = \mathrm{Op}^w a_\delta$ with symbol

$$a_\delta(x,\xi) = a(\delta x,\xi) = t_1(\delta x)\sum_{j=1}^{3}(\cos(a_j\cdot\xi)\sigma_1 - \sin(a_j\cdot\xi)\sigma_2) + \delta\big(M(\delta x) + t_2(\delta x)\sum_{j=1}^{3}\sin(b_j\cdot\xi)\big)\sigma_3.$$

This means that for a sufficiently smooth function $\psi(x)$, we have by definition:

$$H_\delta\psi = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} e^{i(x-y)\cdot\xi} a_\delta(\frac{x+y}{2},\xi)\psi(y)dyd\xi.$$

See the appendices for notation and results on pseudo-differential operators. Note that when $a = a(\xi)$ is independent of x, then $H_\delta = H$ is the unperturbed Haldane model in (2.1).

Here $0 < \delta \ll 1$ models a separation of scales between the scale $O(1)$ of the lattice and the scale $O(\delta^{-1})$ of the macroscopic approximation. Our objective is to solve the following evolution problem over times of order δ^{-1}:

$$i\partial_t\varphi_\delta = H_\delta\varphi_\delta, \qquad \varphi_\delta(t,x) \approx \psi_\delta(t,x) = e^{iK\cdot x}\delta\phi(\delta t,\delta x). \tag{2.5}$$

The wavepackets of interest are those with wavenumbers close to the Dirac point K, which is consistent with the above computations. The *envelope* $\phi(\delta t,\delta x)$ models the macroscopic behavior for large distances $x = \delta^{-1}X$ and long times $t = \delta^{-1}T$ with (T,X) both of order $O(1)$. Our objective is to show that when $\phi(T,X)$ solves a Dirac equation, then $\psi_\delta(t,x)$ approximately solves the microscopic dynamics (2.5) for times of order $t = \delta^{-1}T$ with $T = O(1)$. We consider the normalization $\delta\phi(\delta t,\delta x)$ so that all functions are of L^2-norm of order $O(1)$.

We observe that

$$\begin{aligned}
H_\delta\psi_\delta(t,x) &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} e^{i(x-y)\cdot\xi} a(\delta\frac{x+y}{2},\xi)e^{iK\cdot y}\delta\phi(\delta t,\delta y)dyd\xi \\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} e^{i(x-y)\cdot(K+\delta\zeta)} a(\delta\frac{x+y}{2},K+\delta\zeta)e^{iK\cdot y}\delta\phi(\delta t,\delta y)d\delta y d\zeta \\
&= \frac{e^{iK\cdot x}}{(2\pi)^2}\int_{\mathbb{R}^4} e^{i(\delta x-\delta y)\cdot\zeta} a(\frac{\delta x+\delta y}{2},K+\delta\zeta)\delta\phi(\delta t,\delta y)d(\delta y)d\zeta.
\end{aligned}$$

The above operator is thus naturally written in the phase-space variables $(\delta x,\zeta)$ with a natural expansion in powers of δ in the term $K+\delta\zeta$. To formalize this representation, we introduce

$\xi = K + \delta\zeta$ and recast the operator $H_\delta = \mathrm{Op}^w a_\delta$ using

$$
\begin{aligned}
a_\delta(x,\xi) &= a(\delta x, \xi) \;=\; b_\delta(\delta x, \delta^{-1}(\xi - K)) \\
b_\delta(X,\zeta) &= t_1(X)\sum_{j=1}^{3}(\cos(a_j\cdot(K+\delta\zeta))\sigma_1 - \sin(a_j\cdot(K+\delta\zeta))\sigma_2) \\
&+ \delta\Big(M(X) + t_2(X)\sum_{j=1}^{3}\sin(b_j\cdot(K+\delta\zeta))\Big)\sigma_3 =: \delta(b_0 + \delta b_{1\delta}) \\
b_0(X,\zeta) &= t_1(X)\sum_{j=1}^{3}\Big(\sin(a_j\cdot K)a_j\cdot\zeta\sigma_1 - \cos(a_j\cdot K)a_j\cdot\zeta\sigma_2\Big) \\
&+ \Big(M(X) + t_2(X)\sum_{j=1}^{3}\sin(b_j\cdot K)\Big)\sigma_3 \\
&= t_1(X)\tfrac{3}{2}a(\zeta_2\sigma_1 - \zeta_1\sigma_2) + (M(X) - 3\sqrt{3}t_2(X))\sigma_3 \\
b_{1\delta}(X,\zeta) &= \delta^{-2}(a_\delta(\delta^{-1}X, K+\delta\zeta) - \delta b_0(X,\zeta)) \in S^2
\end{aligned}
$$

with a bound for $b_{1\delta}(X,\zeta)$ in the symbol class S^2 independent of δ; see (C.4) in Appendix C for notation. This bound is clear for $\delta|\zeta| \lesssim 1$ small by Taylor expansion. For $\delta|\zeta| \gtrsim 1$, we have $\delta^{-2} \lesssim |\zeta|^2$ and the bound holds as well.

Dirac approximation. From the above calculation we get

$$
\begin{aligned}
H_\delta\psi_\delta(t,x) &= e^{iK\cdot x}(\mathrm{Op}^w b_\delta[\delta\phi(\delta t,\cdot)])(t,\delta x) \\
&= e^{iK\cdot x}\delta(\mathrm{Op}^w b_0[\delta\phi(\delta t,\cdot)])(t,\delta x) + e^{iK\cdot x}\delta^2(\mathrm{Op}^w b_{1\delta}[\delta\phi(\delta t,\cdot)])(t,\delta x)
\end{aligned}
$$

so that thanks to the macroscopic Dirac equation

$$(D_t + \delta\mathrm{Op}^w b_0)[\delta\phi(\delta t,\cdot)])(t,\delta x) = 0,$$

we have

$$(D_t + H_\delta)\psi_\delta(t,x) = e^{iK\cdot x}\delta^2(\mathrm{Op}^w b_{1\delta}[\delta\phi(\delta t,\cdot)])(t,\delta x). \tag{2.6}$$

The macroscopic equation may equivalently be written as the Dirac equation

$$(D_T + \mathrm{Op}^w b_0)\phi(T,X) = 0.$$

Here we use $T = \delta t$ and $X = \delta x$, which both all $O(1)$ while t and x are $O(\delta^{-1})$.

More precisely, this is, using the anti-commutator notation $\{F,G\} = FG + GF$,

$$\Big(D_T + \frac{3a}{2}\frac{1}{2}\{t_1(X), D_2\sigma_1 - D_1\sigma_2\} + (M(X) - 3\sqrt{3}t_2(X))\sigma_3\Big)\phi(T,X) = 0. \tag{2.7}$$

The use of the anticommutator is necessary to ensure that the Dirac operator is (Hermitian) symmetric (and in fact self-adjoint as an unbounded operator with domain $H^1(\mathbb{R}^2;\mathbb{C}^2)$).

We assume that $t_1(X)$ is a smooth function bounded above and below by positive constants. We also assume that $t_2(X)$ and $M(x)$ are smooth, bounded, functions. We then have the following result:

Theorem 2.1 (Dirac approximation) *Let $\phi_0(X) \in H^2(\mathbb{R}^2;\mathbb{C}^2)$ and $\phi(T,X)$ solution of (2.7) with initial condition $\phi(0,X) = \phi_0(X)$. Let $\varphi_\delta(t,x)$ be the solution of the microscopic equation*

$$(D_t + H_\delta)\varphi_\delta = 0 \tag{2.8}$$

with initial condition $\varphi_\delta(0,x) = e^{iK\cdot x}\delta\phi_0(\delta x)$.

Then for $\tau > 0$, we have uniformly in $0 \le T = \delta t \le \tau$,

$$\Big\|\delta e^{iK\cdot x}\phi(T,\delta x) - \varphi_\delta(\frac{1}{\delta}T, x)\Big\|_{L^2(\mathbb{R}^2_x;\mathbb{C}^2)} = \Big\|e^{i\frac{1}{\delta}K\cdot X}\phi(T,X) - \frac{1}{\delta}\varphi_\delta(\frac{1}{\delta}T,\frac{1}{\delta}X)\Big\|_{L^2(\mathbb{R}^2_X;\mathbb{C}^2)} \le C_\tau\delta. \tag{2.9}$$

Proof. With the above smoothness assumptions on the coefficients (t_1,t_2,M), the solution of the Dirac equation is in $H^2(\mathbb{R}^2;\mathbb{C}^2)$ uniformly in $T \in [0,\tau]$ for sufficiently smooth initial conditions by standard regularity results of Dirac systems of equations. As a result $\mathrm{Op}^w b_{1\delta}\phi(T,\cdot)$ is uniformly bounded in $L^2(\mathbb{R}^2;\mathbb{C}^2)$ for $T \in [0,\tau]$ since $b_{1\delta} \in S^2_{1,0}$. Using (2.6) and (2.8), we obtain by unitarity of the solution operator of the Dirac equation that $(\varphi_\delta - \psi_\delta)(t,\cdot)$ is of order $t\delta^2$ in the $L^2(\mathbb{R}^2;\mathbb{C}^2)$ sense. This gives the result for $t = \delta^{-1}T$ with $T \le \tau$. $\square$

Remark 2.2 (Dirac points and valleys.) *A similar result holds for any valley point K or K' (and their rotations and translates). Note that the constant in front of the term $t_2(X)$ depends on the valley K. The exact same derivation shows that if H_δ is replaced by $H_\delta + \delta V(\delta x)$ for $V(X)$ a smooth function valued in 2×2 Hermitian matrices, then the limiting Dirac operator $\mathrm{Op}^w b_0$ is replaced by $\mathrm{Op}^w b_0 + V(X)$.*

If wavepackets are constructed in the vicinity of N different Dirac points (typically with N an even number), then we obtain a block-diagonal system of size $2N\times 2N$. We mention again that for operators that are periodic on Γ, then only two Dirac points K and K' are possibly different. However, once the periodicity assumption is released, which is the case in the presence of perturbations of δx, then the different Dirac points can no longer be identified. The quasi-momenta of the periodic setting become real momenta of the form $K+q$ in the Dirac model for K any Dirac point and asymptotically $q \in \mathbb{R}^2$.

Band theory for periodic Schrödinger equation. We now move to a brief presentation without details of the derivation of the Dirac equation from a periodic Schrödinger model following works in [21, 22]. We primarily follow [19] and refer to these works for detail.

Start from the scalar Schrödinger equation

$$P_0 = -\Delta + V$$

on $L^2(\mathbb{R}^2;\mathbb{C})$ with V a honeycomb potential, i.e., a real-valued scalar function that is periodic w.r.t. $\Lambda = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2$, even in the sense that $V(-x) = V(x)$, and $V(Rx) = V(x)$ for R rotation by $2\pi/3$. The lattice is constructed as in Figure 1 with $v_1 = a(\sqrt{3},1)^t$ and $v_2 = a(\sqrt{3},-1)^t$, with $a > 0$ chosen for instance to ensure the normalization $\det(v_1,v_2) = 1$. The dual lattice is $\Lambda^* = \mathbb{Z}k_1 + \mathbb{Z}k_2$ with $k_i \cdot v_j = \delta_{ij}$. The corresponding fundamental cells are $\mathbb{L} = \{s\cdot v;\ s \in [0,1)^2\}$ and $\mathbb{L}^* = \{\tau\cdot k,\ \tau \in [0,2\pi)^2\}$.

The A and B lattice points are not distinguished yet. Floquet-Bloch theory [RS78, §XIII] then states the following. Decompose a dual variable into $\xi \to 2\pi w^* + \xi$ with $\xi \in \mathbb{L}^*$ and $w^* \in \Lambda^*$. We then define

$$L^2_\xi = \{u \in L^2_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{C});\ u(x+w) = e^{i\xi\cdot w}u(x),\ w \in \Lambda\}, \quad \xi \in \mathbb{L}^*$$

with inner product $(f,g)_\xi = \int_{\mathbb{L}} \bar{f} g dx$ on a compact fundamental cell. These are functions twisted by $\xi \in \mathbb{L}^*$ at the (fundamental) cell's boundary; they are periodic for $\xi = 0$. More generally, we may replace ξ by $\xi + 2\pi w^*$ for any $w^* \in \Lambda^*$ so L^2_ξ becomes periodic in $\xi \in \mathbb{R}^2$.

The main advantage of the construction is that we may advantageously decompose L^2 as the direct sum of L^2_ξ for $\xi \in \mathbb{L}^*$ (by Fourier transform) and that, moreover, when V is periodic, we observe that P_0 leaves (smooth functions in) L^2_ξ invariant. Furthermore, since $\mathbb{L}$ and $\mathbb{L}^*$ are compact domains in $\mathbb{R}^2$, we obtain that P_0 has compact resolvent (i.e., $(P_0-\lambda)^{-1}$ for $\lambda \in \mathbb{C}\backslash\mathbb{R}$ is compact) so that we can introduce the spectrum

$$\lambda_1(\xi) \le \lambda_2(\xi) \le \ldots \le \lambda_j(\xi) \le \ldots.$$

The eigenvalues are parametrized by $\xi \in \mathbb{T} = \mathbb{L}^*$ a torus and are Λ^*-periodic. The graphs $\xi \to \lambda_j(\xi)$ are called the dispersion surfaces.

The following result is obtained in [22]. A pair $(\xi_*, E_*) \in \mathbb{R}^2 \times \mathbb{R}$ is a **Dirac point** of $P_0 = -\Delta + V$ when: E_* is an eigenvalue of $P_0(\xi_*)$ of multiplicity 2 with orthonormal eigenvectors $\phi_{1,2}$ such that $\phi_1(Rx) = e^{i2\pi/3}\phi_1(x)$ and $\phi_2(x) = \overline{\phi_1(-x)}$ while moreover, for some $j_* \ge 1$ and $\nu_F > 0$, we have

$$\lambda_{j_*} = E_* - \nu_F|\xi - \xi_*| + O(\xi - \xi_*)^2, \quad \lambda_{j_*+1} = E_* + \nu_F|\xi - \xi_*| + O(\xi - \xi_*)^2.$$

This means that two bands labeled by j and $j+1$ touch at the Dirac point (ξ_*, E_*) and that the union of these bands looks like a cone with rotational symmetry and slope the Fermi velocity ν_F in the vicinity of the Dirac point. The Fermi velocity is given by $\nu_F = 2|(\phi_1, \theta \cdot D\phi_2)|$ independent of $\theta \in \mathbb{S}^1$ (thanks to the symmetries of V) where $D = -i\nabla$ in two space dimensions.

The result in [22] states that Dirac points exist *generically* for $P_0 = -\Delta + V$. Moreover, by symmetry such Dirac points come (at least) in pairs

$$\xi_* \in \{\xi_*^A,\ \xi_*^B\} \bmod 2\pi\Lambda^*, \qquad \xi_*^{\rm A} = \frac{2\pi}{3}(2{\rm k}_1 + {\rm k}_2), \qquad \xi_*^{\rm B} = \frac{2\pi}{3}({\rm k}_1 + 2{\rm k}_2).$$

Such points are moreover stable against perturbations of the Hamiltonian that preserve the spatial inversion (parity) and time-reversal symmetry (conjugation).

The next level in the theory is to break the spatial inversion (parity) but not the conjugation invariance and introduce

$$P_\delta = P_0 + \delta W$$

with $W \in C^\infty(\mathbb{R}^2, \mathbb{R})$ with $W(x+w) = W(x)$ for $w \in \Lambda$ but now $W(-x) = -W(x)$. Assume moreover that

$$\theta_F = |(\phi_1, W\phi_1)_{\xi_*}| > 0.$$

[22] shows that if (ξ_*, E_*) is a Dirac point, then a gap opens when $\delta \ne 0$. More precisely, we obtain that

$${\rm dist}(\sigma(P_0(\xi_*)), E_*) = \theta_F \delta + O(\delta^2).$$

We can then add a domain wall $\mathcal{P}_\delta = -\Delta + V + \delta\kappa_\delta W$ with $\kappa_\delta(x) = \kappa(\delta x \cdot k')$ for k' a given direction such that $k' \cdot v = 0$ so that the new potential $V + \delta\kappa_\delta W$ remains periodic with respect to $\mathbb{Z}v$. We assume that $\kappa(t) = {\rm sign}(t)$ for $|t|$ large (domain wall). Also, $v = a_1 v_1 + a_2 v_2$ (for $a_i \in \mathbb{Z}$ relative prime numbers) is the direction of the edge $\mathbb{R}v$. For b_1, b_2 in $\mathbb{Z}$ such that $a_1 b_2 - a_2 b_1 = 1$, we introduce $v' = b_1 v_1 + b_2 v_2$, $k = b_2 k_1 - b_1 k_2$ and $k' = -a_2 k_1 + a_1 k_2$.

This type of construction was first derived in [21]. We follow the presentation in [19]. Define $\zeta_* = \xi_* \cdot v$ the wavenumber along the interface where $\kappa \approx 0$. Then [19] shows that the eigenvalues of $\mathcal{P}_\delta[\zeta_* + \mu\delta]$ near E_* are δ^2 away from those of the Dirac operator

$$H[\mu] = \begin{pmatrix} 0 & \nu_* k' \\ \overline{\nu_* k'} & 0 \end{pmatrix} D_t + \begin{pmatrix} 0 & \nu_* l \\ \overline{\nu_* l} & 0 \end{pmatrix} \mu + \theta_* \kappa(t) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Here, $l = k - (k \cdot k')k'/|k'|^2$ and $\nu_* \in \mathbb{C}\backslash 0$ is such that $\nu_*(\eta_1 + i\eta_2) = 2(\phi_1, (\eta \cdot D)\phi_2)$ (such that the Fermi velocity $\nu_F = |\nu_*|$). We also identify $(l_1, l_2) \equiv l_1 + il_2$ (same for k'). Also, $\theta_* = (\phi_1, W\phi_1)$ is such that $|\theta_*| = \theta_F \neq 0$ by assumption.

A formal multiscale expansion in [21] (see, e.g., [19, §3.1]) roughly goes as follows. The microscopic material remains periodic in the direction v. We may therefore apply a Bloch transform with ζ a one dimensional variable playing a similar role to the two dimensional variable ξ above. Assume that u_δ is an eigenvector of $\mathcal{P}_\delta = -\Delta + V + \delta\kappa_\delta W$ satisfying $u_\delta(x + v) = e^{i\zeta} u_\delta(x)$ where $\zeta = \zeta_* + \mu\delta$. In other words, μ is a rescaled 'momentum' (dual variable) in the direction of the edge. Consider the Ansatz

$$u_\delta(x) = U_\delta(x, \delta k' \cdot x), \quad U_\delta(x, t) = e^{i\mu\delta l \cdot x}\Big(\sum_{j=1}^{2} \alpha_j(t)\phi_j(x) + \delta V_\delta(x, t)\Big) + O(\delta^2)$$

as well as $E_\delta = E_* + \delta\theta + O(\delta^2)$. Plugging the Ansatz into $\mathcal{P}_\delta$, equating like powers of δ, and solving each equation in turn (a classical multi-scale method), we obtain [21, 19] that a necessary condition is

$$(H[\mu] - \theta) \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = 0.$$

The above expressions relate the eigenvectors u_δ and $(\alpha_1, \alpha_2)^t$ as well as the eigenvalues $E_* + \delta\theta$ and θ of the microscopic, and macroscopic models, respectively.

If we Fourier transform back μ to a spatial variable s (orthogonal to t), then we find that the operator takes the form of the two-dimensional Dirac operator (with mass term playing the role of a domain wall):

$$H[\mu] = \nu_1 D_t \sigma_1 + \nu_2 D_s \sigma_2 + \nu_3 \kappa(t) \sigma_3$$

where the matrices $\sigma_{1,2,3}$ (are rotations of the standard Pauli matrices and) satisfy the relation

$$\{\sigma_i, \sigma_j\} := \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij} I_2.$$

Such models generalize those seen in Lecture 1 and describe in particular the edge modes localized in the t variable propagating along the edge in the s variable. Systems of Dirac equations are arguably the simplest model to describe asymmetric transport.

Partial Differential Models. The above derivations justify the use of Dirac operators as macroscopic descriptions of transport in topological insulators. We stress again that the macroscopic models apply to describe the propagation of wavenumbers close to one of the K or K' points; such wavenumbers correspond to Fermi energies close to the energy of the Dirac points. While K and K' points are globally described as elements of a Brillouin zone in the microscopic model, the macroscopic descriptions apply to each Dirac point and are independent from one another. In other words, if excitations are present in the vicinity of N such Dirac points, we obtain in the limit a system of N uncoupled Dirac operators. In the presence of non-periodic perturbations, these Dirac operators are all different, in contrast to the microscopic description where two Dirac point

K describe the same physics because of the periodicity constraint. The wavenumber ξ appearing in the microscopic description are quasi-momenta in the sense that microscopic solutions are periodic in ξ. In contrast, in the macroscopic model where periodicity is broken by the perturbations, the wavenumber ξ may be interpreted as a bona fide momentum.

For lattice models that break the periodicity assumption, a necessary step to account for any form of perturbation, and their detailed (topological) analysis, we refer the reader to the monograph [36].

Many other PDE macroscopic models may be obtained, for instance Bogoliubov de-Genes Hamiltonians in the theory of superconductors or systems of water wave equations to model the equatorial asymmetric transport mentioned in the introductory lecture. The lectures focus on such continuum partial differential (or more generally pseudo-differential) systems of equations.

The main examples of Hamiltonians coming from different applications that we want to consider are as follows. The Dirac model we already encountered:

$$H_D = D_x\sigma_1 + D_y\sigma_2 + m(y)\sigma_3 = \begin{pmatrix} m(y) & D_x - iD_y \\ D_x + iD_y & -m(y) \end{pmatrix}$$

The BdG models for p and d waves:

$$H_p = \Big(\frac{1}{2m}(D_x^2 + D_y^2) - \mu(y)\Big)\sigma_1 + \tfrac{1}{2}\{c(y), D_y\}\sigma_2 + c_0 D_x\sigma_3$$

$$H_d = (\frac{1}{2m}\Big(D_x^2 + D_y^2) - \mu(y)\Big)\sigma_1 + c_0(D_y^2 - D_x^2)\sigma_2 + \tfrac{1}{2}D_x\{c(y), D_y\}\sigma_3$$

Some Hamiltonians appearing in Floquet Topological Insulators and in multi-layer graphene problems; for instance the following three-replica model:

$$H_F = \begin{pmatrix} 1 + D\cdot\sigma & \varepsilon B^*(y) & O \\ \varepsilon B(y) & D\cdot\sigma & \varepsilon B^*(y) \\ O & \varepsilon B(y) & -1 + D\cdot\sigma \end{pmatrix}$$

Finally, the geophysical water wave model for atmospheric flows near the equator:

$$H_W = \begin{pmatrix} 0 & D_x & D_y \\ D_x & 0 & if(y) \\ D_y & -if(y) & 0 \end{pmatrix}$$

In all cases, the coefficients $\mu = \mu(y)$ (in blue online) describe possible domain walls modeling the transition from one insulator in the northern hemisphere to another insulator in the southern hemisphere.

3 Lecture 3.

Classical and quantum Hall effects. We consider elements of the Integer Quantum Hall Effect (IQHE) closely following the theory developed in [1] and the lecture notes on QHE available at [45]. We will also refer to results in [9], which proposes an alternative analysis of the IQHE. This lecture presents the IQHE at a formal level while the next lecture focuses on a mathematical analysis of the effect.

Classical Hall effect. Let us start with the classical Hall effect. The motion of a particle in a magnetic field is (m: mass; v: velocity; B: magnetic field; $-e$: electron charge)

$$m\dot{v} = -ev \times B$$

leading to

$$x(t) = X - R\sin(\omega_B t + \phi), \quad y(t) = Y + R\cos(\omega_B t + \phi), \quad \omega_B = \frac{eB}{m},$$

with (X, Y, R, ϕ) arbitrary. The Drude model is a more precise description in the presence of an electric field and some friction:

$$m\dot{v} = -ev \times B - eE - \tau^{-1}mv.$$

At equilibrium, $\dot{v} = 0$, and defining a current density

$$J = -nev,$$

for n the electron density, we find

$$\begin{pmatrix} 1 & \omega_B\tau \\ -\omega_B\tau & 1 \end{pmatrix} J = \frac{e^2 n\tau}{m} E, \qquad J = \sigma E.$$

This is Ohm's law with σ a 2×2 matrix. Define $\rho = \sigma^{-1}$ to get

$$\rho = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ -\rho_{xy} & \rho_{xx} \end{pmatrix} = \frac{1}{\sigma_{DC}} \begin{pmatrix} 1 & \omega_B\tau \\ -\omega_B\tau & 1 \end{pmatrix}, \qquad \sigma_{DC} = \frac{ne^2\tau}{m}.$$

So, $\rho_{xy} = B/ne$ is independent of τ and predicts current in the direction orthogonal to E that increases linearly with B while $\rho_{xx} = m/(ne^2\tau)$ is constant. This classical Hall effect is not what is observed experimentally for low temperatures where quantum effects are important.

Different behavior in quantum world. In the Integer Quantum Hall Effect (IQHE), one rather observes experimentally that:

$$\rho_{xy} = \frac{h}{e^2}\frac{1}{\nu}, \quad \nu \in \mathbb{Z}.$$

This is so stable that h/e^2 is now defined practically by the value measured in such experiments. Such a resistivity is observed for magnetic fields with transitions of plateaus when, for a fixed density n, we have:

$$B = \frac{hn}{\nu e} = \frac{n}{\nu}\Phi_0, \qquad \Phi_0 = \frac{2\pi\hbar}{e} = \frac{h}{e} \quad \text{the elementary quantum flux.}$$

The values of the magnetic field correspond to ν filled Landau levels. Impurities are then responsible for the plateaus of the conductivity. The degeneracy of Landau levels per unit area is $B\Phi_0^{-1}$ as we will see below. Therefore, it takes an increase of $n\Phi_0 = B$ to fill another Landau level. As n increases from $\nu B/\Phi_0$ to $(\nu+1)B/\Phi_0$, all electrons are localized (by Anderson localization) as they do not have enough energy to get to the next Landau level. As n increases between these levels, the resistivity is quantized to $\Phi_0/(e\nu)$ [9].

When ρ_{xy} is constant on a plateau, σ_{xx} vanishes. The latter becomes large when ρ_{xy} jumps from one plateau to the next.

Landau levels for magnetic Schrödinger equation. The classical Lagrangian is $L = \frac{1}{2}m\dot{x}^2 - e\dot{x} \cdot A$. The canonical momentum is then $p = \partial L/\partial \dot{x} = m\dot{x} - eA$. $m\dot{x} = p + eA$ is the mechanical momentum. The Hamiltonian is then

$$H = \dot{x} \cdot p - L = \frac{1}{2m}(p + eA)^2.$$

The quantum Hamiltonian is obtained by promoting p to the operator $D = -i\nabla$. $\pi = p + eA$ is mechanical momentum operator satisfying the commutation relations

$$[\pi_x, \pi_y] = -ie\hbar B.$$

We can then define

$$\mathfrak{a} = \frac{1}{\sqrt{2e\hbar B}}(\pi_x - i\pi_y), \quad \mathfrak{a}^* = \frac{1}{\sqrt{2e\hbar B}}(\pi_x + i\pi_y)$$

and verify $[\mathfrak{a}, \mathfrak{a}^*] = 1$ while $H = \hbar\omega_B(\mathfrak{a}^*\mathfrak{a} + \frac{1}{2})$. This provides energies (eigenvalues of H)

$$E_n = \hbar\omega_B(n + \frac{1}{2}), \quad n \in \mathbb{N},$$

as we recognize a quantum harmonic oscillator.

Let us be a bit more precise and obtain the eigenstates as well. In the Landau gauge, $A = (0, xB)$ so that $H = \frac{1}{2m}(p_x^2 + (p_y + eBx)^2)$. Then Fourier transforming y to $\hbar k$, we get

$$H_k = \frac{1}{2m}p_x^2 + \frac{m\omega_B^2}{2}(x + kl_B^2)^2$$

for $l_B^2 = \hbar/eB$ magnetic length (roughly $10^{-8}m$). The wavefunctions are given by

$$\psi_{n,k}(x, y) = e^{iky}H_n(x + kl_B^2)e^{-\frac{(x+kl_B^2)^2}{2l_B^2}}.$$

We observe that on $\mathbb{R}^2$, the Landau levels are infinitely degenerate since $k \in \mathbb{R}$ is arbitrary for each level n.

For a domain of extension $A = L_xL_y$, the number of states is heuristically found to be $N = \frac{AB}{\Phi_0}$. Φ_0 is magnetic flux contained in an area $2\pi l_B^2$. Indeed, for a system in a box of size L_y, this quantizes k to $2\pi/L_y$ by Fourier series. For x, we observe that in ways that are very heuristic, we want $0 \leq kl_B^2 \leq L_x$ so that there are L_x/l_B^2 such values of k. Therefore $N = (L_y/2\pi)(L_x/l_B^2) = L_xL_yB/\Phi_0 = AB/\Phi_0$.

The degeneracy of the electron density per Landau level is therefore indeed $B\Phi_0^{-1}$ as mentioned above. This creates an energy barrier to fill the next Landau level (once a level is full, the next electrons entering the system have not choice but to go to the next level since they are Fermions and thus two of them cannot occupy any given state).

Turning on an electric field. In that case:

$$H = \frac{1}{2m}(p_x^2 + (p_y + eBx)^2) + eEx.$$

Then

$$\psi(x, y) = \psi_{n,k}(x + mE/eB^2, y)$$

and energies

$$E_{n,k} = \hbar\omega_B(n + \frac{1}{2}) - eE(kl_B^2 + \frac{eE}{m\omega_B^2}) + \frac{mE^2}{2B^2}.$$

Now the group velocity $v_y = \dfrac{1}{\hbar}\dfrac{\partial E_{n,k}}{\partial k} = -\dfrac{E}{B}$: we thus observe a perpendicular Hall current as in the classical setting.

Landau levels in symmetric gauge. In the symmetric gauge $A = (-By/2, Bx/2)$, the above degeneracy may also be understood by yet another momentum $\tilde{\pi} = p - eA$. This operator is not gauge invariant but in symmetric gauge we have $[\tilde{\pi}_x, \tilde{\pi}_y] = ie\hbar B$ and the otherwise complicated commutators with the $\pi_{x,y}$ all vanish: $[\pi_i, \tilde{\pi}_j] = 0$. We can define operators $\mathfrak{b}$ as above for $\tilde{\pi}$ and then construct

$$|n, m\rangle = \frac{\mathfrak{a}^{*n}\mathfrak{b}^{*m}}{\sqrt{n!m!}}|0, 0\rangle$$

This fills in the degeneracies. This is best written in $z = x + iy$ and

$$\partial = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}), \qquad \bar{\partial} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$$

and then

$$\mathfrak{a} = -i\sqrt{2}(l_B\bar{\partial} + \frac{z}{4l_B}),\ \mathfrak{a}^* = -i\sqrt{2}(l_B\partial - \frac{\bar{z}}{4l_B}),\ \mathfrak{b} = -i\sqrt{2}(l_B\partial + \frac{\bar{z}}{4l_B}),\ \mathfrak{b}^* = -i\sqrt{2}(l_B\bar{\partial} - \frac{z}{4l_B}).$$

We then easily find that the kernel of $\mathfrak{a}\psi_{LLL,0}$ is (up to normalizing constant c)

$$\psi_{LLL,0} = ce^{-|z|^2/4l_B^2}, \quad \psi_{LLL,m} = c\mathfrak{b}^{*m}\psi_{LLL,0} = c(z/l_B)^m e^{-|z|^2/4l_B^2}$$

for the lowest Landau level (LLL). We can apply $\mathfrak{a}^*$ to populate higher Landau levels.

Note that for angular momentum

$$J = i\hbar(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}) = \hbar(z\partial - \bar{z}\bar{\partial}), \qquad J\psi_{LLL,m} = \hbar m\psi_{LLL,m}.$$

These are therefore eigenstates of angular momentum with increasing values as m increases. We also observe that the Hermite functions are primarily supported on concentric rings with increasing radii as m increases.

The degeneracies are now a bit easier to estimate than in the Landau gauge. The wavefunction with momentum m is peaking in intensity on a ring of radius $\sqrt{2m}l_B$. Therefore, for a region $A = \pi R^2$, we have roughly

$$N = R^2/(2l_B^2) = A/e\pi l_B^2 = \frac{BA}{\Phi_0}$$

as 'obtained' in the Landau gauge.

Electrons are Fermions satisfying the Pauli exclusion principle: no two electrons can be in the same state. So, for the LLL (lowest Landau level) a maximum of (roughly) $N = BA/\Phi_0$ electrons can occupy it for a sample of area A. Injecting more electrons into the system means populating the second Landau level until $2N$ electrons are present in the system, and so on. This provides a first element of quantization. It now remains to introduce a notion of conductivity and show that it too is quantized. Rather than injecting a 'linear' electric field E as we did earlier, we consider another geometry in which computations are easier.

Laughlin argument and quantized conductivity. Consider the plane punctured at a point, say 0, where we thread a time-dependent flux $\Phi(t) = \Phi_0/T$ moving from 0 to $\Phi_0 = h/e$. The variations are very slow and take a time $T \gg 1/\omega_B$. Now, such a flux is generated by a magnetic potential $A(t) = \Phi(t)e_\theta/r$ with (e_r, e_θ) the usual basis in polar coordinates. This may be rewritten as $A(t) = \Phi(t)\nabla\mathrm{argz}$ where $z = x + iy = |z|e^{i\mathrm{argz}}$.

We now obtain from the Maxwell-Faraday law that $\partial_t A = -E = \Phi_0\nabla\mathrm{argz}/T$. Associated to it is a difference of potential $\oint E \cdot dr = -\Phi_0/T$ independent of the circle enclosing 0. If we can argue that n electrons have been moved from $r = 0$ to $r = \infty$ during time T, corresponding to a radial current $I_r = -ne/T$, we obtain a resistivity

$$\rho_{xy} = \frac{-\Phi_0/T}{-ne/T} = \frac{\Phi_0}{en}$$

and hence a quantized Hall conductivity in integral multiples of e/Φ_0.

This may be 'verified' heuristically on the Landau levels in symmetric gauge. Maybe more directly, we observe that the Hamiltonian in the presence of $\Phi(t)$ may be written in polar coordinates (r, θ) as

$$H_t = \frac{1}{2m}\Big(-\hbar^2 \frac{1}{r}\partial_r r \partial_r + \big(-i\frac{\hbar}{r}\partial_\theta + \frac{eBr}{2} + \frac{e\Phi(t)}{2\pi r}\big)^2 \Big).$$

In other words, at time T, the operator replaces $-i\partial_\theta$ by $-i\partial_\theta + 1$, that is, m by $m+1$ in the Fourier (series) variable. During the adiabatic transformation from $t = 0$ to $t = T$, we thus have that the mode $m = 0$ has 'disappeared'. This corresponds to a particle (i.e., excitation of the operator) that has disappeared, or moved across the domain to $r = \infty$. Since we assume that n Landau levels are filled, there are n such electrons that disappear and this concludes the heuristic analysis of the quantization of the conductivity.

Note that at time $t = T$, we have in effect replaced the vector potential A by $A + \nabla\chi$ where $\chi = \mathrm{arg} z$. This is a singular gauge transformation corresponding to a gauge transformation of ψ by $U^{-1}\psi = e^{-i\chi}\psi$ for $U(x) = z/|z|$. We thus have

$$H_T = UH_0U^{-1}.$$

The analysis of the Hall conductivity associated to H is therefore obtained by comparing H and UHU^{-1}. The quantized conductivity $\sigma_{xy} = \frac{e}{\Phi_0}\nu$ for $\nu \in \mathbb{Z}$ measures the number of electrons ν moved 'to infinity' when adiabatically transforming H to UHU^{-1}.

Another interpretation for these integers is as follows. Consider a Fermi energy E sitting between Landau levels. Remember that fermions do not like to live in the same state. So a Fermi energy E means that electrons fill in all the energy levels below E. In a finite-size sample of area $A = \pi R^2$, say, we saw that there are therefore $N = BA/\Phi_0$ electrons per Landau level. If we formally define the projector $P = \chi(H < E)$ projecting onto the eigenstates of H with energy less than E, we will obtain that the dimension of the range of that projection operator is νN, where ν is the number of filled Landau levels.

We may now also introduce $Q = \chi(UHU^{-1} < E)$ the corresponding projection for transformed operator. The above Laughlin argument shows that the difference of dimensions of these ranges is exactly equal to ν, the number of filled Landau levels.

Lost particles and lost eigenstates. Note that eigenstates are interpreted in (second-quantization) quantum mechanics as being occupied by electrons when their energy is below the Fermi energy. The reason is based on the Pauli exclusion principle, stating that no two electrons can be in the same state, and on the principle that electrons will occupy eigenstates with the lowest available energy. Therefore, adiabatically transforming H to UHU^* or equivalently P to UPU^* implies that a number of electrons leave the system. More generally, we may also simply observe that a corresponding number of eigenstates was somehow lost during the adiabatic transformation. This interpretation then also applies in other applications than electronics, such as for instance photonics, where the fermions do not have any equivalence.

4 Lecture 4.

Avron-Seiler-Simon theory. Now we can start doing a bit of math. Let H be the magnetic Schrödinger equation $H = (D+A)^2 = (D+A)\cdot(D+A)$ for $B = \nabla \times A$ constant. It admits Landau levels $2B(n+\frac{1}{2})$. As a consequence, it is an *insulator* for any Fermi energy E landing between such levels.

We therefore define as above the important object consisting in projecting H onto all states with energy below E:

$$P = \chi(H \leq E).$$

Since no energy of H resides close to E, the projection is independent of $E+\delta E$ so long as $E+\delta E$ does not cross any Landau level. Moreover, if we replace H by $H+V$ for a reasonable perturbation V, then $\chi(H+V \leq E)$ will also be defined. This now allows us to obtain (quantized) conductivities for classes of operators $H+V$ for which the above explicit calculations (of eigenstates and so on) are no longer available.

Note that P above is defined as a spectral quantity. We know that H is a self-adjoint operator as an unbounded operator with domain $H^2(\mathbb{R}^2)$ in $L^2(\mathbb{R}^2)$. More information is available in [1] on this front.

We can also consider the second projection obtained after the adiabatic transformation:

$$Q = \chi(UHU^{-1} \leq E).$$

As we mentioned above, each of these projections have the heuristic interpretation that they count the number of modes below energy E, hence the number of particles present in each system below energy E. In our operator defined on $\mathbb{R}^2$, as soon as E is above one or more Landau levels, this means that there is an infinite number of such particles since such levels are infinitely degenerate. However, there are some situations in which $\infty - \infty$ is defined and we are fortunate to be in one such case. Heuristically, we want to think of the area $A = \pi R^2$ above as 'very large'. The difference of electrons in the Laughlin argument is given by ν independent of A and hence in the limit of infinitely large A as well.

Indeed, [1] defines the index of a pair of projections P and Q as the excess of the dimension of the range of P compared to that of Q. Moreover, that excess will turn out to: (i) be quantized; (ii) be stable against fluctuations V; and (iii) indeed correspond to the number of electrons that 'disappeared' (or in fact 'appeared') as we apply the singular gauge transformation $U = e^{i\chi}$. We will also observe that for exactly n Landau levels below E, then the excess of dimension is indeed equal to n.

Pair of projections. Let P and Q be orthogonal projections. When they are finite-rank, then $\mathrm{Index}\,(P,Q) \equiv \dim P - \dim Q = \mathrm{Tr}(P-Q)$ computes the difference of dimensions. This is generalized for P and Q such that $P-Q$ is compact as

$$\mathrm{Index}\,(P,Q) := \dim \mathrm{Ker}(P-Q-1) - \dim \mathrm{Ker}(P-Q+1).$$

By compactness assumption, both kernels above are finite dimensional. It is remarkable that the index may be computed in several more 'practical' ways. A first result is

Proposition 4.1 *Assume $(P-Q)^{2n+1}$ trace-class for some $n \in \mathbb{N}$. Then for all $\mathbb{N} \ni m \geq n$*

$$\mathrm{Index}\,(P,Q) = \mathrm{Tr}(P-Q)^{2m+1}.$$

Proof. We first observe that $(P-Q)^2$ commutes with P and with Q. Then

$$(P-Q)^{2n+2}R = (P-Q)^{2n}(R-RSR)$$

for $(R,S) = (P,Q)$ or (Q,P) so that

$$(P-Q)^{2n+3} = (P-Q)^{2n+1} - (P-Q)^{2n}[PQ,QP] = (P-Q)^{2n+1} - [PQ,B]$$

for $B = [Q,(P-Q)^{2n+1}]$ trace class so that, since PQ is bounded, $\mathrm{Tr}[PQ,B]=0$. So the trace is defined and independent of $m \geq n$. Sending $m \to \infty$ and noting that $-1 \leq P-Q \leq 1$ yields the result. Indeed, we may decompose $H = H_1 \oplus H_2 \oplus H_3$ where H_1 is the finite dimensional space where $(P-Q)_{|H_1} = I$, where H_2 is the finite dimensional space where $(Q-P)_{|H_2} = I$, and where H_3 is the orthogonal complement. We then observe that $\|(P-Q)_{|H_3}\| < 1$. The trace on that space therefore asymptotically vanishes as $m \to \infty$. This concludes the derivation. $\square$

Indices of a Fredholm operator and pair of projections. Let $Q = UPU^*$ for U unitary. Then

Proposition 4.2 *Let P and $Q = UPU^*$ be orthogonal projections with U unitary. Assume that $(P-Q)^{2n+1}$ is trace-class. Then $(P-PQP)^{n+1}$ and $(Q-QPQ)^{n+1}$ are trace-class and PUP is a Fredholm operator on* $\mathrm{Ran}P$ *(equivalently $PUP + I - P$ is Fredholm) and*

$$\mathrm{Index}\,(P,Q) = \mathrm{Tr}([P,U]U^*)^{2n+1} = \mathrm{Tr}(P-PQP)^{n+1} - \mathrm{Tr}(Q-QPQ)^{n+1} = -\mathrm{Index}\,PUP.$$

Proof. We observe that $P-Q = [P,U]U^*$ and $(P-Q)^2P = P-PQP = P(P-Q)^2$ so that for instance $(P-PQP)^{n+1} = (P-Q)^{2n+2}P$ and hence is trace-class. This also provides the second equality.

Next, note that $P-PQP = P-PUPU^*P$ as well as $Q-QPQ = U(P-PU^*PUP)U^*$ so that raised to power $n+1$ and using cyclicity of trace for operators of the form UAU^* with A trace-class, we have that the above equals

$$\mathrm{Tr}(P-PUPU^*P)^{n+1} - \mathrm{Tr}(P-PU^*PUP)^{n+1}.$$

The bounded operators PUP and PU^*P are such that $R_1 = P - PUPPU^*P = P(P-Q)P$ and $R_2 = P - PU^*PPUP$ are such that R_1^N and R_2^N are trace-class. We thus apply [27, Proposition 19.1.14] (see also appendix where this Fedosov formula is recalled) to obtain that the index of PU^*P on $\mathrm{Ran}P$ is given $\mathrm{Tr}R_1^N - \mathrm{Tr}R_2^N$. This is also clearly the index of $PU^*P + I - P$ on the full Hilbert space. $\square$

Proposition 4.3 *Assume $Q = U_1PU_1^* = U_2^*RU_2$ with $P-Q$ and $Q-R$ compact. Then*

$$\mathrm{Index}\,(P,R) = \mathrm{Index}\,(P,Q) + \mathrm{Index}\,(Q,R)$$

Proof. This is equivalent to $\mathrm{Index}\,QU_1U_2Q = \mathrm{Index}\,QU_1Q + \mathrm{Index}\,QU_2Q$. We know that $\mathrm{Index}\,QU_1QU_2Q = \mathrm{Index}\,QU_1Q + \mathrm{Index}\,QU_2Q$ from $\mathrm{Index}\,AB = \mathrm{Index}\,A + \mathrm{Index}\,B$. Now $QU_1(I-Q)U_2Q = U_1P(P-Q)U_2Q$ is compact so that we have $\mathrm{Index}\,QU_1U_2Q = \mathrm{Index}\,QU_1QU_1Q$ by stability of the index under compact perturbations. $\square$

Fredholm index as an integral. Let us assume that P is a projection with Schwartz kernel $p(x,y)$ jointly continuous in (x,y) and such that

$$|p(x,y)| \leq \frac{C}{1+|x-y|^\eta} \quad \eta > 2. \tag{4.1}$$

Assume that U is multiplication by a complex-valued function $u(x)$ with $|u(x)| = 1$ differentiable away from a point (say $x = 0$) and such that

$$|u(x+y) - u(x)| \leq C\frac{|x|}{|y|}.$$

The winding number of u about the singularity $x = 0$ is denoted by $N(U)$.

For instance, we may consider smooth modification of

$$u_n(x) = \frac{(x+iy)^n}{|x+iy|^n}$$

and equal to 1 at the origin, say. Then $N(U) = n$.

With these preliminary statements, we obtain the first main result:

Proposition 4.4 *Under the above hypotheses with $Q = UPU^*$ we have that $(P-Q)^p$ is trace-class for $p > 2$ (while $P-Q$ is compact so that the* Index (P,Q) *is defined) and in particular, using $p = 3$,*

$$-\mathrm{Index}\, PUP = \int_{\mathbb{R}^6} p(x,y)p(y,z)p(z,x)(1-\frac{u(x)}{u(y)})(1-\frac{u(y)}{u(z)})(1-\frac{u(z)}{u(x)})dxdydz.$$

Proof. We observe that the Schwartz kernel of $P-Q$ is $p(x,y)(1-\frac{u(x)}{u(y)})$ so that if κ denotes the Schwartz kernel of $(P-Q)^3$, then the above right-hand side is nothing but the integral of $\kappa(x,x)$. So, it only remains to apply Russo's criterion in Lemma A.3 for $d = 2$, $p > 2$, and $n = 3$. See [1, Proposition 3.6] for details, which we do not reproduce here. ∎

Time reversal invariance. When H, hence P is time-reversal symmetric, we have that $p(x,y)$ is real-valued. This implies that the triple product of p's is even (invariant) under conjugation while the triple product of u's is odd as one verifies. For instance $\overline{1-\frac{u(x)}{u(y)}} = 1-\frac{u(y)}{u(x)}$ since $u(x)\bar{u}(x) = 1$; so the product of the three terms picks up a $-$ term under conjugation. Therefore the index has to vanish. It is really the presence of a magnetic field that ensures (besides providing spectral gaps) that the index of PUP is not trivial.

Combes-Thomas-type argument. It remains to verify that (4.1) holds for $P = \chi(H \leq E)$ for E a Fermi energy between Landau levels. It turns out that $p(x,y)$, the integral kernel of P is exponentially decaying in $|x-y|$ provided that E lands in a spectral gap of H. This is a Combes-Thomas estimate, whose derivation is not entirely straightforward and would bring us a bit too far off track. So, we simply refer to [1, Theorem A.1]. We observe there that (4.1) holds for $H = (D+A)^2 + V$ in dimensions $d = 2, 3$ as soon as A and V are compact perturbations of $A_0 = (0, Bx)$ and $V_0 = 0$. For a detailed derivation of Combes-Thomas estimates for functionals of the magnetic Schrödinger and several other operators, we refer the reader to [24].

Such estimate hold for smooth functions $f(H)$, which is not the case for P. However, we assume that E lives in a spectral gap of H. Therefore, $P = P_\varepsilon(H)$ where $P_\varepsilon(x)$ is a smooth function such that $P'_\varepsilon(x)$ is supported inside the spectral gap. Moreover, H is semi-bounded, meaning that there is no spectrum in the vicinity of $-\infty$. As a consequence, we may also replace χ by a function that is smooth (as indicated above) and compactly supported. In other words, we may assume that $P_\varepsilon(x) \in C_c^\infty(\mathbb{R})$ has compact support and is smooth. The Combes-Thomas estimates then imply that the kernel of $P_\varepsilon(H)$ decays exponentially fast away from the diagonal.

Explicit index computation for Magnetic Schrödinger equations. We now want to estimate the above integral when the perturbation $V = 0$. We know that the above integral is independent of any V that does not change the Landau levels significantly. Indeed, thanks to Proposition 4.2, the index is stable against continuous perturbations tV for $0 \le t \le 1$. It is this stability against perturbation V and A that justifies all the efforts we have made.

Still, even when $V = 0$ and $A = (0, Bx)$, the calculations are not entirely trivial as the operator is not invariant by translation because the magnetic potential is not constant (even if B is). However, it turns out that P and the kernel p still satisfy a form of covariance that is sufficient in our context. More precisely, for any shift $a \in \mathbb{R}^2$, we can find a family of unitary (multiplication) transformations $U_a(x)$ such that

$$p(x, y) = U_a(x)p(x - a, y - a)U_a^*(y). \tag{4.2}$$

Indeed, let $T_a f = f(x - a)$ the unitary (on $L^2(\mathbb{R}^2)$) operation of shift by $a \in \mathbb{R}^2$. We verify that

$$T_a A(x) = A(x - a) = (0, B(x - a_1)) = A(x) - \nabla\Lambda_a(x), \quad \Lambda_a(x) = (Ba_1x_2).$$

Here, we use $x = (x_1, x_2)$ and $a = (a_1, a_2)$. Define $U_a(x) = e^{-i\Lambda_a(x)}$. We observe that

$$e^{i\Lambda_a(x)} D e^{-i\Lambda_a(x)} = D - \nabla\Lambda_a(x),$$

so that

$$T_a(D + A)^2 T_{-a} = (D + (T_a A))^2 = U_a^*(D + A)^2 U_a.$$

Here, U_a is the unitary operation of point-wise multiplication by $U_a(x)$. Thus,

$$(U_a T_a)H(U_a T_a)^* = H, \qquad (U_a T_a)f(H)(U_a T_a)^* = f(H)$$

for any Borel function f and hence for P as well. Writing this for Schwartz kernels, (4.2) holds. This implies in particular that

$$p(x, y)p(y, z)p(z, x) = p(0, y - x), p(y - x, z - x)p(z - x, 0)$$

since every object is scalar-valued. It thus remains to integrate

$$\int_{\mathbb{R}^2} (1 - \frac{u(x-a)}{u(x-b)})(1 - \frac{u(x-b)}{u(x-c)})(1 - \frac{u(x-c)}{u(x-a)})dx = 2\pi i N(U)\text{Area}(a, b, c). \tag{4.3}$$

That this remarkable geometric identity holds is a result of Alain Connes and is proved in detail in [1, Lemma 4.4]; see also [36]. We do not reproduce the proof but still want to stress that this identity is central in the explicit calculation of the index. We thus obtain

Theorem 4.5

$$\text{Index}\, PUP = -2\pi i N(U) \int_{\mathbb{R}^4} p(0, x)p(x, y)p(y, x) x \wedge y dx dy$$

with $x \wedge y = x_1 y_2 - x_2 y_1$.

As we mentioned above, for time-reversal invariant operators, $p(x,y)$ is real-valued so that both terms above have to vanish since the l.h.s. is clearly real-valued while the r.h.s. would be purely imaginary.

For the Landau Hamiltonian, we have, as noted in the preceding lecture,

$$H = (D+A)^2 = 2\mathfrak{a}^*\mathfrak{a} + 1, \quad \mathfrak{a} = -i\sqrt{2}(\bar\partial + \frac{z}{4}).$$

We use here $\hbar = B = e = 2m = 1$ so that $\omega_B = 2$ and $l_B = 1$. The Landau levels $E_m = 2m+1$ are populated by

$$|n,m\rangle(z) = (\pi n!(m+1)!)^{-\frac{1}{2}}(\mathfrak{a}^*)^m(z^n e^{-\frac{1}{2}|z|^2}).$$

This provides a projection (check this using that $|n,m\rangle(0) = 0$) onto the mth Landau level given by

$$p_m(0,z) = |m,m\rangle(0)|m,m\rangle(z).$$

The kernel of the projection operator is thus indeed smooth and (at least) exponentially decaying as $|z| \to \infty$.

Lemma 4.6 *We have* $\operatorname{Index} P_m U P_m = -1$.

Proof. (Sketch.) We use [1, Lemma 7.2] to compute the indices and obtain that $\operatorname{Index} P_m U P_m = -1$ for $U = z/|z|$ for each Landau level in [1, Proposition 7.3]. A sketch of the proof (also sketched in the aforementioned reference) goes as follows. For a Landau level m fixed, the eigenvector $|n,m\rangle(z)$ has angular momentum equal to $n-m$. This is clear from the z^n component when $m=0$ and may be verified for other values of m. Now applying $z/|z|$ to it maps n to $n+1$ as we saw earlier in the Laughlin argument. When projected back to the mth Landau level (by application of P_m), we observe that $P_m U P_m$ has the effect of mapping $|n,m\rangle$ to $c|n+1,m\rangle$, where $c = c(n,m) \neq 0$ is a constant (independent of z).

Seeing $P_m U P_m$ as a semi-infinite matrix A in the representation of $P_m L^2(\mathbb{R}^2)$ with basis $n \to |n,m\rangle$ (at fixed Landau level m), we observe that A has vanishing entries except at entries (n,n') for which $n'+1 = n$. This matrix therefore has the same structure as a shift operator (where 1 is replaced by $c_n \neq 0$). Now, if the coefficients c_n are bounded and bounded away from 0, the matrix A is also the representation of a Fredholm operator. As for the shift, we obtain that the kernel of A^* is one-dimensional while that of A is trivial. So, the index is -1 and this concludes the proof. $\square$

Remark 4.7 *As an exercise, check that the required properties on $c_{n,m}$ really hold. Note that we do not use the explicit expression obtained in Theorem 4.5 and the index may be computed directly for the type of Fredholm operator satisfying the hypotheses of [1, Lemma 7.2]. We will see that Theorem 4.5, once generalized to matrix-valued operators, finds applications the explicit computation of indices for Dirac-type operators.*

So, if the Fermi energy is between the mth and the $(m+1)$st level, then for $P = \chi(H < E)$, we have

$$\operatorname{Index} PUP_{|\operatorname{Ran} P} = -m. \tag{4.4}$$

Here, we use in fact that the index is additive (these are not entirely trivial results; see lemma below) or we simply modify the above proof to get the result directly.

This concludes our analysis of the IQHE: When adiabatically moving from H to UHU^{-1} for a relatively large class of Hamiltonians H with spectral gap, then the relative index associated to the pair $P = \chi(H < E)$ and $Q = UPU^*$ modeling adiabatic transport of electrons 'to infinity' is indeed quantized. This corresponds to a quantized Hall conductivity $\sigma_{xy} = \nu e/\Phi_0$ that is stable against perturbations of H. Moreover, ν is the number of Landau levels occupied below the Fermi energy E. This explains the plateaus, their location, and their stability to a large extent.

Additivity of index. That the index is additive may be obtained as follows in our context. Assume that $P = \sum_{j=1}^J P_j$ with P_j orthogonal projectors on a Hilbert space $\mathcal{H}$ such that $P_j P_k = \delta_{jk} P_j$. Define $Q_j = UP_jU^*$ and $Q = UPU^*$ and assume that $Q_j - P_j$ is compact while $(Q_j - P_j)^{2n+1}$ is trace-class. Then we have

Lemma 4.8 *Let P_j for $1 \le j \le J$ be as above and $P = \sum_{j=1}^J P_j$. Then*

$$\mathrm{Index}\, PUP = \sum_{j=1}^{J} \mathrm{Index}\, P_j U P_j.$$

Proof. We have $PUP = \sum_{i,j} P_i U P_j = \sum_i P_i U P_i + K$, where we want to prove that K is compact. Indeed, let $j \neq k$ and consider $P_j U P_k$. Since $P_j - Q_j = [U, P_j]U_j^*$ is compact, then so is $[U, P_j]$ so that $P_j U P_k = P_j [U, P_k]$ is compact (we used $P_j P_k = 0$). This shows that K is compact and hence $\mathrm{Index}\, PUP = \mathrm{Index} \sum_{j=1}^J P_j U P_j$. Now, $P_j U P_j$ act on orthogonal subspaces of $\mathcal{H}$ so that clearly $\mathrm{Index} \sum_{j=1}^J P_j U P_j = \sum_{j=1}^J \mathrm{Index}\, P_j U P_j$. This concludes the derivation. ◻

When $\tilde{P} = I - P$ and $\tilde{Q} = U\tilde{P}U^*$, then $\tilde{P} - \tilde{Q}$ is also compact. Under the assumption that $(P - Q)^{2n+1}$ and $(\tilde{P} - \tilde{Q})^{2n+1}$ are trace-class, we obtain as above that

$$\mathrm{Index}\, PUP + \mathrm{Index}\, \tilde{P}U\tilde{P} = 0.$$

Note that if P projects onto the spectrum below E in a spectral gap, then $\tilde{P}$ projects onto the spectrum above E. Then $(P + \tilde{P})U(P + \tilde{P}) = U$ is a fredholm operator with trivial index.

5 Lecture 5.

Bulk invariant for regularized Dirac operator. We come back to Dirac operators and introduce the *modified* Dirac operator

$$H = D_x \sigma_1 + D_y \sigma_2 + [m(y) + \eta \Delta] \sigma_3.$$

Here, $-\Delta = D_x^2 + D_y^2$ is the usual Laplacian and $\eta > 0$ is a real number small enough that $1 - 2\eta m(y) \ge 0$.

Remark 5.1 *Dirac (and generalized Dirac) operators are by definitions operators H such that H^2 is a Laplace (Beltrami) operator modulo lower-order terms. So, H above is a Dirac operator only when $\eta = 0$. We will refer to the operator with $\eta \neq 0$ as a* modified Dirac operator*.*

Let us start with the *bulk Hamiltonian* with $m(y) = m \neq 0$ constant and assume that $1 - 2\eta m \ge 0$ (to simplify the description of the spectral gap) with $\eta \neq 0$ as well. We mostly follow the presentation in [2].

Self-adjointness and functional calculus. We first show that H is a self-adjoint operator as an operator with domain $\mathfrak{D}(H) \subset L^2(\mathbb{R}^2;\mathbb{C}^2) = \mathcal{H}$. The domain is obtained so that $(H \pm i)\mathfrak{D}(H) = \mathcal{H}$.

This domain of definition is best defined in the Fourier domain since

$$\hat{H}(\xi) = \xi \cdot \sigma + (m - \eta|\xi|^2)\sigma_3$$

where $\xi = (\xi_1, \xi_2)$ is the dual variable to $x = (x_1, x_2)$. This shows that

$$\hat{H}^2 = |\xi|^2 + (m - \eta|\xi|^2)^2 = \eta^2|\xi|^4 + (1 - 2\eta m)|\xi|^2 + m^2 \geq \eta^2|\xi|^4 + m^2.$$

We thus define, with $\mathcal{F}^{-1}$ inverse Fourier transform,

$$\mathfrak{D}(H) = \mathcal{F}^{-1}\{(\eta^2|\xi|^4 + m^2)^{-1}\hat{H}(\xi)\hat{f};\ \hat{f} \in L^2(\mathbb{R}^2;\mathbb{C}^2)\}.$$

Then clearly, $(H \pm i)\mathfrak{D}(H) = \mathcal{H}$. Therefore, H is a self-adjoint operator from $\mathfrak{D}(H)$ to $\mathcal{H}$ that is in fact boundedly invertible. We then verify that the same domain actually makes

$$D_x\sigma_1 + D_y\sigma_2 + [m(y) + \eta\Delta]\sigma_3 + V(x)$$

where $V(x)$ is multiplication operator by a smooth Hermitian matrix compactly supported $V(x)$, a self-adjoint bounded operator on $\mathcal{H}$, and where $m(y)$ is similarly a smooth compact perturbation of m constant. (Exercise: check this using, e.g., the Kato-Rellich criterion recalled in the Appendices).

Bulk invariant. Now that we have a self-adjoint operator, we use the spectral theorem to define the projection operator

$$P(H) = \chi(H - E < 0),$$

for $-m < E < m$. This is the same construction as the one we used for the magnetic Schrödinger equation. Again, $P = P(H)$ is defined spectrally since $x \to \chi(x - E < 0)$, which equals 1 when $x - E < 0$ and 0 otherwise, is a Borel function.

Note that E resides in a spectral gap since $\sigma(H) \cap (-m, m) = \emptyset$. Let $u(x) = \frac{x_1 + ix_2}{|x|}$ the unitary multiplication operator on $L^2(\mathbb{R}^2;\mathbb{C}^2)$ with U the unitary operator of multiplication by $u(x)$ as in the preceding lecture. We know that Index (P, Q) with $Q = UPU^*$ models the adiabatic transition from an initial state to a final one where angular momentum is increased by one unity. Our objective here is therefore to address this effect for a massive Dirac Fermion rather than a magnetic Schrödinger model. We note that the Dirac Fermion has been regularized by $\eta \neq 0$. When the Dirac equation is derived as the macroscopic envelope of a microscopic description, this corresponds to pushing the approximation to second-order rather than first-order. This may provide a justification for η and its sign. Irrespective of how the model was derived, we will see that regularization is necessary to define a two-dimensional topological invariant that resembles the quantized Hall conductivity introduced in the preceding lecture.

Our main first result is as follows.

Theorem 5.2 *Let H be the above bulk Hamiltonian and $P(H) = \chi(H - E < 0)$ for $-m < E < m$. Then $PUP_{\mathrm{Ran}P}$ is a Fredholm operator and we have*

$$-\mathrm{Index}\, PUP_{\mathrm{Ran}P} = \frac{1}{2}\big(\,\mathrm{sign}(m) + \,\mathrm{sign}(\eta)\big). \tag{5.1}$$

Note that the above expression is not even an integer when $\eta = 0$. We also observe that the phase of the insulator is either trivial or not depending on the choice of the sign of the regularization.

The first step in the proof of the above theorem is the following

Proposition 5.3 *The operator* $(P-Q)^3 = U[P,U^*][P,U][P,U^*]$ *is trace-class. If we denote by* $\kappa(x,y)$ *its Schwartz kernel, then*

$$-\mathrm{Index}\, PUP_{\mathrm{Ran}P} = \mathrm{Tr}(P-Q)^3 = \int_{\mathbb{R}^2} \mathrm{tr}\,\kappa(x,x)dx.$$

Proof. We have $K := (P-Q) = [P,U]U^* = U[U^*,P]$. We wish to apply a matrix-valued version of Russo's criterion in Lemma A.3. Let K_{ij} be the components of K for $1 \leq i,j \leq 2$. We prove that $K_{ij} \in \mathcal{I}_3$ the Schatten class with $p=3$. This implies that $(P-Q)^3$ is trace-class and that the trace is given by the trace of the integral on the diagonal of its Schwartz kernel.

Since H is gapped in $(-m,m)$, we may as well replace E by 0 since $\chi(H<E) = \chi(H<0)$ by spectral calculus. We then have an explicit expression for $\hat{H}(\xi)$ as 2×2 matrix in the Fourier domain. If $p(x,y) = p(x-y)$ is the kernel of $P = \frac{1}{2}(I - \frac{H}{|H|})$ using a polar decomposition $H = U|H|$ with U partial isometry and $|H| = (H^*H)^{\frac{1}{2}} = (H^2)^{\frac{1}{2}}$ and $\hat{p}(\xi)$ is its Fourier transform, we find

$$\hat{p}(\xi) = \frac{I}{2} - \frac{1}{2}\frac{\xi\cdot\sigma + (m-\eta|\xi|^2)\sigma_3}{(|\xi|^2 + (m-\eta|\xi|^2)^2)^{\frac{1}{2}}}.$$

Let B be a constant operator. Then $P-Q = (P-B) - U(P-B)U^*$ since U is multiplication by a scalar quantity so that $UBU^* = B$. The Schwartz kernel of the operator B is $B\delta(x-y)$ and hence B in the Fourier domain. We may therefore subtract $B = \frac{1}{2}(I + \mathrm{sign}(\eta)\sigma_3)$ and define $R = P-B$ so that the corresponding kernel is

$$\hat{r}(\xi) = -\frac{1}{2}\frac{\xi\cdot\sigma}{|\xi,m|} - \frac{1}{2}(\mathrm{sign}(\eta) + \frac{m-\eta|\xi|^2}{|\xi,m|})\sigma_3, \quad |\xi,m| = (|\xi|^2 + (m-\eta|\xi|^2)^2)^{\frac{1}{2}}.$$

We find

$$|\xi,m| = |\eta||\xi|^2\sqrt{1 + \frac{1-2\eta m}{\eta^2|\xi|^2} + \frac{m^2}{\eta^2|\xi|^4}}.$$

As a consequence,

$$\hat{r}(\xi) = -\frac{1}{2}\frac{\xi\cdot\sigma}{|\eta||\xi|^2} + O(|\xi|^{-2})$$

for large $|\xi|$. In two-space dimensions, this implies that after inverse Fourier transform of Riesz potentials,

$$r(x) = C\frac{x\cdot\sigma}{|x|^2} + O(|\ln|x||).$$

This shows that $p(x)$ has a leading singularity in the vicinity of $x=0$ of the form $|x|^{-1}$. This is in sharp contrast to the kernel in the magnetic Schrödinger case seen in a preceding lecture, where $p(x)$ is shown to be smooth.

We also observe that for $\alpha = (\alpha_1,\alpha_2)$ with $|\alpha| = \alpha_1 + \alpha_2$ a multi-index, then

$$|\partial^\alpha \hat{p}(\xi)| \leq C_\alpha \langle\xi\rangle^{-|\alpha|}, \quad \langle\xi\rangle = \sqrt{1+|\xi|^2}.$$

This decay of $\hat{p}(\xi)$ comes from the spectral gap assumption on H. The inverse Fourier transform thus satisfies that $|x^\alpha||p(x)|$ is bounded for any multi-index α. Controlling the coefficients C_α more precisely, we may be able to show that $p(x)$ decays exponentially rapidly as $|x|\to\infty$.

Collecting the above results, we obtain that (again, $u(x)$ is scalar-valued while $p(x-y)$ is 2×2 matrix-valued)

$$|p(x-y) - q(x,y)| = |p(x-y)(1 - \frac{u(x)}{u(y)})| \leq C_\beta \min\big(1, \frac{|x-y|}{|y|}\big)\frac{1}{|x-y|\langle x-y\rangle^\beta}$$

for any $\beta \in \mathbb{N}$. The above minimum may be seen as an upper bound for $|1-\frac{u(x)}{u(y)}|$.

Let $K = P - Q$ with 2×2 matrix-valued kernel $k(x,y) = p(x-y) - q(x,y)$. We wish to show (with $\frac{1}{q}+\frac{1}{p}=1$) that

$$\Big(\int_{\mathbb{R}^2}\big(\int_{\mathbb{R}^2}|k(x,y)|^q dx\big)^{\frac{p}{q}}dy\Big)^{\frac{1}{p}} \leq C. \tag{5.2}$$

When the above holds for $p>2$, then K is the Schatten class $\mathcal{I}_p(\mathbb{R}^2)$ and with $p=3$, we thus find that $K^3 = (P-Q)^3 \in \mathcal{I}_1$ is trace-class. Moreover, as in Lemma A.3, the trace $\mathrm{Tr}(P-Q)^3$ may be obtained by integrating the Schwartz kernel of $(P-Q)^3$ along the diagonal.

It thus remains to show (5.2). This estimate is done in detail in [2, Lemma 3.3]. We do not reproduce this proof here and leave the rest of the derivation as an exercise. ⊔⊓

In the magnetic Schrödinger case, we directly computed Index PUP by using a representation of Landau levels in modes with prescribed angular momentum. Here, we use a different, and quite powerful, strategy based on estimating the integral of the Schwartz kernel along the diagonal. This calculation is significantly simplified by the fact that $p(x,y) = p(x-y)$, which comes from the invariance by spatial translation of the modified Dirac operator (or any differential operator with constant coefficients). As in the magnetic Schrödinger case, the geometric identity (4.3) is central. We have

Lemma 5.4 *We have*

$$\mathrm{Tr}(P-Q)^3 = -2\pi i \int_{\mathbb{R}^4} \mathrm{tr}\ p(-x)p(x-z)p(z)\ x\wedge(x-z)\ dxdz.$$

Proof. The trace of $(P-Q)^3$ is given explicitly by

$$\mathcal{T} := \int_{\mathbb{R}^{2\times 3}} p(x_1,x_2)(1-\frac{u(x_1)}{u(x_2)})p(x_2,x_3)(1-\frac{u(x_2)}{u(x_3)})p(x_3,x_1)(1-\frac{u(x_3)}{u(x_1)})dx_1dx_2dx_3.$$

Changing variables $x = x_1$ and $y_j = x_{j+1}-x_1$ for $j=1,2$, we get using the translational invariance of P

$$\mathcal{T} = \int_{\mathbb{R}^{2\times 2}} p(-y_1)p(y_1-y_2)p(y_2)(2\pi i)y_1\wedge y_2 dy_1dy_2 \tag{5.3}$$

using the geometric identity (4.3) written as

$$\int_{\mathbb{R}^2}(1-\frac{u(x)}{u(y_1+x)})(1-\frac{u(y_1+x)}{u(y_2+x)})(1-\frac{u(y_2+x)}{u(x)})dx = 2\pi i(-y_1)\wedge(-y_2).$$

Here, $x\wedge z = x_1z_2 - x_2z_1 = (x-z)\wedge z$ is the (two-dimensional) volume of the parallelogram with vertices 0, x, and z (and $x+z$). This gives the result. ⊔⊓

Chern number. We will come back to a more systematic approach to the topological classification of (vector or principal) bundles. Here, we are faced with the following situation. For $\xi\in\mathbb{R}^2$, we have a family of projectors $\hat{P}(\xi)$. Moreover, as $|\xi|\to\infty$, $\hat{P}(\xi)$ converges to $\frac{1}{2}(I-\sigma_3) = \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)$ in our case of interest. In other words, it is important to see the family $\xi\to\hat{P}(\xi)$ as defined on the one-point compactification of $\mathbb{R}^2$, where all points at infinity are identified with the south pole, say, of a sphere.

For $\hat{P}(\xi)$ such a family of projectors, we define the Chern number

$$\mathrm{Ch}[\hat{P}] = \frac{i}{2\pi}\int_{\mathbb{R}^2}\mathrm{tr}\ \hat{P}[\partial_1\hat{P},\partial_2\hat{P}]d\xi = \frac{i}{2\pi}\int_{\mathbb{R}^2}\mathrm{tr}\ \hat{P}d\hat{P}\wedge d\hat{P}. \tag{5.4}$$

Here, $\partial_j = \dfrac{\partial}{\partial \xi_j}$ while $d\hat{P} = \partial_1 \hat{P} d\xi_1 + \partial_2 \hat{P} d\xi_2$, from which the second equality easily follows (with $d\xi_1 \wedge d\xi_2 = -d\xi_2 \wedge d\xi_1$).

An important property of the above object, which is a topological invariant (as is the index of a Fredholm operator), is that it is always integer-valued. Moreover, from the above derivation, we obtain that

Corollary 5.5 *We have*

$$-\mathrm{Index}\, PUP = \mathrm{Ch}[\hat{P}].$$

Proof. We look at the right-hand side in (5.3) and refer to the Appendices for conventions on Fourier transforms. Then $(2\pi)^2 \int f(-x) g(x) dx = \int \hat{f} \hat{g} d\xi$ for $f = p$ while $g = g_1 * g_2$ with $g_l(x) = x_l p(x)$ for $l = 1, 2$. The right-hand side in (5.3) is then seen to equal the Chern number defined in (5.4). ⊐

It thus remains to compute the Chern number. We will consider several ways to do so in future lectures. Here, we simply estimate the integral defining it above and obtain:

Lemma 5.6 *We have*

$$\mathrm{Ch}[\hat{P}] = \frac{\mathrm{sign}(m) + \mathrm{sign}(\eta)}{2}.$$

Proof. Recall that $\hat{P} = \frac{1}{2}(I - \frac{\hat{H}}{|\hat{H}|})$ so that

$$-2\partial_j \hat{P} = \partial_j \frac{1}{|\hat{H}|}\ \hat{H} + \frac{1}{|\hat{H}|} \partial_j \hat{H}.$$

Since $\mathrm{tr}\, A[B, C] = \mathrm{tr}\, B[C, A]$ for matrices A, B, C, and $[\hat{P}, \partial_j \hat{P}] = \frac{1}{4|\hat{H}|^2}[\hat{H}, \partial_j \hat{H}]$, we observe that we need to compute

$$\mathrm{Ch}[\hat{P}] = \frac{i}{2\pi} \int_{\mathbb{R}^2} \frac{-1}{8|\hat{H}|^3} \mathrm{tr}\, \hat{H}[\partial_1 \hat{H}, \partial_2 \hat{H}] dk.$$

With $\hat{H} = k \cdot \sigma + (m - \eta |k|^2)\sigma_3$, we observe that $\partial_j \hat{H} = \sigma_j - 2\eta k_j \sigma_3$ so that

$$[\partial_1 \hat{H}, \partial_2 \hat{H}] = [\sigma_1, \sigma_2] - 2\eta k_1 [\sigma_3, \sigma_2] - 2\eta k_2 [\sigma_1, \sigma_3],$$

with therefore, using the identity $[\sigma_i, \sigma_j] = 2i\varepsilon^{ijk}\sigma_k$ (for instance $[\sigma_1, \sigma_2] = 2i\sigma_3$) with ε^{ijk} the (Levi-Civita) antisymmetric tensor such that $\varepsilon^{111} = 1$,

$$\mathrm{tr}\, \hat{H}[\partial_1 \hat{H}, \partial_2 \hat{H}] = 4i\big((m - \eta|k|^2) + 2\eta|k|^2\big) = 4i(m + \eta|k|^2).$$

As a consequence,

$$\mathrm{Ch}[\hat{P}] = \frac{1}{4\pi} \int_{\mathbb{R}^2} \frac{m + \eta|k|^2}{(|k|^2 + (m - \eta|k|^2)^2)^{\frac{3}{2}}} dk = \frac{1}{2} \int_0^\infty \frac{(m + \eta r^2) r}{(r^2 + (m - \eta r^2)^2)^{\frac{3}{2}}} dr =: I$$

We verify that the above integral equals $\frac{1}{2}(\,\mathrm{sign}(m) + \mathrm{sign}(\eta))$. Indeed following [31] and introducing

$$\cos\theta = \frac{m - \eta r^2}{(r^2 + (m - \eta r^2)^2)^{\frac{1}{2}}},$$

with $r dr = \frac{1}{2} dr^2$, we observe that

$$I = \int_0^\infty -\frac{\partial \cos\theta}{\partial r^2} \frac{1}{2} dr^2 = -\frac{1}{2} \int_\alpha^\beta d\cos\theta = \frac{1}{2}(\cos\beta - \cos\alpha)$$

where $\alpha = \frac{\pi}{2}(1 - \operatorname{sign}(m))$ is obtained in the limit $r^2 \to 0$ while $\beta = \frac{\pi}{2}(1 + \operatorname{sign}(\eta))$ is obtained in the limit $r^2 \to \infty$.

Such integrals provide a quick way to compute invariants, for instance numerically since the above integrals are absolutely convergent. We will consider other, more practical, ways to compute the invariant that will elucidate the above change of variables, which appears to come out of nowhere in the above derivation. ⌶

The above steps also conclude the proof of Theorem 5.2.

Remark 5.7 (Un-regularized Dirac.) For the un-regularized Dirac operator with $\eta = 0$, we verify that $\hat{P}(\xi)$ remains singular as $|\xi| \to \infty$ and in particular depends on the direction $\hat{\xi} = \frac{\xi}{|\xi|}$. We are not able to apply the Russo criterion in this setting. In fact, using the above calculations, we also observe that $\mathrm{Ch}[\hat{P}]$ is in fact also defined when $\eta = 0$ and takes the value $\frac{1}{2}\operatorname{sign}(m)$; indeed we find that $\cos\theta \to 0$ as $r \to \infty$ in that setting. Therefore, $\mathrm{Ch}[\hat{P}]$ is not integral-valued and may not be defined as a true invariant. Mathematically, this is related to the fact that $\hat{P}(\xi)$ depends on the direction $\hat{\xi}$ as $|\xi| \to \infty$ so that the plane $\mathbb{R}^2$ cannot be compactified to a sphere while maintaining the continuity of $\hat{P}(\xi)$ on the sphere.

The Dirac operator with $\eta = 0$ therefore cannot be assigned a bulk phase, at least not with our interpretation using the adiabatic transformation between P and UPU^*.

The solution to this issue was to regularize the operator with $\eta \neq 0$. However, the above calculations show that the phase depends on the sign of the regularization.

Topological insulators in practice manifest themselves by peculiar, asymmetric, transport properties at interfaces separating insulators in different 'phases'. We are then not so much interested in the individual phases of the insulators as we are in the *phase differences*. Assume a (modified or not) Dirac operator with mass term m_+ in an upper half space and another operator with mass m_- in the lower half space. The asymmetric transport observed along the interface separating this insulators should depend only on the difference of phases. And indeed, we have that

$$\frac{\operatorname{sign}(m_+) - \operatorname{sign}(m_-)}{2} \in \mathbb{Z}$$

is an integer independent of the sign (or even the presence) of the regularization parameter η.

We will show below in many settings that quantized asymmetric transport is indeed characterized by such a difference even when the phases of both insulators cannot be defined unambiguously (i.e., without regularization). Moreover, even if Chern numbers for each insulator may not be defined as an invariant, we will introduce a notion of *bulk-difference invariant*, which is quantized as an integer, and characterizes the phase transition between the insulators. The bulk-edge correspondence then states quite generally that this bulk-difference invariant is directly related to the quantized asymmetric transport observed at the interface.

Once we have obtained the index of a Fredholm operator $P(H)UP(H)$, we observe that any perturbation V of H such that $P(H+V) - P(H)$ is a compact operator ensures that $P(H+V)UP(H+V)$ remains Fredholm with the same index as PUP. We quantify perturbations V such that an index modeling asymmetric transport remains stable in Lecture 8 and in particular in Remark 8.9; see also [2] for details on the bulk theory of the modified Dirac operator.

6 Lecture 6.

Asymmetric transport in one dimensional setting. We now come back to the asymmetric transport observed at the interface separating two topological insulators. The physical observable characterizing this asymmetric transport is σ_I defined in (1.5). It plays a similar role to describe interface transport as $\sigma_B = \sigma_{xy}$ does for two-dimensional bulk invariants (with B standing for Bulk while I stands for Interface).

Before looking at two-dimensional Hamiltonians, which will occupy us for most of the rest of these lectures, we start with operators in one space dimension. Since the domain wall provides confinement and hence generates a wave guide, the notions of asymmetric transport for unconfined one-dimensional models and for laterally confined two-dimensional models are actually quite similar beyond technical details.

One dimensional transport equation. We thus start with the simplest of one dimensional operators, namely $H = D = D_x$ acting on a domain $\mathfrak{D}(D) = H^1(\mathbb{R}) \subset L^2(\mathbb{R})$ and a self-adjoint operator in that sense. The operator D is the quintessential asymmetric transport model. Solving

$$(D_t + D_x)u(t,x) = 0, \qquad u(0,x) = u_0(x)$$

with $D_t = -i\partial_t$ yields $u(t,x) = e^{-iD_x t}u_0(x) = u_0(x-t)$.

Moreover, this asymmetric transport is robust to (Hermitian) perturbations. Let $V(x)$ be a real-valued (measurable) bounded function and define $W(x) = \int_0^x V(y)dy$. Then

$$D_x + V = e^{-iW} D_x e^{iW}$$

so that

$$(D_t + D_x + V)u(t,x) = 0, \qquad u(0,x) = u_0(x)$$

is solved by $u(t,x) = e^{-iW(x)}e^{-iD_x t}[e^{iW}u_0](t,x) = e^{-iW(x)}e^{iW(x-t)}u_0(x-t)$ with probability density $|u|^2(t,x) = |u_0|^2(x-t)$ as in the unperturbed case.

This operator is not time-reversal symmetric, since applying a time reversion to D produces $-D$. As such, it is somewhat unlikely to describe many meaningful physical problems as a stand-alone operator. But it arguably serves as the simplest model of asymmetric transport.

Edge conductivity. Let us come back to the conductivity

$$\sigma_I[H] = \mathrm{Tr}\, i[H,P]\varphi'(H).$$

It turns out that for $P(x)$ and $\varphi(H)$ reasonable functions, then, as we shall derive,

$$2\pi\sigma_I[D_x] = 2\pi\sigma_I[D_x + V] = 1,$$

reflecting the above asymmetric transport and its robustness. Before obtaining such a result, we consider an object that is in fact even more robust than σ_I as it is related to the index of a Fredholm operator also of the form PUP.

First of all, we observe that a commutator acts as a form of (non-commutative) derivation since $[AB,C] = A[B,C] + [A,C]B$. Here, the role of C is played by $P = P(x)$ the operator of multiplication by $P(x)$. In the definition of σ_I, we need some smoothness conditions on P so that $[H,P]$ is defined as a reasonable (bounded) operator when $H = D$. In the form PUP, P will have to be a projection, for instance $P(x) = \theta(x)$ the Heaviside function. Let $\phi(h) \in C_0^\infty(\mathbb{R})$ and consider

$$\begin{aligned}&[H^{n+1},P]\phi(H) = H[H^n,P]\phi(H) + [H,P]H^n\phi(H) \equiv [H^n,P]H\phi(H) + [H,P]H^n\phi(H)\\ &\equiv (n+1)[H,P]H^n\phi(H) = [H,P](H^{n+1})'\phi(H),\end{aligned}$$

invoking some cyclicity, for instance $A \equiv B$ meaning that $\mathrm{Tr}\, A = \mathrm{Tr}\, B$, and iterating the derivation. We will in fact justify such manipulations later. Since polynomials are dense in many spaces functions with compact support, the above can be generalized to

$$[\psi(H), P]\phi(H) \equiv [H, P]\psi'(H)\phi(H).$$

Let us assume that $\psi(H) = U(H) = e^{2\pi i\varphi(H)}$ a unitary operator and that $\phi(H) = U^*(H) = e^{-2\pi i\varphi(H)}$. Then

$$[U(H), P]U^*(H) \equiv [H, P]2\pi i\varphi'(H)e^{2\pi i\varphi(H)}e^{-2\pi i\varphi(H)} = 2\pi i[H, P]\varphi'(H).$$

Therefore, assuming the quantities are defined, and they better be in cases of interest, then

$$\mathrm{Tr}\,[U(H), P]U^*(H) = 2\pi\sigma_I[H].$$

In the right-hand side, assuming that P is a projector, we recognize the trace of $Q - P$ in the Fedosov formula used in Lecture 4, assuming the latter is trace-class, and hence Index $PUP_{\mathrm{Ran}P}$.

Fredholm operator $P(x)U(H)P(x)$. We now look at the operator PUP when $H = D$. Note an important difference with the two-dimensional case treated in the past two lectures. In the latter case, $P(H)$ was a projector constructed spectrally while U was a unitary operator of multiplication by $u(x) = (x_1 + ix_2)/|x|$. Operators of the form PUP with $P(H)$ and $U(x)$ possibly matrix-valued, are useful in even dimensions (where a confined dimension does not count as a dimension). Operators of the form PUP with $P(x)$ and $U(H)$ possibly matrix-valued, are useful in odd dimensions. For more information on these structures and the related ones of Fredholm modules and spectral triples that permeate the literature on the mathematical analysis of topological insulators, see, e.g., [36] and [2] for its applications to systems of Dirac operators.

So for us, $P(x)$ is a projector, for instance the Heaviside function equal to 1 for $x \geq 0$ and 0 for $x < 0$. And $U(H)$ is a unitary operator, which we take of the form

$$U(H) = e^{i2\pi\varphi(H)}$$

with $\varphi(h)$ a smooth function converging to 0 at $-\infty$ and to 1 at $+\infty$. We could consider more general functions such that $e^{i2\pi\varphi(h)}$ has the same limit at $\pm\infty$ and hence is continuous as an object seen on the one–point compactification of $\mathbb{R}$ given by $\mathbb{R} \cup \infty \cong \mathbb{S}^1$ the unit circle.

We now wish to compute the index of PUP for $H = D$ (and hopefully get 1 for φ as above). We will see that the index is defined for a class of functions φ that is larger than the class for which $Q - P = [U, P]U^*$ is trace-class. Proving the latter statement is in fact not entirely straightforward since criteria allowing us to know when an operator is trace-class are not trivial.

An explicit example. Consider the function $\varphi(h) = \frac{1}{2} + \frac{1}{\pi}\arctan h$ so that

$$U = -e^{i2\arctan D} = \frac{iD + 1}{iD - 1} = I + W, \quad W := \frac{2}{iD - 1}.$$

This operator is invariant under spatial translations so that in the Fourier domain it corresponds to multiplication by $\hat{u}(\xi) = 1 + \hat{w}(\xi)$, where

$$\hat{w}(\xi) = \frac{2}{i\xi - 1} = \frac{-2(1 + i\xi)}{1 + \xi^2}, \qquad w(x) = -2\theta(-x)e^x,$$

with $\theta(x)$ the Heaviside function and $w(x)$ the inverse Fourier transform of $\hat{w}$ (Exercise:check).

We thus verify that

$$\hat{u}^*\partial_\xi \hat{u} = \frac{2i}{1+\xi^2}$$

so that

$$w_1[\hat{u}] = \frac{1}{2\pi i}\int_{\mathbb{R}} \hat{u}^*\partial_\xi \hat{u} d\xi = \frac{1}{\pi}\arctan\xi\Big|_{-\infty}^{\infty} = 1.$$

Here, $w_1[\hat{u}]$ is the **winding number** associated to the operator U with Schwartz kernel $u(x-y)$ and Fourier symbol $\hat{u}(\xi)$.

The winding number is typically defined over the unit circle $\mathbb{S}^1 = [0,2\pi]/_{0\sim 2\pi}$. By some stereographic projection, we may map $\mathbb{R}$ to its one-point compactification $\mathbb{S}^1$ by gluing $\pm\infty$ at $\theta = 0$, say. Let π be the corresponding map from $\mathbb{S}^1$ to $\mathbb{R}$. By construction, $\hat{u}(-\infty) = \hat{u}(\infty)$ so that $\pi^*\hat{u}(\theta) := \hat{u}(\pi\circ\theta)$ is continuous on $\mathbb{S}^1$. Moreover $d\pi^*\hat{u} = \pi^* d\hat{u}$ in their respective variables so that w_1 is the usual winding number of $\pi^*\hat{u}(\theta)$.

The winding number w_1 of an invertible continuous function $\hat{u}$ from $\mathbb{S}^1$ to $\mathbb{C}\backslash\{0\}$ is a topological invariant. It captures the number of times the path $(\theta,\hat{u}(\theta))$ winds around the origin in the complex plane as θ runs from 0 to 2π. This invariant is independent of continuous homotopy deformations of $\hat{u}$. We will come back to such invariants and more general degrees of maps.

So, associated to U is an invariant $w_1[\hat{u}(\xi)]$, which is defined when H has constant coefficients and hence acts as a Fourier multiplier in the Fourier domain. As indicated above, we also want to look at the operator $P(x)U(H)P(x)$.

When $\varphi(h) = \frac{1}{2} + \frac{1}{\pi}\arctan h$, we verify using the above construction that $PUPf = 0$, i.e., f is in the kernel of PUP if and only if f properly normalized is given by

$$f(x) = \theta(x)e^{-x}.$$

We also verify that the kernel of PU^*P is trivial (since it is formally composed of non-normalizable functions). This is a function supported on the positive half line $x > 0$ such that $Pf = f$. When P is replaced by UPU^*, then $\text{Index}\,(P,Q) = -\text{Index}\, PUP_{\text{Ran}P}$ morally models the number of modes going from the right of $x = 0$ to the left of $x = 0$. For $H = D$, we expect $\text{Index}\, PUP_{\text{Ran}P} = 1$ and this is indeed the case: the mode $f(x)$ is mapped by the unitary transformation to $Uf(x) = \theta(-x)e^x = f(-x)$ and verifies $PUf = 0$. We also find that

$$Qu = UPU^*Uf = Uf \quad \text{so that} \quad (P-Q+I)Uf = 0.$$

We thus have an explicit expression relating the kernels of $P-Q\pm I$ with those of PUP and PU^*P for this specific example; see the index of pair of projections considered in Lecture 4. The function Uf is mapped by U^* to $U^*Uf = f$. So a function Uf supported on the left of $x = 0$ is mapped to a function f supported on the right of $x = 0$. This corresponds to the transport of one mode from left to right as expected for $H = D$.

Note, however, that the function $f(x)$ is not smooth at the origin. This is related to the fact that $\hat{w}(\xi)$ converges slowly to 0 as $|\xi| \to \infty$, something that will create technical difficulties.

***PUP* as a Fredholm operator.** The Fedosov theory mentioned earlier in our analysis of the magnetic Schrödinger operator and recalled in the appendices shows that PUP is a Fredholm operator as soon as we can show that ($P-Q$ is compact and) an appropriate power of $P-Q$ is trace-class. We first show that $P-Q$ is Hilbert-Schmidt for a large class of functions $\varphi(h)$ including the one considered above.

Since $P-Q=[P,U]U^*=[P,W]U^*$, it is sufficient to show that $[P,W]$ is Hilbert-Schmidt (HS). Being HS is equivalent to having a square-integrable Schwartz kernel (see appendix). Therefore, with $w(x-y)$ the Schwartz kernel of W,

$$\|[P,W]\|_2^2 = \int_{\mathbb{R}^2} (p(x)-p(y))^2 |w(x-y)|^2 dxdy.$$

When the above integral is finite, then $[P,W]$ is HS. This is the case when $w(x)=2\theta(-x)e^x$ and $p(x)=\theta(x)$ although this requires some verification as p does not decay at infinity. We have

$$\int_{\mathbb{R}^2} (p(x)-p(y))^2 |w(x-y)|^2 dxdy = \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} (p(y+z)-p(y))^2 dy \Big) |w(z)|^2 dz.$$

Now

$$\int_{\mathbb{R}} (p(y+z)-p(y))^2 dy = |z|$$

as one verifies (only when y is between 0 and z is the integrand non-zero and then equal to 1). So, the operator is HS when

$$\int_{\mathbb{R}} |z||w(z)|^2 dz < \infty. \tag{6.1}$$

A sufficient criterion is therefore that $w \in L^2_{\frac{1}{2}}(\mathbb{R})$ where

$$L^2_\delta(\mathbb{R}^d) = \{ f \in L^2(\mathbb{R}^d) \ \text{ such that } \ |x|^\delta f(x) \in L^2(\mathbb{R}^d) \}. \tag{6.2}$$

In such cases, we therefore obtain that $P-Q$ is Hilbert-Schmidt and hence that $PUP_{\mathrm{Ran}P}$ is a Fredholm operator.

Index Computation and Stability. How do we compute Index $PUP_{\mathrm{Ran}P}$? We just saw that $P-Q$ was HS so that $(P-Q)^n$ is trace-class for $n\geq 2$ and in particular $n=3$.

Is $P-Q$ itself trace-class? This is highly unlikely although we do not present any proof (since this is not needed eventually). Looking at $[P,W]$, which would also be trace-class, we might expect to compute the trace as the integral

$$\int_{\mathbb{R}} \Big((p(x)-p(y))w(x-y) \Big)_{y=x} dx.$$

While $p(x)-p(y)$ vanishes on the diagonal, the evaluation $w(0)$ is not defined as w precisely has a jump there. Looking at the kernel of $[P,W]U^*$ means (since $U^*=I+W^*$) looking at an integral of the form $\int (p(x)-p(y))w(x-y))\delta(y-x)dydx$, which again is problematic since $w(z)$ is discontinuous at $z=0$.

We develop an alternative strategy based on the stability of the index.

Consider $U_\varphi = e^{i2\pi\varphi(H)}$ for φ as above and $U_\psi = e^{i2\pi\psi(H)}$ with ψ to be constructed. The operator $PU_\varphi P_{\mathrm{Ran}P}$ is Fredholm and so there is an open neighborhood where every Fredholm operator has the same index. We construct U_ψ in that neighborhood. We want for $\|f\|=1$,

$$\|(U_\varphi - U_\psi)f\| = C\|(e^{i2\pi\hat\varphi(\xi)} - e^{i2\pi\hat\psi(\xi)})\hat f\| \leq C \sup_\xi |e^{i2\pi\hat\varphi(\xi)} - e^{i2\pi\hat\psi(\xi)}|\|f\|.$$

Now we choose $\hat\psi$ smooth equal to $\hat\varphi$ on $(-M,M)$ and equal to -1 on $\xi<-M-1$ and equal to 1 on $\xi\geq M+1$, say. For M large enough, the above supremum is as small as necessary. We clearly have that $PU_\psi P$ is Fredholm (on Ran$P$) and hence the indices are the same. A theory for trace-class operators we devise next applies to U_ψ and we get that the index is given by the winding number of $\hat{u}_\psi$. However, that winding number is clearly the same as that of the initial $\hat{u}_\varphi$ since winding numbers are defined quite generally. So the index of $PUP_{\mathrm{Ran}P}$ is 1 when $\pi\varphi(H) = \arctan H$. Again, it is quite unlikely that $[P,U]P^*$ is trace-class so that we cannot apply the Fedosov formula directly. The above result shows that the Toeplitz operator has the same index for ψ a smooth version of φ.

Index Computation and trace estimates. We just saw above how to approximate an operator PUP with $(P-Q)$ H.S. by an operator for which $\hat{w}(\xi)$ is smooth. When the latter holds, we now prove that $P-Q$ is trace class and that

$$\text{Index } PUP_{\mathrm{Ran}P} = \mathrm{Tr}(Q-P) = \mathrm{Tr}[W,P]U^* = \int_{\mathbb{R}^2} (p(x)-p(y))w(x-y)u^*(y-x)dxdy.$$

This is the integral along the diagonal of the Schwartz kernel of the operator $Q-P$.

Trace-class property. We now find a criterion ensuring that $P-Q$ is trace-class with trace given by the integral of the kernel of $P-Q$ along the diagonal. We use trace-class criteria recalled in the appendices.

Lemma 6.1 *Let $p(x) \in \mathfrak{S}[0,1]$ and $w(x) \in C^3(\mathbb{R})$ such that $\langle x\rangle^\beta \partial^\alpha w \in L^1(\mathbb{R})$ for $\beta > 2$ and all $|\alpha| \leq 3$. Then the operator $[P,U] = [P,W]$ with Schwartz kernel $(p(x)-p(y))w(x-y)$ is trace-class with vanishing trace.*

Proof. First, let $\tilde{p} \in \mathfrak{S}[0,1]$ be a smooth switch function. Then $p-\tilde{p}$ is compactly supported. Thus

$$(p(x)-p(y))w(x-y) = (\tilde{p}(x)-\tilde{p}(y))w(x-y) + (p(x)-\tilde{p}(x))w(x-y) - w(x-y)(p(y)-\tilde{p}(y))$$

so that $[P,U]$ may be written as a sum of three corresponding operators. Since $(p(x)-\tilde{p}(x))$ is bounded and compactly supported and $\hat{w} \in L^2_\delta$ for some $\delta > \frac{1}{2}$ by assumption on w, we apply criterion A.5 to the above last two terms and obtain that the operator $[P-\tilde{P},U]$ is trace-class.

We may now assume that $p(x)$ is a smooth function. We write $p(x)-p(y) = p(x)(1-p(y)) - p(y)(1-p(x))$ and consider the operator with kernel $b(x,x-y,y) = (1-p(x))p(y)w(x-y)$. We observe that $b(x,x-y,y)$ indeed vanishes when x is large or when $-y$ is large. The estimates on w allow us to verify (A.3) and hence obtain that $A = [P,W]$ is trace-class.

It remains to show that the trace of $[P,U]$ vanishes. When P is smooth, we apply Lemma A.4 and integrate $(p(x)-p(y))w(x-y))$ along the diagonal, which vanishes. When p is not smooth, we approximate it by a smooth function p_ε. The approximation leads to an error corresponding to kernels of the form $(p-p_\varepsilon)(x)w(x-y)$. Since $p-p_\varepsilon$ is arbitrarily small in L^2_δ for $\delta > \frac{1}{2}$, we may apply criterion A.5 to obtain that the difference of traces for $[P,U]$ and $[P_\varepsilon,U]$ is negligible. This shows that the trace of $[P,U]$ also vanishes for an arbitrary $p(x) \in \mathfrak{S}[0,1]$. ⊓⊔

Note that none of the operators P, W, PW, or WP are trace-class. Still, $[P,W] = [P,U]$ is trace-class with a vanishing trace.

We have obtained that $[P,U]$ was trace-class. Since U^* is bounded, $[P,U]U^*$ is also trace-class. The kernel of the operator is $a(x,z) = \int (p(x)-p(y))u(x-y)u^*(y-z)dy$. When p is smooth, then $a(x,y)$ is jointly continuous in (x,y) so that following Appendix A the trace of $[P,U]U^*$ is given by the integral of $a(z,z)$ over $\mathbb{R}$. When $p(x)$ is not smooth, we may approximate it by a smooth p_ε up to a negligible error and pass to the limit as $\varepsilon \to 0$ to obtain that

$$\operatorname{Tr}[P,U]U^* = \int_{\mathbb{R}^2} (p(x)-p(y))u(x-y)u^*(y-x)dxdy.$$

When $P^2 = P$ is a projector, then we have that the trace also provides an expression for the index of the Fredholm operator:

$$\operatorname{Tr}[P,U]U^* = -\operatorname{Index} PUP_{\operatorname{Ran}P} = \int_{\mathbb{R}^2} (p(x)-p(y))u(x-y)u^*(y-x)dxdy.$$

Note that for $w(x-y) = -2e^{x-y}\theta(y-x)$ for $U = (iH) + 1/(iH-1)$, then $\hat{w}(\xi)$ is not in L^1 and the above Lemma does not apply directly.

Traces and winding numbers. It remains to compute $-\operatorname{Index} PUP_{\operatorname{Ran}P}$. We identify it with the winding number of u as follows:

$$\begin{aligned}\int_{\mathbb{R}^2} (p(x)-p(y))u(x-y)u^*(y-x)dxdy &= \int_{\mathbb{R}}\int_{\mathbb{R}} [p(x)-p(x+X)]dxu(X)u^*(-X)dX \\ = \int_{\mathbb{R}} Xu(X)u^*(-X)dX = \frac{i}{2\pi}\int_{\mathbb{R}} \partial_\xi \hat{u}(\xi)u^*(\xi)d\xi &= \frac{-1}{2\pi i}\int_{\mathbb{R}} \hat{u}^*(\xi)d\hat{u}(\xi) = -w_1[\hat{u}],\end{aligned}$$

the winding number of $\hat{u}$, the Fourier transform of $u(x)$. In the middle, we used a Parseval identity along with $\partial_\xi \to -iX$, to move to an integral over dual Fourier variables.

Winding number and spectral flow. Let H be an operator invariant by translation in x and with Fourier transform the family of operators $\xi \to \hat{H}(\xi)$. The eigenvalues $E_j(\xi)$ of $\hat{H}(\xi)$ may typically be chosen continuous and for $H = D$, we have $\hat{H}(\xi) = \xi = E(\xi)$ with one branch.

The *spectral flow* of H, denoted by $\operatorname{SF}(H)$, is the number of times the branches $E_j(\xi)$ cross a given value, say $E = 0$, with crossing counted positively when $E_j'(\xi) > 0$ and counted negatively when $E_j'(\xi) < 0$. We assume that no $E_j(\xi)$ has 0 as a critical value (i.e., at those values of ξ such that $E_j'(\xi) = 0$, then $E_j(\xi) \neq 0$). The spectral flow is easily seen to be topological insofar as it is independent of continuous deformations of the branches $E_j(\xi)$ so long as the limits at $\pm\infty$, which are assumed different from 0, are kept constant during the deformation.

We recall that $\hat{u}(\xi) = e^{i2\pi\varphi(\xi)}$ since $\hat{H} \equiv \xi$ for $H = D$ with $\varphi(E)$ a continuous function of E going from 0 at $-\infty$ to 1 at $+\infty$. The winding number of $\hat{u}$ may thus be identified with the spectral flow of H.

For $H = D$ so that $E(\xi) = \xi$ the unique branch of spectrum, we clearly observe that $\operatorname{SF}(D) = 1$, which is also the winding number of $\hat{u}$.

Winding number and edge current. Let us come back to the observable modeling edge current:

$$\sigma_I[D] = \operatorname{Tr} i[D,P]\varphi'(D).$$

We now have to assume that $P(x)$ is a smooth(er) function with $P(x) = \theta(x)$ outside of a compact domain, say, so that $P'(x)$ is defined and compactly supported. Then $A = i[D,P]\varphi'(D) = P'(x)\varphi'(D)$ and the trace-class criterion (A.5) applies provided $P'(x) \in L^2_\delta$ and $\varphi'(\xi) \in L^2_\delta$ as well for $\delta > \frac{1}{2}$. When both functions are smooth and compactly supported, these assumptions clearly hold and A is indeed trace-class. Moreover, since the kernel of A is smooth, we may apply (A.2) to obtain, using the notation $\check{\psi}(x)$ for the inverse Fourier transform of φ' that

$$2\pi\sigma_I = 2\pi\mathrm{Tr}A = 2\pi\int_{\mathbb{R}} P'(x)\check{\psi}(0)dx = 2\pi\check{\psi}(0) = \int_{\mathbb{R}}\varphi'(h)dh = \frac{1}{2\pi i}\int_{\mathbb{R}} e^{-2\pi i\varphi}de^{2\pi i\varphi} = 1.$$

So, the asymmetric transport associated to D is again obtained as a winding number associated to $\varphi(\xi)$ (more precisely the winding number of $\hat{u} = e^{i2\pi\varphi}$). This also shows, for this specific example, that $2\pi\sigma_I = \text{Index } PUP_{\mathrm{Ran}P}$. Note, however, that we have different notions of $P(x)$ in these two cases: a smooth version to construct $2\pi\sigma_I$ and a more singular version to construct Index $PUP_{\mathrm{Ran}P}$ since P needs to be a projector. When P is a smooth approximation of $\tilde{P}(x) = \theta(x)$ the Heaviside function, we have to replace Index $PUP_{\mathrm{Ran}P}$ by Index $PUP + I - P$, which is defined and equal to Index $\tilde{P}U\tilde{P}$. The reason is that $P - \tilde{P}$ has compact support and as one verifies $(P - \tilde{P})W$ is a compact perturbation (and even trace-class applying (A.5)). So, in summary,

$$2\pi\sigma_I = 2\pi\mathrm{Tr}\, i[D,P]\varphi'(D) = \mathrm{Index}\, PUP + I - P = \mathrm{Index}\, \tilde{P}U\tilde{P}_{\mathrm{Ran}\tilde{P}} = w_1[\hat{u}] = 1.$$

This is also given by $\mathrm{Tr}(\tilde{Q} - \tilde{P}) = \mathrm{Tr}[U,\tilde{P}]U^* = \mathrm{Tr}[U,P]U^*$ as one verifies. However, there does not seem to be a direct relation between Index $PUP + I - P$ and $\mathrm{Tr}[U,P]U^*$ when P is a smooth function.

Summary. For $U = I + W$ such that $[P,U] = [P,W]$ is Hilbert-Schmidt, we have that $PUP_{\mathrm{Ran}P}$ is a Fredholm operator. Approximating w by $\tilde{w}$ sufficiently smooth, we obtain that Index $PUP_{\mathrm{Ran}P} =$ Index $P\tilde{U}P_{\mathrm{Ran}P}$ by stability of the index. We next obtain that $[P,\tilde{W}]$ and hence $[P,\tilde{U}]\tilde{U}^*$ is trace-class with trace given by the integral along the diagonal of the operators' Schwartz kernel. That integral is then shown to equal the winding number of $\hat{\tilde{u}}$, and hence by stability of the winding number, also that of $\hat{u} = e^{i2\pi\varphi}$.

This provides a method to show that the index associated to $U = \frac{iD+1}{iD-1}$ equals 1 even though $P - Q$ may not be trace-class.

When P is a smoother function, we may define $2\pi\sigma_I = 2\pi\mathrm{Tr}i[H,P]\varphi'(H)$ and obtain that the latter quantity also equals the winding number of $\hat{u}$, which is 1.

7 Lecture 7.

Dirac operator and asymmetric transport. We now consider asymmetric transport properties for the specific example of two dimensional Dirac operators. Since a regularization $\eta\Delta$ is not needed here and would complicate a number of calculations, we assume $\eta = 0$ and consider the model

$$H = D\cdot\sigma + m(y)\sigma_3.$$

As we did in the first lecture, it is convenient to consider the operator QHQ, still called H and given by

$$H = D_x\sigma_3 - D_y\sigma_2 + m(y)\sigma_1 = \begin{pmatrix} D_x & \mathfrak{a} \\ \mathfrak{a}^* & -D_x \end{pmatrix}, \qquad \mathfrak{a} = \partial_y + m(y), \quad \mathfrak{a}^* = -\partial_y + m(y)$$

with $\mathfrak{a}^*$ a formal adjoint to $\mathfrak{a}$. Since H is invariant by translation in the x variable, we consider the partial Fourier transform $\hat{H} = \mathcal{F}_{x\to\xi}H\mathcal{F}^{-1}_{\xi\to x}$ with

$$\hat{H} = \hat{H}(\xi) = \begin{pmatrix} \xi & \mathfrak{a} \\ \mathfrak{a}^* & -\xi \end{pmatrix}, \qquad \hat{H}^2(\xi) = \begin{pmatrix} \xi^2 + \mathfrak{a}\mathfrak{a}^* & 0 \\ 0 & \xi^2 + \mathfrak{a}^*\mathfrak{a} \end{pmatrix}.$$

More precisely, we define the direct sum $H = \mathcal{F}^{-1}\int_{\mathbb{R}}^{\oplus} \hat{H}(\xi)d\xi\mathcal{F}$ over the direct sum $L^2(\mathbb{R}^2;\mathbb{C}^2) = \int_{\mathbb{R}}^{\oplus} L^2_\xi(\mathbb{R};\mathbb{C}^2)d\xi$ where $L^2_\xi(\mathbb{R};\mathbb{C}^2)$ is the space of functions $e^{ix\xi}f(y)$ with $f(y)\in L^2(\mathbb{R};\mathbb{C}^2)$ and $\hat{H}(\xi)$ acts on a domain in $L^2_\xi(\mathbb{R};\mathbb{C}^2)$ with range in the latter space.

We saw in the first lecture that when $e^{-M(y)}\in L^2(\mathbb{R})$ for $M(y)=\int_0^y m(z)dz$, then we could construct modes that were localized close to the axis $x=0$. To construct $U(H)$, we need more information on the spectrum of H and the family $\hat{H}(\xi)$.

To simplify derivations, we assume that $m(y)$ is a bounded Lipschitz function equal to m_+ when $y>L$ and equal to m_- when $y<-L$ for some $m_0=\min\{|m_+|,|m_-|\}>0$ and some L.

Let now φ be a smooth non-decreasing function such that $\varphi'(h)$ is supported in $(-m_0,m_0)$ while $\varphi(-m_0)=0$ and $\varphi(m_0)=1$. As in the preceding lecture, we denote

$$U[H]=e^{i2\pi\varphi(H)},\qquad W[H]=U[H]-I.$$

Let P be the operator of multiplication by $p(x,y)=\theta(x)$ the Heaviside function independently of y. The objective of this lecture is to show the following.

Theorem 7.1 *Let H and φ be as described above. Then $PUP_{\mathrm{Ran}P}$ is a Fredholm operator and*

$$\mathrm{Index}\; PUP_{\mathrm{Ran}P} = \frac{1}{2}\big(\,\mathrm{sign}(m_-) - \,\mathrm{sign}(m_+)\big).$$

Note that this is the same index as $\mathrm{Index}\,\mathfrak{a}$ introduced in the first lecture.

Remark 7.2 *This is a bulk-edge correspondence: The edge asymmetric transport given by the index is quantized and given by the difference of bulk properties of the two bulks $y>0$ and $y<0$.*

The rest of this lecture is devoted to a proof of this result.

By spectral calculus recalled in the appendix, $\mathcal{F}^{-1}f(H)\mathcal{F}=f(\mathcal{F}^{-1}H\mathcal{F})$ for $f\in C_c^\infty(\mathbb{R})$ so that

$$W[H]=\mathcal{F}^{-1}\int_{\mathbb{R}}^{\oplus} W[\hat{H}(\xi)]d\xi\;\mathcal{F}.$$

Note that W has compact support in $(-m_0,m_0)$ by construction of φ. It is therefore sufficient to identify the spectrum of $\hat{H}(\xi)$ in that range and from the expression for $\hat{H}^2$, the spectrum of $\mathfrak{a}^*\mathfrak{a}$.

Lemma 7.3 *The part of the spectrum of $\mathfrak{a}^*\mathfrak{a}$ in the interval $[0,m_0^2)$ is discrete and composed of a finite number of distinct J eigenvalues $0<\lambda_j\le m_0^2$ for $1\le j\le J$ plus a one-dimensional contribution $\lambda_0=0$ when the index ε in the above theorem is $\varepsilon=\pm1$.*

Associated are eigenvalues of $\hat{H}(\xi)$ given by

$$E_m(\xi)=\pm(\lambda_j+\xi^2)^{\frac{1}{2}},\qquad m=(\pm,j)\tag{7.1}$$

and rank-one projectors $\Pi_m(\xi)=\psi_m(\xi,y)\otimes\psi_m(\xi,y)$. The eigenvectors $\psi_m(\xi,y)$ decay rapidly as $|y|\to\infty$ and are real-analytic in ξ in the interval $(-m_0,m_0)$. Associated to $\lambda=0$ is an eigenvalue $E_0(\xi)=\varepsilon\xi$ for $|\xi|<m_0$ and an eigenvector $\psi_0(y)$ independent of ζ and exponentially rapidly decaying as $|y|\to\infty$.

This lemma is proved in [2, Lemma 3.10] for the above Dirac operator as well as the modified one when $\eta\neq0$. In the latter case, ψ_0 also depends on ξ. We avoid these complications by assuming that $\eta=0$. The proof is based on an analysis of the standard Sturm-Liouville operator $\mathfrak{a}^*\mathfrak{a}$. We sketch it here.

Proof. Assume $|m_+|=|m_-|=m_0$ to slightly simplify the presentation. The positive operator

$$\mathfrak{a}^*\mathfrak{a}=-\partial_y^2+m_0^2+v(y)=(-\partial_y^2+m_0^2)(I+(-\partial_y^2+m_0^2)^{-1}v(y))$$

with $v(y)=m^2(y)-m_0^2-m'(y)$, has essential spectrum in $[m_0^2,\infty)$ and discrete spectrum in $[0,m_0^2)$ since $v(y)$ is a bounded function compactly supported and $(-\partial_y^2+m_0^2)^{-1}v(y)$ is a compact operator. We use here the fact that by Weyl's theorem (see below), the essential spectrum of an operator is stable with respect to compact perturbations. So, the spectrum of $\mathfrak{a}^*\mathfrak{a}$ is necessarily discrete in $[0,m_0^2)$. (The discrete spectrum is the spectrum composed of isolated eigenvalues with finite multiplicity. The essential spectrum is the complement of the discrete spectrum in the spectrum.)

We denote by $0<\lambda_j<m_0^2$ for $1\leq j\leq J$ the eigenvalues in that range of energies. It is a standard property of Sturm-Liouville operators that the eigenvalues are simple (although this is not really used in the proof). It is well known that the positive eigenvalues of $\mathfrak{a}^*\mathfrak{a}$ and $\mathfrak{a}\mathfrak{a}^*$ are the same. We already identified the vanishing eigenvalue of $\mathfrak{a}$ when $\varepsilon=-1$ or of $\mathfrak{a}^*$ when $\varepsilon=1$ in the first lecture. This gives the branches (7.1). For the Dirac equation, the spectrum is characterized by

$$\xi\psi_1+\mathfrak{a}^*\psi_2=E_m(\xi)\psi_1,\quad \mathfrak{a}\psi_1-\xi\psi_2=E_m(\xi)\psi_2.$$

Since $|E|>|\xi|$ for $n\neq 0$, this gives a unique normalized eigenvector $\psi_m(\xi,y)=(\psi_{m1},\psi_{m2})^t(y,\xi)$, for instance a normalized version of $\mathfrak{a}\mathfrak{a}^*\psi_2=(E^2-\xi^2)\psi_2$ while $\psi_1=(\xi-E(\xi))^{-1}\mathfrak{a}^*\psi_2$.

Since we are in a one-dimensional setting, it is a classical result that $E_m(\xi)$ and $\psi_m(\xi,y)$ are analytic in ξ [29, Theorems VII.1.7 and VII.1.8]. The exponential decay in y comes from the ordinary differential equation $(-\partial_y^2+m_0^2)\psi_2=0$ satisfied outside of a compact interval. ∎

Theorem 7.4 (Weyl criterion) *Let A and B be self-adjoint operators such that $(A+i)^{-1}-(B+i)^{-1}$ is compact. Then*

$$\sigma_{\rm ess}(A)=\sigma_{\rm ess}(B).$$

We can replace i above by any $\mathbb{C}\ni z\notin\mathbb{R}$. We use it above for $\mathfrak{a}^*\mathfrak{a}+i=\mathfrak{a}_0^*\mathfrak{a}_0+i+v$ so that $(\mathfrak{a}_0^*\mathfrak{a}_0+i)^{-1}=(\mathfrak{a}^*\mathfrak{a}+i)^{-1}(I+v(\mathfrak{a}_0^*\mathfrak{a}_0+i)^{-1})$ and the above applies to $A=\mathfrak{a}^*\mathfrak{a}$ and $B=\mathfrak{a}_0^*\mathfrak{a}_0=-\partial_y^2+m_0^2$.

The above lemma gives us the expression

$$W[H]=\mathcal{F}^{-1}\int_{(-m_0,m_0)}^{\oplus}\sum_m W(E_m(\xi))\Pi_m(\xi)d\xi\ \mathcal{F}.$$

We then have:

Proposition 7.5 *For H and φ as described above, $[P,W]$ is Hilbert-Schmidt and hence $PUP_{{\rm Ran}P}$ is a Fredholm operator.*

Proof. The above sum over m is finite and hence it is enough to show that each term in $[P,W]$ contributes a HS term. Let the operator corresponding to one such m have a Schwartz kernel valued in 2×2 matrices

$$w(x-x',y,y')=\int_{\mathbb{R}}W(E(\xi))\psi(\xi,y)\psi^*(\xi,y')\frac{1}{2\pi}e^{i(x-x')\xi}d\xi$$

so that the contribution in $[P,W]$ has Schwartz kernel

$$k(x,x',y,y')=(p(x)-p(x'))w(x-x',y,y').$$

We wish to show that $\mathrm{tr} k^* k$ is integrable in (x, x', y, y'). As in the one dimensional setting, changing variables to $x, x - x'$ shows that integration in the first variable gives a contribution $|x - x'|$. It remains to integrate the remaining terms in the $x - x', y, y'$ variables using the regularity results recalled in the preceding lemma to conclude that k is indeed in $L^2(\mathbb{R}^4; \mathbb{C}^{2\times 2})$. We leave the details to the reader. $\square$

It now remains to compute Index $PUP_{\mathrm{Ran}P}$. As in the one-dimensional case, this requires showing that $[U, P]U^*$ is a trace-class operator and then use the above spectral decomposition to show that Index $PUP_{\mathrm{Ran}P}$ is given by the sum of the winding numbers of the various branches of absolutely continuous spectrum. The proof involves a combination of results in [2] and [4].

Step 1: Trace-class estimate. The first step is to show that $[P, W]$ is trace-class. We have

Lemma 7.6 *Assume that W has a Schwartz kernel of the form*

$$w\Big(\frac{1}{2}(x + x'), x - x', y, y'\Big)$$

that is sufficiently smooth in all variables and rapidly decaying in the last three variables, say faster than polynomially.

Then $[P, W]$ is trace class for $P = \theta(x - x_0)$ a Heaviside function. Moreover (A.2) *holds for the operators $[P, W]$ and $[P, W]U^*$.*

Proof. For concreteness we assume $x_0 = 0$. Note that for the problems considered in this section, w depends only on $x - x'$ and not on $\frac{1}{2}(x + x')$. The more general result proves useful when we consider operators that are no longer invariant by translation in x. The proof is exactly the same as that of Lemma 6.1 in the one dimensional setting since the smoothness assumptions in the remaining variables (y, y') allows one to apply Lemma A.4 in the appendix as well. $\square$

Step 2: Winding number estimates. (i) Assume we are given a Hamiltonian H such that the operator $W(H)$ has the following Schwartz kernel:

$$w(x - x'; y, y') = \sum_m \int_{\mathbb{R}} W(E_m(\xi))\psi_m(y, \xi)\psi_m^*(y, \xi)\frac{e^{i(x-x')\xi}d\xi}{2\pi} \tag{7.2}$$

for $W(h)$ smooth and compactly supported and for finitely many m with $E_m(\xi)$ and $\psi_m(y, \xi)$ smooth in ξ and rapidly decaying in y (faster than polynomially for concreteness).
(ii) Assume further that $[P, W] = [P, I + W]$ and $[P, W](I + W)$ are trace-class with corresponding traces given by (A.2).

Hypothesis (i) implies that H on the support of $W(h)$ is decomposed as a finite number of smooth branches of absolutely continuous spectrum. The operator $W[H]$ is therefore not trace-class (or compact). Hypothesis (ii) states that $[P, W]$ now properly localizes in the $x-$variable so as to be trace-class. Since $I + W$ is bounded, $[P, W](I + W)$ is trace-class as well. The final assumption (A.2) is that these traces may be computed as integrals of Schwartz kernels along their diagonal. Then the following result states that the index of PUP is given as the sum of the winding numbers of the spectral branches.

Theorem 7.7 (Index and Spectral flow) *Under hypotheses (i) and (ii) above, we have that*

$$\text{Index } PUP_{\mathrm{Ran}P} = \sum_m w_1[e^{i2\pi\varphi\circ E_m}].$$

We recall that w_1 is the winding number

$$w_1[f] = \frac{1}{2\pi i}\int_{\mathbb{R}} f^*(\xi)df(\xi).$$

Proof. The trace of $[P,W]$ is seen to vanish since $(p(x)-p(y))_{|y=x}=0$. The Schwartz kernel of W is given by (7.2). The kernel of W^* is given by $w^*(x'-x;y',y)$ with $w^*_{ij}=\bar w_{ji}$. The kernel of $[P,W]W^*$ is thus given by

$$t(x,x';y,y') = \int_{\mathbb{R}^{d+1}} (\chi(x)-\chi(x"))w(x-x";y,y")w^*(x'-x";y',y")dx"dy",$$

with $d=1$ here where P corresponds to multiplication by $\chi(x)$. Therefore, $T:=\mathrm{Tr}[P,W]W^*$ is computed by $T=\int_{\mathbb{R}^{d+1}} t(x,x;y,y)dxdy$. Using the change of variables $(x,x")\to(z,x")=(x-x",x")$ with $dxdx"=dzdx"$, and computing $\int_{\mathbb{R}}(\chi(x"+z)-\chi(x"))dx"=z$, we obtain

$$T = \mathrm{tr}\int_{\mathbb{R}^{2d+1}} zw(z;y,y')w^*(z;y,y')dzdydy'.$$

Using the Fourier transform from z to ξ yields by Parseval

$$T = \frac{-\mathrm{tr}}{2\pi i}\int_{\mathbb{R}^{2d+1}} \partial_\xi \hat w(\xi;y,y')\hat w^*(\xi;y,y')d\xi dydy',$$

where $\hat w(\xi;\cdot)$ is the component-wise Fourier transform of $w(x;\cdot)$ given by

$$\hat w(\xi;y,y') = \sum_j W(E_j(\xi))\psi_j(y,\xi)\psi^*_j(y',\xi).$$

If we were in one space dimension, this would be the winding number of w. Here, we have the additional difficulty that the eigenvectors possibly depend on ξ as well as on the spatial variable y.

The derivation ∂_ξ applies to $W\circ E_j$ and to $\psi_j(y,\xi)\psi^*_j(y',\xi)$. Consider the latter contribution at fixed ξ, which is

$$\tau(\xi) := \int \sum_{j,k} \mathrm{tr}\partial_\xi[\psi_j(y,\xi)\psi^*_j(y',\xi)]\psi_k(y',\xi)\psi^*_k(y,\xi)dydy'.$$

We show that $\tau(\xi)=0$. Indeed, we distribute ∂_ξ over the product, exchange y and y' in the second contribution to get (dropping the ξ-dependence to simplify notation)

$$\mathrm{tr}\int \sum_{j,k}[\partial_\xi\psi_j(y)\psi^*_j(y')\psi_k(y')\psi^*_k(y) + \psi_j(y')\partial_\xi\psi^*_j(y)\psi_k(y)\psi^*_k(y')]dydy'.$$

Applying traces to these products of rank-one matrices yields (with $\bar\psi$ as a column vector)

$$\tau(\xi) = \int \sum_{j,k}[\partial_\xi\psi_j(y)\cdot\bar\psi_k(y)\bar\psi_j(y')\cdot\psi_k(y') + \partial_\xi\bar\psi_j(y)\cdot\psi_k(y)\psi_j(y')\cdot\bar\psi_k(y')]dydy'.$$

By orthogonality of the eigenvectors, only the terms $j=k$ survive the integration and then $\tau(\xi)=\int\sum_j \partial_\xi|\psi_j(y)|^2dy = \sum_j\partial_\xi\int|\psi_j(y)|^2dy = 0$ since we assume our eigenvectors normalized.

As a consequence,

$$T = \frac{-\mathrm{tr}}{2\pi i} \sum_{j,k} \int \partial_\xi W \circ E_j(\xi) \psi_j(y,\xi) \psi_j^*(y',\xi) W^* \circ E_j(\xi) \psi_k(y',\xi) \psi_k^*(y,\xi) dy dy' d\xi.$$

Taking traces again and integrating in y and y' yields

$$T = \frac{-1}{2\pi i} \sum_j \int \partial_\xi W \circ E_j(\xi) W^* \circ E_j(\xi) d\xi = -\sum_j w_1[W \circ E_j].$$

This concludes the derivation. ☐

Finalizing the proof of the main Theorem. We have seen that for the Dirac operator, $PUP_{\mathrm{Ran}P}$ satisfies the above estimates and in particular $[U,P]U^*$ is trace-class. It then remains to compute the winding number of each branch of spectrum, or equivalently the spectral flow of H, to observe that each branch E_m in (7.1) for $m \neq 0$ has a vanishing spectral flow corresponding to a vanishing winding number. For $m = 0$, we observe that the branch simply does not exist when m_+ and m_- have the same sign, while the branch is linearly decreasing when $m_- < 0 < m_+$ and linearly increasing when $m_+ < 0 < m_-$. This concludes the proof of the theorem.

Remark 7.8 *(i): The proof applies to operators that are invariant with respect to spatial translations in x. (ii): The operator needs to have a sufficiently explicit spectral decomposition so that the winding numbers of their branches of absolutely continuous spectrum (i.e., the spectral flow of the family $\hat{H}(\xi)$) can be estimated.*

We will address (i) in the next lecture, where we show that Index $PUP_{\mathrm{Ran}P}$ *is stable against a large class of perturbations.*

The issue (ii) is more challenging. While an explicit spectral decomposition (in the range $(-m_0, m_0)$) is rather explicit for Dirac operators, such diagonalizations are still significantly more challenging than for bulk operators, where $\hat{H}(\xi,\zeta)$ becomes a matrix whose diagonalization is often much simpler. How one may compute the index of $PUP_{\mathrm{Ran}P}$ or other invariants associated to asymmetric transport from simpler bulk properties is the objective of bulk-edge correspondences. We will develop such a correspondence in a later lecture and show that some calculations indeed tremendously simplify when asymmetric transport invariants may be written in terms of bulk properties.

Edge invariant and spectral flow. The above calculations relating spectral flows and edge invariants generalize in the following way.

Let L^2_ξ be a space of functions $u(x)$ with values in a fixed Hilbert space $\mathcal{H}'$, for instance $L^2(\mathbb{R}^{d-1}_y;\mathbb{C}^p)$ and locally L^2 in $x \in \mathbb{R}$ and such that $u(x+1) = e^{ix\xi}u(x)$. We then have

$$L^2 = \int_{\mathbb{R}}^{\oplus} L^2_\xi d\xi$$

as the space $\mathcal{H} = L^2(\mathbb{R};\mathcal{H}')$.

We assume a Hamiltonian H commuting with translations in x such that

$$H = \mathcal{F}^{-1} \int_{\mathbb{R}}^{\oplus} \hat{H}(\xi) d\xi \ \mathcal{F}$$

with $\hat{H}(\xi)$ acting as an unbounded operator on $\mathcal{H}'$. We assume the spectral decomposition

$$H = \mathcal{F}^{-1} \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}} E_j(\xi)\Pi_j(\xi) d\xi \mathcal{F}$$

with $\xi \to E_j(\xi)$ smooth branches and $\Pi_j(\xi)$ a rank-one operator in the sense that its Schwartz kernel is

$$\Pi_j(\xi; x, x') = \psi_j(\xi)\psi_j^*(\xi)\frac{1}{2\pi}e^{i(x-x')\xi} = \tilde{\Pi}_j(\xi)\frac{1}{2\pi}e^{i(x-x')\xi}$$

with $\psi_j(\xi) \in L^2_\xi$. More precisely, we have that the projector acts on $f \in L^2$ as

$$\Pi_j(\xi)f(x) = \psi_j(\xi)\frac{1}{2\pi}e^{ix\xi}\int_{\mathbb{R}} e^{-ix'\xi}\big(\psi_j(\xi), f(x')\big)_{L^2_\xi} dx' = \psi_j(\xi)\frac{1}{2\pi}e^{ix\xi}\big(\psi_j(\xi), \hat{f}(\xi)\big)_{L^2_\xi}$$

with $\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)dx$ the Fourier transform.

In fact, all we need is to assume the above decomposition locally in the spectral variable. More precisely, for I a fixed energy interval, we assume that for $\Phi \in C^\infty_c$ supported in I, we have

$$\Phi(H) = \sum_{j\in J} \int_{\mathbb{R}} \Phi \circ E_j(\xi)\Pi_j(\xi) d\xi$$

for J a finite number of branches crossing the interval I. In particular, this implies

$$\widehat{\Phi(H)}\, \psi_j = \mathcal{F}\Phi(H)\mathcal{F}^{-1}\psi_j = \Phi \circ E_j\, \psi_j.$$

Assuming all traces below defined and obtained as integrals along diagonals of Schwartz kernels, we want to compute

$$2\pi\sigma_I = \operatorname{Tr} 2\pi i[H, P]\varphi'(H) = \operatorname{Tr} 2\pi i[\Psi(H), P]\varphi'(H)$$

for $\Psi(h) = h$ on the support of $\varphi'(h)$, as we easily see by cyclicity of the trace. In fact, we can start with the latter definition for the invariant with Ψ compactly supported since $H = \Psi(H)$ on the energy range I of interest.

Note that as for earlier derivations in this Lecture, we have to assume that $i[\Psi(H), P]\varphi'(H)$ is indeed a trace-class operator. Sufficient conditions are given in Lemma A.4 in the appendix.

Using the above spectral decomposition, we find the following expression for the Schwartz kernels in the x-variables:

$$\begin{aligned} 2\pi i[\Psi(H), P](x, x') &= \sum_j \int_{\mathbb{R}} 2\pi i \Psi \circ E_j(\xi)[\Pi_j(\xi), P](x, x')d\xi \\ &= \sum_j \int_{\mathbb{R}} 2\pi i \Psi \circ E_j(\xi)\tilde{\Pi}_j(\xi)\frac{1}{2\pi}(P(x') - P(x))e^{i(x-x')\xi}d\xi, \end{aligned}$$

as well as

$$\varphi'(H)(x', x'') = \sum_k \int_{\mathbb{R}} \varphi' \circ E_k(\xi')\tilde{\Pi}_k(\xi')\frac{1}{2\pi}e^{i(x'-x'')\xi'}d\xi'.$$

Therefore, using Tr' for trace on L^2_ξ and Tr for trace on L^2, we compute as earlier

$$\begin{aligned}
&\mathrm{Tr}\, 2\pi i[\Psi(H),P]\varphi'(H)\\
= \;& \mathrm{Tr}' \sum_{j,k}\int_{\mathbb{R}^2} 2\pi i \Psi\circ E_j(\xi)\tilde\Pi_j(\xi)\varphi'\circ E_k(\xi')\tilde\Pi_k(\xi')\frac{1}{2\pi}(P(x')-P(x))e^{i(x-x')\xi}\frac{1}{2\pi}e^{i(x'-x)\xi'}d\xi d\xi' dx' dx\\
= \;& \mathrm{Tr}' \sum_{j,k}\int_{\mathbb{R}^2} 2\pi i \Psi\circ E_j(\xi)\varphi'\circ E_k(\xi')\tilde\Pi_j(\xi)\tilde\Pi_k(\xi')\frac{1}{(2\pi)^2}(P(x')-P(x'+z))e^{iz(\xi-\xi')}d\xi d\xi' dx' dz\\
= \;& \mathrm{Tr}' \sum_{j,k}\int_{\mathbb{R}^2} \Psi\circ E_j(\xi)\varphi'\circ E_k(\xi')\tilde\Pi_j(\xi)\tilde\Pi_k(\xi')\frac{-iz}{2\pi}e^{iz(\xi-\xi')}d\xi d\xi' dz
\end{aligned}$$

since

$$\int_{\mathbb{R}}(P(x+z)-P(x))dx = z \qquad \text{from the derivative in } z \text{ being equal to 1.}$$

Thus,

$$\begin{aligned}
\mathrm{Tr}\, 2\pi i[\Psi(H),P]\varphi'(H) &= \mathrm{Tr}' \sum_{j,k}\int_{\mathbb{R}^2} \Psi\circ E_j(\xi)\varphi'\circ E_k(\xi')\tilde\Pi_j(\xi)\tilde\Pi_k(\xi')\frac{-1}{2\pi}\partial_\xi e^{iz(\xi-\xi')}d\xi d\xi' dz\\
&= \mathrm{Tr}' \sum_{j,k}\int_{\mathbb{R}} \partial_\xi\big(\Psi\circ E_j(\xi)\tilde\Pi_j(\xi)\big)\varphi'\circ E_k(\xi)\tilde\Pi_k(\xi)d\xi
\end{aligned}$$

after integration by parts in ξ an integration in z yielding a term $\delta(\xi-\xi')$.

We now observe that

$$\partial_\xi\tilde\Pi_j\tilde\Pi_k + \partial_\xi\tilde\Pi_k\tilde\Pi_j = 0, \quad \mathrm{Tr}'\partial_\xi\tilde\Pi_j\tilde\Pi_j = 0.$$

Indeed the first one directly comes from $\partial_\xi(\tilde\Pi_k\tilde\Pi_j)=0$ when $j\neq k$. The second one comes from

$$\partial_\xi\tilde\Pi_j^2 = \partial_\xi\tilde\Pi_j\tilde\Pi_j + \tilde\Pi_j\partial_\xi\tilde\Pi_j = \partial_\xi\tilde\Pi_j$$

so that

$$0 = \mathrm{Tr}'(I-\tilde\Pi_j)\tilde\Pi_j\partial_\xi\tilde\Pi_j = \mathrm{Tr}'\tilde\Pi_j\partial_\xi\tilde\Pi_j(I-\tilde\Pi_j) = \mathrm{Tr}'\tilde\Pi_j^2\partial_\xi\tilde\Pi_j$$

and hence the result.

This implies, since $\tilde\Pi_j$ is assumed to be rank-one,

$$\begin{aligned}
\mathrm{Tr}\, 2\pi i[\Psi(H),P]\varphi'(H) &= \sum_j\int_{\mathbb{R}} \partial_\xi\big(\Psi\circ E_j(\xi)\big)\varphi'\circ E_j(\xi)\mathrm{Tr}'\,\tilde\Pi_j(\xi)d\xi\\
&= \sum_j\int_{\mathbb{R}} \partial_\xi\big(\Psi\circ E_j(\xi)\big)\varphi'\circ E_j(\xi)d\xi.
\end{aligned}$$

Thus, replacing φ' by Φ, we have obtained that, generally,

$$\mathrm{Tr}\, 2\pi i[\Psi(H),P]\Phi(H) = \sum_j\int_{\mathbb{R}} \partial_\xi\big(\Psi\circ E_j(\xi)\big)\Phi\circ E_j(\xi)d\xi. \tag{7.3}$$

When $\Psi(H)=H$ on the support of $\Phi(H)=\varphi'(H)$, we find

$$\mathrm{Tr}\, 2\pi i[\Psi(H),P]\varphi'(H) = \sum_j\int_{\mathbb{R}} E_j'(\xi)\varphi'\circ E_j(\xi)d\xi = \sum_j \varphi_j(+\infty)-\varphi_j(-\infty).$$

This shows that the edge current $2\pi\sigma_I$ is given by the sum of the spectral flows of each individual branches of absolutely continuous spectrum.

The above result is more general. Assuming for instance $\Psi(h) = \frac{1}{2\pi i}U(h)$ and $\Phi(h) = U^*(h)$ for $U(h) = e^{i2\pi\varphi(h)}$, we retrieve the result of Theorem 7.7. It is straightforward to observe that the two quantities are the same and therefore that, generally,

$$2\pi\sigma_I = \mathrm{Tr} 2\pi i[H,P]\varphi'(H) = \mathrm{Tr}[U,P]U^*.$$

We will revisit this equality in more general settings in Lecture 9.

The relation between $2\pi\sigma_I$ and the spectral flow of H in an given energy interval is therefore quite general for Hamiltonians that commute with respect to translations in the x-variable.

8 Lecture 8.

Stability under perturbations. We now consider in more detail the stability of topological invariants under perturbations of the Hamiltonian. In particular, we wish to understand for which perturbations V do we have that indices associated to H and $H+V$ agree. We focus on the invariant $P(x)U(H)P(x)$. We thus need a method to address perturbations $U(H+V)$ in the spectral calculus. A useful framework to do so is the Helffer-Sjöstrand formula; see (C.7) in the Appendix and [18, Chapter 8].

Let W be a smooth function with compact support as used in earlier sections and $\tilde{W}$ an almost analytic extension. Let H be a self-adjoint operator and V be $H-$bounded with bound $\alpha < 1$ so that $H+V$ is also self-adjoint (see Appendix). Then,

$$W(H+V) - W(H) = \frac{-1}{\pi}\int_Z \bar{\partial}\tilde{W}(z)\Big((z-H-V)^{-1} - (z-H)^{-1}\Big)d^2z,$$

where Z is a compact domain.

We now prove the following abstract stability result:

Proposition 8.1 *Let H and $H+V$ be self-adjoint operators such that for each ε, there is V_ε such that $\|V-V_\varepsilon\| \le \varepsilon$ (or more generally $\|(V-V_\varepsilon)(z-H)^{-1}\| \le \varepsilon|\Im z|^{-1}$) and $V_\varepsilon(z-H)^{-1}$ is compact in $\mathcal{I}_p$ for some $p<\infty$ with $\|V_\varepsilon(z-H)^{-1}\|_p \le C_\varepsilon|\Im z|^{-q}$ uniformly for $z \in Z$ for some $q \in \mathbb{N}$. We assume that $P(x)U(H)P(x)$ is a bounded Fredholm operator on the range of $P(x)$ (in an underlying Hilbert space $\mathcal{H}$).*

Then $U(H+V) - U(H)$ is compact and $P(x)U(H+V)P(x)$ is a Fredholm operator on the range of P. Moreover, we have the stability:

$$\mathrm{Index}\ P(x)U(H+V)P(x) = \mathrm{Index}\ P(x)U(H)P(x).$$

Proof. We look at the difference of operators $P(x)U(H+V)P(x) - P(x)U(H)P(x) = P(x)(W(H+V) - W(H))P(x)$ (defined on the range of P) which are defined by spectral calculus since all operators are self-adjoint. We recall the formula

$$B^{-1} - A^{-1} = B^{-1}(A-B)A^{-1} = A^{-1}(A-B)B^{-1} \tag{8.1}$$

for A and B invertible, which we may apply for $B = z-H-V$ and $A = z-H$ to find

$$W(H+V) - W(H) = \frac{-1}{\pi}\int_Z \bar{\partial}\tilde{W}(z)(z-H-V)^{-1}V(z-H)^{-1}d^2z.$$

We approximate V by V_ε as above. The above term is therefore by assumption the uniform limit (i.e., the limit in operator norm) of

$$\tilde{W}_\varepsilon = \frac{-1}{\pi} \int_Z \bar{\partial} \tilde{W}(z)(z - H - V)^{-1} V_\varepsilon (z - H)^{-1} d^2 z.$$

To obtain this result, we assume $N \geq q + 1$ for the smooth almost analytic extension $\tilde{W}$ and recall the standard estimate on resolvent operators:

$$\|(z - \tilde{H})^{-1}\| \leq C|\Im z|^{-1},$$

which holds for any self-adjoint operator $\tilde{H}$. With the above assumptions, $\tilde{W}_\varepsilon$ is compact (whereas neither $W(H + V)$ nor $W(H)$ is) and in the Schatten class $\mathcal{I}_p$ with

$$\|\tilde{W}_\varepsilon\|_p \leq C \int_Z |\bar{\partial}\tilde{W}(z)||\Im z|^{-q-1} d^2 z \leq C_\varepsilon < \infty.$$

As a uniform limit of $\tilde{W}_\varepsilon$, we have that $W(H + V) - W(H)$ is indeed compact. This shows that $PU(H + V)P = PU(H)P + K$ with K compact is Fredholm and the equality of the indices. $\square$

Applications in one-space dimension. We assume: (a) H is an unperturbed (matrix-valued) differential operator with constant coefficients with $\hat{H}(\xi) = \mathcal{F}H\mathcal{F}^{-1}$ for $\xi \in \mathbb{R}^d$ with $\hat{R}(z, \xi) = (z - \hat{H}(\xi))^{-1}$ in $L^p(\mathbb{R}^d)$ with $|\Im z|\|\hat{R}(z)\|_p \leq C$ uniformly for $z \in Z$. We also assume: (b) Let V be the operator of multiplication by $V(x)$. (We can look at more general perturbations but this is the one we consider here.) Moreover, $V \in L^\infty_\varepsilon(\mathbb{R}^d)$ (i.e., a bounded function converging to 0 at infinity). Finally, we assume: (c) $P(x)U(H)P(x)$ is a Fredholm operator on the range of $P(x)$.

The applications are typically to be found for $d = 1$, where (c) above is typically verified but the result is independent of dimension otherwise. With this we can prove the following stability result:

Corollary 8.2 *Let H and $H + V$ be self-adjoint with assumptions (a), (b) and (c) above holding. Then $U(H + V) - U(H)$ is compact and $P(x)U(H + V)P(x)$ is a Fredholm operator on the range of P, with moreover, the stability*

$$\text{Index } P(x)U(H + V)P(x) = \text{Index } P(x)U(H)P(x).$$

Proof. We use the criterion on operators of the form $f(x)g(D)$ showing that the operator is in $\mathcal{I}_p$ when both f and g are in $L^p(\mathbb{R}^d)$. By assumptions (a) and (b) this is the case for $V_\varepsilon(x)(z - H(D))^{-1}$ with $q = 1$. We may now apply the above proposition. $\square$

The above result does not directly apply for Dirac operators since the unperturbed Dirac operator with domain wall is not invariant by translation in both variables. But the above result directly applies for general one-dimensional operators and after some modifications to higher-order operators with confinement.

Lemma 8.3 *Let $H = \mathcal{F}^{-1}\hat{H}\mathcal{F}$ be an operator invariant by translation with symbol $\hat{H}(\xi)$ for $\xi \in \mathbb{R}^d$ such that all eigenvalues $\lambda(\xi)$ of $\hat{H}(\xi)$ satisfy $|\lambda(\xi)| \geq C|\xi|^\alpha$ for $C > 0$, $\alpha > 0$, and $|\xi| \geq \xi_0$. Then for $\alpha p > d$, we have $|\Im z|\|\hat{R}(z)\|_p \leq C$ uniformly for $z \in Z$. This applies to $H = D$ and immediate generalizations such as for instance $H = D \oplus D \oplus (-D)$.*

Proof. Above, $|\hat{H}(\xi)|$ is any norm on finite dimensional matrices. We find the bound, with $z=\lambda+i\mu$,

$$|z-\hat{H}|^{-p} \le \left\{ \begin{array}{cl} C|\mu|^{-p} & |\xi|\le\xi_1, \\ C|\xi|^{-\alpha p} & |\xi|>\xi_1. \end{array} \right.$$

This uses the self-adjointness of $\hat{H}$ so that $\hat{H}$ is diagonalizable. Moreover, the bound is independent of λ in a compact set. As soon as $\alpha p>d$, the above function is integrable with integral bounded by $C|\mu|^{-p}$ so that $|\Im z|\|\hat{R}(z)\|_p\le C$ uniformly for $z\in Z$. The theory applies for D with $\alpha=1$. ∎

Consider for instance the operator

$$H = D\oplus D\oplus(-D)$$

acting on $H^1(\mathbb{R})\oplus H^1(\mathbb{R})\oplus H^1(\mathbb{R})$. This is an operator such that $P(x)U(H)P(x)$ has index $1+1-1=1$. Now, let $V(x)$ be a 3×3 matrix-valued multiplication operator such that all components tend to 0 at infinity. Then $H+V$ is a self-adjoint operator as one verifies and the index of $PU(H+V)P$ is also equal to 1. Note that the latter operator is non-trivial in the sense that the perturbation V generates scattering unlike the scalar case $D+V$, which may be transformed to D by gauge transformation (integrating factor in a Lie group $U(1)$ exponentiating a Lie algebra $i\mathbb{R}$ when one looks at it carefully).

The result is also (somewhat) surprising. For a topologically trivial (and time-reversal symmetric) operator such as $D\oplus(-D)+V$, we have the phenomenon of *Anderson localization*, stating that the transmission through a slab where V is supported is exponentially small as V increases (if the above model comes from a wave equation, then V admits off-diagonal terms coupling the two components of the equation). However, Index $PUP=1$ implies quantized asymmetric transport from left to right independent of the level of randomness V. In other words, *the nontrivial topology acts as a quantized obstruction to Anderson localization.* See [3, 36] for references on this topic.

Dirac operator with domain wall. Let us come back to

$$H = D\cdot\sigma + m(y)\sigma_3.$$

We assume that $m(y)$ converges to $m_\pm$ at $\pm\infty$ with $m'(y)$ compactly supported and $|m_+|=|m_-|=m_0>0$. For V a perturbation, we still would like to show that $W(H+V)-W(H)$ is compact. It remains to show that $V_\varepsilon(z-H)^{-1}$ is in $\mathcal{I}_p$ to apply the above proposition. We can in fact bring ourselves to the setting of a constant-coefficient operator by writing $H=H_0+m(y)\sigma_3$ knowing that $m(y)\sigma_3$ is a bounded operator. Then

$$(z-H)^{-1} = (z-H_0)^{-1} + (z-H_0)^{-1}m(y)\sigma_3(z-H)^{-1}$$

We have $m(y)\sigma_3(z-H)^{-1}$ bounded by $C|\Im z|^{-1}$ so that $V(z-H)^{-1}=V(z-H_0)^{-1}B$ with B bounded by $C|\Im z|^{-1}$. We thus apply the above Lemma for H_0 with $\alpha=1$ and $d=2$ to get that $V_\varepsilon(z-H)^{-1}$ satisfies the hypotheses of the above proposition for $p>2$ with $q=2$. Therefore the proposition gives the:

Corollary 8.4 (Stability of Dirac operator with domain wall.) *Let $H=D\cdot\sigma+m(y)\sigma_3$ and assume that $V\in L^\infty_\varepsilon(\mathbb{R}^2;\mathbb{C}^2\times\mathbb{C}^2)$ is Hermitian. Then $H+V$ has an invariant given by the index of the Fredholm operator $P(x)U(H+V)P(x)$ equal to $\frac{1}{2}(\operatorname{sign}(m_-)-\operatorname{sign}(m_+))$.*

The same type of derivation provides stability against variations in the domain wall. Note that such variations are not in $L^\infty_\varepsilon(\mathbb{R}^2)$.

Let $m_1(y)$ be a Lipschitz switch function in $\mathfrak{S}(m_-, m_+)$ as used in the derivation of Proposition 7.5 for instance. We assume $|m_-| \geq m_0$ and $|m_+| \geq m_0$ for $m_0 > 0$.

Let $m_2(y)$ be another Lipschitz switch function in $\mathfrak{S}(m_-, m_+)$. Then we have the result:

Corollary 8.5 (Stability of Dirac operator against domain wall perturbations.) *Let $H_j = D \cdot \sigma + m_j(y)\sigma_3$ with m_j Lipschitz switch functions in $\mathfrak{S}(m_-, m_+)$ with $|m_-| \geq m_0$ and $|m_+| \geq m_0$ for $m_0 > 0$. Let φ a smooth switch function in $\mathfrak{S}[0, 1; -m_0, m_0]$. Then $PU(H_j)P$ are Fredholm operators on* $\mathrm{Ran}P$ *and their indices agree.*

Proof. Define $U_t = U(H_t)$ with $H_t = D \cdot \sigma + m_t\sigma_3$ for $m_t = (1-t)m_1 + tm_2$, $t \in [0, 1]$. We know that PU_tP is Fredholm from Proposition 7.5. We write

$$U(H_t) - U(H_s) = -\frac{1}{\pi}\int_Z \partial\tilde{W}(z)(z - H_t)^{-1}(H_t - H_s)(z - H_s)^{-1}d^2z. \tag{8.2}$$

Since $H_t - H_s = (m_t - m_s)\sigma_3$ is an operator bounded by $C|t-s|$ in the uniform sense, we deduce that $U(H_t) - U(H_s)$ is also bounded by $C|t-s|$. Thus, $t \mapsto PU_tP$ is a continuous family of Fredholm operators. Their index is therefore independent of t. $\square$

This completes the analysis of the asymmetric transport associated to Dirac equations with domain wall. We obtained an invariant, the index of $P(x)U(H)P(x)$, and showed that it was stable against a large class of perturbations V and variations in the domain wall $m(y)$. We have assumed that the domain wall was Lipschitz to simplify the diagonalization of $\mathfrak{a}^*\mathfrak{a}$. Such results may be extended to less regular functions $m(y)$. An alternative method is as follows.

Let $m(y)$ be a function in $L^p(\mathbb{R})$ for $p > 2$ so that $m(y)$ maps $H^1(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$ by Sobolev imbedding. Then $H = D \cdot \sigma + m(y)\sigma_3$ remains a self-adjoint operator with domain $H^1(\mathbb{R}^2; \mathbb{C}^2)$ since $H \pm i$ is still invertible from $L^2(\mathbb{R}^2; \mathbb{C}^2)$ to $H^1(\mathbb{R}^2; \mathbb{C}^2)$. Now consider a sequence of Lipschitz functions $m_k(y)$ that converge to $m(y)$ in $L^p(\mathbb{R})$ knowing that m and m_k agree outside of a compact interval independent of k. Then, defining U_k as the unitary constructed with m replaced by m_k, we have

$$\begin{aligned} U - U_k &= -\tfrac{1}{\pi}\int_Z \bar{\partial}\tilde{W}(z - H)^{-1}(H - H_k)(z - H_k)^{-1}d^2z \\ &= -\tfrac{1}{\pi}\int_Z \bar{\partial}\tilde{W}(z - H)^{-1}(m - m_k)\sigma_3(z - H_k)^{-1}d^2z. \end{aligned}$$

The uniform error $U - U_k$ is therefore bounded by $(m - m_k)\sigma_3(z - H_k)^{-1}$. Since $(z - H_k)^{-1}$ regularizes from L^2 to H^1 (with a bound that blows up in a power of $|\Im z|$) and $m - m_k$ is small as an operator from H^1 back to L^2, we obtain that $U - U_k$ converges to 0 uniformly, and hence so does $[P, U - U_k]$. Since $[P, U_k]$ is compact, we obtain that $[P, U]$ is compact as well so that PUP is a Fredholm operator. Moreover, the index is clearly the same as that of PU_kP. This shows that the index is defined and stable against perturbations of the domain wall in $L^p(\mathbb{R})$ for $p > 2$. In particular, the domain wall may be chosen arbitrarily in $L^\infty(\mathbb{R})$ so long as it takes prescribed values outside of a compact interval.

Summary of topological classification for Dirac operators. Consider the Dirac operator

$$H = D \cdot \sigma + m(y)\sigma_3 + V(x, y) \tag{8.3}$$

with $m(y)$ a domain wall given by a switch function in $\mathfrak{S}(m_-, m_+)$ with $\min(|m_-|, |m_+|) = m_0 > 0$ and with $V(x, y) \in L^\infty_\varepsilon$. We assume $m(y) \in L^p_{\mathrm{loc}}$ for $p > 2$.

Step (i): Self-adjointness. We first observe that H is an unbounded self-adjoint operator on $L^2(\mathbb{R}^2; \mathbb{C}^2)$ with domain $\mathfrak{D}(H) = H^1(\mathbb{R}^2; \mathbb{C}^2)$. This is because $m(y)\sigma_3 + V(x, y)$ maps H^1 to L^2 using the above assumptions.

Let now φ be a smooth switch function in $\mathfrak{S}(0,1,-m_0,m_0)$.

Step (ii): Fredholmness of PUP. Let us first assume $V=0$. We may then approximate $m(y)$ by a sequence of smooth domain walls $m_k(y)$ in $\mathfrak{S}(m_-,m_+)$ and obtain that $U(H_k)$ converges to $U(H)$ uniformly, with an obvious definition for H_k. We proved that $[P,U(H_k)]$ was compact in Proposition 7.5. Therefore, $[P,U(H)]$ is also compact and $PU(H)P$ is thus a Fredholm operator. As in the proof of Corollary 8.5, we have that $m(y)\sigma_3(\lambda-H)^{-1}$ is a bounded operator on $L^2(\mathbb{R}^2;\mathbb{C}^2)$ with the above assumptions on $m(y)$. Note $m(y)$ is not necessarily bounded as a function but is bounded as an operator from H^1 to L^2 while $(\lambda-H)^{-1}$ is bounded from L^2 to H^1 with a bound $C|\Im z|^{-1}$. Therefore Lemma 8.3 applies to $D\cdot\sigma$ and we then use Proposition 8.1 to show that V generates a relatively compact perturbation so that $PU(H+V)P$ is Fredholm.

This shows that for H in (8.3) we have that $PU(H)P$ is a Fredholm operator. We now compute its index. We know that $m(y)$ may be replaced by a smooth approximation m_k without changing the index. We then use Corollary 8.5 to obtain that $\operatorname{Index} PU(H)P = \operatorname{Index} PU(H_0)P$, where $H_0 = D\cdot\sigma + m_0(y)\sigma_3$ for m_0 any smooth switch function in $\mathfrak{S}(m_-,m_+)$.

Step (iii): Trace-class property of $[P,U]$. The next step is to relate the index to the computation of a trace given as an integral (A.2). This is done using Lemma 7.6.

Step (iv): Computing the index as a winding number. We may finally choose m_0 to be a monotone function switching from m_- to m_+ on a compact interval. It remains to compute the index of $PU(H_0)P$. Based on the material introduced, at this stage we need to appeal to the explicit diagonalization of the operator as shown in an earlier lecture. One advantage of assuming that $m'(y)$ has a constant sign is that is directly shows that $\mathfrak{a}\mathfrak{a}^*$ does not admit an discrete spectrum in $[0,m_0^2]$. So, the only relevant part of the spectrum is the mode $E_0(\xi)=\varepsilon\xi$ that appears when $\operatorname{sign}(m_-)\operatorname{sign}(m_+)=-1$.

Remark 8.6 [Stability using Fredholm operator $P(x)U(H)P(x)$]. The procedure presented in this section extends to a large class of problems with the following important caveats.

Steps (i), (ii), and (iii) are functional-analytic and adapt to large classes of problems once we know that the kernel $w(x-x',y,y')$ of W is sufficiently smooth. However, proving this result requires a fair amount of information on the spectrum of $H=D\cdot\sigma+m(y)\sigma_3$, and in particular on its branches of absolutely continuous spectrum that cross the energy interval of interest $(-m_0,m_0)$.

Step (iv) is also typically quite problem-dependent as it requires us to understand the winding number of each of the aforementioned branches of ac spectrum.

Steps (ii)-(iii) will be bypassed using pseudo-differential calculus in lecture 9. Step (iv) will be bypassed when we will have access to a bulk-edge correspondence, whose derivation remains quite technical and is presented in detail in Lecture 10.

Remark 8.7 [Stability using edge conductivity $\sigma_I(H)$]. We showed heuristically that the index of PUP was related to the edge conductivity σ_I in (1.5). We can also ask oneself whether $\sigma_I[H+V]=\sigma_I[H]$. We will show that this is indeed the case but only for a more limited class of perturbations V. Indeed, we always need to show that $[H+V,P]\varphi'(H+V)$ is trace-class, and this is more constraining than showing that $W(H+V)-W(H)$ is compact.

There are two methods showing the stability of σ_I. One is to relate σ_I to the index of PUP. This is done for example in [5, 4]. The other method is to directly show stability by, first showing that the operator is indeed trace-class, and then showing that the difference of traces may be written somehow as the trace of a commutator of trace-class operators. This is the strategy followed in [37].

Remark 8.8 [Stability using bulk conductivity $\sigma_B(H)$]. In two space dimensions $x=(x_1,x_2)$, when we considered the magnetic Schrödinger operator or the bulk regularized Dirac operator, we directly looked at the Fredholm operator $P(H)U(x)P(H)$. Instead, we can introduce a bulk conductivity

$$\sigma_B = \operatorname{Tr} iP(H)[[\Lambda_1, P(H)], [\Lambda_2, P(H)]] \tag{8.4}$$

assuming the above operator is trace-class. Here, Λ_j are switch functions of the variable x_j, i.e., smooth functions such that $\Lambda_j(x_j) = 0$ for $x_j < -M$ and $\Lambda_j(x_j) = 1$ for $x_j > M$ for some $M > 0$. The above is defined as an adiabatic curvature in [1, Section 6]. When H satisfies a covariance against magnetic translations (this is the case when $V = 0$ and B is constant), then it is shown in that reference by an explicit calculation that σ_B is given by the index of $P(H)U(x)P(H)$.

Remark 8.9 [Stability using Fredholm operator $P(H)U(x)P(H)$]. Let us finally comment on the stability of the Fredholm operator $P(H)UP(H)$ in two space dimensions with U multiplication by $u(x) = z/|z|$ where $z = x_1 + ix_2$. To prove stability when H is perturbed to $H + V$, one first has to be careful about domains since the range of $P(H)$ and that of $P(H + V)$ do not need to coincide. We therefore consider

$$T[H_V] := P(H_V)UP(H_V) + I - P(H_V)$$

with $H_V = H + V$ or H since now both operators are posed on the whole Hilbert space constructed after $L^2(\mathbb{R}^2)$, for instance $H^1(\mathbb{R}^2; \mathbb{C}^2)$ for the domains of definition of H and $H + V$. We then wish to show that $P(H + V) - P(H)$ is compact as this clearly implies that $T[H_V]$ is Fredholm when $T[H]$ is and that they have the same index.

Under general hypotheses, we can show that $H + V$ has discrete spectrum in the bulk gap, as we did for the Dirac problem with domain wall. So, possibly up to a small shift, we can also define $P_\varepsilon(H + V) = P(H + V)$ where $P_\varepsilon(h) = P(h)$ on the spectrum of $H + V$ and on the spectrum of H (for this, we may use the spectral theorem recalled in the appendix).

A main remaining issue is then that P does not decay at infinity (as a function) so that the Helffer-Sjöstrand formula does not apply. However, we verify that $P(h) = \frac{1}{2}(1 - \operatorname{sign}(h))$ depends only on the sign of h and not its magnitude. We may therefore replace $P(h)$ by $P(h/\langle h\rangle)$ since $P(h) = P(h/\langle h\rangle)$. We then replace $P(h/\langle h\rangle)$ by a function $P_\eta(h)$ that is smoothed out at $h = 0$ since H and $H + V$ have a spectral gap at 0, is equal to $P(h) = P(h/\langle h\rangle)$ on $[-1, 1]$, and is compactly supported. In other words,

$$P_\eta(H) - P_\eta(H + V) = P(H) - P(H + V)$$

while at the same time $P_\eta \in C_c^\infty(\mathbb{R})$. We may now apply the Helffer-Sjöstrand formula for the function P_η.

Following the last remark, we apply the stability results to the bulk Dirac operator $H_V = D \cdot \sigma + (m - \eta\Delta)\sigma_3 + V$ and obtain the following:

Lemma 8.10 *For the regularized bulk Dirac operator $H = D \cdot \sigma + (m - \eta\Delta)\sigma_3$ with $V \in L^\infty_\varepsilon$, we find that $P(H+V) - P(H)$ is a compact operator on $\mathcal{H}$. As a consequence, $P(H+V)U(x)P(H+V)$ on* $\operatorname{Ran}(H + V)$ *is Fredholm with the same index as the unperturbed operator.*

We leave the proof as an exercise. The first need to show that $\hat{H}_0 = \xi \cdot \sigma + (m + \eta|\xi|^2)\sigma_3$ is elliptic (this is the case both when $\eta = 0$ as we already saw and for $\eta \neq 0$ for different reasons) so that Lemma 8.3 applies.

A different proof needs to be developed for the magnetic Schrödinger operator $H = (D+A)^2+V$ since the 'unperturbed' operator $(D + A)^2$ is not invariant by translation. It is however covariant with respect to magnetic translations. See [24] for stability results in this context.

9 Lecture 9.

Hamiltonians modeled as pseudo-differential operators. We would like to define the operator $P(x)U(H)P(x)$ and the conductivity σ_I for a large class of models and obtain an explicit description of their associated invariants. A convenient tool to do so is to use algebras of pseudo-differential operators. The reason is that many tools are then available to consider questions such as self-adjointness, functional calculus, trace-class criteria, composition of operators, invertibility of operators, and so on. Moreover, we will see that the semi-classical calculus is quite useful in the computation of indices of Fredholm operators.

We will be concerned with two-dimensional problems with one domain wall. The theory essentially extends, albeit with a number of technical difficulties, to higher d-dimensional problems with $d-1$ domain walls. This finds applications in a field sometimes called Higher-Order Topological Insulators (HOTI), which are relatively easy to classify once one presents them with domain walls. See [5] for background and results.

Weyl quantization and symbols. We refer to section C in the Appendix for notation on pseudo-differential operators. We consider Hamiltonians written in terms of a so-called Weyl quantization, i.e., in the form

$$Hf(x) = (\mathrm{Op}^w a)f(x) = \int_{\mathbb{R}^{2d}} \frac{e^{i(x-y)\cdot\xi}}{(2\pi)^d} a(\frac{x+y}{2},\xi)f(y)d\xi dy.$$

The (matrix-valued) function $a(x,\xi)$ is the *symbol* of the (pseudo-differential) operator $H=\mathrm{Op}^w a$. It is defined on phase space $(x,\xi)\in\mathbb{R}^{2d}$ in d space dimensions. The symbol a is assumed to be a smooth function in $C^\infty(\mathbb{R}^2)$ satisfying appropriate bounds under differentiation. Of particular use to us is the class of symbols $a\in S^m$ defined by the infinite number of constraints

$$|\partial_\xi^\alpha \partial_x^\beta a|(x,\xi) \le C_{\alpha,\beta}\langle\xi\rangle^{m-|\alpha|}.$$

The best coefficients $C_{\alpha,\beta}$ serve as bounds on seminorms for a. This allows one to define a Fréchet topology on S^m affording us to compose such operators. In order to invert such operators (modulo compact perturbations) and to obtain an appropriate functional calculus, we assume that our operators are *elliptic*. For symbols in S^m, this means that

$$|a_{\min}(x,\xi)| \ge C\langle\xi\rangle^m - 1$$

where $a_{\min}$ is the minimal singular value of the symbol a.

Examples of operators include $-\Delta = \mathrm{Op}^w(\xi^2)$ with $\xi^2\in S^2$ or more generally $-\nabla\cdot A(x)\nabla = \mathrm{Op}^w(\xi\cdot A(x)\xi)$ with symbol also in S^2 (Exercise: check). An advantage of the Weyl quantization (with $a=a(\frac12(x+y),\xi)$ rather than the left- $a(x,\xi)$ or right- $a(y,\xi)$ quantizations) is that H is symmetric when $a(x,\xi)$ is a Hermitian-valued (real-valued when scalar) function.

For elliptic and symmetric symbols in appropriate symbol classes (such as S^m), we obtain that H is self-adjoint, that the resolvent $R(z)=(z-H)^{-1}$ whenever defined is a pseudo-differential operator (with symbol in S^{-m}), and that $f(H)$ is also a pseudo-differential operator (with symbol in $S^{-\infty}$ for f is smooth and compactly supported).

Pseudo-differential operators and the associated calculus thus offer a natural framework to construct operators such as $[H,P]\varphi'(H)$ and prove for instance that they are trace-class; see section C in the Appendix for additional details.

Example of Dirac operator. Consider the Dirac operator $H = D\cdot\sigma + m(y)\sigma_3 + V(x,y)$ for m and V smooth and bounded. Then $H = \mathrm{Op}^w a$ with $a = \xi\cdot\sigma + m(y)\sigma_3 + V(x,y)$ as 2×2 Hemitian matrices. Then $H \in \mathrm{Op}^w S^1$ and $f(H) \in \mathrm{Op}^w S^{-\infty}$ since H is clearly elliptic (Exercise: prove this). Note that this does not require any specific form of a domain wall for $m(y)$.

The above result does not imply any trace-class property. If we use the order function $\mathfrak{m}(X) = \langle\xi\rangle$, then we deduce that $f(H) \in S^{-\infty}(\langle\xi\rangle)$, i.e., an operator with symbol that decays faster than algebraically as $|\xi| \to \infty$. However, we do not have any information regarding decay of the symbol in the spatial variables.

Let us assume that $V(x)$ is a function that belongs to $S^{-\infty}(\langle x\rangle)$. The PDO calculus indicates that $V(x)f(H)$ belongs to $S^{-\infty}(\langle x\rangle\langle\xi\rangle)$ with now $(\langle x\rangle\langle\xi\rangle)^{-d-1}$ integrable. As a consequence, $V(x)f(H)$ is trace-class with trace given by the integral of its symbol thanks to (C.10). This should be compared with operators of the form $f(x)g(D)$ except that $f(H)$ is a more general operator with non-constant coefficients. The PDO calculus combined with the above functional calculus provides a very efficient tool to verify trace-class properties.

We finally remark that the ellipticity condition (C.8) is crucial in the derivation. While non-elliptic symbols may be invertible, their inverse will no longer map L^2 to H^m as is the case for elliptic operators.

Trace-class estimates. Let us focus on the two-dimensional setting. The spatial variables are now called (x,y) and the Fourier variables (ξ,ζ). We want to look at asymmetric transport for a large class of problems. To do so, we consider an elliptic operator with spectral gap at $|y|$ sufficiently large. We then want to show that $[\psi(H),P]\phi(H)$ is trace-class where ψ is bounded and $\phi \in C_0^\infty(\mathbb{R})$.

The Helffer-Sjöstrand formula (C.7) again plays an important role. So do resolvent identities of the form

$$(z-H_1)^{-1} - (z-H_2)^{-1} = (z-H_1)^{-1}(H_1-H_2)(z-H_2)^{-1} = (z-H_2)^{-1}(H_1-H_2)(z-H_1)^{-1}.$$

Assume that $H_j \in \mathrm{Op}^w S^m$ for $j=1,2$ are elliptic operators. Then as we saw above $(z-H_j)^{-1} \in \mathrm{Op}^w S^{-m}$ and hence $(z-H_j)^{-1} \in \mathrm{Op}^w S(1)$, which is a larger class. Now assume $T \in \mathrm{Op}^w S(\mathfrak{m}_T)$. Using the functional calculus, we obtain that $(z-H_1)^{-1}T(z-H_2)^{-1} =: \mathrm{Op}^w\tau \in \mathrm{Op}^w S(\mathfrak{m}_T)$ as well. Moreover, using (C.9) we obtain that for some C_N and M_N,

$$\mathtt{C}_N(\tau) \le C_N|\Im z|^{M_N} \tag{9.1}$$

uniformly on $z \in Z$ compact subset of $\mathbb{C}$. This is all we need to apply the Helffer-Sjöstrand formula (C.7) to control functionals of H_1 and H_2.

There is one ingredient missing before we can obtain trace-class estimates of operators of the form $[\psi(H),P]\phi(H)$. If H is elliptic, then $\phi(H) \in S^{-\infty}(\langle\xi,\zeta\rangle)$. Here as elsewhere, $\langle a,b\rangle = \sqrt{1+|a|^2+|b|^2}$. This handles the Fourier variables. It remains to obtain decay in the spatial variables as well in order to use the criterion (C.10). We will show that decay in the spatial variable x comes from the commutator $[\psi(H),P]$. When H is the Dirac operator, then $i[H,P] = P'\sigma_3$, which is indeed in $S(\langle x\rangle^{-\infty})$.

Operators with domain walls. It remains to obtain decay in y. This is the role of the *domain wall*. As in the definition of σ_I, we will then need to choose $\phi(H)$ with ϕ supported inside the bulk gap generated by the domain wall. To set this up, we introduce the hypothesis:

[H1] Let $H = \mathrm{Op}^w\sigma \in \mathrm{Op}^w S^m$ with $m>0$ be an *elliptic* operator, i.e., (C.8) holds. We assume the existence of $H_\pm = \mathrm{Op}^w\sigma_\pm \in \mathrm{Op}^w S^m$ elliptic and with symbols $\sigma_\pm(\xi,\zeta)$ independent of (x,y).

Moreover, we assume the existence of $L > 0$ such that $\sigma(x, y, \xi, \zeta) = \sigma_+(\xi, \zeta)$ when $y > L$ and $\sigma(x, y, \xi, \zeta) = \sigma_-(\xi, \zeta)$ when $y < -L$. Finally, we assume the existence of a *bulk* spectral gap $[E_1, E_2]$ in the sense that $E - \sigma_\pm$ is invertible for each $E \in [E_1, E_2]$. This is equivalent to the fact that $[E_1, E_2]$ is a spectral gap from $H_\pm$.

The hypothesis $m > 0$ is important as we will see. Ellipticity is also necessary as the above functional calculus would not hold without it. Ellipticity is also crucial to establish any type of Fredholm property. We then prove the main result of this lecture:

Proposition 9.1 *Assume H satisfies the above hypothesis [H1]. Let $\phi \in C_c^\infty(\mathbb{R})$ with compact support in (E_1, E_2) and ψ_j be either a polynomial or a bounded function in $C^\infty(\mathbb{R})$ for $j = 1, 2$. Finally, let P be a smooth switch function from 0 to 1. Then*

$$\phi(H) \in \mathrm{Op}^w S(\langle y, \xi, \zeta\rangle^{-\infty}) \qquad \textit{and} \qquad \psi_1(H)[\psi_2(H), P]\phi(H) \in \mathrm{Op}^w S(\langle x, y, \xi, \zeta\rangle^{-\infty}).$$

Moreover, $[\phi(H), P]$ and $\psi_1(H)[\psi_2(H), P]\phi(H)$ are trace-class with traces given by (C.10)*. Finally,* $\mathrm{Tr}\ [\phi(H), P] = 0$.

Proof. We first show that $[\psi_2(H), P] \in \mathrm{Op}^w S(\langle \xi, \zeta\rangle^s \langle x\rangle^{-\infty})$ for some $\mathbb{R} \ni s < \infty$. Indeed, we write using $1 = P + (1 - P)$

$$[A, P] = (1 - P(x))AP(x) - P(x)A(1 - P(x)),$$

with $P(x) \in \mathrm{Op}^w(\langle x_-\rangle^{-\infty})$ while $1 - P(x) \in \mathrm{Op}^w \langle x_+\rangle^{-\infty})$. As a consequence using the composition rule, we find that if $A \in \mathrm{Op}^w S(\mathfrak{m}_A)$, then $[A, P] \in \mathrm{Op}^w S(\mathfrak{m}_A \langle x\rangle^{-\infty})$. Since we may choose $\mathfrak{m}_A = \langle \xi, \zeta\rangle^s$ for some $s < \infty$ by assumption on ψ_2, this proves this first step.

Now, since H is elliptic, we obtain from the paragraph following (C.8) that $I + H^2 = (i + H)(-i + H)$ is invertible and that moreover, $(I + H^2)^{-1} \in S^{-2m}$ with $-2m < 0$. Let $\phi_t(H) = (1 + H^2)^t \phi(H)$ with $\phi_t \in C_c^\infty(\mathbb{R})$ for any $t \in \mathbb{R}_+$.

Consider the operators $H_\pm$. By assumption, they have a gap in $[E_1, E_2]$. Indeed, $H_\pm = \mathcal{F}^{-1}\sigma_\pm \mathcal{F}$ so that the (necessarily absolutely continuous) spectrum of $H_\pm$ is the same as that of $\sigma_\pm$ (seen as a multiplication operator), i.e., $\sigma(H_\pm) = \mathrm{Ran}\,\sigma_\pm$, which does not intersect $[E_1, E_2]$ by assumption.

As a consequence, $\phi(H_\pm) = \phi_t(H_\pm) = 0$. Choosing L large enough that $P(y)$ is constant on each connected component of $|y| > L$, we have

$$\phi(H) = P(y)(I + H^2)^{-t}(\phi_t(H) - \phi_t(H_+)) + (1 - P(y))(I + H^2)^{-t}(\phi_t(H) - \phi_t(H_-)).$$

The above first term is

$$T_+ = -\frac{1}{\pi} \int_Z \bar{\partial}\tilde{\phi}(z) P(y)(I + H^2)^{-t}(z - H)^{-1}(H - H_+)(z - H_+)^{-1} d^2 z.$$

Using (9.1), we observe that $P(y)(I+H^2)^{-t}(z-H)^{-1}(H-H_+)(z-H_+)^{-1}$ is a PDO in $\mathrm{Op}^w S(\mathfrak{m})$ with $\mathfrak{m} = \langle \xi\rangle^{-2mt}\langle y_+\rangle^{-\infty}\langle y_-\rangle^{-\infty}$ with bounds on the seminorms of the symbol given by $C_N|\Im z|^{-M_N}\mathfrak{m}$. For each N, the above integral is then defined since $|\bar{\partial}\tilde{\phi}(z)| \le C_M |\Im z|^M$ for every $M \ge 0$. Thus $T_+ \in \mathrm{Op}^w S(\langle \xi, \zeta\rangle^{-2mt}\langle y_+\rangle^{-\infty}\langle y_-\rangle^{-\infty})$ for every $t > 0$ and hence in $\mathrm{Op}^w S(\langle y\rangle^{-\infty}\langle \xi, \zeta\rangle^{-\infty})$. The same result applies to the second component T_- so that $\phi(H) \in \mathrm{Op}^w S(\langle y, \xi, \zeta\rangle^{-\infty})$.

Collecting the bounds for $\phi(H)$ with symbol in $S(\langle y\rangle^{-\infty}\langle \xi, \zeta\rangle^{-\infty})$ and $[\psi_2(H), P]$ with symbol in $S(\langle x\rangle^{-\infty}\langle \xi, \zeta\rangle^s)$ for some $s < \infty$ while $\psi_1(H) \in \mathrm{Op}^w\langle \xi, \zeta\rangle^s$ for some $s < \infty$ as well, we get the result for $\psi_1(H)[\psi_2(H), P]\phi(H)$ by composition. It is then clear that $\psi_1(H)[\psi_2(H), P]\phi(H)$ is trace-class with trace given by an integral of the symbol as given in (C.10).

The operator $[\phi(H),P]$ has symbol in $S(\langle x,y,\xi,\zeta\rangle^{-\infty})$ and is thus trace-class for the same reasons. Let $w(x,x',y,y')(P(x)-P(x'))$ be its Schwartz kernel. We know from (C.10) that the trace is given as the integral of the Schwartz kernel along the diagonal, which clearly vanishes. $\square$

Let H satisfy [H1]. Applying the above proposition to a switch function $\varphi(E)\in C^\infty(\mathbb{R})$ in $\mathfrak{S}(0,1;E_1,E_2)$ (see (1.6)), we obtain that $i[H,P]\varphi'(H)$ and $[U(H),P]U^*(H)$ are both trace-class. Here, $U(h)=e^{i2\pi\varphi(h)}$ so that $U=I+W$ with W compactly supported in (E_1,E_2).

We may apply this result to a Dirac operator of the form

$$H=D\cdot\sigma+m(y)\sigma_3+V(x,y)$$

for $V(x,y)$ an arbitrary smooth, bounded Hermitian-valued multiplication operator that vanishes for $|y|\geq L$. There is also no difficulty in replacing $D\cdot\sigma$ by $\Gamma_{ij}(x,y)D_i\sigma_j$ for $\Gamma(x,y)$ a uniformly invertible matrix with bounded coefficients such that $\Gamma(x,y)=\Gamma_\pm$ constant invertible matrices on each connected component of $|y|>L$.

Edge invariants. Recall that asymmetric transport has been quantized using two objects: one is $\sigma_I=\mathrm{Tr}\, i[H,P]\varphi'(H)$ in (1.5). The other one is the Fredholm operator $PUP_{\mathrm{Ran}P}$ whose index is given by $\mathrm{Tr}[U,P]U^*$. Note that the two spatial profiles $P(x)$ are different in these two cases. In the former case, P has to be a smooth function for the commutator to be defined as a bounded operator. In the latter case, P has to be a projector $P^2(x)=P(x)$ and hence not a smooth function. In fact, we have not proved that $[U,P]U^*$ was trace-class for P a projector. Yet, this is the case and the two invariants are in fact equal to one-another.

Theorem 9.2 *Let H be an operator satisfying [H1]. Let $P_1(x)$ be a smooth switch function in $\mathfrak{S}(0,1)$ and $P(x)$ be a projector in $\mathfrak{S}(0,1)$. Let φ be a smooth non-decreasing switch function in $\mathfrak{S}(0,1;E_1,E_2)$. Let $U(h)=e^{i2\pi\varphi(h)}=I+W(h)$.*

Then $[U(H),P]U^*(H)$ is a trace-class operator. Moreover, $P(x)U(H)P(x)_{|\mathrm{Ran}P}$ is a Fredholm operator and we have

$$2\pi\sigma_I=\mathrm{Tr}2\pi i[H,P_1]\varphi'(H)=\mathrm{Tr}[U(H),P_1]U^*(H)=\mathrm{Tr}[U(H),P]U^*(H)=\mathrm{Index}\ PUP_{\mathrm{Ran}P}. \quad (9.2)$$

Proof. The above proposition shows that $i[H,P_1]\varphi'(H)$ is trace-class and the first equality is then a definition. The proposition also implies that $[U(H),P_1]U^*(H)$ is trace-class. We now first prove that $[U(H),P]U^*(H)$ is trace class and the third equality above.

By hypothesis $P-P_1$ is compactly supported. Let $\chi(x)$ be a smooth compactly supported function such that $\chi(x)=1$ on the support of $P-P_1$. Then

$$[U(H),P]=[U(H),P_1]+[U(H),P-P_1]=[W(H),P_1]+W(H)\chi(x)(P-P_1)(x)-(P-P_1)\chi(x)W(H).$$

We know from the above proposition that $[W(H),P_1]$ is trace-class. Since $W(H)\in\mathrm{Op}^wS(\langle y,\xi,\zeta\rangle^{-\infty})$ and $\chi(x)\in\mathrm{Op}^wS(\langle x\rangle^{-\infty})$, by composition calculus, the operators $W(H)\chi(x)$ and $\chi(x)W(H)$ are also trace-class. Since multiplication by $P-P_1$ is a bounded operator, we conclude that $W(H)(P-P_1)(x)$ and $(P-P_1)(x)W(H)$ as well as $[U(H),P]$ are trace-class and so is $[U(H),P]U^*(H)$. Now,

$$\mathrm{Tr}W(P-P_1)=\mathrm{Tr}W(P-P_1)\chi=\mathrm{Tr}\chi W(P-P_1)=\mathrm{Tr}(P-P_1)\chi W=\mathrm{Tr}(P-P_1)W.$$

where we use that $\mathrm{Tr}AB=\mathrm{Tr}BA$ when A is trace-class and B is bounded. Thus, $\mathrm{Tr}[U,P-P_1]=0$ and so

$$\begin{aligned}\mathrm{Tr}[U,P-P_1]U^* &= \mathrm{Tr}[W,P-P_1]W^*=\mathrm{Tr}W(P-P_1)W^*-\mathrm{Tr}(P-P_1)WW^*\\ &= \mathrm{Tr}(P-P_1)W^*W-\mathrm{Tr}(P-P_1)WW^*=0.\end{aligned}$$

Indeed $UU^* = U^*U = I = I+W+W^*+W^*W = I+W+W^*+WW^*$ so that $WW^* = W^*W$. This shows that $\mathrm{Tr}[U(H), P_1]U^*(H) = \mathrm{Tr}[U(H), P]U^*(H) = \mathrm{Index}\ PUP_{\mathrm{Ran}P}$, the last equality coming from the Fedosov formula and more precisely Lemma B.6 since $P - Q = [P, U]U^*$ is trace-class.

It remains to show the second equality, namely that the two invariants are really one and the same when σ_I is defined (the index is defined in much greater generality as we saw).

Using the results of Proposition 9.1, $[H^n, P]\phi(H)$ is trace-class and, using only $\mathrm{Tr}AB = \mathrm{Tr}BA$ when A is trace-class and B bounded, we get

$$\begin{aligned}\mathrm{Tr}[H^n, P]\phi &= \mathrm{Tr}H^{n-1}[H, P]\phi + \mathrm{Tr}[H^{n-1}, P]H\phi \\ &= \mathrm{Tr}[H, P]H^{n-1}\phi + \mathrm{Tr}H^{n-2}[H, P]H\phi + \mathrm{Tr}[H^{n-2}, P]H^2\phi = \mathrm{Tr}[H, P]nH^{n-1}\phi,\end{aligned}$$

where the last step is obtained by induction. So, for any polynomial $Q(H)$, we have

$$\mathrm{Tr}[Q(H), P]\phi(H) = \mathrm{Tr}[H, P]Q'(H)\phi(H). \tag{9.3}$$

Let ψ be a compactly supported smooth function equal to 1 on the supports of W and ϕ. We have

$$[(W - Q)\phi\psi, P] = (W - Q)\phi[\psi, P] + (W - Q)[\phi, P]\psi + [(W - Q), P]\phi\psi.$$

By Proposition 9.1, all terms are trace-class and the left-hand side has a vanishing trace. Thus

$$\mathrm{Tr}[(W - Q), P]\phi = \mathrm{Tr}(W - Q)(\phi[P, \psi] + [P, \phi]\psi) = \mathrm{Tr}(W - Q)\phi[P, \psi] + \mathrm{Tr}(W - Q)\psi[P, \phi].$$

We use the cyclicity of the trace and that two functionals of H commute. Let W_p be a polynomial approximation of W such that $(W - W_p)\phi$ and $(W' - W_p')\phi$ go to zero uniformly as operators. The same holds when ϕ is replaced by ψ. Therefore using (9.3),

$$\mathrm{Tr}[W, P]\phi - \mathrm{Tr}[H, P]W'\phi = \lim_{p\to\infty}(\mathrm{Tr}[W_p, P]\phi - \mathrm{Tr}[H, P]W_p'\phi) = 0.$$

We now choose $\phi = W^*$ and use $\mathrm{Tr}[W, P] = 0$ to obtain

$$\mathrm{Tr}[U, P]U^* = \mathrm{Tr}[W, P]W^* = \mathrm{Tr}[H, P]W'W^* = \mathrm{Tr}[H, P]U'U^* - \mathrm{Tr}[H, P]W' = 2\pi\sigma_I - \mathrm{Tr}[H, P]W',$$

where we used $U'(H)U^*(H) = 2\pi i\varphi'(H)$. So, it remains to show that $\mathrm{Tr}[H, P]W' = 0$. Let $\psi_1^2 + \psi_2^2 = 1$ with $\psi_1 = 1$ on the support of W. Since $[W, P]$ is trace-class,

$$\mathrm{Tr}[W, P]\psi_2^2 = \mathrm{Tr}\psi_2[W, P]\psi_2 = \mathrm{Tr}(\psi_2 WP\psi_2 - \psi_2 P\psi_2) = 0$$

since $W\psi_2 = 0$. Now, since $W'\psi_2^2 = 0$,

$$\mathrm{Tr}[H, P]W' = \mathrm{Tr}[H, P]W'\psi_1^2 = \mathrm{Tr}[W, P]\psi_1^2 = \mathrm{Tr}[W, P] - \mathrm{Tr}[W, P]\psi_2^2 = 0.$$

This concludes the proof. $\square$

Stability of the edge invariant. The above result provides a strong stability property for σ_I. We now know that it is quantized with $2\pi\sigma_I \in \mathbb{Z}$. It is therefore immune to a very large class of perturbations once it is defined, i.e., once $[H, P]\varphi'(H)$ is indeed a trace-class operator. Let us mention again that Index $PUP_{\mathrm{Ran}P}$ only requires $P - Q$ to be compact to be a Fredholm operator. It therefore enjoys stronger stability than σ_I.

However, when $i[H, P]\varphi'(H)$ is trace-class, both indices are defined and agree as we showed above. We therefore want to use the stability of the Fredholm operator PUP to obtain that of the edge invariant $2\pi\sigma_I$.

Let a_0 and a_1 be two elliptic symbols in S^m. For $t \in [0, 1]$, we define $a_t = (1 - t)a_0 + ta_1 = a_0 + t(a_1 - a_0)$ and $H_t = \mathrm{Op}^w a_t$. We thus have $H_t - H_s = \mathrm{Op}^w(a_t - a_s) = (t - s)\mathrm{Op}^w(a_1 - a_0)$. We wish to show that the index $PU(H_t)P$ is independent of t. This will imply $2\pi\sigma_I[H_1] = 2\pi\sigma_I[H_0]$.

Proposition 9.3 (Stability of σ_I) *Let H_t be defined as above for $t \in [0,1]$. Assume that* [H1] *is satisfied for each H_t, $t \in [0,1]$ with (E_1, E_2) chosen uniformly in t. Assume φ in Theorem 9.2 chosen with support in $[E_1, E_2]$. Then $2\pi\sigma_I[H_t] = \mathrm{Index}\, PU(H_t)P$ is* independent *of $t \in [0,1]$.*

Proof. We verify that $H_t - H_s = (t-s)(H_1 - H_0)$. We now write as before

$$U(H_t) - U(H_s) = -(t-s)\frac{1}{\pi}\int_Z \bar\partial\tilde W(z)(z-H_t)^{-1}(H_1-H_0)(z-H_s)^{-1}d^2z.$$

We now use $(z-H_s)^{-1} = (i-H_s)^{-1}(1+(i-z)(z-H_s))^{-1}$. We know that $\|(z-H_t)^{-1}\| \le |\Im z|^{-1}$ and have $\|(1+(i-z)(z-H_s))^{-1}\| \le C|\Im z|^{-1}$ on Z as well.

Assume $\|(H_1-H_0)(i-H_s)^{-1}\|$ bounded uniformly in s. Then from the above integral, $\|U(H_t) - U(H_s)\| \le C|t-s|$ from the usual properties of the almost analytic extension $\tilde W(z)$ (showing that $|\bar\partial\tilde W(z)|$ is bounded by $C|\Im z|^q$ for $q=2$ here). This implies that $t \to PU(H_t)P$ is a continuous path of Fredholm operators since [H1] holds for each t. The results recalled in the Appendix then show that the index is independent of $t \in [0,1]$. The result on the stability of σ_I is then a consequence of Theorem 9.2.

It remains to find a bound for $N = \|(H_1-H_0)(i-H_s)^{-1}\|$. This is a consequence of a Calderón-Vaillancourt theorem (see [18, Theorem 7.11] and [28] for details and proofs for this central result) stating that the latter operator is bounded by a seminorm of its symbol in $S(1)$. This in turn is implied by bounds on the seminorms of the symbols of $H_1 - H_0$ and $(i-H_s)^{-1}$ according to (C.6). By assumption, $H_1 - H_0$ has symbol bounded in each seminorm of S^m. Thanks to (C.9), the operator $(i-H_s)^{-1}$ has symbol bounded in any seminorm in S^{-m}. Moreover, the bound is clearly continuous in s and hence uniformly bounded in $s \in [0,1]$. Thus, the symbol of $(H_1-H_0)(i-H_s)^{-1}$ is bounded in any seminorm of $S(\langle\xi\rangle^{-m}\langle\xi\rangle^m) = S(1)$ uniformly in s. This shows that N above is bounded and concludes the proof of the proposition. ∎

We consider the following applications of the preceding result. Let a be an elliptic symbol (possibly matrix-valued) in S^m such that [H1] holds. We denote by b an arbitrary symbol in S^{m-1} compactly supported in (x,y). Then we have

Corollary 9.4 *Let $a \in S^m$ be elliptic as above and $b \in S^{m-1}$ have compact support in (x,y). Let $a_t = a + tb$. Then [H1] holds for $H_t = \mathrm{Op}^w(a+tb)$ and $\sigma_I[H_1] = \sigma_I[H_0]$.*

Let a be as above and $a_0(x,y,\xi,\zeta) = a(0,y,\xi,\zeta)$. Then $\sigma_I[H_1] = \sigma_I[H_0]$ where $H_0 = \mathrm{Op}^w a_0$ satisfies [H1].

Let $0 < h_1 \le 1$ and $0 < h_2 \le 1$ and define $a_h(x,y,\xi,\zeta) = a(x,y,h_1\xi,h_2\zeta)$ while $H_h = \mathrm{Op}^w a_h$. Then for every multi-index h as above, H_h satisfies [H1] and $\sigma_I[H_t]$ is independent of t.

Proof. For the first result, we observe that $a + tb$ satisfy [H1] with a possible redefinition of L since $a + tb$ remains elliptic for each $t \in [0,1]$. It remains to apply the above Proposition. This result shows that the edge conductivity is stable with respect to compact perturbations b that are negligible for large (ξ,ζ) compared to a.

For the second result, we define $a_t = (1-t)a_0 + ta$. Such a symbol clearly satisfies [H1] for each $t \in [0,1]$ and the result follows from the proposition. This result shows that we may replace $a(x,y,\xi,\zeta)$ by a symbol that is independent of x. The resulting operator $H_0 = \mathrm{Op}^w a_0$ is therefore invariant by translation, which tremendously simplifies expressions for $\varphi'(H_0)$ for instance.

For the last result, we observe that the symbols are elliptic for each $h_1 > 0$ and $h_2 > 0$. Moreover, the operators $\sigma_\pm(h_1\xi, h_2\zeta)$ have the same collective gap independent of (h_1,h_2) since they are independent of (ξ,ζ). Assume $h_2 = 1$ for concreteness. We have that the symbol is elliptic for all $h_1 > 0$. We slightly modify the proof of the above proposition and assume that

$H_t = \mathrm{Op}^w a(x,y,t\xi,\zeta)$ for $h_0 \le h \le 1$. Then $U(H_t) - U(H_s)$ is bounded by $\|(H_t - H_s)(i - H_s)^{-1}\|$ as before. The latter is bounded by a seminorm of $a(x,y,t\xi,\zeta) - a(x,y,s\xi,\zeta)$. We have

$$a(x,y,t\xi,\zeta) - a(x,y,s\xi,\zeta) = (t-s)\int_0^1 \partial_\xi a(x,y,[ut+(1-u)s]\xi,\zeta)du.$$

This is clearly bounded in any seminorm in S^m. This shows that $U(H_t) - U(H_s)$ is bounded. The same result applies to h_1 fixed and h_2 varying.

This final result shows that the index is independent of rescaling of the dual variables. In particular, the result still holds when $h = h_2$ is arbitrarily small. This is the *semiclassical* limit. In that limit, we have an asymptotic expansion of the resolvent $(z - H_h)^{-1}$ in powers of h. This allows us to better understand the non-commutativity in $\varphi'(H)$. This will be a crucial step in the next lecture when we tackle the bulk-edge correspondence. $\square$

Going back to the Dirac operator (8.3). We already know how to classify such an operator. The above PDO calculus allows one to bypass the verification that $[P,U]$ is trace-class as we did using Lemma 7.6. Indeed, we may continuously transform the operator in (8.3) to the operator H with a smooth domain wall $m(y)$ and with $V = 0$; see discussion following (8.3). It remains to compute the index of H with a smooth domain wall. We thus replace Lemma 7.6 in Step (iii) there by Theorem 9.2 to obtain that $[P,U]$ is trace-class. The computation of the index in Step (iv) is still unmodified and appeals to an explicit diagonalization of H in the appropriate energy range and computing the winding number of each branch of absolutely continuous spectrum. Bypassing this step is one of the objectives of the next lecture.

What we have gained with the PDO calculus is the identification of the index of $PU(H)P$ with the edge conductivity $2\pi\sigma_I[H]$. Note that such a result only holds for sufficiently smooth operators H, albeit ones that do include perturbations V. For general Dirac operators (8.3) for which PUP is indeed a Fredholm operator, it is unlikely that $[H,P]\varphi'(H)$ is trace-class.

10 Lecture 10.

Classification by Domain Walls. This lecture introduces a classification of partial differential operators by means of domain walls. We construct another Fredholm operator $F(H)$ (affine in H) whose index is shown to admit an explicit expression as an integral of its symbol. This is a Fedosov-Hörmander formula implementing an Atiyah-Singer index theory in Euclidean space. Moreover, it turns out that the index of $F(H)$ and that of $PU(H)P$ agree. This is a *bulk-edge* correspondence whose derivation is everything but simple.

The bulk-edge correspondence is useful in the sense that the computation of the index of F is often significantly simpler than that of $PU(H)P$. We will see several methods to perform such computations in subsequent lectures.

One dimensional setting. We saw in an earlier lecture that $H = D$ was the ultimate model for asymmetric transport along an interface. We also obtained that $P(x)U(H)P(x)$ was a Fredholm operator with index equal to 1 for a large choice of profiles $\varphi \in \mathfrak{S}(0,1)$ by spectral flow computation. Indeed, the spectral flow associated to $E(\xi) = \xi$ equals 1. We now introduce the following operator

$$F = H - ix = \frac{1}{i}\mathfrak{a}, \quad \mathfrak{a} = \partial_x + x.$$

We already know that the annihilation operator $\mathfrak{a}$ is topologically non-trivial. It is a Fredholm operator on $D(F)$ to $L^2(\mathbb{R})$ with $D(F)$ the space of square integrable functions such that Df and

xf are square-integrable as one readily verifies. The claim is that the index of F is easy to compute from its symbol. We observe that $F = \mathrm{Op}^w a$ with $a(x,\xi) = \xi - ix$. Thus $da = d\xi - idx$ and we observe that

$$\mathrm{Index}\, F = \frac{1}{2\pi i}\int_{|x,\xi|=R} a^{-1}da = 1. \tag{10.1}$$

This result is independent of $R > 0$. This formula is one of the simplest Fedosov-Hörmander formulas [27, Theorem 19.3.1]. What is remarkable in the above formula is that the analytical index of F, given by the difference of dimensions of kernels and co-kernels of F, may be computed simply from the symbol $a(x,\xi)$ of F. This is conceptually simpler than evaluating a spectral flow, which requires an entire diagonalization of H on the support of $\varphi'(h)$.

The curve of integration $\{|x,\xi| = R\}$ may be replaced by any (non-self-intersecting) curve $\mathcal{C}$ winding once around the origin. Indeed, we verify that

$$d(a^{-1}da) = da^{-1}\wedge da + a^{-1}d^2a = -a^{-1}daa^{-1}\wedge da + 0 = 0$$

so that we can integrate $d(a^{-1}da)$ on the volume separating two such curves $\mathcal{C}$ and obtain that the integrals of $a^{-1}da$ along each curve agree as an application of the Stokes theorem. [Exercise: check details].

So we might as well assume that $\mathcal{C} = \{|x,\xi| = 1\}$ the unit circle in the (x,ξ) plane. Restricted to that circle and introducing polar coordinates ($x = R\cos\theta, \xi = R\sin\theta$), we find that $(d\xi - idx)_{|\mathcal{C}} = i^*(d\xi - idx) \equiv d\xi - idx = Re^{i\theta}d\theta = e^{i\theta}d\theta$ where $i : \mathcal{C} \to \mathbb{C}$ is the inclusion and i^* is the measure pullback, which effectively restricts a measure on $\mathbb{C}$ to one on $\mathbb{S}^1 = \mathcal{C}$. We also have $a^{-1} = (\xi + ix)/(x^2+\xi^2) = ie^{-i\theta}$ on $\mathcal{C}$ (i.e., possibly somewhat pedantically but these notions are very useful, $i^*a^{-1} = a^{-1}\circ i = ie^{-i\theta}$; as is often the norm we denote by i both inclusion and $\sqrt{-1}$). Thus $a^{-1}da = id\theta$ whose integral along $\mathcal{C}$ is $2\pi i$ and hence the above result.

Rather than an unbounded domain wall x, we may in fact introduce the operator

$$F = D - i\mathtt{m}(x) = -i\mathfrak{a}, \qquad \mathfrak{a} = \partial_x + \mathtt{m}(x),$$

with $\mathtt{m}(x)$ a smooth non-decreasing switch function in $\mathfrak{S}(-2,2)$ such that $\mathtt{m}(x) = x$ for $|x| \le 1$ (to simplify). The operator F is now self-adjoint with domain $D(F) = H^1(\mathbb{R})$. The symbol of F is $a(x,\xi) = \xi - i\mathtt{m}(x)$ which is invertible as soon as $x \neq 0$. The integral (10.1) still equals 1 since $\mathtt{m}(x) = x$ on the support of $\mathcal{C}$. We can then show that F is a Fredholm operator and that the index is also given by (10.1) (simply by estimating both sides and verifying they are equal). This may also be deduced from a modified version of [27, Theorem 19.3.1].

Two-dimensional classification. Consider now a bare operator $H_0 = D\cdot\sigma = D_x\sigma_1 + D_y\sigma_2$. As it stands, the operator is not insulating. However, we use the structure of Pauli matrices (two-dimensional Clifford algebra) $\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}$ to add a *first domain wall* $m(y)\sigma_3$ and define

$$H_1 = H_0 + m(y)\sigma_3 = D_x\sigma_1 + D_y\sigma_2 + m(y)\sigma_3.$$

We saw how to classify such an operator in the preceding lectures by means of a Fredholm operator $P(x)U(H)P(x)$ and an edge conductivity $\sigma_I = \mathrm{Tr}\, i[H,P]\varphi'(H)$.

Another classification is based on the same confinement as in the one-dimensional case. Adding a *second domain wall*, we define

$$F = H_1 - im_2(x) = D_x\sigma_1 + D_y\sigma_2 + m(y)\sigma_3 - i\mathtt{m}(x).$$

Here, $\mathtt{m}(x)$ is chosen as above. Thanks to the choice of 'matrices' $(\sigma_1, \sigma_2, \sigma_3, i)$, we can verify that F is indeed a Fredholm operator from $H^1(\mathbb{R}^2; \mathbb{C}^2)$ to $L^2(\mathbb{R}^2; \mathbb{C}^2)$. Moreover, defining

$$a(x, y, \xi, \zeta) = \xi\sigma_1 + \zeta\sigma_2 + m(y)\sigma_3 - i\mathtt{m}(x),$$

we have that (at least when $m(y) = y$ and $\mathtt{m}(x) = x$ [27, Theorem 19.3.1])

$$\operatorname{Index} F = \frac{1}{24\pi^2} \int_{\mathbb{S}^3_R} \operatorname{tr}(a^{-1} da)^{\wedge 3} \tag{10.2}$$

where $\mathbb{S}^3_R$ is the sphere of radius R in phase space $(x, y, \xi, \zeta) \in \mathbb{R}^4$ (oriented such that $d\xi \wedge dx \wedge d\zeta \wedge dy > 0$) where R is chosen large enough that a^{-1} is defined for $|(x, y, \xi, \zeta)| \geq R$. We wil verify that $d(\operatorname{tr}(a^{-1}da)^{\wedge 3}) = 0$ as a volume form so that we may apply the Stokes theorem and obtain that the integral is independent of R for R sufficiently large (in fact any $R > 0$ suffices for the Dirac operator since the *topological charge* of the operator is concentrated at $(x, y, \xi, \zeta) = 0$). This is a Fedosov-Hörmander formula [27, Theorem 19.3.1] generalizing the above one-dimensional formula.

So again, we observe that the index of F may be computed from an explicit integral of an explicit symbol a. This raises three questions:

(i) is the index stable ?

(ii) is the integral easy to compute?

(iii) is the index interesting physically?

The answer to (i) is yes in great generality as in the preceding lecture. We have that $\operatorname{Index}(F+V) = \operatorname{Index} F$ for a large class of perturbations V to H. However, the current existing derivations of such a result require that the mass terms tend to infinity at infinity in an appropriate way [27, Theorem 19.3.1]. This problem is also considered in [5] and we will not develop it further in this lecture.

The answer to (ii) is also yes. It is reasonably straightforward to compute the explicit integral numerically. In many settings of interest, the integral further simplifies although there does not seem to be an explicit rule to compute it for general operators. For instance, the computations significantly simplify for operators with a Clifford structure (i.e., Dirac-type operators). In particular, assume that

$$H_0 = h_1(D)\sigma_1 + h_2(D)\sigma_2 \quad \text{ and define } \quad h(\xi, \zeta, x, y) = (h_1, h_2, m(y), \mathtt{m}(x))(\xi, \zeta, x, y).$$

Then it turns out that the index of F is equal to the topological degree of the map $h/|h|$ from the sphere $\mathbb{S}^3_R$ to the unit sphere $\mathbb{S}^3$. Indeed, $|h|$ does not vanish on $\mathbb{S}^3_R$ by assumption on the domain walls. There are reasonably explicit formulas for the topological degree of a map from $\mathbb{S}^3$ to itself. Such maps are classified by $\pi_3(\mathbb{S}^3) \cong \mathbb{Z}$. This will be analyzed in greater detail in the next lecture. For operators without any Clifford structure, the computations are less immediate. We will see general methods to compute these indices in a subsequent lecture as well.

Finally, the answer to (iii) is also positive as an application of a *bulk-edge correspondence*. The latter says that $\operatorname{Index} F = 2\pi\sigma_I(H_1)$, at least when $E_1 < 0 < E_2$, i.e. energy 0 is in a bulk band gap. So $\operatorname{Index} F$ captures asymmetric transport along the x-axis before we impose the last confinement using $\mathtt{m}(x)$. Following [5, 46], we call the index of F a topological charge and the deformations leading to the above formula a *topological charge conservation.*

The above classification extends to arbitrary dimensions [5].

Bulk-Edge Correspondence. We do not address the theory of the above operator F directly. The reason is that formula (10.2) applies to operators F with symbols in appropriate classes that in particular have symbols that become constant as (x, y, ξ, ζ) become infinite. This is not the case for F when the domain walls are bounded. For instance for (y, ξ, ζ) large, we still have that x may not be large and the symbol therefore varies in x. One way to 'fix' this issue is to assume that the domain walls are linear in (x, y). In this setting, the inverse of the symbol is small in all variables (x, y, ξ, ζ) and hence does satisfy the hypotheses that allow one to derive (10.2). See [5] for more details.

However, what we really want is to show that $2\pi\sigma_I[H]$ *is* given by the right-hand-side in (10.2). This is a *topological charge conservation* (TCC) leading to a *bulk-edge correspondence* that relates σ_I to an integral that involves only the symbol of H and, as we will see, in fact only the symbols of $H_\pm$, which are bulk invariants. We do not need that (10.2) represent the index of F, even if morally this is the case and this is why we presented it that way.

The main result of this section is:

Theorem 10.1 *Let a_1 be a symbol in S^m with $H = H_1 = \mathrm{Op}^w a_1$ and let [H1] and the assumptions of Theorem 9.2 hold. Let $\alpha \in (E_1, E_2)$ and define $F = H_1 - \alpha - ix = \mathrm{Op}^w a$ with $a = a_1 - \alpha - ix$. Then*

$$2\pi\sigma_I[H] = \frac{1}{24\pi^2}\int_{\mathbb{S}^3_R} \mathrm{tr}(a^{-1}da)^{\wedge 3} \in \mathbb{Z}. \tag{10.3}$$

Here, R is large enough that a^{-1} is defined for $|(x, y, \xi, \zeta)| \geq R$.

Proof. The proof is long and split into several parts. A number of computational details are left to the reader. We first part uses a time honored method in the identification of invariants: continuously deform the invariants on each side of an equality to an expression that is sufficiently simple that the identification is possible. The first part makes such preliminary calculations.

Part 1. Let a_1 be the symbol given above and $a_0(x, y, \xi, \zeta) = a_1(0, y, \xi, \zeta)$ with $H_0 = \mathrm{Op}^w a_0$. We have seen in Corollary 9.4 that the edge conductivity was the same for both operators. Now the right-hand side is also a topological invariant classifying invertible (matrix-valued) functions a mapping the sphere $\mathbb{S}^3_R$ to $GL(\mathbb{C})$, the space of invertible matrices. Such mappings in $\pi_3(GL(\mathbb{C})) \cong \mathbb{Z}$ (an element of the Bott periodicity result) are classified by homotopy classes. From the index theorem [27, Theorem 19.3.1], we know that the right-hand-side of (10.3) is an integer. That integer is therefore independent of continuous deformations of a on $\mathbb{S}^3_R$, for instance the deformation $(1-t)a_0 + ta$. This shows that we may replace H and a by H_0 and a_0 in establishing (10.3).

We therefore assume that $a_1 = a_1(y, \xi, \zeta)$ does not depend on the variable x. Still using Corollary 9.4, we know that we may continuously deform $a(y, \xi, \zeta)$ to $a(y, \xi, h\zeta)$ for $0 < h \leq 1$ and that $\sigma_I[H_h]$ is independent of h. For the same reason as above, the right-hand side in (10.3) is also independent of h but this is in fact a property we will not need. The main reason for using the invariance in h is that as $h \to 0$, we approach a *semi-classical* regime, where computations of functionals of operators such as $\varphi'(H)$ become more explicit.

Part 2. We now define $a(x, y, \xi, \zeta) = a_1(y, \xi, \zeta) - \alpha - ix$. Since H is invariant by translations in x, we have

$$H = \mathcal{F}^{-1}\int_{\mathbb{R}}^{\oplus} \hat{H}(\xi)d\xi\mathcal{F}, \qquad \hat{H}(\xi) = \mathrm{Op}^w a_1(y, \xi, \zeta).$$

Here, ξ is a parameter and the Weyl quantization (C.2) operates only in the variables (y, ζ). We know that $2\pi\sigma_I[H]$ may be written as an integral of the Schwartz kernel of $2\pi i[H, P]\varphi'(H)$ along the diagonal. Using the Fubini theorem, and denoting by Tr_y the integration in the (y, y') variables, we have

$$\begin{aligned}
2\pi\sigma_I &= 2\pi\mathrm{Tr}_y \int_{\mathbb{R}} i[H,P](x,x')\varphi'(H)(x-x')dx'dx \\
&= 2\pi\mathrm{Tr}_y \int_{\mathbb{R}} -x'H(x')\varphi'(H)(x-x')dx' = \mathrm{Tr}_y \int_{\mathbb{R}} \partial_\xi \hat{H}(\xi)\varphi'(\hat{H}(\xi))d\xi,
\end{aligned}$$

where we have used as before $\int_{\mathbb{R}}(P(x-x') - P(x))dx = -x'$ first and the Parseval equality second. Here $[H,P](x,x')$ denotes the (distribution-valued) Schwartz kernel of the operator $[H,P]$. The above duality product (written as an integral) is defined since $\varphi'(H)$ has a smooth Schwartz kernel. Alternatively, we may replace $[H,P]$ by $[\Phi(H),P]$ with Φ compactly supported, smooth, and $\Phi(h) = h$ on the support of $\varphi'(h)$. This may be verified by cyclicity of the trace using a function ϕ such that $\phi\varphi' = \varphi'$ and $\phi\Phi = H\phi$. The above derivation applies with $\partial_\xi\Phi(\hat{H}) = \partial_\xi\hat{H}$ eventually.

Note that formally, $\partial_\xi\hat{H}(\xi)\varphi'(\hat{H}(\xi))d\xi = d(\varphi \circ \hat{H}(\xi))$. This reflects the fact that $2\pi\sigma_I$ may be computed as a spectral flow of branches of spectrum of $\hat{H}$. However, we do not pursue that route to prove (10.3).

Part 3. Define the symbol $s = s(Y,\xi)$ such that $\mathrm{Op}^w(s) = \partial_\xi\hat{H}\varphi'(\hat{H})$. Here $Y = (y,\zeta)$ are the variables in the Weyl quantization and ξ acts as a parameter. We know from results of the last section that $\partial_\xi\hat{H}\varphi'(\hat{H})$ is indeed a PDO with symbol in $S(\langle\xi,Y\rangle^{-\infty})$.

We now wish to use the invariance of $2\pi\sigma_I$ with respect to the rescaling $\zeta \to h\zeta$. Sending $h \to 0$ means considering the semi-classical regime, where non-commutating operators *almost* commute since $i[hD_x, x] = h$ is small when h is. See below for the notation we use on semi-classical calculus.

We then define H_h and $\hat{H}_h$ so that ζ is replaced by $h\zeta$ in the symbols; i.e., $\hat{H} = \mathrm{Op}^w a_1(\zeta)$ while $\hat{H}_h = \mathrm{Op}^w a_1(h\zeta) = \mathrm{Op}^w_h a_1(\zeta)$ using the convenient semiclassical Weyl quantization notation

$$\mathrm{Op}^w_h(a)\psi(y) = \frac{1}{2\pi h}\int_{\mathbb{R}^2} e^{i\frac{1}{h}(y-y')\cdot\zeta} a(\frac{y+y'}{2},\zeta;h)\psi(y')dy'd\zeta.$$

Compared to the standard Weyl quantization with $h = 1$, the semi-classical version is defined after change of variables $\zeta \to \zeta/h$ while the resulting symbol is allowed to depend on h more generally. Note that $\mathrm{Op}^w_h(1) = Id$ the identity operator. See section D in the appendix for notation.

With this notation, we thus have $\mathrm{Op}^w_h(s) = \partial_\xi\hat{H}_h\varphi'(\hat{H}_h)$ with now $s(Y,\xi;h)$. We observe that the symbol of $\partial_\xi\hat{H}_h\varphi'(\hat{H}_h)$ depends non-trivially on h.

In this setting, the trace is given by the following formula:

$$2\pi\sigma_I = \frac{1}{2\pi h}\int_{\mathbb{R}^3} \mathrm{tr}\ s(Y,\xi;h)dYd\xi.$$

We know from Corollary 9.4 that $2\pi\sigma_I$ is *independent* of $h > 0$. We will be using the Helffer-Sjöstrand formula to represent $\varphi'(\hat{H}_h)$ in terms of the resolvent of $\hat{H}_h$. We thus introduce (yet another) convenient symbol

$$\sigma_z = z - a_1(y,\xi,\zeta), \qquad \sigma_{\alpha+i\omega} = -a(\omega,y,\xi,\zeta)$$

where we parametrize the complex plane by $z = \lambda + i\omega$. Note that the role of x in the definition of a the symbol of the Fredholm operator F is now played by the imaginary part ω of the complex number z.

We have $\partial_\xi\hat{H}_h = -\mathrm{Op}^w_h(\partial_\xi\sigma_z)$ by definition of σ_z. Define the symbol $\varsigma(Y,\xi;h)$ so that $\varphi'(\hat{H}_h) = \mathrm{Op}^w_h\varsigma$. Then $s = -\partial_\xi\sigma_z\sharp_h\varsigma$ by composition calculus recalled in (D.3). What is needed from this calculus here is

$$a\sharp_h b = ab - \frac{i}{2}\{a,b\}h + O(h^2) \tag{10.4}$$

where $\{a,b\} = \partial_\zeta a \partial_y b - \partial_y a \partial_\zeta b$ is the Poisson bracket. Here $O(h^2)$ is in the sense of $S(\mathfrak{m}_1\mathfrak{m}_2)$ if $a \in S(\mathfrak{m}_1)$ while $b \in S(\mathfrak{m}_2)$.

We know from the semiclassical functional calculus ([18, Proposition 8.7] and following paragraph there) that s and ς admit the semiclassical expansion

$$s = s_0 + hs_1 + O(h^2), \qquad \varsigma = \varsigma_0 + h\varsigma_1 + O(h^2),$$

where $O(h^2)$ is in the sense of $S(\langle y,\xi,\zeta\rangle^{-\infty})$ in both cases. Thus, using Proposition D.4,

$$2\pi\sigma_I = \frac{1}{2\pi}\int_{\mathbb{R}^3} \mathrm{tr}\ s_1(Y,\xi)dYd\xi = \lim_{\mathrm{R}\to\infty} 2\pi\sigma_{I\mathrm{R}}, \qquad 2\pi\sigma_{I\mathrm{R}} := \frac{1}{2\pi}\int_{(-\mathrm{R},\mathrm{R})^3} \mathrm{tr}\ s_1(Y,\xi)dYd\xi.$$

For any $\varepsilon > 0$ and any $\mathrm{R} \geq \mathrm{R}_0(\varepsilon)$ large enough, then $|2\pi\sigma_I - 2\pi\sigma_{I\mathrm{R}}| < \varepsilon$.

We thus focus on an integral approximation of $2\pi\sigma_{I\mathrm{R}}$. Using (10.4), we have

$$s_1 = -\partial_\xi\sigma_z\varsigma_1 + \frac{i}{2}\{\partial_\xi\sigma_z, \varsigma_0\}.$$

Applying the Helffer-Sjöstrand formula, we have

$$\varsigma = -\frac{1}{\pi}\int_Z \bar\partial\tilde\varphi'(z)\mathrm{tr}\, r_z dYd\xi d^2z$$

where $\mathrm{Op}_h^w r_z = (z-\hat H_h)^{-1}$ is the symbol of the resolvent operator. Writing $r_z = r_{z0} + hr_{z1} + O(h^2)$, we have from Proposition D.4 a corresponding expansion for ς with components

$$\varsigma_j = -\frac{1}{\pi}\int_Z \bar\partial\tilde\varphi'(z)\mathrm{tr}\, r_{zj} dYd\xi d^2z \tag{10.5}$$

for $j = 0, 1$. We observe that $(z - \hat H_h)^{-1}(z - \hat H_h) = I$ so that by the composition calculus, $r_z\sharp_h\sigma_z = 1$. Using (10.4) (see also Proposition D.3) yields:

$$r_z\sigma_z + \frac{h}{2i}\{r_z, h_z\} + O(h^2) = 1, \qquad r_z = \sigma_z^{-1} + \frac{ih}{2}\{\sigma_z^{-1},\sigma_z\}\sigma_z^{-1} + O(h^2).$$

We remain formal in the above asymptotic expansions since we use the precise results obtained in Propositions D.3 and D.4 based on the estimate (D.4) for the symbol of the resolvent operator.

Thus, keeping terms that are independent of h, we have

$$2\pi\sigma_{I\mathrm{R}} = \frac{-1}{2\pi^2}\int_{(-\mathrm{R},\mathrm{R})^3\times Z} \bar\partial\tilde\varphi'(z)\,\mathrm{tr}\left(-\partial_\xi\sigma_z\frac{i}{2}\{\sigma_z^{-1},\sigma_z\}\sigma_z^{-1} + \frac{i}{2}\{\partial_\xi\sigma_z,\sigma_z^{-1}\}\right)dYd\xi d^2z.$$

Since $2\pi\sigma_{I\mathrm{R}}$ converges to $2\pi\sigma_I$, we obtained so far that

$$2\pi\sigma_I = \frac{i}{4\pi^2}\int_{\mathbb{R}^3\times Z} \bar\partial\tilde\varphi'(z)\,\mathrm{tr}\,\tau\, dYd\xi d^2z, \qquad \tau = \partial_\xi\sigma_z\{\sigma_z^{-1},\sigma_z\}\sigma_z^{-1} - \{\partial_\xi\sigma_z,\sigma_z^{-1}\}.$$

We recall that $\tilde\varphi'(z)$ vanishes outside of $Z \subset \mathbb{C}$. Let $Z = Z_+ \cup Z_-$ with $Z_\pm = Z \cap \{\pm\omega > 0\}$. Since $z \to \tau(z)$ is analytic, $\bar\partial\tilde\varphi'(z)\tau(z) = \bar\partial(\tilde\varphi'(z)\tau(z))$. Thus after integration by parts on $Z_\pm$ (Stokes theorem) using $\bar\partial = \frac12(\partial_\lambda + i\partial_\omega)$ and using that $\tilde\varphi'(\lambda) = \varphi'(\lambda)$, we find that

$$2\pi\sigma_I = \frac{1}{8\pi^2}\int_{\mathbb{R}^3\times\mathbb{R}} \varphi'(\lambda)\mathrm{tr}\left[\tau(z)\right]_{\lambda-0i}^{\lambda+0i} dYd\xi d\lambda = \lim_{\mathrm{R}\to\infty}\frac{1}{8\pi^2}\int_{(-\mathrm{R},\mathrm{R})^3\times\mathbb{R}} \varphi'(\lambda)\mathrm{tr}\left[\tau(z)\right]_{\lambda-0i}^{\lambda+0i} dYd\xi d\lambda.$$

Here $[\tau(z)]_{\lambda-i0}^{\lambda+i0} = \lim_{\omega\downarrow 0}\tau(\lambda+i\omega) - \tau(\lambda-i\omega)$. Since σ_z^{-1} is defined for (Y,ξ) sufficiently large, we obtain that $[\tau(Y,\xi,z)]_{\lambda-i0}^{\lambda+i0} = 0$ there. A Poisson bracket is in divergence form since

$$\partial_\zeta a \partial_y b - \partial_y a \partial_\zeta b = \partial_\zeta(a\partial_y b) - \partial_y(a\partial_\eta b).$$

The integral of such terms over $(-\mathrm{R},\mathrm{R})^3$ thus involves only boundary terms where $|(Y,\zeta)| \geq \mathrm{R}$ is chosen large enough. This shows that the term $\{\partial_\xi\sigma_z, \sigma_z^{-1}\}$ in the definition of τ is in divergence form and hence does not contribute to $2\pi\sigma_I$. As a consequence,

$$2\pi\sigma_I = \int_{\mathbb{R}} \varphi'(\lambda) I(\lambda) d\lambda, \qquad I(\lambda) = \lim_{R\to\infty} \frac{1}{8\pi^2} \int_{|(Y,\xi)|\leq R} \mathrm{tr}\, \Big[\sigma_z^{-1}\partial_\xi\sigma_z\{\sigma_z^{-1},\sigma_z\}\Big]_{\lambda-0i}^{\lambda+0i} dY d\xi.$$

Part 4. We will obtain that $I(\lambda)$ is independent of λ so that $2\pi\sigma_I = I$ in fact. We now use

$$\{\sigma_z^{-1},\sigma_z\} = \sigma_z^{-1}\partial_y\sigma_z\sigma_z^{-1}\partial_\zeta\sigma_z - \sigma_z^{-1}\partial_\zeta\sigma_z\sigma_z^{-1}\partial_y\sigma_z,$$

to obtain that

$$\sigma_z^{-1}\partial_\xi\sigma_z\{\sigma_z^{-1},\sigma_z\} dY d\xi = \frac{1}{3}(\sigma_z^{-1} d\sigma_z)^{\wedge 3},$$

with the orientation $d\xi\wedge dy\wedge d\zeta > 0$. Since $d(\mathrm{tr}(\sigma_z^{-1}d\sigma_z)^{\wedge 3}) = 0$ where σ_z is seen as a function of (ω,ξ,y,ζ) and since σ_z^{-1} is defined for (Y,ξ) large, we may close together the two discs at $\omega = \pm 0$ to obtain a closed surface in $\mathbb{R}^4$ and then by Stokes' theorem deform the closed surface to a sphere S_R with sufficiently large radius in (ω,ξ,y,ζ). We then obtain that

$$I(\lambda) = \frac{1}{24\pi^2}\int_{\mathbb{S}_R^3} \mathrm{tr}(\sigma_z^{-1}d\sigma_z)^{\wedge 3}$$

with positive orientation $d\omega\wedge d\xi\wedge dy\wedge d\zeta > 0$. This is a winding number that is stable with respect to continuous changes of σ_z, and in particular is independent of λ so long as σ_z^{-1} is defined. This shows that $I(\lambda)$ is independent of λ and that

$$2\pi\sigma_I = \frac{1}{24\pi^2}\int_{\mathbb{S}_R^3} \mathrm{tr}(\sigma_z^{-1}d\sigma_z)^{\wedge 3} = \frac{1}{24\pi^2}\int_{\mathbb{S}_R^3} \mathrm{tr}(\sigma_{\alpha+i\omega}^{-1}d\sigma_{\alpha+i\omega})^{\wedge 3}$$

This concludes the proof of the theorem since $a(\omega,y,\xi,\zeta) = -\sigma_{\alpha+i\omega}(y,\xi,\zeta)$. ☐

Bulk-invariant formulation. We now recast (10.3) to a bulk-edge correspondence form. We verify that $d(\mathrm{tr}(a^{-1}da)^{\wedge 3}) = 0$ so that as an application of the Stokes theorem, we have

$$2\pi\sigma_I[H] = \frac{1}{24\pi^2}\Big(\int_{y=-R}\mathrm{tr}(a^{-1}da)^{\wedge 3} - \int_{y=+R}\mathrm{tr}(a^{-1}da)^{\wedge 3}\Big), \tag{10.6}$$

deforming the sphere into an integral in (x,ξ,ζ) over the planes $y = \pm R$. Note that a depends directly on the symbols of the bulk operators $H_\pm$ on these planes, and more precisely $a_\pm = \alpha - i\omega - \sigma_\pm(\xi,\zeta)$, where $\sigma_\pm$ are the symbols of the bulk operator $H_\pm$. This shows that the topological charge conservation (10.3) may also be interpreted as a bulk-edge correspondence with the edge conductivity σ_I directly related to the properties of the bulk Hamiltonians $H_\pm$. This is a striking reduction in complexity. The next lectures present tools to compute the invariant explicitly.

11 Lecture 11.

Computation of Bulk-difference invariants. This lecture leaves the realm of Fredholm operators in non commutative geometry (what is meant by this is that the algebras of operators we considered were generated by operators of multiplication by x and differentiation D_x and they do not commute $[x, D_x] = i$) and focuses on the computation of invariants in commutative geometry (what is meant now is that we are dealing with algebras generated by the coordinates of manifolds such as ξ_1 and ξ_2 and those commute $[\xi_1, \xi_2] = 0$; there will still be plenty of matrices involved and those obviously still do not commute).

The invariants we are interested in classify manifolds and objects constructed on top of them such as vector or principal bundles. Our base manifold is in fact $\mathbb{R}^2$, which is topologically trivial according to most classifications. The domain $\mathbb{R}^2$ models the dual Fourier variables (ξ_1, ξ_2) in applications to (two-dimensional) topological insulators. However, $\mathbb{R}^2$ may be compactified in a number of ways. One way is one-point compactification where $\mathbb{R}^2 \cup \{\infty\} \cong \mathbb{S}^2$: we add a point at infinity and identify the union with the sphere by stereographic projection. It turns out that this compactification is not always useful. Another compactification looks at two copies of $\mathbb{R}^2$, concretely corresponding to the two insulating phases of interest, above and below an interface separating them. We can then wrap each copy of the plane around a half sphere and glue them together along the equator; see figure below. The point of these compactifications is that we may define continuous objects on the resulting sphere, which now has an interesting non-trivial topology $\pi_2(\mathbb{S}^2) \cong \mathbb{Z}$. We consider this construction of *bulk-difference* invariants in more detail now and show how they are related to invariants we presented in past lectures and how we may compute them.

Bulk-difference invariant. We saw above how $2\pi\sigma_I$ is related to the *difference* in (10.6). Each integral $\int_{y=\pm R}$ is defined in terms of bulk quantities but may not be a bulk invariant. Indeed, for the Dirac operator, each term is of the form $\frac{1}{2}\,\mathrm{sign}(m) \notin \mathbb{Z}$.

However, the *difference* of the above bulk quantities (not quite invariants) *is* indeed an invariant: the *bulk-difference invariant*. We now formalize this notion of bulk-difference invariant and relate it to (10.6) so the latter may be computed explicitly.

Consider a Hamiltonian H acting on a domain $\mathfrak{D}(H) \subset L^2(\mathbb{R}; \mathbb{C}^n)$ and invariant by translation $H = \mathcal{F}^{-1}\hat{H}(\xi)\mathcal{F}$ with $\xi \in \mathbb{R}^2$. The Hamiltonian is therefore represented by the family $\xi \to \hat{H}(\xi)$ with $\hat{H}(\xi)$ an $n \times n$ Hermitian matrix. It may therefore be diagonalized

$$\hat{H}(\xi) = \sum_{j=1}^{n} E_j(\xi)\Pi_j(\xi)$$

where $E_j(\xi)$ is the jth branch of (absolutely continuous) spectrum and $\Pi_j(\xi)$ is the corresponding one-dimensional eigenspace. For the Dirac problem with constant mass term m, we find that $n = 2$ and $E_\pm(\xi) = \pm\sqrt{|\xi|^2 + m^2}$. We leave it as an exercise to find the projectors $\Pi_\pm(\xi)$.

We will assume to simplify that all projectors are rank-one and the eigenvalues simple (and hence not crossing).

Let J be a subset of $\{1, \ldots, n\}$ and define $\Pi(\xi) \equiv \Pi_J(\xi) = \sum_{j\in J} \Pi_j(\xi)$. Then $\xi \mapsto \mathrm{Ran}\Pi(\xi)$ defines a (trivial) vector bundle over $\mathbb{R}^2$ diffeomorphic to $\mathbb{R}^2 \times \mathbb{C}^J$. If we now 'replace' $\mathbb{R}^2$ by a closed manifold M of dimension two and ξ parametrizes M, then $\xi \mapsto \mathrm{Ran}\Pi(\xi)$ may be a non-trivial vector bundle in the sense that it cannot be written globally as $M \times V$ for V a vector space of dimension J. Topological invariants are able to capture such obstructions to triviality. One such invariant is the Chern number we already encountered, which may be written as

$$\tilde{c}[\Pi] = \frac{i}{2\pi}\int_M \operatorname{tr}\Pi d\Pi \wedge d\Pi = \frac{i}{2\pi}\int_M \operatorname{tr}\Pi[\partial_i\Pi, \partial_2\Pi] d\xi \in \mathbb{Z}. \tag{11.1}$$

When M is not compact, for instance $M = \mathbb{R}^2$, then the above integral may still be defined but may not be an integer anymore. We already observed this for the Dirac problem, where we found that $\tilde{c}[\Pi_1] = \frac{1}{2}\operatorname{sign}(m)$. The reason is that the behavior of the integrand as $|\xi| \to \infty$ depends on the direction of ξ.

As a standing hypothesis, we *assume* that the integral in (11.1) is defined as a Lebesgue integral. For the Dirac problem, we observe that the integrand decays as $|(\xi,\zeta)|^{-3}$ at infinity so that the above integral indeed converges absolutely.

One-point compactification. The simplest way to compactify $\mathbb{R}^2$ is to add the point at infinity $\mathbb{S}^2 \cong \mathbb{R}^2 \cup \{\infty\}$. While the compactification is always defined, it is useful to define topological invariants if $\Pi(\xi)$ mapped to $\mathbb{S}^2$ remains *continuous*. This is the case when $\Pi(\xi)$ is independent of $\xi/|\xi|$ as $|\xi| \to \infty$. This applies to the *regularized* Dirac operator $D_x\sigma_1 + D_y\sigma_2 + (m + \eta\Delta)\sigma_3$ with symbol $\xi\sigma_1 + \zeta\sigma_2 + (m - \eta|\xi|^2)\sigma_3$ (with $|\xi|^2 = \xi^2 + \zeta^2$) where the term $\eta|\xi|^2$ dominates for $|\xi|$ large and is independent of $\xi/|\xi|$. This explains why we were able to compute a Chern number associated to a bulk phase for the *regularized* Dirac operator. When $\eta = 0$, it turns out that Π strongly depends on $\xi/|\xi|$ and the one-point compactification yields a non-continuous object on the sphere. We do not expect any stable topological classification in such a setting.

Circle compactification. The notion of *bulk-difference* invariant is one way to restore the integrality of the Chern number when the one-point compactification fails in the cases that interest us here, namely the computation of edge currents. In this setting, we actually have two Hamiltonians $\hat{H}^\pm(\xi)$ corresponding to the two half spaces (say north and south of an interface given by the $x-$axis). Both Hamiltonians may be diagonalized

$$\hat{H}^\pm(\xi) = \sum_{j=1}^n E_j^\pm(\xi)\Pi_j^\pm(\xi).$$

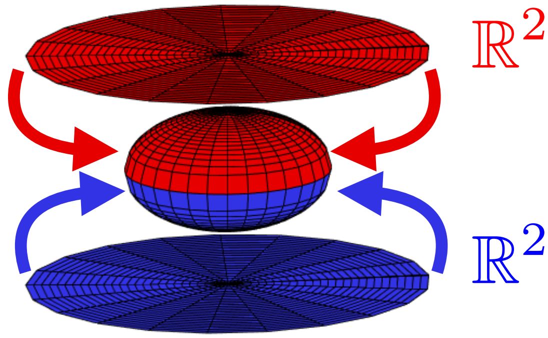
For J fixed, we may again define $\Pi^\pm(\xi) = \sum_{j\in J}\Pi_j^\pm(\xi)$. We now have two families of projectors parametrized by $\xi \in \mathbb{R}^2$. The bulk-difference invariant is constructed by projecting Π^+ onto the northern half sphere, Π^- onto a southern half sphere, and both projectors being glued along the equator. If the resulting object may be continuously defined in the vicinity of the equator the sphere, then we have a continuous family of projectors on the *closed manifold* $\mathbb{S}^2 = M$ and (11.1) now becomes an integer (although we have not proved that yet).

Constrution of bulk-difference invariant. More precisely, decompose $\xi = |\xi|\hat{\xi}$ and *assume* that

$$\lim_{|\xi|\to\infty}\Pi^+(|\xi|\hat{\xi}) = \lim_{|\xi|\to\infty}\Pi^-(|\xi|\hat{\xi}), \qquad \forall\hat{\xi} \in \mathbb{S}^1. \tag{11.2}$$

We assume that both limits exist and agree for each $\hat{\xi}$ on the equator $\mathbb{S}^1$. Let us now consider the stereographic projections mapping $\phi = (x,y,z) \in \mathbb{S}^2 \subset \mathbb{R}^3$ to the two planes $P_\pm \cong \mathbb{R}^2$ as

$$(x,y) = \frac{\xi}{\sqrt{1+|\xi|^2}}, \quad z = \frac{\pm 1}{\sqrt{1+|\xi|^2}}, \qquad \xi \in P_\pm$$



and then defining $\mathbb{S}^2 \cong (P_+ \sqcup P_-)/\sim$ where the equivalence relation $\sim$ identifies the points $(x, y, z = \pm 0)$ for (x, y) along the equator $\mathbb{S}^1$. We denote by π the inverse map from $\mathbb{S}^2$ to the two planes $P_\pm$. We then define the pulled-back projectors $\pi^*\Pi(\phi) = \Pi(\pi(\phi))$ (still called $\Pi(\phi)$ to simplify). By naturality of the pullback, we also have $\pi^*(d\Pi) = d(\pi^*\Pi)$ in their respective coordinates so that we may *define the bulk-difference Chern invariant*:

$$c[\Pi] = \frac{i}{2\pi}\int_{\mathbb{S}^2} \operatorname{tr} \Pi d\Pi \wedge d\Pi = \frac{i}{2\pi}\int_{\mathbb{R}^2} \operatorname{tr}\Big(\Pi^-[\partial_1\Pi^-, \partial_2\Pi^-] - \Pi^+[\partial_1\Pi^+, \partial_2\Pi^+]\Big) d\xi. \tag{11.3}$$

Note that the orientation on the two planes is inherited from that on the sphere with the bottom (blue) plane having the same orientation as the sphere.

For the Dirac problem, we find that

$$c[\Pi_1] = \frac{1}{2}(\operatorname{sign}(m_-) - \operatorname{sign}(m_+))$$

as obtained earlier. In other words, while the 'Chern number' on each plane may not be defined as an invariant with integral value, the difference of two such 'Chern numbers' is indeed a topological invariant, the Chern bulk-difference invariant. This reflects the fact that it is easier to define phase transitions than absolute phases.

Additivity of Chern numbers. For $\hat{H}(\xi)$ as above, we define $c_i = c[\Pi_i]$ the (first) Chern number associated to the ith projector when $J = \{i\}$.

An important property of these numbers is their additivity:

$$c[\Pi_i + \Pi_{i+1}] = c[\Pi_i] + c[\Pi_{i+1}]. \tag{11.4}$$

This property is not entirely immediate since c_i depends non-linearly on Π_i. More generally, if J and K are disjoint subsets of $\{1, \ldots, n\}$, then $c[\Pi_J + \Pi_K] = c[\Pi_J] + c[\Pi_K]$. A proof is as follows:

Proof. Let P and Q be two orthogonal projectors with $PQ = QP = 0$. Differentiating $P^2 = P$ and $PQ = 0$ yields $PdP = dP(I - P)$ while $PdQ = -dPQ$. We want to compute

$$\operatorname{tr}(P+Q)d(P+Q)d(P+Q) = \operatorname{tr} PdPdP + QdQdQ + PdPdQ + QdQdP + (PdQ + QdP)d(P+Q).$$

Using the above and $dPdQ = d(PdQ)$, we find

$$\operatorname{tr} PdPdQ = \operatorname{tr} dP(I-P)dQ = d\operatorname{tr}(PdQ) + \operatorname{tr} dPdPQ = d\operatorname{tr}(PdQ) + \operatorname{tr} QdPdP.$$

Therefore, with $\eta = \operatorname{tr}(PdQ)$,

$$\operatorname{tr} PdPdQ + QdQdP = \operatorname{tr} Qd(P+Q)dP + d\eta.$$

Now,

$$\begin{aligned}
&\mathrm{tr}Qd(P+Q)dP = \mathrm{tr}Qd(P+Q)dPQ = \mathrm{tr}Qd(P+Q)(-P)dQ = \mathrm{tr}[Q(P-I)+Q]dPdQ = 0\\
&\mathrm{tr}PdQd(P+Q) = \mathrm{tr}PdQd(P+Q)P = \mathrm{tr}(-dP)Qd(P+Q)P = \mathrm{tr}dPdQ[P-(I-Q)P] = 0.
\end{aligned}$$

It remains to integrate over the sphere with $\int_{\mathbb{S}^2} d\eta = 0$ by Stokes' theorem to obtain $\int \mathrm{tr}(P+Q)d(P+Q)d(P+Q) = \int \mathrm{tr}PdPdP + \int \mathrm{tr}QdQdQ$ as was to be demonstrated. ◻

From the definition of the numbers, we observe that $c[I_n] = 0$ so that $\sum_{j=1}^n c_j = 0$. Note that the integrality of the Chern numbers was neither established nor necessary in the above derivation.

For the Dirac operator, this means that $c_2 = c[\Pi_2] = -c_1$: projection onto the energies above a gap results in a Chern number opposite to that obtained from projections onto energies below the gap.

Green function bulk-difference winding number. While the above Chern number already appeared in the analysis of the asymmetric transport associated to the Dirac operator, in the last lecture, we obtained a topological charge conservation involving a seemingly different invariant associated to a map $a : \mathbb{S}^3 \to GL(\mathbb{C})$. We now show that these two invariants are in fact very much related.

Let $z = \alpha + i\omega \in \mathbb{C}$ and define the Green function

$$G = G_\alpha(\omega, \xi) = (z - \hat{H}(\xi))^{-1} = \sum_{j=1}^n (z - E_j(\xi))^{-1} \Pi_j(\xi), \tag{11.5}$$

as an application of the spectral theorem (for matrices). In other words, G is the symbol of the resolvent operator in the Fourier variables. This is a map from $\mathbb{R}^3$ to $GL(\mathbb{C})$ the space of invertible matrices.

We can also define a bulk-difference invariant based on $G^\pm$ corresponding to two insulators. Unlike the case of projectors, we observe that G_α tends to 0 as $\omega \to 0$ as well as when $(\xi, \zeta) \to 0$ for elliptic operators. We therefore need to normalize G before it may be glued at infinity along an equator for otherwise we would obtain an extension of G on $\mathbb{S}^3$ that would be singular on the equator (and no longer in $GL(\mathbb{C})$). We thus define

$$\tilde{G}(\omega, \xi) = \sum_{j=1}^n \Big(\frac{z - E_j(\xi)}{|z - E_j(\xi)|} \Big)^{-1} \Pi_j(\xi). \tag{11.6}$$

We observe that $\tilde{G} \in U(\mathbb{C})$ is now a unitary matrix.

As we did for the projectors, we *assume* that

$$\lim_{|\omega,\xi| \to \infty} \tilde{G}^+(|\omega, \xi|\theta) = \lim_{|\omega,\xi| \to \infty} \tilde{G}^-(|\omega, \xi|\theta), \qquad \forall \theta \in \mathbb{S}^2. \tag{11.7}$$

We then define $\mathbb{S}^3 \cong (V_+ \sqcup V_-)/\sim$ for the two spaces $V_\pm \cong \mathbb{R}^3$ with the equivalence relation $\sim$ identifying the two spheres at infinity parametrized by $\theta \in \mathbb{S}^2$. We may then define the corresponding invariant, which we will call a *bulk-difference generalized winding number*

$$\begin{aligned}
W_\alpha &= \frac{1}{24\pi^2} \int_{\mathbb{S}^3} \mathrm{tr}(\tilde{G}^{-1} d\tilde{G})^{\wedge 3} \\
&= \frac{1}{24\pi^2} \int_{\mathbb{S}^3} \mathrm{tr}(G^{-1} dG)^{\wedge 3} = \frac{1}{24\pi^2} \int_{\mathbb{R}^3} \mathrm{tr}\Big(((G^-)^{-1} dG^-)^{\wedge 3} - ((G^+)^{-1} dG^+)^{\wedge 3} \Big).
\end{aligned} \tag{11.8}$$

Note that W_α is defined with an opposite sign convention compared to [4]. Note, however, that (11.8) is equivalent to

$$-W_\alpha = \frac{1}{24\pi^2} \int_{\mathbb{S}^3} \mathrm{tr}(GdG^{-1})^{\wedge 3}. \tag{11.9}$$

The latter expression, which may easily be verified, is more convenient in practice since $G^{-1} = z - \hat{H}(\xi)$ is easily differentiated when the Hamiltonian is known. Note also that $\partial_\omega G^{-1} = i$.

Proof. We now justify the equalities in (11.8). The last equality comes from the change of measure from $\mathbb{S}^3$ to $\mathbb{R}^3$ given by the stereographic projections. As for the projectors, we *assume* that the above integrals over $\mathbb{R}^3$ converge absolutely. This is again the case for the Dirac operator for instance.

The second equality is based on the following homotopy invariance showing that the above integral, which is the same as the one that appears in the Fedosov-Hörmander formula, is homotopy stable. This is based on the fact that $\omega = \mathrm{tr}(A^{-1}dA)^{\wedge 3}$ is a closed three-form on $GL(n, \mathbb{C})$; i.e., $d\mathrm{tr}(A^{-1}dA)^{\wedge 3} = 0$. This was used to apply the Stokes theorem in a deformation of the Fedosov-Hörmander formula. More generally, let a_t mapping $[0,1] \times \mathbb{S}^3$ to $GL(n, \mathbb{C})$. Then

$$\mathrm{tr}(a_t^{-1}da_t)^{\wedge 3} = a_t^* \mathrm{tr}(A^{-1}dA)^{\wedge 3}$$

is closed on $[0,1] \times \mathbb{S}^3$ for the same reason; see remark below showing that in cohomology, $a_0^* = a_1^*$, which means that for the above three-form ω, we have $a_1^*\omega = a_0^*\omega + d\eta$ for a two-form η. Therefore, as an application of the Stokes theorem, we find that

$$\int_{\mathbb{S}^3} \mathrm{tr}(a_1^{-1}da_1)^{\wedge 3} = \int_{\mathbb{S}^3} \mathrm{tr}(a_0^{-1}da_0)^{\wedge 3}.$$

Alternatively, we may consider a being equal to a_t on the sphere of radius $R+t$ and apply Stokes' theorem on the annulus between $\mathbb{S}^3_R$ and $\mathbb{S}^3_{R+1}$. We can then consider a being given by a_1 on the whole annulus and apply Stokes' theorem again. This clearly implies that a_0 and a_1 are homotopic. It remains to define $a_t = \sum_{j=1}^n \big(\frac{z-E_j(\xi)}{|z-E_j(\xi)|^{1-t}}\big)^{-1}\Pi_j(\xi)$ with $a_0 = G$ while $a_1 = \tilde{G}$. This shows the equality in (11.8). ◻

Remark 11.1 *Let a_0 and a_1 homotopic smooth maps from M to N with M and N smooth closed manifolds. Above, $M = \mathbb{S}^3$ and $N = GL(n, \mathbb{C})$. For $t \in [0,1]$, $i_t : M \to M \times [0,1]$ defined by $i_t(x) = (x,t)$. Let $a : M \times [0,1] \to N$ be the homotopy map such that $a_j = a \circ i_j$ for $j = 0, 1$.*

For each p, there exists by [30, Lemma 17.9], a homotopy operator $h : \Omega^p(M \times [0,1])$ to $\Omega^{p-1}(M)$ such that for each $\tilde{\omega} \in \Omega^p(M \times [0,1])$, then

$$h(d\tilde{\omega}) + d(h\tilde{\omega}) = i_1^*\tilde{\omega} - i_0^*\tilde{\omega}.$$

Now let $\omega \in \Omega^p(N)$ be a closed $p-$form. In the above application, this is $\omega = \mathrm{tr}(A^{-1}dA)^{\wedge 3}$. Then we have that $\tilde{\omega} = a^*\omega$ is a closed $p-$form on $M \times [0,1]$ since $da^*\omega = a^*d\omega = 0$. Using the above homotopy operator, we thus have

$$a_1^*\omega - a_0^*\omega = i_1^*a^*\omega - i_0 * a^*\omega = dha^*\omega + hda^*\omega = d\eta$$

is an exact $p-$form on M with $\eta = ha^*\omega$ here. By the Stokes theorem, this means that for a top-degree form,

$$\int_M a_1^*\omega = \int_M a_0^*\omega.$$

Relation among the invariants. Both $c[\Pi]$ and W_α are based on the same Hamiltonian $\hat{H}(\xi)$. It turns out that they are directly related as follows:

Proposition 11.2 *Let α be a global spectral gap and G_α be the bulk-difference invariants constructed as above. Let $c_i = c[\Pi_i]$ be the bulk-difference Chern invariants. Then*

$$W_\alpha = \sum_{E_i<\alpha} c_i = -\sum_{E_i>\alpha} c_i. \tag{11.10}$$

Proof. A direct proof of the above result is shown in detail in [4]. The proof shows that the equality holds at the level of the integrands, in the sense that $\int_{\mathbb{R}_\omega} \operatorname{tr}(G^{-1}dG)^3 = 12\pi i \sum_{h_i<\alpha} \operatorname{tr}\Pi_i d\Pi_i d\Pi_i$ as an equality of two-forms after integration in ω.

To slightly simplify the proof, we first use the homotopy invariance of the Chern and winding numbers as follows. First, up to a shift in the Hamiltonian from $\hat{H}$ to $\hat{H}-\alpha$, we may assume that $\alpha=0$. Let then $\Pi_-(\xi) = \sum_{E_j<0} \Pi_j(\xi)$, which is defined unambiguously since 0 is in a spectral gap of $\hat{H}$. The projector Π_- is not affected by the transformation from $E_j(\xi)$ to $\operatorname{sign}(E_j(\xi))$. Let us now define (recall $\alpha=0$ where we have a global spectral gap)

$$G_t = \sum_{j=1}^n (i\omega - |E_j(\xi)|^{-t} E_j(\xi))^{-1} \Pi_j(\xi), \qquad G_1 = (i\omega+1)^{-1}\Pi_-(\xi) + (i\omega-1)^{-1}\Pi_+(\xi)$$

for $t\in[0,1]$ with $\Pi_+ = I - \Pi_-$. We verify that $G_t \in GL(\mathbb{C})$ for all $t\in[0,1]$ and that using the proof of (11.8), we have

$$\int_{\mathbb{S}^3_R} \operatorname{tr}(G_1^{-1}dG_1)^{\wedge 3} = \int_{\mathbb{S}^3_R} \operatorname{tr}(G_0^{-1}dG_0)^{\wedge 3}.$$

Let us relabel $G=G_1$, with now a modified Hamiltonian with normalized eigenvalues $|E_j(\xi)|=1$. This result underlines the fact that the topological character of $\hat{H}$ is all encoded in the projectors Π_j so long as a spectral gap is maintained for all ξ, i.e. $E_j(\xi)$ bounded away from 0.

Consider in (11.8) the partial integral in ω of the term involving G^- (called G below) and giving the two-forms:

$$I = \int_{\mathbb{R}_\omega} \operatorname{tr}(G^{-1}dG)^3 = -\int_{\mathbb{R}_\omega} \operatorname{tr}(GdG^{-1})^3$$

with now $G^{-1} = i\omega - \sigma$ with $\sigma = \Pi_+ - \Pi_-$ so that $\partial_\omega G^{-1} = i$. Thus

$$-I = 3i\Big(\int_{\mathbb{R}} \operatorname{tr}(i\omega-\sigma)[(i\omega-\sigma)^{-1}\partial_\xi\sigma, (i\omega-\sigma)^{-1}\partial_\zeta\sigma]d\omega\Big)d\xi d\zeta.$$

For α and β in $\{-1,1\}$, we have

$$\int_{\mathbb{R}} (i\omega-\alpha)^{-2}(i\omega-\beta)^{-1}d\omega = \frac{\pi}{4}(\operatorname{sign}(\alpha) - \operatorname{sign}(\beta)).$$

With $(i\omega-\sigma)^{-j} = (i\omega-1)^{-j}\Pi_+ + (i\omega+1)^{-j}\Pi_-$, we find

$$-I = \frac{3\pi i}{2}\operatorname{tr}\,\big([\Pi_+\partial_\xi\sigma, \Pi_-\partial_\zeta\sigma] - [\Pi_-\partial_\xi\sigma, \Pi_+\partial_\zeta\sigma]\big)d\xi d\zeta.$$

Now $\Pi_\pm^2 = \Pi_\pm$ so that $\Pi_\pm\partial_x\Pi_\pm\Pi_\pm = 0$. This shows

$$[\Pi_+\partial_\xi\sigma, \Pi_-\partial_\zeta\sigma] - [\Pi_-\partial_\xi\sigma, \Pi_+\partial_\zeta\sigma] = \Pi_+[\partial_\xi\sigma, \partial_\zeta\sigma] - \Pi_-[\partial_\xi\sigma, \partial_\zeta\sigma] = \sigma[\partial_\xi\sigma, \partial_\zeta\sigma].$$

Finally $\partial_x \sigma = \partial_x \Pi_+ - \partial_x \Pi_- = 2\partial_x \Pi_+$ so that

$$-I = 12\pi i \text{ tr } \Pi_+[\partial_\xi \Pi_+, \partial_\zeta \Pi_+] d\xi d\zeta.$$

It remains to divide by $24\pi^2$ and integrate in the (ξ, ζ) variables to obtain the components in (11.10) involving the lower plane in (11.3). The integral over the upper plane is treated similarly. This concludes the derivation. $\square$

Bulk-edge correspondence. We may now come back to (10.6). Assume the hypotheses of Theorem 10.1 and the gluing conditions (11.2) for the bulk operator $H_\pm = \text{Op}^w \sigma_\pm$ that appear in [H1]. Then, for α in a spectral gap,

$$2\pi\sigma_I[H] = W_\alpha = \sum_{E_j < \alpha} c_j \tag{11.11}$$

where the Chern numbers c_j are constructed from the bulk operators $H_\pm$ as shown earlier.

This is one of the most important result of these lecture notes. It states that for a large class of perturbed elliptic partial differential models with domain wall, the asymmetric transport $2\pi\sigma_I$ is not only quantized but also given by an explicit expression involving only the Chern numbers associated to the bulk operators $H_\pm$.

For the Dirac operator, this is again

$$2\pi\sigma_I = \frac{1}{2}(\text{ sign}(m_-) - \text{ sign}(m_+)).$$

The derivation now entirely bypasses the need to diagonalize the operator H and compute the winding number of each branch of absolutely continuous spectrum. This is an important simplification in Floquet Topological Insulators and in gated twisted bilayer graphene models where the bulk invariants can be computed explicitly while a diagonalization of the interface Hamiltonian seems totally hopeless; see [6, 8].

12 Lecture 12.

A rapid survey of several topological invariants and their computations. We now look at a number of ways to compute the Chern numbers and winding numbers that appear in the preceding lectures. We already saw a way to compute a Chern number by estimating an integral of the form $\int_{\mathbb{R}^2} \text{tr}\Pi d\Pi d\Pi$. In the calculation of that integral, we found a way to write the integrand in divergence form so as to relate it to boundary terms that were easier to estimate. We will see that this is not a coincidence. To motivate the explicit results that we obtain, we now go over a tour of several classical objects that were introduced to classify topologically various spaces and maps between them. The spaces we consider are all smooth manifolds; for instance Lie groups such as $GL(n, \mathbb{F})$ with $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ and their subgroups, homogeneous spaces such as the spheres $\mathbb{S}^2$, or Euclidean spaces $\mathbb{R}^2$. The maps involve maps among such spaces as well as sections of fiber bundles, i.e, mappings from the base manifold to vector spaces or Lie groups.

The presentation below is based on material in [30], [20], [33], and [34]. See also [13].

Homotopy theory. Classification of manifolds or maps means identifying 'similar looking' manifolds or maps. For manifolds, this may mean finding a smooth map between them that is bijective and with smooth inverse. It turns out that this is often difficult to achieve. A method that has shown great success is homotopy theory.

In what follows, M and N will always denote smooth manifolds.

Two continuous maps f and g from M to N are said to be *homotopic* if there is a continuous map h from $M \times [0,1]$ to N so that $h(0) = f$ and $h(1) = g$. In other words, the images of f and g in N can be continuously deformed from one to the other. We may then introduce *homotopy classes of equivalence* with f and g belonging to the same class if they are homotopic. Symmetry, transitivity, and reflexivity of the notion of equivalence are readily verified. The classes of equivalence from M to N are denoted by $[M, N]$ while the class associated to f is denoted by $[f]$.

Two spaces M and N are then *homotopic* if we can find $f : M \to N$ and $g : N \to M$ such that both $f \circ g$ and $g \circ f$ are homotopic to the identity on N and M, respectively. As an important example, $\mathbb{R}^d$ is homotopic to the point 0 with f mapping x to 0 and g mapping 0 to $0 \in \mathbb{R}^d$. Then $f \circ g$ is identity on the point while $g \circ f : x \to 0$ is homotopic to identity on $\mathbb{R}^d$ using the map $h(t, x) = tx$. Spaces may therefore be homotopic even though they are far from diffeomorphic. This generates a more flexible and easier to compute classification.

Fundamental group. Deciding whether two spaces are homotopic is still not easy. A fruitful way to garner information on a manifold N is to look at functions defined on or with images in N. In the latter case, consider maps from the unit circle $\mathbb{S}^1$ to N with a base point $p \in N$ such that the loop parametrized by $\mathbb{S}^1$ starts and ends at p. This forms a group on the classes of equivalences $[f]$ of maps from $\mathbb{S}^1$ to N. Indeed, we identify maps with images in N that are homotopic. Moreover, we have a group structure by concatenation: the map $[f][g]$ first loops from p to p according to g followed by the loop associated to f. This forms indeed a map from p to p and one verifies that it is $[fg]$.

The group $\pi_1(N)$ is called the *fundamental group*. It is not necessarily abelian (commutative). Consider the figure 8 (although this is not quite a manifold). Then looping around the upper or lower circle of the figure generates non homotopic loops. Moreover, going around the top circle first and the bottom circle second is not homotopic to the reverse operation.

We can show that $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$ counting the number of times the loop winds around the origin.

Higher-order homotopy groups. We can similarly look at maps from $\mathbb{S}^n$ to N. Seeing the sphere as a cube with all boundary points identified to a base point p, we may see that homotopy classes of equivalences of such maps also form a group. If ΩN is the space of loops on N, then the space of loops on ΩN may be identified with $\pi_2(N)$ and so on recursively, so that the group structure is clear.

The groups $\pi_n(N)$ are always abelian (commutative) for $n \geq 2$ as may be (relatively easily) verified. We have that $\pi_m(\mathbb{S}^m) = \mathbb{Z}$ essentially by construction. We observe that $\pi_n(\mathbb{S}^1) = \{0\}$ for $n \geq 2$ since any loop can be continuously deformed to the point p. More generally, we find that $\pi_m(\mathbb{S}^n) = \{0\}$ for $m < n$ as well. However, $\pi_{n+1}(\mathbb{S}^n) = \mathbb{Z}_2$ for $n \geq 3$ and $\pi_{n+2}(\mathbb{S}^n) = \mathbb{Z}_2$ for $n \geq 2$. These are difficult results to obtain.

For tori, we have $\pi_1(\mathbb{T}^n) = \mathbb{Z}^n$ for the $n-$dimensional torus. Each loop around the one-dimensional holes defines an independent generator. The order in which these loops are covered does not matter and the group is abelian. We have $\pi_m(\mathbb{T}^n) = \{0\}$ for $m \geq 2$ so that there are no holes of higher-order than one in tori. Note that even though the cardinality of $\mathbb{Z}$ is that of $\mathbb{Z}^n$, as groups they are not isomorphic when $n \geq 2$ (the latter requires n generators).

Of interest to us are the elements in the *Bott periodicity theorem* stating that

$$\pi_k(U(n)) \cong \pi_k(SU(n)) \cong \left\{ \begin{array}{cc} \{0\} & k \ \text{even} \\ \mathbb{Z} & k \ \text{odd} \end{array} \right.$$

for $n \geq \frac{1}{2}(k+1)$ (stability) while in the real case

$$\pi_k(O(n)) \cong \pi_k(SO(n)) \cong \begin{cases} \{0\} & k = \ 2,4,5,6 \bmod 8 \\ \mathbb{Z}_2 & k = \ 0,1 \bmod 8 \\ \mathbb{Z} & k = \ 3,7 \bmod 8 \end{cases}$$

for $n \geq k+2$.

While we do not really consider the real case in detail here, it is important in general classifications of topological insulators based on symmetries satisfied by Hamiltonians. Of importance to us is the fact that $\pi_3(GL(n,\mathbb{C})) \cong \pi_3(U(n)) \cong \mathbb{Z}$ for $n \geq 2$. This encodes the Chern numbers and non-trivial indices we have encountered so far, and in particular in the form of an integral over a sphere in the Fedosov-Hörmander formula.

Differential forms. While the above groups provide topological information regarding relevant maps such as those in $\pi_3(U(n))$, computing the invariant of a class $[f]$ is difficult. It turns out that differential forms acting on a given manifold X also offer fruitful information. While the above groups are maps from spheres to the manifold of interest, we now consider a dual notion mapping the manifold of interest X to $\Lambda^* T^* M$, the space of (alternating tensor valued) differential forms on M.

We may want to recall some properties of differential forms and of pullbacks here. Following [30], for M a manifold and $T^*_p M$ the cotangent space space of M at p, we denote by $\Lambda^k T^* M$ the alternating $k-$tensors given as the union over all points $p \in M$ of $\Lambda^k(T^*_p M)$. A section of $\Lambda^k T^* M$ is a continuous tensor field which to each point $p \in M$ associates an alternating tensor $\Lambda^k(T^*_p M)$. The vector space of such smooth $k-$forms is then denoted by $\Omega^k(M) = \Gamma(\Lambda^k T^* M)$. The union over $0 \leq k \leq d$ of these vector spaces is called $\Omega^*(M)$. We verify that $\Lambda^k T^* M = \{0\}$ when $k > n$.

The wedge product (exterior product) on alternating tensors extends to a pointwise product on $\Omega^*(M)$ with $(\omega \wedge \eta)_p = \omega_p \wedge \eta_p$ a $k+l$ form if ω is a $k-$ form and η is a $l-$form. In a smooth chart, any $k-$form may be written

$$\omega = \sum_I \omega_I dx^I, \qquad \omega_I = \omega(\partial_{x^{i_1}}, \ldots \partial_{x^{i_k}})$$

for $I = \{i_1 < \ldots < i_k\}$ a subset of $\{1, \ldots, d\}$ of cardinality k.

Pullback. Let $f : M \to N$ be a smooth map with ω a differential form on N. Then the pullback $f^*\omega$ is defined as the differential form on M such that

$$(f^*\omega)_p(X_1, \ldots X_k) = \omega_{f(p)}(df(X_1), \ldots, df(X_k)),$$

where $df_p(X)$ is the differential of f at p applied to the vector field X. For $f : M \to N$ smooth, we have that f^* is a linear map from $\Omega^k(N)$ to $\Omega^k(M)$ that commutes with the wedge product $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$. In any smooth chart, we have

$$f^*(\omega_I dy^I) = \omega_I \circ f \ df^I.$$

For a map $f : \mathbb{R}_x \to \mathbb{R}_y$, we have for zero forms (functions) $f^*u(x) = u \circ f(x)$ and for one-forms $f^*(u(y)dy)(x) = u(f(x))df(x) = f^*u(x)df(x)$. The latter is also $f^*u(x)f'(x)dx$, a formula we now generalize.

Pullback formula for top-degree forms. We will mostly need the notion of pullback on volume (top-degree) forms, which is important in the definition of a topological invariant called the degree of a map. Let $f : M \to N$ be a smooth map between $d-$manifolds with coordinates x^i and y^j on U and V. Then on $U \cap f^{-1}(V)$, we have

$$f^*(udy) = u \circ f(\det Df)dx = f^*u(\det Df)dx,$$

where Df is the Jacobian matrix of f in these coordinates. This comes from the change of variables $dy = (\det Df)dx$ whereas the function $u(y)$ is pulled back to $f^*u(x) = u \circ f(x)$. Note that pullbacks reverse the direction of the map.

Exterior derivative. The exterior derivative d is defined on functions in a coordinate patch as $df = \sum_i \partial_{x^i} f dx^i$ or for a smooth vector field X by $df(X) = \sum_i \partial_{x^i} f X_i$. This generalizes to differential forms on a smooth manifold M with or without boundary as follows. There are unique linear *exterior differentiation* operators d from $\Omega^k(M)$ to $\Omega^{k+1}(M)$ for all k such that

(i): If $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$, then $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$;

(ii): $d \circ d = 0$;

(iii): For $f \in \Omega^0(M) = C^\infty(M)$, df is the differential of f such that $df(X) = Xf$ for X a vector field.

(iv): For $f : M \to N$ smooth and $\omega \in \Omega^k(N)$, then $f^*(d\omega) = d(f^*\omega) \in \Omega^{k+1}(M)$.

The last property is called the *naturality* of the pullback, which was essentially designed to commute with differentials. In coordinates, we have with compact notation

$$f^*(d(udx^I)) = f^*(du \wedge dx^I) = d(u \circ f) \wedge d(x^I \circ f) = d(u \circ f \wedge d(x^I \circ f)) = d(f^*(udx^I)).$$

Recall that in three dimensions, the gradient, curl, and divergence operators may be identified with exterior derivation from functions to one-forms, one-forms to two-forms, and two-forms to three-forms identified with functions so that $d^2 = 0$ implies curl$\circ$grad$= 0$ and div$\circ$curl $= 0$.

Stokes' theorem [30, Theorem 16.25]. Let M be an oriented smooth $n-$manifold with corners and let ω be a smooth compactly supported $(n-1)-$form on M. Then

$$\int_M d\omega = \int_{\partial M} \omega. \tag{12.1}$$

This far-reaching generalization of the fundamental theorem of calculus implies that the integration of exact forms (those of the form $d\omega$) only involves boundary data.

De Rham cohomology groups. A differential form ω is said to be *closed* when $d\omega = 0$. It is said to be *exact* when $\omega = d\eta$ for some differential form η.

Consider the equation $d\omega = 0$. Such equations appear naturally in many systems of partial differential equations such as Maxwell's equations. As such, the above equation admits a lot of solutions since any exact differential form $d\eta$ is in the kernel of d: $d^2\eta = 0$. Are there other solutions? The answer depends on the topology of the manifold M over which the differential forms are defined. While a differential form is always exact *locally*, topological obstructions exist to have globally exact forms. The de Rham cohomology groups capture these obstructions.

Let M be a smooth manifold with or without boundary. Recall that $d : \Omega^p(M) \to \Omega^{p+1}(M)$ is a linear operator so that we may define

$$\begin{array}{rclcl} Z^p(M) & = & \mathrm{Ker}\big(d : \Omega^p(M) \to \Omega^{p+1}(M)\big) & = & \{\text{closed p-forms on M}\} \\ B^p(M) & = & \mathrm{Ran}\big(d : \Omega^{p-1}(M) \to \Omega^p(M)\big) & = & \{\text{exact p-forms on M}\}. \end{array}$$

These are vector spaces and we may now define their quotients of closed forms modulo exact forms:

$$H^p(M) = Z^p(M)/B^p(M). \tag{12.2}$$

This quotient is in fact a real linear space called the pth de Rham cohomology group. We have $H^p(M) = \{0\}$ when $0 < p$ or $p > n$. The above definition means that $H^p(M) = 0$ iif every closed $p-$form is exact.

For ω a $p-$form, we denote by $[\omega]$ its cohomology class of $p-$forms ω' such that $\omega - \omega' = d\eta$ for some $(p-1)-$form η. Then ω and ω' are said to be cohomologous.

The above groups in fact admit a ring structure using the wedge product, which maps $H^p(M) \times H^q(M)$ to $H^{p+q}(M)$. Indeed the wedge product maps differential forms according to the above grading. Moreover, one verifies that the wedge product of exact/closed forms is still exact/closed so the product descends to cohomology with $[a] \wedge [b] = [a \wedge b]$.

Induced cohomology. For any smooth map $f : M \to N$, we observe that the pullback f^* maps $Z^p(N)$ to $Z^p(M)$ by naturality and linearity $0 = f^*0 = f^*d\omega = df^*\omega$ as well as $B^p(N)$ to $B^p(M)$ also by naturality $f^*\omega = f^*d\eta = df^*\eta$. It therefore induces a cohomology map from $H^p(N)$ to $H^p(M)$ also denoted by f^* and defined by $f^*[\omega] = [f^*\omega]$. With this notation I^* denotes the identity map on $H^p(M)$ while $(g \circ f)^* = f^* \circ g^*$ for composition. From these properties, we deduce that diffeomorphic smooth manifolds have isomorphic de Rham cohomology groups. However, the cohomology groups, as the homotopy groups, are shared by manifolds that are far from diffeomorphic as we now see.

Useful corollaries are as follows. The cohomology of a disjoint union of spaces is given by the direct product space of the individual cohomologies so that we can focus on connected components. The cohomology in degree zero of a connected manifold is $H^0(M) \cong \mathbb{R}$ the space of constant functions (since there are no $-1-$forms and $df = 0$ on a connected manifold implies that f is constant).

Homotopy operator. We start with a very useful map called a homotopy operator. For $t \in [0,1]$, define $i_t(x) : M \to M \times [0,1]$ by $i_t(x) = (x,t)$. Note that $M \times [0,1]$ is a manifold with corners when M is a manifold with boundary. All results presented here apply to such manifolds. At any rate, M and $M \times [0,1]$ are homotopic manifolds and i_0^* and i_1^* are homotopic maps from M to itself. The following lemma relates the pullbacks i_0^* and i_1^* on differential forms as follows:

Lemma 12.1 *[30, Lemma 17.9]. For any smooth manifold M, there exists a homotopy operator $h : \Omega^*(M \times [0,1]) \to \Omega^{*-1}(M)$ between the two maps $i_0^*, i_1^* : \Omega^*(M \times [0,1]) \to \Omega^*(M)$ such that*

$$h(d\omega) + d(h\omega) = i_1^*\omega - i_0^*\omega.$$

In fact, the operator is defined as in a Poincaré lemma by $h\omega = \int_0^1 i_t^*(\partial_s \lrcorner\, \omega) dt$ with ∂_s the vector field of differentiation in the variable t.

This shows that for ω closed (i.e., $d\omega = 0$), we find that $i_1^*\omega - i_0^*\omega = d(h\omega)$ is exact. Therefore, in cohomology, as operators from $H^p(M \times [0,1])$ to $H^p(M)$, we have $i_1^* = i_0^*$.

As a consequence, if $f, g : M \to N$ are homotopic smooth maps, then in cohomology from $H^*(N)$ to $H^*(M)$, we have $f^* = g^*$. Indeed if h is the homotopy linking f and g, then $f^* = (h \circ i_0)^* = i_0^* h^* = i_1^* h^* = (h \circ i_1)^* = g^*$.

Homotopy invariance. An important consequence is that cohomology groups are homotopy invariants in the following sense.

Theorem 12.2 *[30, Theorem 17.11]. If M and N are homotopy equivalent smooth manifolds with or without boundary, then the de Rham cohomology groups are isomorphic: $H^p(M) \cong H^p(N)$ with isomorphism induced by any smooth homotopy invariance $f : M \to N$.*

Indeed, for $f : M \to N$ and $g : N \to M$ smooth, the above states that in cohomology, $f^* \circ g^* = (g \circ f)^* = (I_M)^*$ by using the above equality in cohomology, which is $I_{H^*(M)}$. Similarly, $g^* \circ f^* = I_{H^*(N)}$ so that for instance $f^* : H^*(N) \to H^*(M)$ is an isomorphism with inverse g^*.

As a consequence, we obtain that any manifold M contractible, i.e, with an identity map homotopic to a constant map, has trivial cohomology $H^p(M) = \{0\}$ for $p \geq 1$. In particular the Poincaré lemma states that for U star shaped open domain in $\mathbb{R}^n$ (or a half space) then $H^p(U) = \{0\}$ for $p \geq 1$. As a consequence, any point p in a smooth manifold M has a neighborhood on which each closed form is exact. Since $\mathbb{R}^n$ (as well as half spaces) is star-shaped, then $H^p(\mathbb{R}^n) = \{0\}$ for $p \geq 1$.

Examples of cohomology groups [30]. Cohomology groups are easier to compute than homotopy groups. In particular, there is a strong relationship between the cohomology groups of manifolds and of unions and intersections. Of importance for us is the cohomology of spheres, which is entirely understood (unlike homotopy groups) and given by

$$H^p(\mathbb{S}^n) \cong \begin{cases} \mathbb{R} & p = 0 \ \text{ or } \ p = n \\ \{0\} & 0 < p < n. \end{cases} \tag{12.3}$$

In particular, any smooth orientation form (of top degree n) is a basis for $H^n(\mathbb{S}^n) \cong \mathbb{R}$. As a result, $\omega \in \Omega^n(\mathbb{S}^n)$ is exact if and only if $\int_{\mathbb{S}^n} \omega = 0$. Indeed $[\omega_1] = [\omega_2]$ iif $\omega_1 - \omega_2 = d\eta$ which by Stokes means $\int_{\mathbb{S}^n} \omega_1 = \int_{\mathbb{S}^n} \omega_2$ so that $[\omega_1] = 0$ iif ω_1 is exact iif $\int_{\mathbb{S}^n} \omega_1 = 0$.

For punctured Euclidean space, we find that $H^p(\mathbb{R}^{n+1}/\{0\}) \cong H^p(\mathbb{S}^n)$ as above by (homotopically) retracting the punctured space to the corresponding sphere.

Compactly supported cohomology. It is interesting to observe that for non-compact manifolds, the behavior of differential forms at infinity have an influence on the cohomology groups. This shows the importance of appropriately compactifying non-compact domains whenever possible.

We have the following modified Poincaré lemma: Let ω be a compactly supported closed $n-$form on $\mathbb{R}^n$ for $n \geq 1$ such that $\int_{\mathbb{R}^n} \omega = 0$. Then there exists η *compactly supported* such that $d\eta = \omega$.

The novelty compared to the standard Poincaré lemma is that η may be chosen with compact support. We may then define a different cohomology, namely the pth compactly supported de Rham comology as

$$H^p_c(M) = \mathrm{Ker}\big(d : \Omega^p_c(M) \to \Omega^{p+1}_c(M)\big) \ / \ \mathrm{Ran}\big(d : \Omega^{p-1}_c(M) \to \Omega^p_c(M)\big)$$

where $\Omega^p_c(M)$ denotes the compactly supported smooth $p-$forms on M. Then we have

$$H^p(\mathbb{R}^n) \cong \begin{cases} \mathbb{R} & p = 0 \\ \{0\} & 0 < p \leq n \end{cases} \qquad H^p_c(\mathbb{R}^n) \cong \begin{cases} \{0\} & 0 \leq p < n \\ \mathbb{R} & p = n. \end{cases} \tag{12.4}$$

This shows that the two cohomology groups based on arbitrary or compactly supported differential forms do not agree in general for non-compact manifolds. Of course, the two notions are identical on compact manifolds. Note that the top cohomology groups $H^n_c(\mathbb{R}^n) \cong H^n(\mathbb{S}^n) \cong \mathbb{R}$, which are relevant for us here, are non-trivial and agree for $\mathbb{R}^n$ and $\mathbb{S}^n$. This is a general phenomenon:

Top cohomology and integration. Assume that M is an oriented smooth $n-$manifold (with $n \geq 1$). Then there is a linear map $I : \Omega_c^n(M) \to \mathbb{R}$ given by

$$I(\omega) = \int_M \omega. \tag{12.5}$$

Then in cohomology, $I : H_c^n(M) \to \mathbb{R}$ is an isomorphism [30, Theorem 17.30]. In particular, $H_c^n(M) \cong \mathbb{R}$.

The result therefore applies to compact orientable smooth manifolds.

For orientable connected non-compact smooth $n-$ manifolds, we can show as for $\mathbb{R}^n$ that $H^n(M) = \{0\}$. For non-orientable manifolds, one can show that $H^n(M) \cong H_c^n(M) = \{0\}$ as well. So, orientation is necessary to obtain a top-degree non-trivial topology in cohomology.

Regular values and Sard's theorem. We first recall the Sard theorem. If $f : M \to N$ is a smooth map, a point $p \in M$ is a *regular point* of f if its differential $df_p : T_pM \to T_{f(p)}N$ is surjective (maximal rank). It is a *critical point* otherwise.

A point $c \in N$ is a *regular value* of f if every point of the pre-image $f^{-1}(c)$ of c is a regular point of f, and a *critical value* otherwise.

The *Sard theorem* states that critical values of a smooth map form a set of measure zero in N. If M and N have the same dimension, the result holds for C^1 functions.

Consequences of Sard's theorem include the Whitney embedding theorem stating that every smooth $n-$manifold with or without boundary admits a proper smooth embedding into $\mathbb{R}^{2n+1}$. Following this is the Whitney approximation theorem stating that a continuous map from M to $\mathbb{R}^k$ can be approximated by a smooth one. To generalize this to maps from M to N there is the difficulty that a smoothed out version of f may not longer map to N. To fix this, we may construct a tubular neighborhood of a manifold using the embedding theorem and the notion of normal bundle NM. With this, there is then the Whitney approximation theorem for $f : N \to M$ with M smooth without boundary. Then f is homotopic to a smooth map.

Degree Theory. We arrive at an important tool in the explicit computation of topological invariants associated to some (but not all) maps of interest in topological insulators.

Theorem 12.3 *Suppose M and N are connected oriented smooth $n-$manifolds and let $f : M \to N$ be a smooth map. Then there is a unique* integer $\deg f$ *called the* degree of f *such that both conditions hold:*
(i) For every smooth $n-$form ω on N we have

$$\int_M f^*\omega = \deg f \int_N \omega.$$

(ii) If $y \in N$ is a regular value of F, then

$$\deg f = \sum_{x \in f^{-1}(y)} \operatorname{sign}(J_f(x))$$

where $J_f(x)$ is the Jacobian determinant of f at x, i.e., the determinant of the differential df_x.

It is useful to recall the main steps of the derivation of such a result. We saw above that two smooth $n-$forms on either M or N are cohomologous iif they have the same integral. Let $\int_N \theta = 1$

and $k = \int_M f^*\theta$. For ω arbitrary, then $[\omega] = [a\theta]$ with $a = \int_N \omega$. Thus $[f^*\omega] = [af^*\theta]$ so that $ak = k\int_N \omega$ so that (i) holds for a unique real number k independent of ω.

Now let $q \in N$ be a regular value of f with $f^{-1}(q) = \{x_1, \ldots, x_m\}$ isolated points in M (possibly $m = 0$). By the implicit function theorem, there are open $U_i \ni x_i$ such that F is a diffeomorphism from U_i to $f(U_i)$ and the U_i are disjoint. On the closed $K = M\backslash(\cup_i U_i)$, $f_{|K}$ is closed in N and disjoint from q. Shrinking and adapting U_i if necessary, we have a map f from $\sqcup_i U_i$ to W a connected neighborhood of q such that each restriction $f_{|U_i}$ to W is a diffeomorphism.

Now take ω a $n-$form on $W \subset N$ such that $\int_W \omega = 1$. Thus from above

$$\deg f = \int_M f^*\omega = \sum_i \int_{U_i} f^*\omega = \sum_i \mathrm{sign}((J_f)_{U_i}) \int_W \omega = \sum_i \mathrm{sign}((J_f)_{U_i}) = \sum_{x \in f^{-1}(y)} \mathrm{sign}(J_f(x)),$$

where the third equality comes from a change of variables formula.

The three main arguments are: (i) the fact that compactly supported n-forms are cohomologous when they share the same integral; (ii) the Sard theorem stating that almost every value is *not* a critical value of a map f; and (iii) the construction of measures with supports that localize in the vicinity of points of interest. This remarkable result shows that the top cohomology invariant in fact takes *integral* values in a very natural way.

Let us mention important corollaries. Let M, N and P be compactly connected oriented smooth $n-$manifolds. If f and g are smooth maps then $\deg f \circ g = \deg g \deg f$. If f is a diffeomorphism, then the degree of f is 1 if f is orientation preserving and -1 if it is orientation reversing. Finally, if two maps f_0 and f_1 are homotopic, then they have the same degree.

For the converse, let us mention the *Hopf theorem*, stating that maps from a closed oriented compact n-manifold to $\mathbb{S}^n$ are homotopic if and only if they have the same degree. The degree therefore completely classifies such maps topologically. However, this is not true in general for compact orientable manifolds. For instance maps from $\mathbb{T}^2$ to itself given by $f(w, z) = (w, z)$ and $g(w, z) = (z, \bar{w})$ share the same degree but are not homotopic.

Let us also mention that we can define the degree of any *continuous* map by letting the degree be the degree of any smooth map homotopic to F. The Whitney approximation theorem guarantees the existence of such maps. The above results then also apply to continuous maps.

We finally mention the following extension with its own caveats: For f a proper map on non-compact connected oriented smooth $n-$manifolds, a degree may be defined as above for smooth compactly supported $n-$forms as a finite sum of signs of Jacobians. Since properness implies that F is infinite at infinity, this is reasonable. However, the degree of a proper map is no longer a homotopy invariant. For instance, $f, g : \mathbb{C} \to \mathbb{C}$ given by $f(z) = z$ and $g(z) = z^2$ have different degrees although they are homotopic. So the notion of a degree is a lot less powerful than in the compact case.

Degree at regular and critical values. Consider the situation of M an open subset of a smooth n-manifold M_0 with $\bar{M} = M \cup \partial M$, f a map from M to another smooth n-manifold N and $q \in N\backslash f(\partial M)$ a *regular value* of f. The above result then defines the degree of such a map f from M to N. To emphasize the role of q, we call as in [34] such an integer $\deg(f, M, q)$. Now, even if q is a critical value, we can find regular values q' in any neighborhood of q. Moreover, we verify that $\deg(f, M, q')$ is then independent of the choice of q' sufficiently close to q. As a result, we *define* $\deg(f, M, q)$ as this integer $\deg(f, M, q')$. Of course, (ii) above needs to be estimated at such a regular value.

If $K \subset \bar{M}$ such that $q \notin f(K)$, then we verify from the expression of the degree as an integral the *excision* result $\deg(f, M, q) = \deg(f, M\backslash K, q)$.

Also immediate from the integral formulation of the degree is the following result. Assume M and N of dimension n and M' and N' of dimension m. Assume f and f' with degrees at $q \in N$ and $q' \in N'$. Then

$$\deg(f \times f', M \times M', (q, q')) = \deg(f, M, q) \deg(f', M', q'). \tag{12.6}$$

This result is a direct consequence of

$$\int_{M \times M'} (f \times f')^* \mu \cdot \mu' = \int_M f^* \mu \int_{M'} f'^* \mu'.$$

It is for instance reasonably straightforward to apply when M and M' are cubes (rather than spheres).

Gauss-Bonnet theorem. For M a compact smooth oriented surface (i.e., $n = 2$) without boundary and $N = \mathbb{S}^2$, we may define the Gauss map f which to $p \in M$ associates its unit normal at p. We then have that for μ the area measure on $\mathbb{S}^2$, then $f^*\mu(p) = K(p)A$ with $K(p)$ the Gaussian curvature and A the area element in M. Therefore

$$\deg(f, M, \mathbb{S}^2) = \frac{\int_M f^* \mu}{\int_{\mathbb{S}^2} \mu} = \frac{1}{4\pi} \int_M K(p) dA.$$

This is the Gauss-Bonnet formula stating that the integral of curvature on a closed surface is 4π times an integer. We verify that this integer is 1 when M is (homotopic to) a sphere and 0 when it is (homotopic to) a torus. More generally, the integer equals $2 - 2g$ when M is a sphere with g handles (with genus g).

Index of Gauss map. Let f be continuous from the unit ball $\bar{B}$ to $\mathbb{R}^n$ with $f(\partial B) \in \mathbb{R}^n \backslash \{0\}$ so that $\deg(f, B, 0)$ is defined. Consider the Gauss map $\psi : \partial B \to \mathbb{S}^{n-1}$ defined by $\psi(x) = f(x)/|f(x)|$. Then we have the result

$$\deg \psi = \deg(f, B, 0). \tag{12.7}$$

See [34] for a proof.

Index of a vector field. We now want to express the degree of a vector field as an explicit integral. Let $Q \subset U \subset \mathbb{R}^n$ with Q a closed smooth hyper-surface. Let $\xi(x) = (\xi_1(x), \ldots \xi_n(x))$ be a vector field on U. We denote by $Q \ni x \to \psi(x) = \xi(x)/|\xi(x)| \in \mathbb{S}^{n-1}$ the Gauss map associated to the vector field $\xi_{|Q}$ restricted to the hyper-surface Q.

Let now Ω denote the natural normalized volume $(n-1)$-form on $\mathbb{S}^{n-1}$. Then we have $\deg \psi = \int_Q \psi^* \Omega$ by definition. We would like to obtain a more explicit expression. For this, we assume that Q is locally given by the equations $x^i = x^i(u^1, \ldots, u^{n-1})$ for $1 \leq i \leq n$ and define at each point in Q the $n \times n$ matrix L with entries $L_{1j} = \xi^j$ for the first row and $L_{kj} = \partial_{u^{k-1}} \xi^j$ for the rows $2 \leq k \leq n$. With this, we have [20, Section 14]

$$\deg \psi = \deg_Q \xi = \frac{1}{\gamma_n} \int_Q \frac{1}{|\xi|^n} \det L du^1 \wedge \ldots \wedge du^{n-1}. \tag{12.8}$$

Here, γ_n is the volume of $\mathbb{S}^{n-1}$. In dimensions $n = 2$ and $n = 3$, the above formula simplifies to

$$\deg_Q \xi = \frac{1}{2\pi} \oint_Q \frac{dt}{|\xi|^2} \Big(\xi^1 \frac{d\xi^2}{dt} - \xi^2 \frac{d\xi^1}{dt} \Big), \qquad \deg_Q \xi = \frac{1}{4\pi} \int_Q \frac{du \wedge dv}{|\xi|^3} (\xi, [\partial_u \xi, \partial_v \xi]),$$

respectively. These formulas, written in terms of vector fields, find applications for Hamiltonians of the form $h \cdot \Gamma$ with Γ generators of a representation of a Clifford algebra.

Principal bundle notation. The degrees of vector fields find applications in topological insulators modeled by operators of the form $h(x,\xi)\cdot\Gamma$ with Γ a vector of matrices that appear in representations of Clifford algebras. Dirac operators and several operators in super-conductor theory belong to this class.

However, the class of operators that may be classified using the Fedosov- Hörmander formula are more general than those representable in a Clifford algebra. They are more generally maps from a sphere of dimension $2d-1$ in a cotangent bundle of dimension $2d$ to a group of invertible matrices $GL(n,\mathbb{C})$ with a dimension that is typically not $2d-1$ (although it is for operators $h(x,\xi)\cdot\Gamma$ explaining why top degree cohomology is useful). Similarly, the Chern numbers involve a map from a dual space of dimension d to a space of projectors onto vector spaces of dimension $\mathbb{C}^n$.

We now collect useful information mostly from [33]; see also [11, Chapters 1 & 2] for an introduction to principal fiber bundles, connections, and curvature forms.

Manifold. A smooth $n-$manifold is a topological space locally diffeomorphic to a subset of $\mathbb{R}^n$. It is characterized by a finite family of open sets $\{U_i\}$ covering M (i.e. $\cup_i U_i = M$) and homeomorphisms φ_i from U_i to $\varphi_i(U_i)\subset\mathbb{R}^n$. When $U_i\cap U_j\neq\emptyset$, the transition maps $\psi_{ij}=\varphi_i\circ\varphi_j^{-1}$ are smooth from $\varphi_j(U_i\cap U_j)$ to $\varphi_i(U_i\cap U_j)$.

Here, U_i is a patch, (U_i,φ_i) a chart while the family over i is an atlas. Any other atlas such that the union with the first one is still an atlas is called compatible. The equivalence class of such atlases is a differentiable structure, here a smooth one since all transition functions are assumed smooth.

The same notion generalizes to manifolds with a boundary with $\mathbb{R}^n$ above replaced by H^n the half space with last component $x^n\geq 0$.

Fiber bundle. This is a manifold that looks locally like a product of topological spaces. A fiber bundle is an object (E,π,M,F,G) with M a base manifold, $\pi: E\to M$ a surjective projection onto the base manifold, F a manifold called the fiber, and G a Lie group called the structure group and acting on F on the left. Finally, for U_i an open cover of M, we need the existence of local trivializations which are diffeomorphisms $\phi_i: U_i\times F\to\pi^{-1}(U_i)$ with $\pi\circ\phi_i(p,f)=p$. (This shows that locally E looks like the direct product $U_i\times F$.)

Finally, we need the following compatibility conditions. Let $\phi_{i,p}(f):=\phi_i(p,f)$. Then $\phi_{i,p}:F\to F_p$ is a diffeomorphism. When $U_i\cap U_j\neq\emptyset$, we require the transition functions $t_{ij}(p)=\varphi_{i,p}^{-1}\circ\varphi_{j,p}$ from F to F to belong to the structure group G. This means that $t_{ij}: U_i\cap U_j\to G$ implies $\phi_j(p,f)=\phi_i(p,t_{ij}(p)f)$.

There are many examples of fiber bundles and many operations on them that we do not review here. An important class of fiber bundles is that of vector bundles where F is a vector space. The tangent space is a vector bundle with structure group $G=GL(\mathbb{R},n)$ for instance.

Principal bundle. We need another piece of structure. A principal bundle is a fiber bundle where $F\cong G$ and is often denoted by $P\xrightarrow{\pi}M$ or $P(M,G)$. In addition to the left action defined by the above transition functions, we also add a structure of right action $R_g u = ug$ of G on $\pi^{-1}(U_i)\cong U_i\times G$. This action may be defined locally but is globally defined, transitive (i.e. for any u_1 and u_2 in $\pi^{-1}(p)$, there is a such that $u_1=u_2a$), and free (i.e., if $ua=u$ for $u\neq e$ then $a=e$).

Example: The Magnetic monopole. This is a relevant example of a principal bundle for us. Let $M=\mathbb{S}^2$ and $G=U(1)\cong\mathbb{S}^1$. We denote by U_N and U_S an open covering of $\mathbb{S}^2$ and to simplify assume the overlap along the equator (as opposed to an open neighborhood of it to conform to the above hypotheses; things are easily adaptable to that case). Then U_N is parametrized by $0 \leq \theta \leq \frac{\pi}{2}$ and $0 \leq \varphi < 2\pi$ while U_S is parametrized by $\frac{\pi}{2} \leq \theta \leq \pi$ and $0 \leq \varphi < 2\pi$.

The local trivializations are given by $\phi_+^{-1}(u) = (p, e^{i\alpha_+})$ on U_N and $\phi_-^{-1}(u) = (p, e^{i\alpha_-})$ on U_S with $e^{i\alpha_\pm} \in G = U(1)$. The transition function $t_{+-}(\theta = \frac{\pi}{2}, \phi)$ along the equator may be chosen arbitrarily in $U(1)$. Choosing $t_{+-}(\phi) = e^{in\phi}$ for $n \in \mathbb{Z}$ defines a principal bundle.

Note that the right action is multiplication by $g = e^{i\Lambda}$ at each point p so that $\phi_\pm^{-1}(ug) = (p, e^{i(\alpha_\pm + \Lambda)})$. The right action may be interpreted as a $U(1)-$gauge transformation.

Moreover, the choice of n has topological implications. When $n = 0$, we obtain that $P = \mathbb{S}^2 \times \mathbb{S}^1$ a trivial product bundle. As soon as $n \neq 0$, the bundle P is *twisted* and what comes next is to a large extent material necessary so that we can classify such bundles topologically.

Connection. An important piece of structure we need to add on principal bundles is that of a connection. This plays two roles. The first important one is that as the name indicates it helps to connect fiber elements at different points of the base manifold M. In the absence of a Euclidean structure, it is indeed not possible to compare fiber elements at different points on M in a natural way. A second motivation is that the connection is a one-form, i..e, a differential form. This will help us to construct a two-form, the curvature form, as well as higher-order forms until we get a top-order forms. We have seen such forms were useful in the definition of topological degrees. They are also useful in the definition of Chern numbers, which is our main object of interest.

Unfortunately, the notion of connection is rather technical as it involves Lie algebra-valued differential forms. We first recall some aspects of Lie theory. The left and right actions are defined on G by $L_g h = gh$ while $R_g h = hg$. L_g induces a pushforward L_{g*} (the differential of L_g) on tangent spaces $T_h(G) \to T_{gh}(G)$ and the left-invariant vector fields $L_{g*}X_h = X_{gh}$ form a vector space $\mathfrak{g}$ called the Lie algebra of G. Since $X \in \mathfrak{g}$ may be specified by its value at the unit e, we have $\mathfrak{g} \cong T_eG$ the space of tangent vectors to G at e. Associated to a Lie algebra is a Lie bracket $[\cdot, \cdot]$ (the commutator) under which it is closed. There is a set $\{T_\alpha\}$ of generators of the (finite dimensional) Lie algebra and then $[T_\alpha, T_\beta] = f^\gamma_{\alpha\beta} T_\gamma$ for $f^\gamma_{\alpha\beta}$ the structure constants of the Lie algebra. For $G = U(1)$, then $\mathfrak{g} \cong i\mathbb{R}$ for instance and there is a unique generator 1. On G, we have a natural $\mathfrak{g}$-valued one-form, the Maurer-Cartan one form, defined by $\omega = g^{-1}dg$ (when G is a matrix Lie group). This is a map from T_gG to T_eG defined more explicitly by $\omega_g(v) = (L_{g^{-1}})_* v$. We then verify that $L_g^*\omega = \omega$ while $R_g^*\omega = Ad_{g^{-1}}\omega = g^{-1}\omega g$.

Let $P(M, G)$ be the principal bundle and $G_p = \pi^{-1}(p)$ the fiber at $p = \pi(u)$ for some $u \in P$. Then T_uP is the tangent space of P at u. The vertical subspace V_uP of T_uP is the subspace tangent to G_p at u. Heuristically, V_uP is the space of tangent vectors pointing in directions that leave the base point $p \in M$ invariant, whence the name vertical where M is supposed to be horizontal. The union of the T_uP is $TU = \ker(d\pi)$ the kernel of the tangent mapping $d\pi : TP \to TM$.

Now $T_uP = V_uP \oplus H_uP$ for various complements H_uP called horizontal subspaces. The role of a connection is precisely to identify one such H_uP. It takes the form of an element $\omega \in \Omega^1(P) \otimes \mathfrak{g} = C^\infty(P, T^*P \otimes \mathfrak{g})$, i.e., a one-form on P with values in the Lie algebra $\mathfrak{g}$ of the Lie group G. It needs to satisfy the following compatibility conditions. First, $Ad_g(R_g^*\omega) = \omega$ where $Ad_g h = ghg^{-1}$. Second, if $\xi \in \mathfrak{g}$ and X_ξ is the vector field on P differentiating the G action on P by ξ, then $\omega(X_\xi) = \xi$. The latter states that the range of ω is VP the union of the vertical spaces V_uP. Thus, ω determines a bundle map $v : TP \to TV$ with v a projection onto TV. Its kernel is identified as the horizontal complement HV such that $TP = TV \oplus TH$. Note that when X is a point $\{x\}$, then the above Maurer-Cartan form is the unique connection on $\{x\} \times G$.

Local expression. A local section $\sigma : M \supset U \to P$ provides a local trivialization of P. In this trivialization, we may consider the pullback $s^*\omega$ of the connection to obtain a one-form on U with values in $\mathfrak{g}$. If $g : U \to G$ is a smooth map, then $sg(x) = s(x)g(x)$ is a new section. One then verifies that

$$(sg)^*\omega = g^{-1}s^*\omega g + g^{-1}dg \tag{12.9}$$

with d the exterior differentiation, and that ω is uniquely determined by the above family of $\mathfrak{g}-$valued one-forms. Similarly, if $\mathcal{A}_i = \sigma_i^*\omega \in \Omega^1(U_i) \otimes \mathfrak{g}$ for a trivialization on U_i, then on $U_i \cap U_j$ and with $\mathcal{A}_j = \sigma_j^*\omega \in \Omega^1(U_j) \otimes \mathfrak{g}$, then ω is a connection if and only if we have for all such (i,j) that

$$\mathcal{A}_j = t_{ij}^{-1}\mathcal{A}_i t_{ij} + t_{ij}^{-1}dt_{ij}. \tag{12.10}$$

Conversely, if we can define local connections that satisfy the above covariance under change of local section, then a *global connection* ω may be constructed with $s^*\omega$ equal to the given set of local connections. See [33, Chapter 10].

Parallel transport. The connection allows us to transport fiber elements along a curve as follows. Let $\gamma(t)$ for $t \in [0,1]$ be a curve on M with $p = \gamma(0)$. Let $u = (p,f) \in \pi^{-1}(p)$. Then there exists a map $\Gamma : [0,1] \to P$ such that $\Gamma(0) = (p,f)$ and $\Gamma(t) = (\gamma(t), f(t))$ for $f(t)$ in the fiber $\pi^{-1}(\gamma(t))$. This map Γ is a lift of the curve γ from the base manifold M to the principal bundle P. It is given by solving the (uniquely solvable) systems of ordinary differential equations $\omega(\frac{d}{dt}\Gamma) = 0$ with initial condition $\Gamma(0) = (p,f)$. The transport from (p,f) to $(\gamma(t), f(t))$ is called parallel transport.

In a local expression $\mathcal{A} = \sigma^*\omega$, the equation for the lifted curve is $\frac{df}{dt} = -\mathcal{A}(\frac{d}{dt}\gamma, f) = \mathcal{A}(\frac{d}{dt})f$, which provides a differential equation for the fiber component $f(t)$. See [33, Theorem 10.2].

Curvature form. The curvature form of a principal bundle with group structure G is the $\mathfrak{g}$-valued two-form defined by

$$\Omega = d\omega + \frac{1}{2}[\omega \wedge \omega] = d\omega + \omega \wedge \omega \in \Omega^2(P) \otimes \mathfrak{g}.$$

Here, d is exterior derivative on the bundle P (not only on the base M). We note in passing that parallel transporting along a closed curve (loop) generates a subgroup of G called the group of holonomies. This holonomy is captured by the integral of the curvature two-form over the surface generated by the loop. See references for details.

In a local form where $\mathcal{A} = \sigma^*\omega$, we may define $\mathcal{F} = \sigma^*\Omega$ and obtain that

$$\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}, \tag{12.11}$$

where d is now the exterior derivative on M. The above also means that for X and Y vector fields on TM, then

$$\mathcal{F}(X,Y) = d\mathcal{A}(X,Y) + [\mathcal{A}(X), \mathcal{A}(Y)]$$

where $[\cdot,\cdot]$ is the Lie bracket on $\mathfrak{g}$. More precisely, on a chart U with coordinates $x^\mu = \varphi(p)$, we have the relations

$$\mathcal{A} = \mathcal{A}_\mu dx^\mu, \quad \mathcal{F} := \frac{1}{2}\mathcal{F}_{\mu\nu}dx^\mu \wedge dx^\nu, \quad \mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu].$$

For two overlapping charts $U_i \cap U_j$ with $\mathcal{F}_i$ and $\mathcal{F}_j$ the curvatures on these charts, then on $U_i \cap U_j$ they need to verify the compatibility conditoin

$$\mathcal{F}_j = t_{ij}^{-1}\mathcal{F}_i t_{ij} = \mathrm{Ad}_{t_{ij}^{-1}}\mathcal{F}_i. \tag{12.12}$$

A connection ω written locally as $\mathcal{A} = g^{-1}dg$ is called *pure gauge*. In that case, $\mathcal{F} = 0$. Conversely, when $\mathcal{F} = 0$, then $\mathcal{A}$ may be chosen locally as $g^{-1}dg$.

Covariant derivative. Associated to the one-form $\mathcal{A}$, we may also introduce a (local) covariant derivative $\nabla = d + \mathcal{A}$. We then verify that $\mathcal{F} = \nabla^2$. Indeed

$$(d+\mathcal{A})^2 = d^2 + d\mathcal{A} + \mathcal{A}d + \mathcal{A}\wedge\mathcal{A} = (d\mathcal{A}) + \mathcal{A}\wedge\mathcal{A} = \mathcal{F},$$

since $\mathcal{A}$ is a one-form so that as operators $d\mathcal{A} + \mathcal{A}d = (d\mathcal{A})$.

Example: $U(1)$ gauge theory and magnetic monopoles. When $G = U(1)$, which is Abelian with scalar Lie algebra $\mathfrak{u}(1) = i\mathbb{R}$, then $\mathcal{F} = d\mathcal{A}$ since $\mathcal{A}\wedge\mathcal{A} = 0$ then. We deduce the Bianchi identity $d\mathcal{F} = 0$ (more generally, the Bianchi identity is $D\mathcal{F} := d\mathcal{F} + [\mathcal{A},\mathcal{F}] = 0$). Consider again the magnetic monopole with $M = \mathbb{S}^2$ and $P(\mathbb{S}^2, U(1))$ a $U(1)-$bundle. We defined the charts $\mathbb{S}^2_+ = U_N$ and $\mathbb{S}^2_- = U_S$ above. Consider a general transition function $t_{+-}(\phi) =: t(\phi) = e^{i\varphi(\phi)}$ with $\varphi : \mathbb{S}^1 \to \mathbb{R}$. This is the most general transition function on the equator with values in $\mathbb{S}^1$. Such maps are classified in $\pi_1(U(1)) = \mathbb{Z}$ by their degree $\deg t$, which here takes the form of the winding number $\frac{1}{2\pi}\int_0^{2\pi} t^{-1}(\phi)t'(\phi)d\phi$.

Assume $\mathcal{A}_\pm$ defined on U_N and U_S. The above compatibility condition is

$$\mathcal{A}_+ = t^{-1}\mathcal{A}_- t + t^{-1}dt = \mathcal{A}_- + id\varphi, \qquad d\varphi = -i(\mathcal{A}_+ - \mathcal{A}_-).$$

If we *choose* $\mathcal{A}_\pm = \pm ig(1 \mp \cos\theta)d\phi$, then the only compatible choice for $t(\phi)$ is $d\varphi = 2gd\phi$. Since $t(\phi)$ is globally defined on the equator, then $\int_0^{2\pi}\varphi' d\theta = \varphi(2\pi) - \varphi(0) \in 2\pi\mathbb{Z}$. Then so is $\int_0^{2\pi} 2gd\phi = 4\pi g$.

This implies that $2g \in \mathbb{Z}$. Thus the above *choice* of $\mathcal{A}_\pm$ is compatible as a connection with the fiber bundle structure only if $2g \in \mathbb{Z}$. If $\mathcal{A}$ represents a magnetic potential and $\mathcal{F} = d\mathcal{A}$ the corresponding magnetic field, then the total flux is

$$\Phi = \int_{\mathbb{S}^2} \mathcal{F} = \int_{\mathbb{S}^2_+} d\mathcal{A}_+ + \int_{\mathbb{S}^2_-} d\mathcal{A}_- = \int_{\mathbb{S}^1} \mathcal{A}_+ - \mathcal{A}_- = 4\pi g.$$

In this model (this is a model; nobody forces anyone to believe the principal bundle structure), the flux is *quantized* in integer multiples of 2π. If g is interpreted as an electric charge, this shows that if a magnetic monopole exists (with the integer then not vanishing), then electric charge is quantized. Magnetic monopoles do not seem to have been convincingly detected.

Characteristic classes. Let $M(k,\mathbb{C})$ be the $k \times k$ complex matrices and S^r the vector space of symmetric $r-$linear functions on $M(k,\mathbb{C})$. Let S^* be the formal sum over r. The product of two symmetric functions may be defined by symmetrization. For G a Lie group with algebra $\mathfrak{g}$, we define $S^r(\mathfrak{g})$ as the restriction of the above to $\mathfrak{g}$. Then P is said to be invariant if for each $g \in \mathfrak{g}$, P is invariant against replacing $A_i \in \mathfrak{g}$ by $g^{-1}A_i g$.

The symmetrized trace is such an invariant. Let $I^r(G)$ denote such invariants. Then $P(A)$ means $P(A,\dots,A)$ and is an invariant polynomial of degree r in A. For example, $\mathrm{tr}A^r$ is such an invariant polynomial. We may retrieve the multi-linear object by polarization.

On a principal bundle $P(M,G)$. We extend the invariant polynomials from $\mathfrak{g}$ to $\mathfrak{g}$-valued p-forms on M by

$$P(A_1\eta_1,\dots,A_r\eta_r) = \eta_1\wedge\ldots\wedge\eta_r P(A_1,\dots,A_r)$$

for multi-linear objects and $P(A\eta) = \eta \wedge \ldots \wedge \eta P(A)$ for polynomials.

In particular, we may consider $P(\mathcal{F})$ since we have seen that $\mathcal{F}$ was transformed into $g^{-1}\mathcal{F}g$ when changing local trivializations and P is invariant under such changes. Then we have

Theorem 12.4 (Chern-Weil) *For P an invariant polynomial and $\mathcal{F}$ a curvature, then $dP(\mathcal{F}) = 0$ and for $\mathcal{F}$ and $\mathcal{F}'$ corresponding to different connections $\mathcal{A}$ and $\mathcal{A}'$, we have $P(\mathcal{F}) - P(\mathcal{F}')$ is exact. More precisely, if $\tilde{P}$ is the multilinear form associated to P, then*

$$P_r(\mathcal{F}') - P_r(\mathcal{F}) = d\Big[r \int_0^1 dt \tilde{P}_r(\mathcal{A}' - \mathcal{A}, \mathcal{F}_t, \ldots \mathcal{F}_t)\Big]$$

with $\mathcal{F}_t = d\mathcal{A}_t + \mathcal{A}_t \wedge \mathcal{A}_t$ for $\mathcal{A}_t = \mathcal{A} + t(\mathcal{A}' - \mathcal{A})$. The object under brackets is called the transgression $TP_r(\mathcal{A}', \mathcal{A})$.

The invariant polynomial is closed and its integral over M is independent of the connection $\mathcal{A}$ and thus depends only on P. For E a principal bundle or associated vector bundle, we denote by $\chi_E(P)$ the characteristic class defined by the polynomial invariant P.

We have a (Weil) homomorphism $P \to \chi_E(P)$ from $I^*(G)$ to $H^*(M)$. We also have naturality $\chi_{f^*E} = f^*\chi_E$ of bundle pullbacks. This implies that characteristic classes of trivial bundles are trivial. Characteristic classes are therefore subsets of the cohomology classes. Their main interest is that the invariants associated to such classes may be computed explicitly as integrals of differential forms.

We can then define several examples of polynomials such as the Chern classes and Chern characters, whose explicit expressions simplify specific calculations.

Chern classes. For some groups such as $GL(n, \mathbb{C})$, we can show that the above invariant polynomials are the only topological invariants respecting the symmetries used to construct them. A convenient form of polynomials is that of Chern classes, for $E \xrightarrow{\pi} M$ a complex vector bundle or $P(M, G)$ a principal bundle with structure group $G \subset GL(k, \mathbb{C})$, then we define the *total Chern class*

$$c(\mathcal{F}) = \det\Big(I + \frac{i\mathcal{F}}{2\pi}\Big) = 1 + c_1(\mathcal{F}) + c_2(\mathcal{F}) + \ldots \qquad (12.13)$$

where $c_j(\mathcal{F}) \in \Omega^{2j}(M)$ is called the *jth Chern class*. Such classes vanish for $2j > m$ so that the above is a finite sum. From the Chern-Weil theorem, $c_j(\mathcal{F})$ is closed so that $[c_j(\mathcal{F})] \in H^{2j}(M)$ the de Rham cohomology groups. The first two Chern classes are given by

$$c_1(\mathcal{F}) = \frac{i}{2\pi}\mathrm{tr}\mathcal{F}, \qquad c_2(\mathcal{F}) = \frac{-1}{8\pi^2}[\mathrm{tr}\mathcal{F} \wedge \mathrm{tr}\mathcal{F} - \mathrm{tr}(\mathcal{F} \wedge \mathcal{F})].$$

Chern number. For M a closed compact manifold of dimension $2n$ and G a Lie group, the nth Chern class is a $2n$-form and we define the nth *Chern number* as

$$c_n = \int_M c_n(\mathcal{F}) \in \mathbb{R}.$$

This number is a topological invariant of M (for a fixed group G) in the sense that it is independent of the connection $\mathcal{A}$ that generates it. Moreover, the chosen normalization makes that it takes integral values as we now see for c_1 in a specific case.

Example: Chern number for $P(\mathbb{S}^2, U(1))$. We come back to the setting of the magnetic monopole. We find

$$c_1 = \frac{i}{2\pi}\int_{\mathbb{S}^2}\mathcal{F} = \frac{i}{2\pi}\int_{\mathbb{S}^2_+\cup\mathbb{S}^2_-}\mathcal{F} = \frac{i}{2\pi}\int_{\mathbb{S}^1}\mathcal{A}_- - \mathcal{A}_+ = \frac{1}{2\pi i}\int_{\mathbb{S}^1} t^{-1}dt = \deg\varphi = \sum_{\phi\in\varphi^{-1}(0)} J_\varphi(\phi) \tag{12.14}$$

with $t = t_{+-} = e^{i\varphi(\phi)} \in U(1)$ so that $\mathcal{A}_+ - \mathcal{A}_- = t^{-1}dt$ and assuming that 0 is a regular value of $\varphi(\phi)$. We thus obtain that c_1 is indeed an integer.

The data required to compute the first Chern number is therefore the topological class of the transition function t. We recall that the Lie algebra of $U(1)$ is represented by $i\mathbb{R}$ so that $\mathcal{A}$ is a standard one-form on $\mathbb{S}^2$ (with complex-valued coefficients). We saw that for $\mathcal{A}_\pm = \pm ig(1\mp\cos\theta)d\phi$ on $\mathbb{S}_\pm$, then we needed $2g \in \mathbb{Z}$ for the above to define a connection and then the degree of t is $2g$ as well.

Hamiltonians with rank-one projectors. This result generalizes to several settings of interest in topological insulators. A Chern number appeared in the computation of the *bulk-difference invariant* for the Dirac operator represented by the family of Hamiltonians $\hat{H}_\pm = \xi\sigma_1 + \zeta\sigma_2 + m_\pm\sigma_3$. We found a Chern number equal to $\frac{1}{2}(\,\mathrm{sign}(m_-) - \,\mathrm{sign}(m_+))$.

More generally, consider a family of Hamiltonians $\hat{H}(\xi) = \sum_j E_j(\xi)\Pi_j(\xi)$, where $\xi \in \mathbb{S}^2$ after circle compactification. Assume that $\Pi(\xi) := \Pi_j(\xi)$ is rank-one, i.e., $E_j(\xi)$ is a simple eigenvalue for all $\xi \in \mathbb{S}^2$. We defined the Chern number associated to Π as

$$c[\Pi] = \frac{i}{2\pi}\int_{\mathbb{S}^2} \mathrm{tr}\Pi d\Pi \wedge d\Pi.$$

We now show that $c[\Pi]$ is in fact c_1 above for an appropriate connection.

Adiabatic connection. Since $\Pi(\xi)$ is rank-one, it may be expressed as $\Pi(\xi) = \psi(\xi)\otimes\psi(\xi)$ for $\psi(\xi)$ a normalized vector in some space $\mathbb{C}^n$. The normalized eigenvector $\psi(\xi)$ is defined up to a phase $g(\xi) \in U(1)$. However, we may not be able to construct a continuous phase globally on $\mathbb{S}^2$. This is why the notion of a principal bundle $P(\mathbb{S}^2, U(1))$ is the right one for us here. Note that $\Pi(\xi)$ is defined uniquely as the one-dimensional eigenspace of a smoothly varying energy level associated to a smooth Hamiltonian $\hat{H}(\xi)$. While $\psi(\xi)$ may be defined continuously locally, there may be topological obstructions to a global construction. The Chern class and the Chern number precisely characterize this obstruction.

Assume $\psi(\xi)$ given smoothly on $\mathbb{S}_+$ for instance. This is always possible as $\mathbb{S}_+$ is contractible. For the constant $\sigma(\xi) = e \equiv 1$ and all other sections $\sigma_g(\xi) = g(\xi)$, *define*

$$\mathcal{A}_g = (\psi g, d(\psi g)) = (\psi g, d\psi g + \psi dg) = (\psi, d\psi) + g^{-1}dg = \mathcal{A}_e + g^{-1}dg$$

where $(\cdot,\cdot)$ is the standard inner product on $\mathbb{C}^n$ and where we used that $g(\xi) \in \mathbb{C}$ (in fact $ig \in \mathbb{R}$) and that $\|\psi(\xi)\| = 1$ for all $\xi \in \mathbb{S}_+$. We thus observe that (12.9) holds for any section on $\mathbb{S}_+$, as well as on $\mathbb{S}_-$ similarly. As a result, a global connection exists in $\Omega^1(\mathbb{S}^2)\otimes(i\mathbb{R}) \cong \Omega^1(\mathbb{S}^2)$ if we can glue these objects on $\mathbb{S}_1 = \mathbb{S}^2_+ \cap \mathbb{S}^1_-$.

Assume $\psi_\pm(\xi)$ to be continuous eigenvectors on $\mathbb{S}_\pm$ and define $t : \mathbb{S}^1 \to U(1)$ by $\psi_+ = t\psi_-$. The existence of t is guaranteed by the fact that $\psi_\pm(\xi)$ solve the same (simple) eigenvalue problem and are therefore defined up to a multiplicative phase in $U(1)$. As a consequence, our definition of $\mathcal{A}_g$ *defines a connection* on $P(\mathbb{S}^2, U(1))$ and (12.14) applies with $\mathcal{A} = \mathcal{A}_e$. This is the *adiabatic* connection, although we will not comment on the terminology here.

Computation of the Chern number. It remains to relate this number to $c[\Pi]$. Using that $\mathcal{A}_e = \psi^* d\psi = -d\psi^*\psi$, we verify that

$$\begin{aligned}\mathcal{F}_e = \mathrm{tr}\Pi d\Pi d\Pi &= \mathrm{tr}\psi\psi^* d(\psi\psi^*) d(\psi\psi^*) = (\psi, [\psi\psi^* d(\psi\psi^*) d(\psi\psi^*)]\psi) \\ &= d\mathcal{A}_e + 2\mathcal{A}_e \wedge \mathcal{A}_e - \mathcal{A}_e \wedge \mathcal{A}_e = d\mathcal{A}_e + \mathcal{A}_e \wedge \mathcal{A}_e = d\mathcal{A}_e = (d\psi, d\psi),\end{aligned}$$

since $\mathcal{A}_e \wedge \mathcal{A}_e = 0$ for scalar 1-forms, so that indeed $c_1 = c[\Pi]$.

So the recipe to compute a two-dimensional bulk-difference Chern number is: (i) construct $\psi_\pm(\xi)$ a continuous family of normalized eigenvectors of $\Pi(\xi) = \Pi_m(\xi)$; (ii) construct the transition function $t \in U(1)$ on $\mathbb{S}^1$; and (iii) compute the winding number of t as in (12.14).

Remark 12.5 *We see sometimes the definition of a 'connection' $A = \Pi d\Pi$ that may provide a curvature on a different fiber bundle than $P(\mathbb{S}^2, U(1))$. Note that we do not take traces here. Indeed,* $\mathrm{tr}\Pi d\Pi = \mathrm{tr}\Pi^2 d\Pi = \mathrm{tr}\Pi d\Pi\Pi = 0$ *since $\Pi d\Pi\Pi =$ for Π a projector. This comes from $\Pi^2 = \Pi$ so that $\Pi d\Pi + d\Pi\Pi = d\Pi$ or $d\Pi\Pi = (I - \Pi)d\Pi$ so the result follows. Now*

$$F = dA + A \wedge A = d(\Pi d\Pi) + \Pi d\Pi\Pi d\Pi = d\Pi \wedge d\Pi + 0.$$

This is not quite $\Pi d\Pi d\Pi$ and so $\mathrm{tr}F = 0$ in many problems of interest such as the Dirac operator. We can also define $\tilde{A} = \Pi d\Pi - d\Pi\Pi = [\Pi, d\Pi]$ and obtain that $\tilde{F} = d\tilde{A} + \tilde{A}\wedge\tilde{A} = 2d\Pi\wedge d\Pi - d\Pi \wedge d\Pi = d\Pi \wedge d\Pi$ as well. Note that the above A is defined globally since Π is defined globally. When such a connection can be defined globally on $\mathbb{S}^2$ rather than the bundle P, we deduce that P is trivial. See [35] for constructions of adiabatic connections and curvatures for larger groups than $U(1)$.

Bulk-difference Chern number for the Dirac operator. Consider the Dirac operator $\hat{H} = \xi\sigma_1 + \zeta\sigma_2 + m\sigma_3$. Define $z = \xi + i\zeta$ so that

$$\hat{H}(z) = \begin{pmatrix} m & \bar{z} \\ z & -m \end{pmatrix}$$

admits two eigenvalues $E_{1,2} = \pm\sqrt{|z|^2 + m^2}$. Eigenvectors of $\hat{H}(z)$ solve

$$(m - E)\psi_1 + \bar{z}\psi_2 = 0, \quad z\psi_1 - (m + E)\psi_2 = 0.$$

We thus have two different choice of phases

$$\psi_+ = \frac{1}{\sqrt{|z|^2 + (E - m)^2}} \begin{pmatrix} \bar{z} \\ E - m \end{pmatrix}, \qquad \psi_- = \frac{1}{\sqrt{|z|^2 + (E + m)^2}} \begin{pmatrix} m + E \\ z \end{pmatrix}.$$

In order for ψ associated to $E = E_1(z) = -\sqrt{|z|^2 + m^2}$ to be continuously defined for $z \in \mathbb{C}$, we choose the phase ψ_+ when $m > 0$ (since then $|E - m| \geq 2|m|$) and the phase ψ_- when $m < 0$ (since then $|E + m| \geq 2|m|$) . This ensures that the normalization constant never vanishes and is continuously defined on $\mathbb{C}$.

When $m > 0$ in the northern hemisphere and $m < 0$ in the southern hemisphere, we thus have $\psi = \psi_-$ in the southern hemisphere and $\psi = \psi_+$ in the northern hemisphere. In the limit $|z| \to \infty$, using $z = |z|e^{i\phi}$, we deduce that

$$\psi_-(\phi) = \frac{1}{(2|z|^2)^{\frac{1}{2}}} \begin{pmatrix} |z| \\ z \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{i\phi} \end{pmatrix}, \qquad \psi_+(\phi) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\phi} \\ 1 \end{pmatrix}.$$

Thus $t(\phi)=e^{-i\phi}$ so that the bulk-difference Chern number is found to be $c_1=c[\Pi_1]=-1$. We would similarly find that $c_1=c[\Pi_1]=+1$ if m was negative in the northen hemishere and positive in the southern hemisphere. In both cases, we have $c_2=c[\Pi_2]=-c_1$.

Note that when m is the same mass term in both hemispheres, we choose $\psi_-=\psi_+$ and the bundle is trivial, with then $c_1=0$ as expected.

This provides a simpler computation of the invariant than an estimate of the integral of the curvature. It also explains why said integral can always be written in divergence form since the curvature form is exact.

Computations for Hamiltonians with Clifford algebra structure. We generalize the above computation to

$$\hat{H}(\xi)=h(\xi)\cdot\sigma$$

with eigenvalues $E(\xi)=\pm|h(\xi)|$. We will see another general method to compute invariants for such Hamiltonians. The spectral gap assumption means that $|h|\geq h_0>0$ on $\mathbb{S}^2$. Above, h is obtained either from a single Hamiltonian with point compactification or from a pair of Hamiltonians with circle compactification. Using the notation $z=h_1+ih_2$, the eigenvectors are solution of

$$(h_3-E)\psi_1+\bar{z}\psi_2=0,\qquad z\psi_1-(h_3+E)\psi_2=0$$

and hence given by

$$\psi_+(\xi)=\frac{1}{\sqrt{(h_3+E)^2+|z|^2}}\begin{pmatrix}E+h_3\\ z\end{pmatrix},\qquad \psi_-(\xi)=\frac{1}{\sqrt{(h_3-E)^2+|z|^2}}\begin{pmatrix}\bar{z}\\ E-h_3\end{pmatrix}$$

Note that we could exchange the roles of the different components h_j.

If z never vanishes, then ψ_+ is defined globally on $\mathbb{S}^2$ and the resulting Chern class $[c_1(\mathcal{F})]=0$. In the vicinity of those points where $z=0$, then $|h_3|\geq h_0>0$ is bounded below so that either $E+h_3$ or $E-h_3$ is bounded away from 0. Assume to simplify that $|E+h_3|\geq h>0$ on $\mathbb{S}^2_+$ and $|E-h_3|\geq h>0$ on $\mathbb{S}^2_-$. We may then define the connection $\mathcal{A}$ with $\mathcal{A}_\pm=(\psi_\pm,d\psi_\pm)$ and a transition function $t=e^{i\varphi}$ along the equator as in preceding problems. Then $c_1=\deg\varphi$. In practical problems, we need to identify the transition function t in order to compute the invariant. For many problems, this is relatively straightforward.

Let us implement this for the single Hamiltonian $\hat{H}=\xi\sigma_1+\zeta\sigma_2+(m-\eta|(\xi,\zeta)|^\alpha)$ for $\alpha>1$ (for instance $\alpha=2$ as in an earlier lecture) and $\eta\neq 0$. Denote $z=\xi+i\zeta$. Thus, $h_1=\xi$, $h_2=\zeta$, and $h_3=m-\eta|z|^\alpha$. The parameter belongs to the base manifold $\mathbb{S}^2$ with point compactification of $\mathbb{R}^2$ so that $z=0$ corresponds to the north pole and $|z|\to\infty$ to the south pole. In this setting with $\alpha>1$, we observe that ψ_+ converges to $(1,0)^t$ as $|z|\to\infty$ while ψ_- converges to $(0,1)^t$. We thus have continuous functions near the north and south poles on the sphere.

Assume $m\eta<0$. Then we verify that one of the vectors ψ_+ or ψ_- is globally defined on $\mathbb{S}^2$ so that $[c_1(\mathcal{F})]=0$ in that case.

Assume E, m, and η all positive. We then choose $\psi_\pm$ on $\mathbb{S}^2_\pm$. The equator is reached for $|z|=z_0>0$ where h_3 is constant. There, we observe that $t=e^{i\theta}$ with a winding number equal to $c_1=1$. More generally, we find $c_1=\frac{1}{2}(\operatorname{sign}(m)+\operatorname{sign}(\eta))$. For $E<0$, we would obtain $c_1=-\frac{1}{2}(\operatorname{sign}(m)+\operatorname{sign}(\eta))$ this is the opposite sign of the Chern number we obtain in Lemma 5.6. The reason is that the orientation on $\mathbb{R}^2$ and on $\mathbb{S}^2$ are opposite with the above convention that 0 in the plane is mapped to the north pole. Accounting for this orientation mismatch, we retrieve the result in Lemma 5.6. In order for the orientations to match, we would need to equate the point 0 in the plane to the south pole.

Computations for higher-order differential operators. Let $z=\xi+i\zeta$ again and consider the two Hamiltonians $\hat H_\pm=\Re z^n\sigma_1+\Im z^n\sigma_2\pm m\sigma_3$. We could similarly consider the single regularized Hamiltonian replacing m by $m-\eta|z|^{n+1}$. The calculations obtained above when $n=1$ apply verbatim and we get that the Chern number $c_1[\Pi_+]=n$ while $c_1[\Pi_-]=-n$ when $m>0$.

We may introduce the corresponding Hamiltonian

$$H=\mathcal{F}^{-1}\Re z^n\mathcal{F}\sigma_1+\mathcal{F}^{-1}\Im z^n\mathcal{F}\sigma_1+m(y)\sigma_3+V$$

with $m(y)$ a domain wall and $V(x,y)$ compactly supported. We then verify that hypothesis [H1] is satisfied for such an elliptic problem and that by the bulk-edge correspondence, the edge invariant $2\pi\sigma_I[H]=-n$ when $m_\pm=\pm|m|$. When $n=2$, the leading term in the operator is $(D_x^2-D_y^2)\sigma_1+2D_xD_y\sigma_2$.

Topological fluid wave model Consider now the Hamiltonian

$$H=\begin{pmatrix}0 & D_x & D_y\\ D_x & 0 & if(y)\\ D_y & -if(y) & 0\end{pmatrix},\qquad \hat H_\pm(\xi)=\begin{pmatrix}0 & \xi & \zeta\\ \xi & 0 & if_\pm\\ \zeta & -if_\pm & 0\end{pmatrix} \tag{12.15}$$

The left Hamiltonian is written with a domain wall $f(y)$ while the right Hamiltonians correspond to two bulk Hamiltonians with term $f_\pm$. Physically, $f(y)$ is a (real-valued) Coriolis force that takes a different sign in the northern and southern hemispheres. A reasonable assumption for the above flat-earth with spatial coordinates (x,y) model is $f(y)=y$. The operator applies to three-vectors (h,u,v) with h atmosphere thickness (in a linearized setting) and (u,v) velocity field.

We are interested in computing the bulk-difference invariant associated to the above families $\hat H_\pm(z)$, where again $z=\xi+i\zeta$. Let us assume $f_\pm=\pm f$. Then, diagonalizing $\hat H_+$ gives three eigenvalues $E_0=0$, $E_\pm=\pm\sqrt{|z|^2+f^2}$. Associated to the non-vanishing and vanishing eigenvalues are respective eigenvectors

$$\psi^+(z)=c(z)\begin{pmatrix}|z|\\ |z|^{-1}E\xi+if|z|^{-1}\zeta\\ |z|^{-1}E\zeta-if|z|^{-1}\xi\end{pmatrix},\qquad \psi_0^+=\frac{1}{\sqrt{|z|^2+f^2}}\begin{pmatrix}if\\ \zeta\\ -\xi\end{pmatrix}$$

with $c(z)$ normalizing constants. We have $\psi^-(z)$ defined similarly with f replaced by $-f$. As it stands, the above eigenvector is continuous except at $z=0$. Let us write $z=|z|e^{i\phi}$ as usual. We verify the following: when $f>0$, then $e^{-i\phi}\psi^+$ is continuous at $z=0$, where it is equal to $2^{-\frac12}(0,1,-i)^t$. When $f<0$, however, we find that $e^{-i\phi}\psi^+$ is continuous at $z=0$, where it equals $2^{-\frac12}(0,1,i)^t$.

Consider the branch $E>0$ first. Assume $f_+=f>0$ so that $f_-=-f<0$. We thus choose $\psi_+=e^{-i\phi}\psi^+$ on $\mathbb{S}^2_+$ and $\psi_-=e^{i\phi}\psi^-$ on $\mathbb{S}^2_-$. Since both ψ^+ and ψ^- converge to $3^{-\frac12}(1,\cos\phi,\sin\phi)^t$ as $|z|\to\infty$ along the equator, we observe that $t=e^{-2i\phi}$ and in fact $t=e^{-2i\phi}\,\mathrm{sign}(f)$ if $f<0$ is also taken into account. This shows that $c_1[\Pi_+]=-2\,\mathrm{sign}(f)$ for the branch with positive energy.

When $E<0$, we find similarly that $c_1[\Pi_-]=2\,\mathrm{sign}(f)$ since in the vicinity of $z=0$, E now converges to $-|f|$ rather than $+|f|$.

It is a general property that the sum of Chern numbers over all separated energy bands should vanish. This implies that $c_1[\Pi_0]=0$. This may be verified directly from the expression of $\psi_0^\pm$, which both converge to $(0,|z|^{-1}\zeta,-|z|^{-1}\xi)$ as $|z|\to\infty$ so that $\psi^\pm$ glued together on $\mathbb{S}^2$ along the equator for a continuous globally defined eigenvector on $\mathbb{S}^2$. This implies that the corresponding Chern number $c_1[\Pi_0]=0$.

Let us finish this section by a remark: The above operator H is not elliptic in $H^1(\mathbb{R}^2;\mathbb{C}^3)$ since $\hat{H}$ admits a full branch of flat spectrum at 0. As a consequence, we are not allowed to apply the bulk-edge correspondence equating $2\pi\sigma_I$ with the above bulk-difference invariant equal to 2 when $\mathrm{sign}(f)>0$, say. It turns out that $2\pi\sigma_I(H)$ can indeed be shown to equal 2 when $f(y)=y$, at least in the absence of any perturbation. This is based on a diagonalization of the above Hamiltonian and a computation of the spectral flow of each branch of absolutely continuous spectrum. The derivation mimics the one we developed for the Dirac operator. The same derivation based on spectral flows shows that $2\pi\sigma_I=1$, and not 2, when $f(y)=\mathrm{sign}(y)$. The presence of essential spectrum at 0 is a sufficiently strong deviation from our ellipticity assumption that the bulk-edge correspondence fails in some cases. It is not clear for which profiles of the Coriolis force $f(y)$ the bulk-edge correspondence $2\pi\sigma_I=2$ applies or not.

Chern numbers and degree theory for Clifford algebra Hamiltonians. We revisit the computation of the Chern numbers for Hamiltonians of the form $\hat{H}(\xi)=h(\xi)\cdot\sigma$ with $\sigma_{1,2,3}$ the standard Pauli matrices. We presented above a computation for such a number with a choice of phase based on the sign of $E+h_3$. We now look at a more explicit, different, form of the computation. It may be useful to recall the Fedosov-Hörmander formula. There, the symbol a maps $\mathbb{S}^3$ to $GL(n,\mathbb{C})$. For the above hamiltonians, $n=2$ and we have seen that $a\in GL(n,\mathbb{C})$ may be homotopy-transformed to a matrix in $U(2)$. An additional homotopy transformation would bring the determinant of a to 1 and hence to an element in $SU(2)$, which has (real-)dimension 3, as does $\mathbb{S}^3$. In such a setting, a maps a sphere to a manifold of the same dimension. Maps from manifolds of the same degree have an invariant defined by their degree. Since a is naturally related to h, it is reasonable to expect that the degree of h will play a direct role in the computation of the Chern number.

To carry out this derivation we recall that for the above Hamiltonian and $\hat{P}=\frac{1}{2}(I-\hat{H}/|h|)$,

$$\frac{i}{2\pi}\mathrm{tr}\hat{P}d\hat{P}\wedge d\hat{P}=\frac{i}{2\pi}\frac{-1}{8|h|^3}\mathrm{tr}\hat{H}[\partial_1\hat{H},\partial_2\hat{H}]d\xi d\zeta$$

with $|\hat{H}(\xi)|=|h(\xi)|$. The above derivation only uses the fact that $\hat{P}=\frac{1}{2}(I-\hat{H}/|h|)$ and the cyclicity of the trace, as well as the fact that $(\hat{H})^2=|h|^2I_2$. Now

$$[\partial_1\hat{H},\partial_2\hat{H}]=\sum_{i,j}\partial_1h_i\partial_2h_j[\sigma_i,\sigma_j]=\sum_{i,j,k}\partial_1h_i\partial_2h_j2i\varepsilon^{ijk}\sigma_k$$

so that, using that Pauli matrices have vanishing traces,

$$\mathrm{tr}\hat{H}[\partial_1\hat{H},\partial_2\hat{H}]=\sum_{i,j,k}4i\varepsilon^{ijk}\partial_1h_i\partial_2h_jh_k.$$

This is

$$\frac{i}{2\pi}\mathrm{tr}\hat{P}d\hat{P}\wedge d\hat{P}=\frac{1}{4\pi}\frac{d\xi d\zeta}{|h|^3}h\cdot(\partial_1h\wedge\partial_2h).$$

In this we recognize the integrand in (12.8) that appears in the computation of the degree of the vector field $h(\xi)$ or the degree of the Gauss map $h(\xi)/|h(\xi)|$. This shows that for the projection onto negative energies,

$$c_1[\Pi_-]=\deg_{\mathbb{S}^3}h=\deg h/|h|$$

For the Dirac operator with $h=(\xi,\zeta,m_\pm)$ and $h/|h|=(\xi,\zeta,m_\pm)/\sqrt{|z|^2+m_\pm^2}$ in $\mathbb{S}^2_\pm$ glued along $\mathbb{S}^1$ by continuity since $m_\pm$ is irrelevant there.

It remains to compute the index. When $m_\pm$ have the same sign, then one of the points $(0,0,\pm 1)$ is never attained while the other one is attained twice with a Jacobian having a different sign. The index therefore vanishes. When $m_\pm$ have different signs, then $(0,0,1)$, say, is attained only once. Estimating the Jacobian there, which is proportional to identity, then provides (with our orientation conventions) $c_1[\Pi_-] = \mathrm{sign}(m_-) = -\,\mathrm{sign}(m_+)$. This is consistent with earlier calculations.

From $(\xi,\zeta,m)\cdot\Gamma$ **to** $h\cdot\Gamma$**.** We saw in earlier paragraphs two methods to compute the Chern numbers of operators of the form $h\cdot\sigma$ for σ the Pauli matrices. We also saw how to compute the Chern numbers (there is one for each energy band) when Γ are the Gellmann matrices in (12.15). Let $\mathcal{F} = \mathrm{tr}\Pi d\Pi d\Pi$ the curvature obtained for $h_0 = (\xi,\zeta,f)$. Consider now a more general vector $h = h(\xi,\zeta,f)$. Then $\Pi_h = \Pi\circ h$ with $\Pi = \Pi_{h_0}$. As a consequence, $\mathcal{F}_h = h^*\mathcal{F}$ with $\mathcal{F} = \mathcal{F}_{h_0}$. Since $\mathcal{F}$ is a two-form on the sphere after compactification, we find that

$$c_1[h] = \int_{\mathbb{S}^2} \mathcal{F}_h = \int_{\mathbb{S}^2} h^*\mathcal{F} = \deg h \int_{\mathbb{S}^2} \mathcal{F} = \deg h\ c_1[h_0].$$

Thus computing the Chern numbers for h_0 is sufficient to obtain them for all vector fields h by applying degree theory. In other words, the generalization from h_0 to h applies not only for Clifford algebra structures but rather for all Hamiltonians based on a matrix structure for which $c_1[h_0]$ may be computed.

Bulk-difference invariant and isotropic Hamiltonians. Consider the Hamiltonian $h_0\cdot\Gamma$ given above with $h_0 = (\xi,\zeta,m)$. We know that $\mathcal{F} = d(\psi, d\psi)$ so that $d\mathcal{F} = 0$. As a consequence, rather than integrating it along the two planes generating the bulk-difference invariant, we can use Stokes' theorem and obtain that

$$\int_{P_+\cup P_-} \mathcal{F} = \int_{\mathbb{S}^2} \mathcal{F}$$

where $\mathcal{F}$ is not $\pi^*\mathcal{F}$ (with π stereographic projections) anymore but rather the curvature obtained directly from the operator $h_0\cdot\Gamma$.

Combining the two preceding remarks, we observe that the Chern numbers of $h\cdot\Gamma$ are given by the degree of the vector field h times the Chern numbers of the Hamiltonian $h_0\cdot\Gamma$, and that the latter can be obtained by integration along the two-sphere $\xi^2+\zeta^2+m^2=1$. The operator $h_0\cdot\Gamma$ is therefore more symmetrical in the three variables (ξ,ζ,m). The case $\Gamma = \sigma_{1,2,3}$ are the spin 1/2 matrices on $\mathbb{C}^2$. The case Γ in (12.15) corresponds to the spin 1 matrices on $\mathbb{C}^3$.

Spin s **Hamiltonians.** Following [39], we may now compute the Chern number when $H = h_0\cdot\Gamma$ with Γ the spin-s spin matrices on $\mathbb{C}^{2s+1}$. When $s=\frac12$, then $\Gamma = \sigma_{1,2,3}$ the Pauli matrices. When $s=1$, we may choose the matrices in (12.15). More generally, $\Gamma_{1,2,3}$ may be chosen with Γ_3 a diagonal matrix $\mathrm{Diag}(s, s-1, \ldots, -s)$. We then observe that $(d\psi, d\psi)$ is invariant by rotation so that it is sufficient to compute it for ψ an eigenvector of Γ_3. In the vicinity of such a point, we have $H = z\Gamma_- + \bar z\Gamma_+ + \Gamma_3$ up to $O(|z|^2)$ terms. This then involves the commutator $[\Gamma_1,\Gamma_2]|m\rangle$ where $|m\rangle$ is the eigenvector of Γ_3 corresponding to eigenvalue m. Since $[\Gamma_1,\Gamma_2] = 2i\Gamma_3$, we deduce as in [39] that the curvature is $-im$. Integrating over the sphere and multiplying by $i/2\pi$ gives $c_1[\Pi_m] = 2m$.

We retrieve the invariants $(-1,1)$ for the Dirac problem with $s=\frac12$ and the invariants $(-2,0,2)$ for the geophysical problem with $s=1$.

13 Lecture 13.

Asymmetric transport, scattering theory, and integral formulations. This lecture looks at the relation between asymmetric transport and a general scattering theory that more quantitatively models transport along the interface. The material follows closely [7].

Dirac operator with linear domain wall. We consider exclusively the Dirac operator

$$H = D_x\sigma_3 - D_y\sigma_2 + m(y)\sigma_1$$

with a domain wall $m(y) = y$ and perturbations $H_V = H + V$ for V a Hermitian-valued multiplication operator by $V(x, y)$ which we assume smooth, bounded, and with compact support in the x-variable.

Both the perturbed and unperturbed operators may be shown to be unbounded operators from $\mathfrak{D}(H)$ to $L^2(\mathbb{R}^2;\mathbb{C}^2)$ where $\mathfrak{D}(H) = \{\psi \in L^2 \text{ such that } y\psi \in L^2 \text{ and } \nabla\psi \in L^2\}$. In this setting H_0 is *self-adjoint* and *elliptic*. The pseudo-differential calculus we presented for $m(y)$ a bounded domain wall extends to this setting. This is done in detail in [5], where generalizations to higher-dimensional problems are also presented. We thus obtain the same classification as in the setting of bounded domain walls and find that

$$2\pi\sigma_I[H] = 2\pi\sigma_I[H_V] = -1.$$

This may be obtained by applying a bulk-edge correspondence relating the edge invariant to the topological charge $\mathrm{Index}\, F$ where $F = H_V - ix$. The latter is then a Fredholm operator in an appropriate topology and its index is easily found to equal -1 by an application of the Fedosov-Hörmander formula. Alternatively, one may explicitly diagonalize H_0 and compute the invariant $\mathrm{Index}\, P(x)U(H_0)P(x)$ by spectral flow as we did in an earlier lecture. Note that we did not say anything about the support of φ' defining σ_I. In fact, since the domain wall $m(y) = y$ has infinite range, it turns out that *all* energies are not allowed to propagate into the bulk and are confined in the vicinity of $y = 0$. Therefore, φ' may be chosen arbitrarily (still integrating to 1 and for concreteness with compact support).

Integral formulation. While the topological classification of operators is useful in practice, then so is a more quantitative description of transport and how the asymmetry materializes in concrete examples. In particular, we wish to understand the quantitative effect of the perturbation V on transport properties. This explains the choice of H above: it is topologically nontrivial and yet admits a relatively explicit inversion.

Let $E \in \mathbb{R}$. What we mean by inversion is that

$$(H - E)^{-1} := \lim_{\varepsilon\downarrow 0}(H - (E + i\varepsilon))^{-1} \tag{13.1}$$

exists and admits an explicit kernel (Green's function), at least for all energies E that avoid a bad set of measure 0.

Note that the self-adjoint operator H admits (purely) absolutely continuous spectrum in the whole of $\mathbb{R}$. Invertibility of $(H - E)$ thus requires some care, which we address below in detail. Let us assume $(H - E)^{-1}$ available. Let $\psi_{\rm in}$ be a (plane wave in x) solution of $(H - E)\psi_{\rm in} = 0$. We wish to compute a solution ψ of $(H + V - E)\psi = 0$. Decomposing $\psi = \psi_{\rm in} + \psi_{\rm out}$, we obtain

$$(H + V - E)\psi_{\rm out} = -V\psi_{\rm in}.$$

This is solved as follows. Define $\rho=(H-E)\psi_{\rm out}$ so that $\psi_{\rm out}=(H-E)^{-1}\rho$ according to (13.1). Then

$$\rho+V(H-E)^{-1}\rho=-V\psi_{\rm in}.$$

We have replaced solving for ψ by an integral equation for ρ. From the above equation, it is obvious that ρ, if defined, has support on the support of V. If the latter has compact support for instance, then so does ρ. This is an important simplification with practical signification. Instead of inverting $H+V-E$ on an infinite domain, we can now discretize the above equation on a finite domain to obtain numerical approximations of $\psi_{\rm out}$. We will see that this will eventually provide approximations for $\sigma_I[H+V]=-1$ as well.

Spectral decomposition. We write in the partial Fourier domain

$$H=\mathcal{F}^{-1}\int_{\mathbb{R}}^{\oplus}\hat H(\xi)d\xi\mathcal{F},\qquad \hat H(\xi)=\xi\sigma_3-D_y\sigma_2+y\sigma_1=\begin{pmatrix}\xi & \mathfrak{a}\\ \mathfrak{a}^* & -\xi\end{pmatrix},\qquad \mathfrak{a}=\partial_y+y.$$

We recognize in $\mathfrak{a}$ a standard representation of the annihilation operator while $\mathfrak{a}^*:=-\partial_y+y$ is the creation operator. It is useful in this context to realize that

$$\hat H(\xi)^2=\begin{pmatrix}\xi^2+\mathfrak{a}\mathfrak{a}^* & 0\\ 0 & \xi^2+\mathfrak{a}^*\mathfrak{a}\end{pmatrix}$$

is block-diagonal. Since $(\hat H-z)(\hat H+z)=\hat H^2-z^2$, we observe that

$$(\hat H-z)^{-1}=(\hat H-z)(\hat H^2-z^2)^{-1}.$$

In other words, if we can compute the Green's function of the scalar operators in $(\hat H^2-E^2)$, then applying $\hat H-E$ gives the Green's function of $\hat H-E$. More generally, we observe that $\hat H^2$ has a compact resolvent and hence discrete spectrum for each $\xi\in\mathbb{R}$. This property thus also holds for $\hat H$.

The quantum harmonic oscillator. This is the operator $\mathfrak{a}^*\mathfrak{a}=-\partial^2+y^2-1$. Define the family of Hermite functions

$$\varphi_0(y)=\pi^{-\frac14}e^{-\frac12 y^2},\quad \varphi_n(y)=a_n(\mathfrak{a}^*)^n\varphi_0(y)\tag{13.2}$$

with a_n chosen so that $\|\varphi_n\|_{L^2(\mathbb{R})}=1$. Then we have

$$\mathfrak{a}^*\mathfrak{a}\varphi_n=2n\varphi_n,\qquad \mathfrak{a}\varphi_n=\sqrt{2n}\varphi_{n-1},\quad \mathfrak{a}^*\varphi_n=\sqrt{2n+2}\varphi_{n+1}.$$

Absolutely continuous spectrum of H. Let M be the union of the following indices m. For $n=0$, we define $m=(0,-1)$ while for each $n\ge1$ and $\varepsilon=\pm1$, we define $m=(n,\varepsilon)$. With this convention, $(0,1)\notin M$ while any other pair $(n,\pm1)\in M$. Define next for each $\xi\in\mathbb{R}$ and $m=(n,\varepsilon)\in M$,

$$E_m(\xi)=\varepsilon\sqrt{\xi^2+n^2},\qquad \phi_m=c_m\begin{pmatrix}\mathfrak{a}\varphi_n\\ (E_m-\xi)\varphi_n\end{pmatrix},\quad c_m^{-2}=2n+|E_m-\xi|^2,\qquad n\ge1\tag{13.3}$$

and

$$E_0(\xi)=-\xi,\qquad \phi_0=\begin{pmatrix}o\\ \varphi_0\end{pmatrix},\qquad m=(0,-1).\tag{13.4}$$

We verify that the above family of eigenvectors $\phi_m(\xi)$ forms a basis of $L^2(\mathbb{R};\mathbb{C}^2)$. More precisely, for $M \ni m = (\varepsilon, n)$ and $M \ni q = (\eta, p)$, then $(\phi_m, \phi_q)_{L^2(\mathbb{R};\mathbb{C}^2)} = 0$ when $n \neq q$ while they span the same two-dimensional space as $\varphi_{n-1}(0,1)^t$ and $\varphi_n(1,0)^t$ when $n = p$ while $\varepsilon \neq \eta$. The spectrum of H is therefore composed of one branch of simple absolutely continuous spectrum for each $m \in M$.

Propagating and evanescent modes. The above combination of Hermite functions provides a spectral decomposition of $\hat{H}$ for each $\xi \in \mathbb{R}$. What we are interested in now is solving $\hat{H} - E$ at a fixed value of $E \in \mathbb{R}$. We therefore introduce for $m = (n, \varepsilon) \in M$,

$$\xi_m(E) = \varepsilon(E^2 - 2n)^{\frac{1}{2}} = \varepsilon \begin{cases} \sqrt{E^2 - 2n}, & E^2 \geq 2n \\ i\sqrt{2n - E^2}, & E^2 \leq 2n. \end{cases}$$

Associated to $\xi_m(E)$ is the same eigenvector $\phi_m = \phi_m(y; E)$ as defined in (13.3), which in the physical domain provides a generalized eigenvector

$$\psi_m(x, y; E) = e^{i\xi_m x}\phi_m(y; E),$$

solutions of $(H - E)\psi_m = 0$. We observe that for each $E \neq \sqrt{2n}$, there is a finite number of real-valued solutions $\xi_m(E)$ corresponding to propagating modes and a countable number of imaginary solutions ξ_m corresponding to evanescent modes. Figure 2 presents the branch $E_0(\xi) = -\xi$ and the

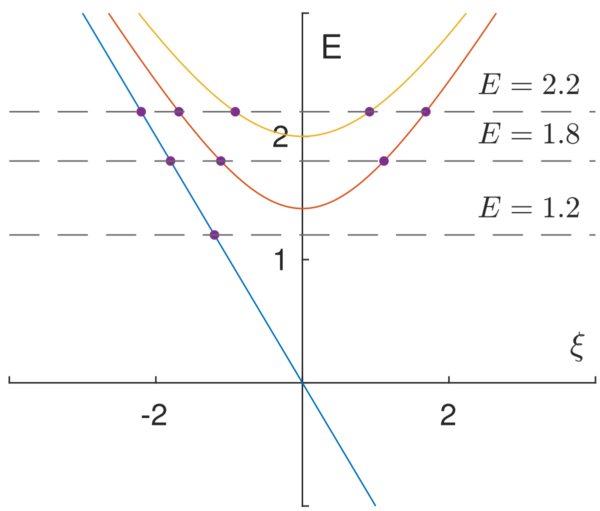


Figure 2: Three branches of absolutely continuous spectrum $\xi \to E_m(\xi)$.

branches $E_m(\xi)$ for $m = (1, 2; +1)$. For energies $0 < E < \sqrt{2}$, only E_0 corresponds to a propagating mode while five propagating modes exist for $2 < E < \sqrt{6}$. The group velocity $\partial_\xi E_m$ indicates the direction of propagation of the corresponding mode. For all energies $E^2 \neq 2n$, we observe that there is always one more mode going to the left than to the right. This reflects the asymmetric transport $2\pi\sigma_I(H) = -1$ and its interpretation as a spectral flow.

In the presence of a perturbation V, the propagating and evanescent modes become coupled. Far away from the support of V, however, only propagating modes are present since exponentially growing solutions are not allowed and exponentially decaying solutions negligible.

Outgoing conditions. We saw that H admitted a countable number of branches of single-valued absolutely continuous spectrum. Inversion of $(H-E)$ is therefore not guaranteed since $\mathbb{R} \ni E \notin \rho(H)$ the resolvent set of H. With appropriate *outgoing conditions*, however, the problem is indeed well posed.

Consider first the Laplace operator in one space dimension D^2, which admits two branches of simple absolutely continuous spectrum on $(0,\infty)$ (and a bad point at $E=0$ where the two branches meet). Consider the problem $(D^2-z)u=0$ for $z\in\mathbb{C}$ and the corresponding time-dependent Schrödinger equation $(D^2+D_t)v=0$. Let $z\in\mathbb{C}\backslash\mathbb{R}_+$. We observe that $v(t,x)=e^{-izt}u(x)$ generates solutions to both equations when $u(x)=e^{\pm i\sqrt{z}x}$ where $\sqrt{z}$ is defined with a branch cut along $\mathbb{R}_+=[0,\infty)$, the spectrum of D^2, and with the convention that $\sqrt{z\pm i0^+}=\pm\sqrt{z}+i0^+$ for $z\in\mathbb{R}_+$.

Any above plane wave solves the partial differential equations. Any linear combination of the form $u(x)=e^{\pm i\sqrt{z}|x|}$ also solves the equation for $|x|\geq R>0$. This corresponds to $v(t,x)=e^{i\sqrt{z}(\pm|x|-\sqrt{z}t)}$. So, for z real-valued, $v_+(t,x)=e^{i\sqrt{z}(|x|-\sqrt{z}t)}$ is *outgoing*, since it is written as a function of $(|x|-ct)$ with $c>0$ while $v_-(t,x)=e^{i\sqrt{z}(-|x|-\sqrt{z}t)}$ is *incoming*, as a function of $(|x|+ct)$ still with $c>0$.

Consider the solution with outgoing conditions $u_+(x)=e^{i\sqrt{z}|x|}$. Such a solution makes sense physically if and only if $\sqrt{z}=\sqrt{z}+i0^+=\sqrt{z+i0^+}$ so that the above plane waves remain bounded as $|x|\to\infty$. This shows that a choice of outgoing conditions consists in choosing plane waves with *positive* vanishingly small imaginary component. One could similarly choose ingoing conditions by choosing a *negative* vanishingly small imaginary component. This choice, called a *limiting absorption principle* needs to be made in order for the Green's function of a problem to be uniquely defined.

For $z\in\mathbb{C}\backslash\mathbb{R}_+$, the Green's function of the problem $(D^2-z)G=\delta_0$ is given by $G(x,z)=(2\sqrt{z})^{-1}ie^{i\sqrt{z}|x|}$. For $z\in\mathbb{R}_+$, two possible solutions are obtained by limiting absorption as z approaches the positive real axis. The Green's function with *outgoing* radiation condition is given by the same expression $G(x,z)=(2\sqrt{z})^{-1}ie^{i\sqrt{z}|x|}$ but now with $z\in\mathbb{R}_+$. It is the limit $\lim_{\varepsilon\downarrow 0}G(x,z+i\varepsilon)$. An equally valid solution mathematically would be the solution with incoming radiation condition $(2i\sqrt{z})^{-1}e^{-i\sqrt{z}|x|}$, obtained as the limit $\lim_{\varepsilon\downarrow 0}G(x,z-i\varepsilon)$ for $z\in\mathbb{R}_+$. Note that at $z=0$, the problem becomes even more degenerate and we may choose the Green's function to be $-\frac{1}{2}|x|$ up to the addition of any linear solution. In this limit, we no longer have any notion of incoming or outgoing solutions. In what follows, we avoid such bad points.

For Klein-Gordon equations (or wave equations with $m=0$), it is more natural to look at the problems $(D^2+m^2-z^2)u=0$ and $(D^2+m^2-D_t^2)v=0$ with $u=e^{i\sqrt{z^2-m^2}|x|}$ since $\Im\sqrt{z}\geq 0$ for all $z\in\mathbb{C}$ in our convention and $v_\pm=e^{\pm izt+i\sqrt{z^2-m^2}|x|}$ solutions away from $x=0$. It is in the sign in $v_\pm$ that we select incoming or outgoing planewaves, which has no effect on the stationary solution u. The corresponding outgoing Green's function is therefore $(2\sqrt{z^2-m^2})^{-1}ie^{i\sqrt{z^2-m^2}|x|}$, obtained for instance as the limit $\lim_{\varepsilon\downarrow 0}G(x,z+i\varepsilon)$ for $z\in\mathbb{R}$ and $z^2\geq m^2$. This is the choice of outgoing conditions we made in (13.1). Alternatively, we could choose *incoming* solutions $u_-=e^{-i\sqrt{z^2-m^2}|x|}$ or equivalently use a convention of the square root such that $\Im\sqrt{z}\leq 0$.

Green's function. To compute the effect of the perturbation V, we first need to invert the unperturbed operator $(H-E)$ or more precisely as recalled above $(H-(E+i0^+))^{-1}$. Alternatively, each time we have a choice between plane waves of the form $e^{\pm i\xi|x|}$, we make the choice of outgoing conditions such that $\pm\xi>0$.

The explicit construction of the Green's function with outgoing conditions solution of

$$(H-E)G=\delta(x-x_0)\delta(y-y_0)I$$

goes as follows. We first observe that by translation, the Green's function is a function of $x - x_0$ and thus set $x_0 = 0$. Next, we observe that

$$G = (H+E)(H^2 - E^2)^{-1}\delta(x)\delta(y-y_0)I = (H+E)\mathrm{Diag}(G_+, G_-)$$

where $G_\pm$ are Green's functions of

$$(-\partial_x^2 + \partial_y^2 + y^2 \pm 1 - E^2)G_\pm = \delta(x-x_0)\delta(y-y_0).$$

In other words, it is sufficient to construct the outgoing Green's functions for the above scalar problems. Expanding G_- in the basis of Hermite functions $\varphi_n(y)$ gives

$$G_- = \sum_n G_{-,n}(x)\varphi_n(y), \qquad (D_x^2 - E^2 + 2n)G_{-,n}(x) = \delta(x)\varphi_n(y_0).$$

The latter are given by solutions of the one-dimensional Laplace operator we mentioned above:

$$G_{-,n}(x) = \frac{\varphi_n(y_0)}{2\sqrt{|E^2 - 2n|}} \begin{cases} e^{-\sqrt{2n-E^2}|x|} & 2n > E^2 \\ ie^{i\sqrt{E^2-2n}|x|} & 2n < E^2 \end{cases}$$

where we assume $E^2 \neq 2n$ for an $n \in \mathbb{N}$ and choose the solution with outgoing radiation conditions. Define $\theta_n = i(E^2 - 2n)^{\frac{1}{2}}$ for convenience. Then

$$G_-(x,y;y_0) = \sum_{n\geq 0} \frac{-e^{\theta_n|x|}}{2\theta_n}\varphi_n(y)\varphi_n(y_0), \qquad G_+(x,y;y_0) = \sum_{n\geq 0} \frac{-e^{\theta_{n+1}|x|}}{2\theta_{n+1}}\varphi_n(y)\varphi_n(y_0)$$

the latter being obtained replacing $2n$ by $2n+2$ throughout.

The Green's function of $H - E$ with outgoing radiation conditions (since applying the differential operator $H + E$ does not modify what solution is outgoing or incoming) is thus given by

$$G(x,y;y_0) = \begin{pmatrix} (D_x + E)G_+ & \mathfrak{a}G_- \\ \mathfrak{a}^*G_+ & (-D_x + E)G_- \end{pmatrix}(x,y;y_0).$$

Numerical illustrations of the components of this 2×2 matrix for different values of the energy level E are presented in [7]. The solutions $G_\pm$ display logarithmic singularities in the vicinity of (x_0, y_0) as one expects from the Green's function of a Laplace-type problem in two space dimensions. What is striking is that G is essentially supported in a wave guide in the vicinity of $y = 0$ and that it is asymmetric in x. In particular, for $0 < E^2 < 2$, where only one propagating mode is present, the Green's function is essentially supported along the negative axis $x < 0$; see [7].

Integral equation for the density ρ. Assume $E \in \mathbb{R}$ with $E^2 \neq 2n$ for $n \in \mathbb{N}$ and consider the perturbed problem $(H + V - E)\psi = 0$ decomposing $\psi = \psi_{\rm in} + \psi_{\rm out}$, where $\psi_{\rm in}$ is a plane wave solution of $(H - E)\psi_{\rm in} = 0$. Defining $\mathcal{G} = (H - E)^{-1}$ with Schwartz kernel the above Green's function, we write $\psi_{\rm out} = \mathcal{G}\rho$ and find the integral equation

$$\rho + V\mathcal{G}\rho = (I + V\mathcal{G})\rho = f \tag{13.5}$$

with source term $f = -V\psi_{\rm in}$. There is no reason a priori for the operator $I + V\mathcal{G}$ to be invertible. However, by ellipticity of H, we find that $\mathcal{G}$ maps the Sobolev space $H^k(\mathbb{R}^2;\mathbb{C}^2)$ to $H^{k+1}(\mathbb{R}^2;\mathbb{C}^2)$. Assuming V smooth, bounded, and compactly supported, we therefore obtain that $V\mathcal{G}$ is a compact operator on $H^k(\mathbb{R}^2;\mathbb{C}^2)$ for any $k \geq 0$. In fact, by (faster than) exponential decay of the Green's function in y, compact support of V in x (and not in y) is sufficient to obtain the same result. We thus have the following result.

Theorem. The integral equation (13.5) with $f \in L^2(\mathbb{R}^2;\mathbb{C}^2)$ admits a unique solution as soon as -1 is not in the spectrum of the compact operator $V\mathcal{G}$. This is the case for all energies E except for a possible set of measure 0.

We already showed that $V\mathcal{G}$ was compact. Since $H+V$ is self-adjoint, $V\mathcal{G}$ cannot have -1 as an eigenvalue as soon as E is purely imaginary. Since $E \to V\mathcal{G}$ is analytic away from the set $E^2 = 2n$ by inspection of the above Green's function, by the analytic Fredholm theory, -1 can only be an eigenvalue of $V\mathcal{G}$ for a discrete set of real-valued energies E.

We showed in [7] the existence of non-trivial solutions of $(I + v\mathcal{G})\rho = 0$ for an appropriate class of perturbations of the form $V(x) = V_0\chi_{[0,L]}$ with V_0 and L appropriately chosen. Thus, -1 can indeed be in the spectrum of $V\mathcal{G}$. This corresponds to eigenvalues (normalized in $L^2(\mathbb{R}^2;\mathbb{C}^2)$) of the differential problem $(H+V)\psi = 0$. See [7] for details.

Unperturbed Edge conductivity. Let us come back to the computation of $\sigma_I(H)$ and $\sigma_I(H+V)$, which we know from past analyses equal -1.

We start with $\sigma_I(H)$ with H the unperturbed Dirac operator. Then

$$\varphi'(H) = \mathcal{F}^{-1}\int_{\mathbb{R}}^{\oplus} \varphi'(\hat{H}(\xi))\mathcal{F} = \sum_{m\in M} \mathcal{F}^{-1}\int_{\mathbb{R}}^{\oplus} \varphi'(E_m(\xi))\Pi_m(\xi)\mathcal{F}$$

thanks to the above spectral decomposition, where the sum over m is finite for an compactly supported function $\varphi'(E)$ and $\Pi_m(\xi) = \phi_m(\xi)\otimes\phi_m(\xi)$ is a rank-one projection operator.

For the Dirac operator, $i[H,P] = P'(x)\sigma_3$ with therefore a Schwartz kernel given by $P'(x)\sigma_3\delta(x-x')\delta(y-y')$. Thus

$$2\pi\sigma_I = 2\pi\mathrm{Tr}P'(x)\sigma_3\varphi'(H) = \sum_m \int \varphi'(E_m(\xi))2\pi(\psi_m(x,\xi), P'(x)\sigma_3\psi_m(x,\xi))d\xi$$

where $(\cdot,\cdot)$ is inner product on $L^2(\mathbb{R}^2;\mathbb{C}^2)$. Here, we used the explicit expressions of the Fourier and inverse Fourier transforms (this is where a factor 2π cancels out) and the expression $\psi(x,\xi) = (2\pi)^{-\frac{1}{2}}e^{ix\xi}\phi_m(\xi)$. Thus

$$2\pi\sigma_I = \sum_m \int \varphi'(E_m(\xi))\int_{\mathbb{R}} P'(x)\tilde{j}_m(x,\xi)d\xi dx,$$

where $\tilde{j}_m(x,\xi) := 2\pi(\psi_m(x,\xi),\sigma_3\psi_m(x,\xi))_y = (\phi_m(\xi),\sigma_3\phi_m(\xi))_y$ with $(\cdot,\cdot)_y$ the standard inner product on $L^2(\mathbb{R}_y;\mathbb{C}^2)$. The current $\tilde{j}_m(x,\xi)$ is obviously independent of x for plane waves $\psi(x,\xi) = e^{ix\xi}\phi_m(\xi)$ so that, since $P'(x)$ integrates to 1, for any $x_0 \in \mathbb{R}$,

$$2\pi\sigma_I = \sum_m \int_{\mathbb{R}} \varphi'(E)\tilde{j}_m(x_0,\xi_m)\Big|\frac{\partial\xi_m}{\partial E}\Big|dE$$

where we used the change of variables $\xi_m(E) = \varepsilon\sqrt{E^2-2n}$ for $m = (n,\varepsilon)$. Note that the sum involves only propagating modes since ξ is real valued. We define the normalized currents

$$j_m(x,E) = \Big|\frac{\partial\xi_m}{\partial E}\Big|\tilde{j}_m(x,\xi_m) = \Big|\frac{\partial\xi_m}{\partial E}\Big|(\psi_m(x,\xi),\sigma_3\psi_m(x,\xi))_y.$$

The conductivity then takes the form

$$2\pi\sigma_I = \sum_m \int_{\mathbb{R}} \varphi'(E) j_m(x_0, E) dE = \sum_m j_m(x_0, E_0)$$

again independent of $x_0 \in \mathbb{R}$ as well as independent of E since $\varphi'(E)$ may be chosen with support arbitrarily close to any energy E_0 outside of a bad set where $\partial\xi_m/\partial E$ is not defined (infinite).

Perturbed Edge conductivity. In the presence of a perturbation V, the generalized eigenvector $\psi_m(x,\xi)$ is replaced by $\psi_m^V(x,\xi)$ solution of $(H+V-E)\psi_m^V = 0$. Note that $\psi_m^V = \psi_{\rm in} + \psi_{\rm out}$ for an appropriate $\psi_{\rm in}$ and a $\psi_{\rm out} = \mathcal{G}\rho$ with ρ solution of the above integral formulation.

For V compactly supported, we expect (a proof for such results is currently being finalized) that the absolutely continuous spectrum of H, at least away from a set of bad points, is unitarily equivalent to that of $H+V$. This implies the existence of a unitary operator U such that $\psi_m^V = U\psi_m$. In addition to the absolutely continuous spectrum, we expect $H+V$ to have discrete spectrum with a countable number of eigenvalues with finite multiplicity and only accumulation point at infinity. Assuming the absence of singular continuous spectrum (proof being finalized), this implies the decomposition

$$\sigma_I = \sum_m \int \varphi'(E_m(\xi))(\psi_m^V(x,\xi), i[H+V,P]\psi_m^V(x,\xi))d\xi + \sum_p \varphi'(E_p)(\psi_p^V, i[H+V,P]\psi_p^V)$$

where $\psi_m^V(\xi)$ are the generalized eigenvectors associated to the *same* generalized eigenvalue $E_m(\xi)$ as $\psi_n(\xi)$ in the unperturbed case (this is what the unitary operator U provides) and where ψ_p^V corresponds to point spectrum of $(H+V-E_p)\psi_p^V = 0$.

We first show that the point spectrum does not contribute to the edge conductivity. Indeed, the current is

$$\begin{aligned}(\psi_p^V, [H+V,P]\psi_p^V) &= ((H+V)\psi_p^V, P\psi_p^V) - (\psi_p^V, P(H+V)\psi_p^V) \\ &= E(\psi_p^V, P\psi_p^V) - E(\psi_p^V, P\psi_p^V) = 0\end{aligned}$$

Therefore, we find that

$$2\pi\sigma_I = \sum_m \int \varphi'(E)\Big|\frac{\partial \xi_m}{\partial E}\Big|(\psi_m^V(\xi_m), 2\pi i[H,P]\psi_m^V(\xi_m))dE. \tag{13.6}$$

Since the edge conductivity is independent of the choice of $\varphi \in \mathfrak{S}[0,1]$ and $P \in \mathfrak{S}[0,1]$, we obtain that for E_0 outside of a set of measure 0 and for any $x_0 \in \mathbb{R}$, we have

$$2\pi\sigma_I = \sum_m j_m(x_0, E_0), \qquad j_m(x,E) = \Big|\frac{\partial \xi_m}{\partial E}\Big|(\psi_m^V(x_0, \xi_m(E)), 2\pi\sigma_3\psi_m^V(x_0, E)). \tag{13.7}$$

This is a formula that can be tested numerically. For several choices of E, x_0, and V, we obtained that $j_m(x,E)$ depended on the choice of (E,V) (though not on x_0 since the problem has a current conservation law; see below). However, the sum over all components m provided a numerical verification that $2\pi\sigma_I(H+V)$ was equal to -1 within an error of 10^{-12}. Indeed, an advantage of the above variational formulation is that it allows us to use accurate quadratures to estimate the application of $\mathcal{G}$ with essentially arbitrary accuracy.

Far Field Scattering matrix formalism. Assume $V(x,y)$ a potential compactly supported on $(-L,L)$ in x. Let E be a fixed energy outside of $E^2 = 2n$ and $E = E_p$ the discrete eigenvalues $\mathrm{Ker}(H+V-E_p) \neq \{0\}$. For $x < -L$ and $x > L$, only propagating modes solutions of $(H-E)\psi = 0$ are allowed. For $k = k(E)$, we find $k+1$ modes $\psi_{\rm in}$ propagating from right to left and k modes $\psi_{\rm in}$ propagating from left to right; see Figure 2. Each solution $(H+V-E)\psi = 0$ with $\psi = \psi_{\rm in} + \psi_{\rm out}$ for $\psi_{\rm in}$ one of the incoming modes (satisfying incoming radiation conditions) generates a scattered component $\psi_{\rm out}$ with outgoing conditions.

We can then define a scattering matrix

$$S = \begin{pmatrix} R_+ & T_- \\ T_+ & R_- \end{pmatrix}$$

with R_+ a $k \times (k+1)$ matrix of reflection coefficients of modes from left to left, R_- a $(k+1) \times k$ matrix of reflection coefficients of modes from right to right, T_+ a $(k+1) \times (k+1)$ transmission matrix of modes from left to right, and finally T_- a $k \times k$ transmission matrix of modes from right to left.

The scattering matrix is a unitary matrix. We then verify that $\mathrm{tr}T_+^*T_+$ corresponds to the total amount of current propagating from left to right and $\mathrm{tr}T_-^*T_-$ the current propagating from right to left. In the absence of scattering $V = 0$, then obviously $\mathrm{tr}T_+^*T_+ = k$ while $\mathrm{tr}T_-^*T_- = k+1$.

The main result of this section is that the edge conductivity is given, both in the presence and in the absence of fluctuations, by

$$2\pi\sigma_I = \mathrm{tr}T_+^*T_+ - \mathrm{tr}T_-^*T_- = -1.$$

In the presence of strong fluctuations, Anderson localization ensures that $\mathrm{tr}T_+^*T_+$ converges to 0, so that asymptotically no transmission from left to right occurs. In the same regime, we have $\mathrm{tr}T_-^*T_-$ converging to -1. This is therefore an obstruction to Anderson localization, albeit one that is quantized. Out of the $k+1$ modes that may transmit from right to left in the absence of fluctuations, only one mode, a linear combination of the above $k+1$ modes that depends on V, is guaranteed to transmit. In such a setting, asymmetric transport is topologically protected, as we have demonstrated many times, while backscattering is not. Only when $k = 1$, i..e, for $|E|^2 < 2$ in our example, do we have that backscattering is suppressed.

See [7] for additional details of the derivation. This part will be made more explicit and more rigorous once the spectral analysis of the problem is complete.

14 Lecture 14.

This lecture looks at applications of the above theoretical results to models of transport in gated twisted bilayer graphene.

Twisted Bilayer graphene (tBLG). To model transport in a gated tBLG, we start from the following macroscopic Bistritzer-MacDonald 4×4 bulk model

$$H_\eta := \begin{pmatrix} \Omega + D_x\sigma_1 + \eta D_y\sigma_2 & \varepsilon V^* \\ \varepsilon V & -\Omega + D_x\sigma_1 + \eta D_y\sigma_2 \end{pmatrix}. \tag{14.1}$$

This a model for low-energy transport of wavenumbers close to K when $\eta = 1$ and close to K' when $\eta = -1$. Here, η is valley number. The top 2×2 system corresponds to the top layer of the bi-layer system while the bottom 2×2 system corresponds to the bottom layer. We assume a electrostatic potential difference between the two layers due to vertical gating equal to $2\Omega\in\mathbb{R}$. When the two layers are far apart, they do not interact and $\varepsilon=0$. When they are closer to one-another, they interact in a complicated manner. Then $V\in\{A,A^*\}$ models an interlayer coupling term with strength $0\neq\varepsilon\in\mathbb{R}$, where $A=\begin{pmatrix}0&1\\0&0\end{pmatrix}$. Here, A^* and A correspond to whether the triangle has AB or BA stacking geometry respectively. We do not consider the derivation of this model from a periodic Schrödinger equation or from a tight-binding model. Such derivations have been obtained elsewhere in the literature.

Our objective here is to analyze the topological properties of the above Hamiltonian H. We will assume $\eta=1$. It turns out that the topology of the valley $\eta=-1$ is opposite that of the valley $\eta=1$. As a consequence, when inter-valley coupling is present and strong, the overall system is topologically trivial. We assume here no coupling between valleys.

A macroscopic moiré structure appears in tBLG and generates transitions from the BA to the AB configuration. We focus on this transition by introducing a domain wall $m(y)\in\mathfrak{S}(0,1,-y_0,y_0)$ and a transition coupling matrix

$$V(y)=m(y)A+(1-m(y))A^*.$$

We then consider the edge Hamiltonian

$$H:=\begin{pmatrix}\Omega+D_x\sigma_1+\eta D_y\sigma_2 & \varepsilon V^*(y)\\ \varepsilon V(y) & -\Omega+D_x\sigma_1+\eta D_y\sigma_2\end{pmatrix}. \tag{14.2}$$

For $y>0$ large, the system is in a BA configuration while it is in an AB configuration for $-y>0$ large.

The main objective of this lecture is to establish the following:

Theorem 14.1 *Assume $\Omega\varepsilon\neq 0$. Then for $\varphi'(H)$ supported in the interval $(-E_0,E_0)$ for $E_0=\frac{|\Omega\varepsilon|}{\sqrt{4\Omega^2+\varepsilon^2}}$, we have that $[H,P]\varphi'(H)$ is trace-class and that moreover, the following bulk-edge correspondence holds:*

$$2\pi\sigma_I=-2\,\mathrm{sign}(\eta\Omega). \tag{14.3}$$

The above invariant is independent of the sign of $\varepsilon\neq 0$. The support of φ' depends on (ε,Ω) in a non-trivial manner. The outcome of this results is that if inter-valley coupling is neglected, then the transition between the AB and BA insulating phases generates asymmetric transport with two topologically protected modes; see right panel in Fig. 3.

This result will be obtained by applying the theories of Lectures 10 and 11. To apply the results of Lecture 10, we first need to show that the operator satisfies hypothesis [H1]. That the operator is elliptic with a symbol in S^1 is clear. This is true for any value of (ε,Ω) and $\eta=\pm1$. We also need to verify that when $m=0$ and when $m=1$, the bulk operators have a spectral gap in the vicinity of $E=0$. This will be confirmed in the next result. Applying the results of Lecture 11 and in particular Theorem 10.1, we obtain that the invariant (14.3) is given by the integral that appears in the Fedosov-Hörmander formula. We know using the notion of bulk-difference invariants that this term may be computed as an integral of a curvature corresponding to the two bulk phases.

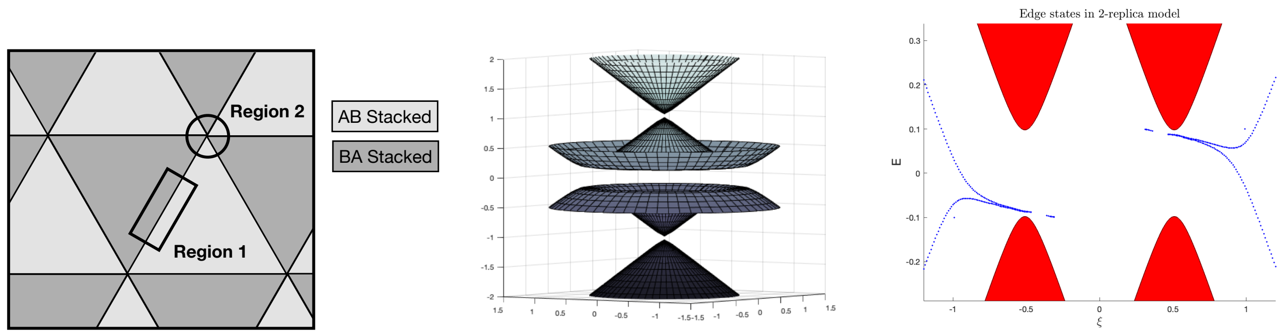


Figure 3: Asymmetric transport in gtBLG. Left: Region 1 in a moiré pattern models the transition between BA and AB stakings. Middle: bulk dispersion relation $\xi \to E_j(\xi)$ for $j = 1, 2, 3, 4$ in the presence of a spectral gap at $E = 0$. Right: dispersion relation of the edge Hamiltonian (14.2) with 2 non-trivial branches of absolutely continuous spectrum crossing $E = 0$ downward when $\Omega > 0$ as expected from Theorem 14.1.

Bulk curvature integral. We consider first the case $\Omega > 0$ and $\varepsilon > 0$ while $\eta = 1$ and observe that the bulk Hamiltonian for $y > y_0$ is given in the Fourier domain by

$$\hat{H}_+(\xi) = \begin{pmatrix} \Omega & \bar{\xi} & 0 & 0 \\ \xi & \Omega & \varepsilon & 0 \\ 0 & \varepsilon & -\Omega & \bar{\xi} \\ 0 & 0 & \xi & -\Omega \end{pmatrix}.$$

We identify here $\xi = (\xi_1, \xi_2)^t$ with its expression as a complex number $\xi = \xi_1 + i\xi_2$ so that $\bar{\xi} = \xi_1 - i\xi_2$. When $\xi = 0$, we observe that $\pm\Omega$ are two eigenvalues associated to the eigenvectors e_1 and e_4 while $\pm\sqrt{\Omega^2 + \varepsilon^2}$ are the remaining two eigenvalues with eigenspaces supported on the span of (e_2, e_3). The constraint $\det(H - E) = 0$ on the eigenvalues is given explicitly by

$$(|\xi|^2 - (E - \Omega)^2)(|\xi|^2 - (E + \Omega)^2) = \varepsilon^2(E^2 - \Omega^2).$$

The four eigenvalues E_j for $1 \le j \le 4$ come in pairs with opposite signs, are all different, and are given explicitly via

$$E_\pm^2 = (\Omega^2 + \frac{1}{2}\varepsilon^2 + |\xi|^2) \pm \sqrt{(4\Omega^2 + \varepsilon^2)|\xi|^2 + \frac{1}{4}\varepsilon^4},$$

where to simplify notation we denote $-E_+ = E_1$, $-E_- = E_2$, $E_- = E_3$ and $E_+ = E_4$. We observe that E_-^2 is large for $|\xi|$ large while $E = 0$ implies $(|\xi|^2 - \Omega^2)^2 + \varepsilon^2\Omega^2 = 0$ which is impossible, and hence the presence of a spectral gap. More precisely, we obtain that $|\xi|^2 \to E_-^2$ is a convex function with a minimum equal to

$$E_{\min}^2 = \frac{\Omega^2\varepsilon^2}{4\Omega^2 + \varepsilon^2} \quad \text{and attained at} \quad |\xi|^2 = \frac{2\Omega^2(2\Omega^2 + \varepsilon^2)}{4\Omega^2 + \varepsilon^2}.$$

The spectral gap about energy $E = 0$ (this is the only existing gap) is thus indeed of size $\frac{2|\Omega\varepsilon|}{\sqrt{4\Omega^2+\varepsilon^2}}$. Note that the spectral gap is 'large' only when both ε and Ω are 'large'.

The corresponding equations for the eigenvectors for $E = E_j$ are

$$(\Omega - E)u_1 + \bar{\xi}u_2 = 0 \quad \xi u_1 + (\Omega - E)u_2 + \varepsilon u_3 = 0$$

$$\varepsilon u_2 - (\Omega+E)u_3 + \bar\xi u_4 = 0, \quad \xi u_3 - (\Omega+E)u_4 = 0.$$

We focus on $j=3,4$ since we are interested in computing the Chern numbers of the branches above the spectral gap. Associated to u^j is a projector $\Pi^j_+ = u_j \otimes u_j$ (where $+$ here refers to the half space $y>y_0$). We recall that the Chern number we wish to compute is

$$W^j_+ = \frac{i}{2\pi}\int_{\mathbb{R}^2} (du^j, du^j) = \frac{i}{2\pi}\int_{\mathbb{R}^2} d(u^j, du^j) = \frac{i}{2\pi}\lim_{R\to\infty}\int_{R\mathbb{S}^1} (u^j, du^j).$$

as an application of the Stokes theorem, with $R\mathbb{S}^1$ the circle centered at the origin with radius equal to R. With $A^j=(u^j,du^j)$ the connection associated to u^j, we observe that we need to integrate it over a 'circle at infinity'. However, in order for the Stokes theorem to apply, we also need to ensure that $u^j(\xi)$ is continuously defined and continuously differentiable for $\xi\in\mathbb{R}^2$. The construction is different for $j=3$ and $j=4$.

For $j=4$, we construct

$$u^4_+(\xi) = c_4\Big(\frac{\bar\xi}{E_4-\Omega}(\Omega+E_4-\frac{|\xi|^2}{\Omega+E_4}), (\Omega+E-\frac{|\xi|^2}{\Omega+E_4}), \varepsilon, \frac{\varepsilon\xi}{\Omega+E_4}\Big)^t.$$

The reason is that $E_4=E_+>\Omega$ so that $(E_4-\Omega)^{-1}$ is uniformly bounded as one easily verifies. Thus, u^4_+ and du^4_+ are clearly continuously defined. Moreover as $|\xi|\to\infty$, and decomposing $\xi=|\xi|\hat\xi$, we observe that u^4_+ converges to

$$u^4_+(\hat\xi) = c_\infty(\hat{\bar\xi}\beta, \beta, \varepsilon, \varepsilon\hat\xi), \quad \beta = 2\Omega+\sqrt{4\Omega^2+\varepsilon^2}, \quad c_\infty = (2\varepsilon^2+2\beta^2)^{-\frac12}.$$

We use the same notation for vectors $u^j_\pm(\xi)$ and their limits $u^j_\pm(\hat\xi)$ as $|\xi|\to\infty$ to simplify notation. This results from the expansion $0<E^2_\pm = |\xi|^2 \pm \sqrt{4\Omega^2+\varepsilon^2}|\xi| + O(1)$. We leave the details to the reader. We also observe that $du^4_+(\xi)$ converges to $du^4_+(\hat\xi)$ in the same limit so that

$$W^4_+ = \frac{i}{2\pi}c_\infty^2\int_{\mathbb{S}^1}\big((\beta e^{-i\theta},\beta,\varepsilon,\varepsilon e^{i\theta}),(-i\beta e^{-i\theta},0,0,i\varepsilon e^{i\theta})\big)d\theta = \frac{i}{2\pi}c_\infty^2(-i\beta^2+i\varepsilon^2)2\pi = \frac{\beta^2-\varepsilon^2}{2(\varepsilon^2+\beta^2)}.$$

This is a somewhat surprising result: If the curvature integral was computed over a compact domain, it would take integral values. For the Dirac operator, we obtained an integral given by half integers. Here, we observe that the curvature integral takes a continuum of values as (Ω,ε) vary. An explanation comes from the fact that the energy branches $\xi\to E_{3,4}(\xi)$ satisfy $E_4(\xi)-E_3(\xi)$ bounded as $\xi\to\infty$ and thus remain intertwined in that limit.

Let us look at the second positive eigenvalue $E_3=E_->0$. The (smooth) function $\xi\to E_-(\xi)-\Omega$ vanishes at $\xi=0$ (and in fact at two other values of ξ). Thus we may no longer divide by $(\Omega-E)$ in the construction of $u(\xi)$, which we want to be smooth. However, it is easy to verify that $\xi^{-1}(E_3(\xi)-\Omega)$ is a smooth function. Eliminating u_2 and u_4 from the above system for u, we obtain

$$u^3_+(\xi) = c_3\Big(\varepsilon, \frac{E_3-\Omega}{\bar\xi}\varepsilon, -(\xi-\frac{(\Omega-E_3)^2}{\bar\xi}), \frac{-\xi}{\Omega+E_3}(\xi-\frac{(\Omega-E_3)^2}{\bar\xi})\Big).$$

We verify that $\xi\to u^3_+(\xi)$ is a continuous and continuously differentiable function. This converges as $|\xi|\to\infty$ to

$$u^3_+(\hat\xi) = c_\infty(\varepsilon, \varepsilon\hat\xi, -\beta\hat\xi, -\beta\hat\xi^2).$$

Following the same procedure as above, we deduce after application of the Stokes theorem to compute the curvature integral that

$$W^3_+ = \frac{-\varepsilon^2-3\beta^2}{2(\varepsilon^2+\beta^2)}, \qquad W_+ = W^3_+ + W^4_+ = -1.$$

So, the sum of the curvature integrals corresponding to both positive energy branches equals -1. This -1 would equal $-1/2$ for the Dirac case and nothing guarantees that W_+, which is a half of a bulk-difference invariant, takes integral values.

Consider now the second bulk Hamiltonian for $y < -y_0$, which in the Fourier domain is

$$\hat{H}_-(\xi) = \begin{pmatrix} \Omega & \bar{\xi} & 0 & \varepsilon \\ \xi & \Omega & 0 & 0 \\ 0 & 0 & -\Omega & \bar{\xi} \\ \varepsilon & 0 & \xi & -\Omega \end{pmatrix}.$$

We observe that the equation for the eigenvalues E is the same as in the previous case. For the largest positive eigenvalue, one finds an associated eigenvector

$$u_-^4(\xi) = c_4\Big(\Omega + E_4 - \frac{|\xi|^2}{\Omega + E_4}, \frac{\xi}{E_4 - \Omega}(\Omega + E_4 - \frac{|\xi|^2}{\Omega + E_4}), \frac{\hat{\xi}}{\Omega + E_4}\varepsilon, \varepsilon\Big).$$

In the limit $|\xi| \to \infty$, we have

$$u_-^4(\hat{\xi}) = c_\infty(\beta, \beta\hat{\xi}, \hat{\bar{\xi}}\varepsilon, \varepsilon), \quad \text{so that} \quad W_-^4 = \frac{\varepsilon^2 - \beta^2}{2(\varepsilon^2 + \beta^2)}.$$

For the eigenvalue $0 < E_3 = E_-$, we find

$$u_-^3(\xi) = c_3\Big(\frac{E_3 - \Omega}{|\xi|}\varepsilon, \varepsilon\frac{\bar{\xi}}{\Omega + E_3}(\frac{(\Omega - E_3)^2}{\xi} - \bar{\xi}), \frac{(\Omega - E_3)^2}{\xi} - \bar{\xi}\Big).$$

In the limit $|\xi| \to \infty$,

$$u_-^3(\hat{\xi}) = c_\infty\big(\hat{\bar{\xi}}\varepsilon, \varepsilon, -\beta\hat{\bar{\xi}}^2, -\beta\hat{\bar{\xi}}\big) \quad \text{so that} \quad W_-^3 = \frac{\varepsilon^2 + 3\beta^2}{2(\varepsilon^2 + \beta^2)} \quad \text{and} \quad W_- = W_-^3 + W_-^4 = 1.$$

An application of the bulk-edge correspondence then states that

$$2\pi\sigma_I = W_+ - W_- = -2$$

and this proves the theorem for ε, Ω, and η positive since hypothesis [H1] of Lecture 9 is now satisfied and hence Theorem 10.1 applies.

Symmetries and bulk curvature computations. To address how W changes depending on system parameters Ω, ε and η, define

$$\hat{H}_\pm(\xi; \Omega, \varepsilon, \eta) = \Omega\sigma_3 \otimes I_2 + I_2 \otimes (\xi_1\sigma_1 + \eta\xi_2\sigma_2) + \frac{\varepsilon}{2}(\sigma_1 \otimes \sigma_1 \pm \sigma_2 \otimes \sigma_2),$$

and the following symmetry operators

$$S_1 = \sigma_1 \otimes I_2, \qquad S_2\psi(\xi) = \psi(-\xi), \qquad S_3 = \sigma_1 \otimes \sigma_1.$$

We let $W(\Omega, \varepsilon, \eta)$ denote the above bulk-difference invariant so that for $\Omega, \varepsilon, \eta > 0$, by the above calculation we have $W(\Omega, \varepsilon, \eta) = -2$. We verify the relations

$$\begin{aligned} S_1\hat{H}_\pm(\xi; \Omega, \varepsilon, \eta)S_1 &= \hat{H}_\mp(\xi; -\Omega, \varepsilon, \eta), \\ S_2\hat{H}_\pm(\xi; \Omega, \varepsilon, \eta)S_2 &= -\hat{H}_\pm(\xi; -\Omega, -\varepsilon, \eta), \\ S_3\hat{H}_\pm(\xi; \Omega, \varepsilon, \eta)S_3 &= \hat{H}_\pm(\xi; -\Omega, \varepsilon, -\eta). \end{aligned}$$

Since $(S_j\psi, d(S_j\psi)) = (S_j\psi, S_j d\psi) = (\psi, d\psi)$ for $j \in \{1,3\}$, and likewise $\int(\psi, d\psi) = \int(S_2\psi, dS_2\psi)$, the connection is invariant to the symmetry operations. However, when the sign of the Hamiltonian changes, i.e. if $S_j H S_j = -\tilde{H}$, the invariant of H is minus the invariant of $\tilde{H}$ as the order of the bands is reversed, and the sum of the four connnections (including positive and negative energies) is zero. We thus conclude

$$W(\Omega,\varepsilon,\eta) = -W(-\Omega,\varepsilon,\eta) = W(\Omega,-\varepsilon,\eta) = W(-\Omega,\varepsilon,-\eta) = -W(\Omega,\varepsilon,-\eta).$$

This concludes the proof of theorem 14.1.

Bulk-difference invariant. While not necessary to apply Theorem 10.1, we would like to recast the above computation of $2\pi\sigma_I$ as a bulk-difference invariant. Following Lecture 11, we wish to glue projectors from the upper half space $+$ with projectors from the lower half space $-$ continuously along the 'circle at infinity'. Upon inspection, we observe that the vectors $u_+^j(\hat{\xi})$ and $u_-^j(\hat{\xi})$ are *not* defined up to a multiplicative phase in $U(1)$ for $j = 3,4$. However, defining $v_-^j(\xi) = \sigma_2 \otimes I_2 u_-^j(\xi)$, we obtain in the limit $|\xi| \to \infty$ that

$$\begin{aligned}
v_-^3(\hat{\xi}) &= \sigma_2 \otimes I_2 u_-^3(\hat{\xi}) = ic_\infty\big(\beta\hat{\bar{\xi}}^2, \beta\hat{\bar{\xi}}, \varepsilon\hat{\bar{\xi}}, \varepsilon\big) = i\hat{\bar{\xi}} c_\infty\big(\beta\hat{\bar{\xi}}, \beta, \varepsilon, \varepsilon\hat{\xi}\big) = i\hat{\bar{\xi}} u_+^4,\\
v_-^4(\hat{\xi}) &= \sigma_2 \otimes I_2 u_-^4(\hat{\xi}) = -ic_\infty\big(\varepsilon\hat{\bar{\xi}}, \varepsilon, -\beta, -\beta\hat{\xi}\big) = -i\hat{\bar{\xi}} c_\infty\big(\varepsilon, \varepsilon\hat{\xi}, -\beta\hat{\xi}, -\beta\hat{\xi}^2\big) = -i\hat{\bar{\xi}} u_+^3.
\end{aligned}$$

Defining the projectors $\Pi_+^j(\xi) = u_+^j \otimes u_+^j$ and $\tilde{\Pi}_-^j(\xi) = v_-^j \otimes v_-^j = \sigma_2 \otimes I_2 (u_-^j \otimes u_-^j)\sigma_2 \otimes I_2$, we observe that Π_+^3 and $\tilde{\Pi}_-^4$ have the same limits for each $\hat{\xi} \in \mathbb{S}^1$ as $|\xi| \to \infty$. Of course, the curvature $\mathrm{tr}\Pi d\Pi \wedge d\Pi$ associated to $\tilde{\Pi}_-^j(\xi)$ is the same as that associated to $\Pi_-^j(\xi) = u_-^j \otimes u_-^j$ (equivalently, invariants are the same for the Hamiltonian $\hat{H}_-$ and the (same) Hamiltonian (written in a different basis) $\sigma_2 \otimes I_2(\hat{H}_-)\sigma_2 \otimes I_2$.

With these constructions, the results of Lecture 11 guarantee that

$$W_+^3 - W_-^4 \in \mathbb{Z}, \qquad \text{and} \qquad W_+^4 - W_-^3 \in \mathbb{Z}.$$

We verify that both equal $-\operatorname{sign}(\eta\Omega)$ and that their sum equals $-2\operatorname{sign}(\eta\Omega)$ as stated in the theorem.

It is a peculiar result that the higher (positive) energy of the BA stacking (where $y > y_0$) may be glued with the lower (positive) energy of the AB stacking (where $y < -y_0$) to provide an integral-valued bulk-difference invariant while the lower energy of the BA stacking may be glued with the higher energy of the AB stacking.

Appendix

Below we collect some of the results on functional analysis, operator theory, spectral theory, pseudo-differential calculus and semiclassical calculus we need in the text.

A Linear operators and compact operators

We collect some information on linear operators, bounded operators, and trace-class operators. The material is taken mostly from [27, Chapter 19] and [18].

Schwartz Kernel. Let X and Y be open sets in $\mathbb{R}^d$. Let $\mathfrak{D}(Y) = C_c^\infty(Y)$ be the set of smooth test functions with compact support and $\mathfrak{D}'(X)$ the dual space of $\mathfrak{D}(X)$, i.e., the space of distributions on X. Then for every linear map A from $\mathfrak{D}(Y)$ to $\mathfrak{D}'(X)$, there is a distribution $a \in \mathfrak{D}'(X \times Y)$ called the Schwartz kernel of the operator A such that

$$\langle Au, v\rangle_{\mathfrak{D}'(X),\mathfrak{D}(X)} = \langle a, u \otimes v\rangle_{\mathfrak{D}'(X\times Y),\mathfrak{D}(X\times Y)}.$$

Any such kernel a also defines a corresponding operator A. When the distribution a is sufficiently smooth, we write

$$Au(x) = \int_Y a(x,y)u(y)dy.$$

We also use this notation when a is an arbitrary distribution, knowing that the above relation is then meant by the notation.

Bounded operators. For $\mathcal{H}$ a Hilbert space, we denote by $\mathcal{B}(\mathcal{H})$ the algebra of bounded (linear continuous) operators. This is in fact a C^*-algebra since $A \in \mathcal{B}(\mathcal{H})$ implies that the adjoint $A^* \in \mathcal{B}(\mathcal{H})$, where A^* is defined thanks to the Riesz representation, and for the uniform norm we have $\|AA^*\| = \|A\|^2$. It is an algebra since it is stable under operator multiplication.

Banach spaces of compact operators. Let H_j be separable Hilbert spaces.

Definition A.1 *The space $\mathcal{I}_2(H_1, H_2)$ of* **Hilbert-Schmidt** *operators is defined as bounded operators T such that*

$$\|T\|_2^2 = \sum \|Te_i\|^2 < \infty$$

for $\{e_i\}$ complete orthonormal basis of H_1. The norm is independent of the choice of bases.

The space $\mathcal{I}_1(H_1, H_2)$ of **trace-class** *operators is defined as $T = T_2^* T_1$ with $T_j \in \mathcal{I}_2(H_j, H_0)$, or equivalently as the space of operators with norm below finite*

$$\|T\|_1 = \sup \sum |(Te_j, f_j)| = \inf \|T_1\|_2 \|T_2\|_2$$

with supremum taken over all orthonormal systems e_j and f_j in H_1 and H_2.

When $H_1 = H_2 = H$, and $T \in \mathcal{I}_1(H,H)$ we define the **trace** *as*

$$\text{Tr } (T) = \text{Tr } (T|H) = \sum (Te_j, e_j).$$

The trace is uniquely defined and a linear form of norm 1 on $\mathcal{I}_1(H,H)$.

If $T_1 \in \mathcal{I}_1(H,H)$ and S is a continuous bijection from H to H_2 then $T_2 = ST_1S^{-1} \in \mathcal{I}_1(H_2,H_2)$ and Tr $(T_1|H) =$ Tr $(T_2|H_2)$.

For $H = H_1 = H_2$ a Hilbert space and T a compact operator on H, we define λ_i as the singular values of T, i.e., the non-negative real numbers such that λ_i^2 are the eigenvalues of the (diagonalizable) compact operator T^*T. The singular values $\lambda_i \to 0$ as $i \to \infty$. We can introduce a more refined structure.

Definition A.2 *Let $1 \leq p < \infty$. We denote by $\mathcal{I}_p(H)$ the space of compact operators T bounded for the following Schatten $p-$norm*

$$\|T\|_p := \Big(\sum_i \lambda_i^p\Big)^{\frac{1}{p}} < \infty.$$

The space $\mathcal{I}_p(H)$ is a Banach space for this norm. Moreover, it is an ideal in the space of bounded operators in the sense that if $T \in \mathcal{I}_p(H)$ and $S \in \mathcal{B}(H)$, then ST and TS belong to $\mathcal{I}_p(H)$.

We may denote by $\mathcal{I}_\infty(H)$ the space of compact operators, which is also an ideal in the space of bounded operators (TS and ST are compact if one is compact and the other is bounded).

The space $\mathcal{I}_2(H)$ is the space of Hilbert-Schmidt operators. The space $\mathcal{I}_1(H)$ is the space of trace-class operators. This is consistent with earlier definitions. Moreover, if $T \in \mathcal{I}_p$ and $S \in \mathcal{I}_q$ for $\frac{1}{p} + \frac{1}{q} = 1$ and $1 \leq p \leq \infty$, then TS and ST are in $\mathcal{I}_1$ with Hölder inequality

$$\|TS\|_1 \leq \|T\|_p \|S\|_q.$$

Criteria for trace-class and other Schatten-class operators. For $\mathcal{H}$ a Hilbert space, we defined $\mathcal{I}_p(\mathcal{H})$ the Schatten class of compact operators with singular values in $l^p(\mathbb{N})$. We consider here only $\mathcal{H} = L^2(\mathbb{R}^d)$.

It is useful to have criteria for when an operator is in $\mathcal{I}_p$. For Hilbert-Schmidt operators from $X \subset \mathbb{R}^d$ to $Y \subset \mathbb{R}^d$, we have the criterion that A with Schwartz kernel $a(x,y)$ is a Hilbert-Schmidt operator if and only if

$$\|A\|_2^2 = \int_{X\times Y} |a|^2(x,y)dxdy < \infty.$$

For $p > 2$, we have the following criterion of [38] modified in[25] to remove an unnecesary L^2 constraint:

Lemma A.3 (Russo's criterion) *Assume T bounded on $L^2(\mathbb{R}^d)$ for $d \geq 1$ with integral kernel $t(x,y)$ while $t^*(x,y)$ is the kernel of the adjoint operator T^*. Let $p > 2$ with conjugate $q = \frac{p}{p-1}$ ($\frac{1}{p} + \frac{1}{q} = 1$.) Assume $\|t\|_{q,p}$ and $\|t^*\|_{q,p}$ are bounded, where*

$$\|t\|_{q,p} = \Big(\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} |t(x,y)|^q dx\Big)^{\frac{p}{q}}\Big)^{\frac{1}{q}}.$$

Then T is a compact operator in the Schatten class $\mathcal{I}_p$ of operators such that the singular values belong to l^p. Moreover, $\|T\|_p \leq \|t\|_{q,p}^{\frac{1}{2}} \|t^*\|_{q,p}^{\frac{1}{2}}$. For $n \geq p$ an integer, we have that T^n is trace-class with $\|T^n\|_1 \leq \|t\|_{q,p}^{\frac{1}{2}} \|t^*\|_{q,p}^{\frac{1}{2}}$ as well.

Moreover, if T^n has integral kernel $k(x,y)$, then the trace of T^n is obtained as the following integral along the diagonal

$$\operatorname{Tr} T^n = \int_{\mathbb{R}^d} k(x,x)dx.$$

Trace-class operators. For $1 \leq p < 2$, the above criterion does not apply. We present a number of estimates allowing us to show that operators are trace-class and obtain estimates on the corresponding traces.

A general strategy to obtain trace-class properties is as follows:

(i) For rank-one T with kernel $\phi(x)\psi(y)$, we have $\|T\|_1 = \|\phi\|\|\psi\|$;
(ii) For $\mu(\xi)$ a non-negative measure on Ξ, we have that for $t(x,y) = \int_\Xi t(x,y;\xi)d\mu(\xi)$ that $\|T\|_1 \leq \int_\Xi \|T(\xi)\|_1 d\mu(\xi)$ since $\mathcal{I}_1$ is a Banach space. Here, $t(x,y;\xi)$ is the Schwartz kernel of the operator $T(\xi)$.

For instance, we may decompose $\mathbb{R}^{2d}$ into cubes by applying a partition of unity to get operators with Schwartz kernels $\phi_m(x)t(x,y)\phi_n(y)$. With $\psi_m(x)$ smooth compactly supported and equal to 1 on the support of $\phi_m(x)$, then further decompose $\phi_m(x)t(x,y)\phi_n(x)$ into rank-one operators, for instance by using a Fourier transform to get the expression $\psi_m(x)\psi_n(y)e^{i\zeta x}e^{i\tau y}\alpha(\zeta,\tau)$, which is the kernel of a rank-one operator. Then $d\mu(\xi)$ above integrates in n, m, ζ, and τ. If the integral is controlled, then the initial operator T is trace-class.

The criterion from [18, Chapter 9] follows the above decomposition for operators with kernels that are sufficiently smooth. Consider an operator A with Schwartz kernel $a(x,y)$ for $x, y \in \mathbb{R}^d$. Then for $a(x,y)$ sufficiently smooth in the following sense, we have A trace-class and

$$\|A\|_1 \leq C \sum_{|\alpha| \leq 2d+1} \|\partial^\alpha_{x,y} a\|_{L^1(\mathbb{R}^{2d})}. \tag{A.1}$$

Moreover, we have

$$\operatorname{Tr} A = \int_{\mathbb{R}^d} a(x,x)dx. \tag{A.2}$$

An application of the above result to several operators that appear in these notes is the following. We use the notation $x = (x', x_d)$.

Lemma A.4 *Let A be an operator with Schwartz kernel $a(x,y) = b(x', y', x_d, y_d - x_d, y_d)$ with $b \in C^{2d+1}(\mathbb{R}^{2d+1})$, bounded, with the estimates for $|\alpha| \leq 2d+1$,*

$$A_\alpha := \sup_{(x_d,y_d)\in\mathbb{R}^2} \int_{\mathbb{R}^{2d-2}} \langle x_d - y_d\rangle^\gamma |\partial^\alpha b|(x', y', x_d, y_d - x_d, y_d)dx'dy' < \infty \tag{A.3}$$

for some $\gamma > 2$ and such that $b(x', y', x_d, y_d - x_d, y_d) = 0$ when $x_d \geq x_0$ and when $y_d \leq y_0$. Then A is trace-class with $\|A\|_1 \leq C \sup_{|\alpha|\leq 2d+1} A_\alpha$ and trace given by the integral of a along its diagonal.

When $d = 1$, the above estimates are $A_\alpha = \sup_{(x,y)\in\mathbb{R}^2} \langle x - y\rangle^\beta |\partial^\alpha b|(x, y - x, y) < \infty$. We recall that $\langle x\rangle = \sqrt{1 + |x|^2}$. The same result holds when instead of the above condition, we have $b(x', y', x_d, y_d - x_d, y_d) = 0$ when $x_d \leq x_0$ and when $y_d \geq y_0$.

Proof. The proof is based on the criterion (A.1). We observe that $\partial^\beta a$ involves a sum of terms of the form $\partial^\alpha b$ with $|\alpha| = |\beta|$. Now

$$\begin{aligned}
\|\partial^\alpha b\|_{L^1(\mathbb{R}^{2d}} &= \int_{\mathbb{R}^{2d}} \langle x_d - y_d\rangle^\gamma |\partial^\alpha b|(x', y', x_d, y_d - x_d, y_d)\langle x_d - y_d\rangle^{-\gamma} dxdy \\
&\leq \int_{x_d \leq x_0} \int_{y_d \geq y_0} A_\alpha \langle x_d - y_d\rangle^{-\gamma} dx_d dy_d \leq CA_\alpha
\end{aligned}$$

when $\gamma > 2$. We may then apply criterion (A.1) to conclude. ∎

For an operator $T \in \mathcal{I}_1(\mathbb{R}^d)$, a criterion allowing us to compute the trace as an integral is given in [14]. For instance, (A.2) holds when $a(x,y)$ is jointly continuous in (x,y). This requires that we already know that the operator is trace-class. When the operator is known to be positive definite, then criteria to show that it is trace class are also given in that paper.

Operators of the form $f(x)g(D)$**.** Let $x \in \mathbb{R}^d$ and $D = -i\nabla_x$. We consider operators of the form $A = f(x)g(D)$ with f and g bounded so that A is bounded on $L^2(\mathbb{R}^d)$. The kernel of A is $f(x)\check{g}(x-y)$ with g the Fourier transform of $\check{g}$. We follow [40, Chapter 4].

For $\alpha \in \mathbb{Z}^d$, let Δ_α be the unit cube centered at α and χ_α its indicatrix function. Define $l^q(L^p)$ as the set of functions for which

$$\|f\|_{p,q} = \Big(\sum_\alpha \|f\chi_\alpha\|_p^q\Big)^{\frac{1}{q}} < \infty$$

with $\|\cdot\|_p$ the $L^p(\mathbb{R}^d)$ norm here (not a Schatten-class norm). In other words, we assume that f is in $L^p(\Delta_\alpha)$ for each cube and the L^p norms are $q-$summable. Then we have the following results:

- Assume $f \in L^p(\mathbb{R}^d)$ and $g \in L^p(\mathbb{R}^d)$ with $2 \le p < \infty$. Then $A = f(x)g(D)$ is in $\mathcal{I}_p$ and

  $$\|A\|_p \le (2\pi)^{-\frac{d}{p}}\|f\|_p\|g\|_p.$$

  Here, we use $\|\cdot\|_p$ both for the operator Schatten norm and for the L^p norm of functions.

- A partial converse is this: If f and g are non-zero and $A \in \mathcal{I}_2$, then $f \in L^2(\mathbb{R}^d)$ and $g \in L^2(\mathbb{R}^d)$.

- Assume f and g are in $l^p(L^2)$ for $1 \le p \le 2$. Then $A \in \mathcal{I}_p$ and

  $$\|A\|_p \le C_p\|f\|_{2,p}\|g\|_{2,p}. \tag{A.4}$$

- A partial converse is: If f and g are both non-zero and $A \in \mathcal{I}_1$, then both f and g are in $l^1(L^2(\mathbb{R}^d))$.

- A criterion obtained from the previous one (e.g., Hilbert Schmidt) and the fact that the uniform limit of compact operators is compact is as follows.
  Let $L^\infty_\varepsilon(\mathbb{R}^d)$ be the space of $L^\infty(\mathbb{R}^d)$ functions such that for all $\varepsilon > 0$, there is some $R(\varepsilon) > 0$ such that $|f(x)| \le \varepsilon$ for all $|x| \ge R$ (bounded functions converging to 0 at ∞). Assume then that $f(x) \in L^\infty_\varepsilon$ and $g(\xi) \in L^\infty_\varepsilon$. Then $f(x)g(D)$ is compact.

- Let f and g belong to $L^2_\delta(\mathbb{R}^d)$ for $\delta > \frac{d}{2}$. Then $A \in \mathcal{I}_1$ with

  $$\|A\|_1 \le C\|f\|_{L^2_\delta}\|g\|_{L^2_\delta}. \tag{A.5}$$

  Here, $L^2_\delta = \{f; \langle x\rangle^\delta f \in L^2\}$ with as usual $\langle x\rangle = \sqrt{1+|x|^2}$. We have $L^2_\delta \subset l^1(L^2)$

These results hold for specific operators of the form $f(x)g(D)$ but are otherwise extremely precise.

B Fredholm operators

Taken mostly from [27, Chapter 19]. See also [32] for relatively straightforward proofs of several results.

Definition B.1 *A bounded linear operator T from B_1 to B_2 Banach spaces is Fredholm when dim Ker T is finite and TB_1 (is closed and) has finite codimension. Then*

$$\text{Index } T = \text{dim Ker } T - \text{dim Coker } T.$$

Let $\mathcal{F}(B_1, B_2)$ the space of Fredholm operators in $\mathcal{B}(B_1, B_2)$, the space of bounded operators, and $\mathcal{K}(B_1, B_2)$ the space of compact operators in $\mathcal{B}(B_1, B_2)$.

Recall that $\text{Coker}T = B_2/\text{Ran}T$ the quotient space, which is a vector space defined as the space of $[\phi]$ for $\phi \in B_2$ where $[\phi]$ is the space of $\psi \sim \phi$ in B_2 where the latter relation holds iif $\phi - \psi \in TB_1 = \text{Ran}T$. By assumption, $\text{Coker}T$ is finite dimensional and hence closed and this may be used to show that TB_1 is closed as well.

Theorem B.2 *The set of Fredholm operators $\mathcal{F}(B_1, B_2)$ is open, dim Ker T is upper semi-continuous (i.e., for S small enough, dim Ker $(T + S) \leq$ dim Ker T), and Index T is constant in each component.*

If $T_1 \in \mathcal{F}(B_1, B_2)$ and $T_2 \in \mathcal{F}(B_2, B_3)$ then $T_2T_1 \in \mathcal{F}(B_1, B_3)$ and we have the logarithmic law

$$\text{Index}T_2T_1 = \text{Index}T_1 + \text{Index}T_2$$

The latter can be obtained by continuity (in t) of the Fredholm operators

$$\begin{pmatrix} I_2 & 0 \\ 0 & T_2 \end{pmatrix} \begin{pmatrix} I_2 \cos t & I_2 \sin t \\ -I_2 \sin t & I_2 \cos t \end{pmatrix} \begin{pmatrix} T_1 & 0 \\ 0 & I_2 \end{pmatrix}.$$

Theorem B.3 *For $T \in \mathcal{F}(B_1, B_2)$ and $K \in \mathcal{K}(B_1, B_2)$ compact, then $T + K \in \mathcal{F}(B_1, B_2)$ and* Index $(T + K) =$ Index T.

If $T \in \mathcal{B}(B_1, B_2)$ and $S_j \in \mathcal{B}(B_2, B_1)$ with $TS_2 = I_2 + K_2$ and $S_1T = I_1 + K_1$ for K_j compact, then T and S_j are Fredholm and Index $T = -$Index S_j, $j = 1, 2$.

More generally, if $t \in I$ is compact and $S_tT_t - I$ and $T_tS_t - I$ are uniformly compact (i.e., they send the unit Ball in B_1 to the same relatively compact subset of B_2), then T_t and S_t are Fredholm, dim Ker T_t and dim Ker S_t are upper semi-continuous and Index $T_t = -$Index S_t *is locally constant and constant on connected components of I.*

Index of Fredholm operator as a trace. We now state some results recasting the index of Fredholm operators as a trace. This is often referred to as a Fedosov or Calderón-Fedosov formula.

Theorem B.4 (Index-Trace relation) *Let $T \in \mathcal{B}(H, H)$ and $S \in \mathcal{B}(H, H)$ with*

$$R_1 = I - ST, \qquad R_2 = I - TS.$$

Assume R_1^N and R_2^N trace-class for $N > 0$. Then T and S are Fredholm and

$$\text{Index } T = -\text{Index } S = \text{Tr } R_1^N - \text{Tr } R_2^N.$$

When $N = 1$, then

$$\text{Index } T = \text{Tr } R_1 - \text{Tr } R_2 = \text{Tr } (TS - ST) = \text{Tr } [T, S]$$

where the cyclicity of the trace does not apply as TS and ST are not trace-class. When $N > 1$, we construct $S' = S(I_2 + R_2 + \dots R_2^{N-1})$ and show that $TS' = I_2 - R_2^N$ while $S'T = I_1 - R_1^N$ using that $S(TS)^kT = (ST)^{k+1}$ and a few more tricks, so that the proof reduces to the case $N = 1$.

Remark B.5 *An operator T is Fredholm if and only if we can find S such that $ST = I + K_1$ and $TS = I + K_2$ with K_1 and K_2 compact operator. The above result shows that S (possibly replaced by S') may be chosen so that K_1 and K_2 are trace-class. In this case,* Index $T = -$Index $S =$ Tr$[T, S] =$ Tr$(R_1 - R_2)$.

Moreover, an operator T is Fredholm with index 0 if and only if we can find K such that $T+K$ is invertible.

So, in some sense, Fredholm operators with index 0 and operators invertible modulo a compact operator and Fredholm operators with arbitrary index have left and right inverses modulo different compact operators. This is Atkinson's criterion for Fredholm operators. We (relatively easily) verify that the compact operators may be replaced by finite rank operators. We sometimes see the terminology: T is Fredholm if and only if it admits a partial inverse.

We can in fact show (see [32]) that the partial inverse S may be chosen so that $TST = T$; in this case $I-ST$ and $I-TS$ are finite rank projectors. In the same vein of results, if T is Fredholm and $T = U|T|$ is its polar decomposition (with U a partial isometry), then we verify that $I-U^*U$ and $I-UU^*$ are projections onto kernels of T and T^* respectively and that $\mathrm{Index}\, T = \mathrm{Tr}(UU^*-U^*U)$. *In other words, the index is independent of $|T| = (T^*T)^{\frac12}$.*

Let P be an orthogonal projector and U a unitary operator defined on a Hilbert space. We will often use the above result with $T = PUP + (I-P)$ and $S = PU^*P + (I-P)$. Indeed, we find

$$TS = PUPU^*P + (I-P), \quad ST = PU^*PUP + (I-P),$$

so that

$$R_1 = P(I-U^*PU)P = -P[U^*,P]UP, \quad R_2 = P(I-UPU^*)P = -P[U,P]U^*P.$$

Therefore, as soon as $[U,P]$ is a compact operator, we find that T and S are Fredholm as recalled in the above remark.

Lemma B.6 *Let U and P be defined as above and assume $[U,P]$ and $[U^*,P]$ trace-class. Then, we have:*

$$\mathrm{Index} PUP_{|\mathrm{Ran} P} = -\mathrm{Index} PU^*P_{|\mathrm{Ran} P} = \mathrm{Tr}[U,P]U^*. \tag{B.1}$$

Proof. Indeed, we deduce from the Fedosov formula that

$$\mathrm{Index} PUP_{|\mathrm{Ran} P} = \mathrm{Index} PUP + I - P = \mathrm{Tr} R_1 - R_2.$$

For A and B bounded, we have the cyclicity $\mathrm{Tr} AB = \mathrm{Tr} BA$ when at least A or B is trace class and $\mathrm{Tr} A + B = \mathrm{Tr} A + \mathrm{Tr} B$ when both A and B are trace-class. Therefore, since $P^2 = P$,

$$\mathrm{Tr} R_1 - R_2 = \mathrm{Tr}(-[U^*,P]UP + UPU^*P - P) = \mathrm{Tr}(-[U^*,P]UP + UP[U^*,P] + UPU^* - P)$$

The first two terms cancel by cyclicity and by additivity, this is $\mathrm{Tr}(UPU^*-P) = \mathrm{Tr}[U,P]U^*$. $\square$

C Pseudo-differential calculus

Fourier transform. We use the convention for the $d-$dimensional Fourier transform

$$\hat f(\xi) = \mathcal{F}_{x\to\xi} f(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)dx, \quad f(x) = \mathcal{F}^{-1}_{\xi\to x} \hat f(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi} \hat f(\xi) d\xi. \tag{C.1}$$

Thus, the Fourier transform of $f * g(x) = \int_{\mathbb{R}^d} f(x-y)g(y)dy$ is $\hat f(\xi)\hat g(\xi)$ and the Parseval relation may be written as $(2\pi)^d \int_{\mathbb{R}^d} f(-x)g(x)dx = \int_{\mathbb{R}^d} \hat f(\xi)\hat g(\xi)d\xi$. The latter also hold for matrix-valued functions f and g component-wise.

In the Fourier domain, $D_j f$ becomes $\xi_j \hat f(\xi)$ while $xf(x)$ becomes $-D_j \hat f(\xi) = i\partial_{\xi_j} \hat f(\xi)$.

Pseudo-differential operators. Let us start with a partial differential operator with constant coefficients $P(D)$ with P a polynomial. We may then write in the Fourier domain

$$P(D)f(x) = (\mathcal{F}^{-1}P(\xi)\hat{f}(\xi))(x) = \int_{\mathbb{R}^{2d}} \frac{e^{ix\cdot\xi}}{(2\pi)^d} P(\xi)e^{-iy\cdot\xi}f(y)dyd\xi.$$

Now, let us assume that $P(x,D)$ is a polynomial in D with coefficients that depend on x, i.e., a partial differential operator with possibly non-constant coefficients. Then as above,

$$P(x,D)f(x) = (\mathcal{F}^{-1}P(x,\xi)\hat{f}(\xi))(x) = \int_{\mathbb{R}^{2d}} \frac{e^{ix\cdot\xi}}{(2\pi)^d} P(x,\xi)e^{-iy\cdot\xi}f(y)dyd\xi.$$

Here $P(x,\xi)$ is the symbol of the operator $P(x,D)$. We observe that the above expression is not symmetrical in (x,y). The operators are sums of terms of the form $a(x)P(D)$. But the differential operator may also be written as a sum of terms of the form $P(D)a(x)$, which would have a symbol $P(\xi,y)$. We define a symmetrized version called the *Weyl quantization* and defined by

$$(\mathrm{Op}^w a)f(x) = \int_{\mathbb{R}^{2d}} \frac{e^{i(x-y)\cdot\xi}}{(2\pi)^d} a(\frac{x+y}{2},\xi)f(y)d\xi dy. \tag{C.2}$$

Such integrals are defined as standard Lebesgue integrals when a and f decay sufficiently rapidly in their respective variables, which is not the case for differential operator. For such well defined integrals, we may formally perform an arbitrary number of integrations by parts and *define* the resulting integrals as *oscillatory integrals* even when they no longer make sense as Lebesgue integrals. See [26] for the details.

Now, as important as differential operators are, their inverses (when these exist) are not differential operators. We would like to define classes of operators that include such inverses (whenever defined). When $P(D)$ is a differential operator such as $1-\Delta$ on Euclidean space with symbol $1+|\xi|^2$, the inverse is easily seen to be $(1+|\xi|^2)^{-1}$ in the Fourier domain. So we observe that if we we accept to have symbols $a(x,\xi)$ that are no longer polynomial in ξ, then (C.2) may be seen as a *definition* of a *quantization*, which in our context is a map from a space of symbols (classical observables) to a space of operators (which when restricted to bounded domains with appropriate boundary conditions admit discrete, quantized spectra unlike the continuously defined symbols; whence the name quantization). Operators constructed by (C.2) are called *pseudo-differential* operators (PDOs)

There is a large literature on algebras of symbols giving rise to algebras of operators and their applications to partial differential equations. In our context of analysis of Topological Insulators using PDE models, these classes of PDOs are useful as they allow us to define composition (multiplication) of operators, construct approximate inverses for *elliptic* PDOs, obtain criteria for the functional calculus, and finally obtain criteria of compactness, and in particular being trace-class, which are relatively straightforward to manipulate.

We introduce a number of classes of symbols and their associated PDOs that allow us to analyze the asymmetric transport associated to topological insulators. We primarily follow [18, 26].

Classes of symbols. Note that any operator A defined on $\mathcal{S}(\mathbb{R}^d)$ the space of Schwartz function and with range in $\mathcal{S}'(\mathbb{R}^d)$ the space of Schwartz distributions may be written in the form (C.2). Indeed such an operator admits a Schwartz kernel $K_A(x,y)$ as a distribution in $\mathcal{S}'(\mathbb{R}^d\times\mathbb{R}^d)$. We assume A and K_A scalar-valued to simplify notation but the same results hold for matrix-valued operators with K_A in $\mathcal{S}'(\mathbb{R}^d\times\mathbb{R}^d)\otimes\mathbb{M}_n(\mathbb{C})$. The Weyl symbol of A may then be defined as

$$a(x,\xi) = \int_{\mathbb{R}^d} e^{-iy\cdot\xi}K_A(x+\frac{y}{2},x-\frac{y}{2})dy \ \in \mathcal{S}'(\mathbb{R}^d\times\mathbb{R}^d). \tag{C.3}$$

With such a choice, we verify that $A = \mathrm{Op}^w a$.

However, arbitrary distributions $a(x,\xi)$ generate too large a class of symbols for, e.g., the composition of operators to be possible. The choice of class of symbols is problem-dependent even if some classes appear quite frequently. We define a few classes that will be useful for our purposes.

Class of symbols S^m. We first start with functions such that $a(x,\xi) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d) \otimes \mathbb{M}_n(\mathbb{C})$. We say that $a \in S^m$ when for each multi-index $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ and $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{N}^d$, we have the following bound on the semi-norm

$$|\partial_\xi^\alpha \partial_x^\beta a|(x,\xi) \leq C_{\alpha,\beta} \langle \xi \rangle^{m-|\alpha|} \tag{C.4}$$

for some constant $C_{\alpha,\beta}$. Here $|\alpha| = \alpha_1 + \ldots + \alpha_d$, and $|\cdot|$ is a norm on $\mathbb{M}_n(\mathbb{C})$.

We observe that if $a(x,\xi)$ is the symbol of a partial differential operator of order m with smooth coefficients, then indeed $a \in S^m$. The class of operators associated to symbols in S^m by (C.2) is called $\mathrm{Op}^w S^m$. Again, any partial differential operator of order m with smooth coefficients is in $\mathrm{Op}^w S^m$. In particular, the Dirac operator with smooth coefficients is in S^1 while the Laplace operator (and the modified Dirac operator involving the term $\eta\Delta$) is in S^2.

We define by S^∞ the union of all S^m over $m \in \mathbb{N}$ and by $S^{-\infty}$ the intersection of all S^m over $m \in \mathbb{N}$. We observe that S^∞ is a graded algebra in the sense that if $a \in S^m$ and $b \in S^n$, then ab and ba are in S^{m+n}. Similarly, we have an algebra of operators $\mathrm{Op}^w S^\infty$. The class of symbols S^m may be assigned a Fréchet topology by taking the best constants $C_{\alpha,\beta}$ above. We refer the reader to [18, 26] for additional information on the topology of these classes of symbols.

Order functions and associated symbols. We will also need the notion of *order function* $\mathfrak{m} : \mathbb{R}^{2d} \to [0,\infty)$ such that there exists constants C_0 and N_0 such that for all $X = (x,\xi) \in \mathbb{R}^{2d}$ and all $Y \in \mathbb{R}^{2d}$, we have

$$\mathfrak{m}(X) \leq C_0 \langle X - Y \rangle^{N_0} \mathfrak{m}(Y).$$

Here $\langle X \rangle = \sqrt{1 + |X|^2}$. Examples of order functions include $\langle X \rangle^p$ for $p \in \mathbb{R}$, $X_+ = \max(X, 0)$ (defined element-wise on each coordinate), or $X_- = -(-X)_+$. When $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are order functions, then so is $\mathfrak{m}_1\mathfrak{m}_2$.

Associated to an order function is a class of symbols $S(\mathfrak{m})$ defined as all $a(x,\xi) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d) \otimes \mathbb{M}_n(\mathbb{C})$ such that for all $\alpha \in \mathbb{N}^{2d}$,

$$|\partial^\alpha a(X)| \leq C_\alpha \mathfrak{m}(X) \quad \forall X \in \mathbb{R}^{2d}. \tag{C.5}$$

Again, choosing the best constants C_α provides a Fréchet topology for $S(\mathfrak{m})$. We use $S^{-\infty}(\mathfrak{m}) = \cap_{m \in \mathbb{Z}} S((\mathfrak{m})^m)$ as a convenient notation.

The operators constructed by (C.2) from $a \in S(\mathfrak{m})$ are denoted by $\mathrm{Op}^w S(\mathfrak{m})$. Such operators, as well as those in $\mathrm{Op}^w S^m$, are called *pseudo-differential operators* (PDO). These families include partial differential operators but are much larger and have better composition and invertibility properties as we will see.

Composition calculus. One major advantage of defining such operators is the following composition property. Let $a \in S(\mathfrak{m}_1)$ and $b \in S(\mathfrak{m}_2)$. Then the composed operator $\mathrm{Op}^w a \mathrm{Op}^w b \in \mathrm{Op}^w S(\mathfrak{m}_1\mathfrak{m}_2)$ and there is $c \in S(\mathfrak{m}_1\mathfrak{m}_2)$ such that $\mathrm{Op}^w a \mathrm{Op}^w b = \mathrm{Op}^w c$. In light of (C.3), we can always define c such that (C.2) holds for $\mathrm{Op}^w a \mathrm{Op}^w b$. The interesting result is that we also obtain that $c \in S(\mathfrak{m}_1\mathfrak{m}_2)$. There is also an explicit formula for c; see [18].

A similar result holds for $a \in S^m$ and $b \in S^n$ with $c \in S^{m+n}$ then and is a generic property satisfied by all interesting classes of PDOs.

In both cases of calculus above, an important stability property for us is the following. Let n count the multi-indices (α, β) (living in a countable space) appearing in the above descriptions of the seminorms and let $C_n(a)$ and $C_n(b)$ be the corresponding constants for the symbols a and b. Let $\mathtt{C}_N(a) = \max_{n\leq N} C_n(a)$ and the same expression for $\mathtt{C}_N(b)$. Then for each N, there exists $N_a = N_a(N)$ and $N_b = N_b(N)$ and C_N independent of (a, b, c) such that

$$\mathtt{C}_N(c) \leq C_N \mathtt{C}_{N_a}(a)\mathtt{C}_{N_b}(b). \tag{C.6}$$

In other words, the mapping from (a, b) to c is continuous in the Fréchet topologies. This control of the seminorms also applies when $a \in S^m$ and $b \in S^n$ with control of $c \in S^{n+m}$ or when $a \in S(\mathfrak{m}_1)$ and $b \in S(\mathfrak{m}_2)$, in which case we have control of the seminorms of $c \in S(\mathfrak{m}_1\mathfrak{m}_2)$.

Helffer-Sjöstrand formula. A very useful framework to compute and estimate functionals of a given operator is to recast it in terms of resolvent operators. This is the objective of the following Helffer-Sjöstrand formula; see [18, Chapter 8] for details and background.

Let $f \in C_0^\infty(\mathbb{R})$ be a smooth function vanishing at infinity. Then we have the existence of a smooth *almost analytic extension* $\tilde{f}(z)$ for $z \in \mathbb{C}$ such that

$$\tilde{f}(\lambda) = f(\lambda), \quad \lambda \in \mathbb{R}, \qquad |\bar{\partial}\tilde{f}(z)| \leq C_N |\Im z|^N, \quad \forall N \in \mathbb{N},$$

where for $z = \lambda + i\mu$

$$\bar{\partial} = \frac{1}{2}(\partial_\lambda + i\partial_\mu), \qquad \partial = \frac{1}{2}(\partial_\lambda - i\partial_\mu)$$

is the Cauchy-Riemann operator. The extension may be chosen with compact support (arbitrarily close to the real axis) in $\mathbb{C}$ and smooth. The extension is not unique. Since we will not need its explicit form beyond the aforementioned properties, we refer to [18, Chapter 8] and [16] for additional details. The advantage of introducing this extension is that it allows us to describe the functional calculus in terms of resolvent operators. More precisely, we have for H an unbounded self-adjoint operator the *Helffer-Sjöstrand formula*

$$f(H) = \frac{-1}{\pi} \int_{\mathbb{C}} \bar{\partial}\tilde{f}(z)(z - H)^{-1} d^2 z \tag{C.7}$$

where $d^2 z = d\lambda d\mu$ is Lebesgue measure on $\mathbb{C}$.

Note that for a $C_c^1(\mathbb{R}^2; \mathbb{C})$ function, we have Pompieu's formula

$$f(z) = \int_{\mathbb{C}} \frac{-1}{\pi(\zeta - z)} \bar{\partial} f(\zeta) d^2\zeta,$$

or in other words $\frac{1}{\pi z}$ is the Green's function of the Cauchy-Riemann operator $\bar{\partial}$. The almost analytic extensions of a real-valued functions f allow us to write the similar-looking formula (C.7) with an extension that remains close to $f(\lambda)$ for $|\Im z|$ small (with $\bar{\partial}\tilde{f}$ arbitrarily small near the real axis) while vanishing for $|\Im z|$ large.

Ellipticity, Self-adjointness, Resolvents and Functional Calculus. We now cite results from [12] and [18] to define an appropriate functional calculus. We are asking the following question. Assume $a \in S(1)$ or $a \in S^m$ and $f \in C_0^\infty(\mathbb{R})$. Then is $f(\mathrm{Op}^w a)$ a PDO and if so in which class? We saw above the Helffer-Sjöstrand formula (C.7) showing that $f(\mathrm{Op}^w a)$ may be written in terms of the resolvent operator $(z - \mathrm{Op}^w a)^{-1}$. So, a first question that needs answer is what we can say of the latter operator.

It turns out that $(z - \mathrm{Op}^w a)^{-1}$ is indeed a PDO. This comes as an application of a Beal's criterion, which relates stability properties of a to stability properties of $\mathrm{Op}^w \partial_X^\alpha a$. In particular, if the latter is a bounded operator on $L^2(\mathbb{R}^d)$ for $|\alpha| \leq 2d+1$, then $\|a\|_\infty$ is bounded [18, Proposition 8.2].

To apply such a result to the resolvent, we need good invertibility properties. For $a \in S^m$ with $m > 0$, we assume that a is Hermitian-valued and *elliptic*. The latter means that if $a_{\min}(x, \xi)$ denotes the smallest singular value of $a(x, \xi)$, then we assume that

$$|a_{\min}(x, \xi)| \geq C\langle\xi\rangle^m - 1 \tag{C.8}$$

for some constant $C > 0$. In other words, a is invertible as soon as $|\xi|$ is sufficiently large and all its singular values are of order $\langle\xi\rangle^m$.

Then [12] shows that $\mathrm{Op}^w a$ is a *self-adjoint* operator with domain $H^m(\mathbb{R}^d)$, the Sobolev space of order m, and that moreover, the *resolvent* $R_z(a) = (z - \mathrm{Op}^w a)^{-1}$ defined for $z \in \mathbb{C}$ with $\Im z \neq 0$ is itself a PDO with symbol in S^{-m}, i.e., $R_z(a) = \mathrm{Op}^w r_z$ with $r_z \in S^{-m}$. For later use, we also need bounds that are uniform in z. Let N again count multi-indices and $\mathtt{C}_N(r_z)$ be the constant of the Nth seminorm associated to r_z. Then there exist M_N and C_N such that

$$\mathtt{C}_N(r_z) \leq C_N |\Im z|^{-M_N}, \tag{C.9}$$

uniformly in z on a compact domain $Z \subset \mathbb{C}$ (with the constant C_N possibly depending on Z). The choice of M_N may be made a bit more explicit. But such a bound is sufficient for the moment. We may then use the Helffer-Sjöstrand formula (C.7) for $f \in C_0^\infty(\mathbb{R})$ to show that $f(\mathrm{Op}^w a) \in \mathrm{Op}^w S^{-\infty}$; see [12]. The proof requires that the almost analytic extension satisfy $|\bar\partial \tilde f| \leq C_N |\Im z|^{-N}$ for each $N \geq 0$.

We recall that for $m \in \mathbb{R}$, $H^m(\mathbb{R}^d)$ is the space of functions such that the Fourier transform $\hat f$ (defined as a distribution when $m < 0$) is in an appropriate weighted L^2 space, namely $\langle\xi\rangle^m \hat f(\xi) \in L^2(\mathbb{R}^d)$. When $\mathrm{Op}^w a$ is matrix-valued, we mean that each operator-valued component of the matrix is an element in $H^m(\mathbb{R}^d)$.

These results are not easy to derive. And they are quite powerful. We know from the spectral theorem that $f(H)$ is a bounded operator. The above results shows that it is in fact an operator in $\mathrm{Op}^w S^{-\infty}$. The symbol of $f(H)$ is a complicated functional of that of H. But such a symbol exists and belongs to $S^{-\infty}$. These results are unfortunately not entirely standard; see [12, 18] and references cited there for background and details.

Trace-class criterion. Another important result for us is a trace-class criterion modeled after (A.1) and (A.2) [18, Chapter 9]. Assume that $\mathfrak{m} \in L^1(\mathbb{R}^{2d})$ and that $|\partial^\alpha a(x, \xi)| \leq C_\alpha \mathfrak{m}(x, \xi)$ for all $|\alpha| \leq 2d+1$. Then $\mathrm{Op}^w a$ is a trace-class operator and

$$\mathrm{Tr}\ \mathrm{Op}^w a = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} \mathrm{tr}\, a(x, \xi) dx d\xi, \qquad \|\mathrm{Op}^w a\|_1 \leq C \max_{|\alpha| \leq 2d+1} C_\alpha\ \|\mathfrak{m}\|_{L^1(\mathbb{R}^{2d})}. \tag{C.10}$$

In other words, all symbols in $S(\mathfrak{m})$ with $\mathfrak{m}$ integrable generate trace-class operators.

D Semiclassical calculus

Semi-classical calculus. In these lectures, we use a number of results from semiclassical calculus presented in detail in [18, 47]. We collect the ones we need here. Several results are presented for scalar symbols for concreteness. They all extend to the matrix-valued case.

Semiclassical operators are pseudodifferential operators with symbols of the form $a(x, h\xi; h)$, which depend on a semiclassical parameter $0 < h \leq 1$. We are primarily interested in understanding the behavior of such operators as $h \to 0$. To do so, it is convenient to introduce semi-classical pseudo-differential operators as follows. See [18, Chapter 7] for details.

Semi-classical Weyl quantization, symbol classes and calculus. Given a parameter $h \in (0,1]$ and a symbol $a(x,\xi;h) \in \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d) \otimes \mathbb{M}_n$, we define the semiclassical Weyl quantization of a as:

$$\mathrm{Op}_h^w(a)\psi(x) := \frac{1}{(2\pi h)^d} \int_{\mathbb{R}^{2d}} e^{i(x-y)\cdot\xi/h} a(\frac{x+y}{2}, \xi; h)\psi(y)dyd\xi, \qquad \psi \in \mathcal{S}(\mathbb{R}^d) \otimes \mathbb{C}^n. \tag{D.1}$$

We have $\mathrm{Op}^w a = \mathrm{Op}_1^w a$ for $h = 1$.

Let $\mathfrak{m}$ be an order function and k and δ real numbers. We say that $a \in S_\delta^k(\mathfrak{m})$ if for every $\alpha \in \mathbb{N}^{2d}$, there exists $C_\alpha > 0$ such that

$$|\partial^\alpha a(X;h)| \leq C_\alpha \mathfrak{m}(X) h^{-k-\delta|\alpha|} \tag{D.2}$$

uniformly for all $X \in \mathbb{R}^{2d}$ and $h \in (0,1]$. We write $S(\mathfrak{m})$ for $S_0^0(\mathfrak{m})$ and $S(\mathfrak{m}^{-\infty})$ to denote the intersection over $s \in \mathbb{N}$ of $S(\mathfrak{m}^{-s})$.

By [18, Chapter 7], we know that if $a \in S(\mathfrak{m}_1)$ and $b \in S(\mathfrak{m}_2)$, then $\mathrm{Op}_h^w(c) := \mathrm{Op}_h^w(a)\mathrm{Op}_h^w(b)$ is a pseudo-differential operator, with

$$c(x,\xi;h) = (a\sharp_h b)(x,\xi;h) := \Big(e^{i\frac{h}{2}(\partial_x\cdot\partial_\zeta - \partial_y\cdot\partial_\xi)} a(x,\xi;h)b(y,\zeta;h)\Big)\Big|_{y=x,\zeta=\xi} \tag{D.3}$$

and $c \in S(\mathfrak{m}_1\mathfrak{m}_2)$. We write $A \in \mathrm{Op}_h^w(S(\mathfrak{m}))$ to mean that $A = \mathrm{Op}_h^w(a)$ for some $a \in S(\mathfrak{m})$.

Resolvents and functional calculus. See [12] and [18, Chapter 8]. By [12], we know that if $a \in S^m$ is Hermitian-valued and satisfies $|a_{\min}(x,\xi)| \geq c\langle \xi, \zeta\rangle^m - 1$ for some $c > 0$, then for all $\Im(z) \neq 0$, $R_z := (z - \mathrm{Op}^w(a))^{-1}$ is a bijection of $L^2(\mathbb{R}^d) \otimes \mathbb{C}^n$ onto $\mathcal{H}^m$, and $R_z = \mathrm{Op}^w(r_z)$ with $r_z \in S^{-m}$. This result applies for $h = 1$ and by simple rescaling uniformly in $h_0 \leq h \leq 1$ for any fixed $h_0 > 0$.

We want a more detailed expression for r_z in the semiclassical regime as $h \to 0$, i.e. for $r_z(h)$ defined as $(z - \mathrm{Op}_h^w(a))^{-1} = \mathrm{Op}_h^w(r_z)$. Defining $z =: \lambda + i\omega$, we apply [18, Proposition 8.4] to obtain that in the semiclassical regime, the kernel of the resolvent operator satisfies

$$|\partial_x^\alpha \partial_\xi^\beta r_z(x,\xi;h)| \leq C_{Z_0,\alpha,\beta}\Big(1 + \frac{\sqrt{h}}{|\omega|}\Big)^{2d+1} |\omega|^{-(|\alpha|+|\beta|)-1}, \qquad z \in Z_0 \tag{D.4}$$

for any compact set $Z_0 \subset \mathbb{C}$. This result applies for $0 < h \leq h_0$ sufficiently small.

Let $0 \leq \delta \leq \frac{1}{2}$. Then we observe that $r_z \in S_\delta^\delta(1)$ uniformly in z on a compact domain such that $|\omega| = |\Im z| \geq Ch^\delta$.

Semiclassical functional calculus. We now write an asymptotic expansion for functions of operators in the semiclassical regime. We start with the following result for exponentials of quadratic functions.

Proposition D.1 *Suppose $u \in S_\delta^k(\mathfrak{m})$ for some $0 \leq \delta < \frac{1}{2}$ and $k \geq 0$. Let $A(\xi) = \frac{1}{2}\langle Q\xi, \xi\rangle$ be a non-degenerate real quadratic form on $\mathbb{R}^d$. Then for $\frac{d}{2} < s \in \mathbb{N}$ and all $N \in \mathbb{N}$, we have*

$$\Big|e^{ihA(D)}u(x) - \sum_{j\leq N-1} \frac{(ihA(D))^j}{j!}u(x)\Big| \leq Ch^{N-k-\delta(2N+s)}\mathfrak{m}(x).$$

This is an adaptation of [47, Chapters 3 &4].

Let $\sigma \in S(\mathfrak{m})$ and $H_h := \mathrm{Op}_h^w(\sigma)$. For $\lambda + i\omega = z \in \mathbb{C} \setminus \{\omega = 0\}$, define as above the symbol $r_z = r_z(x,\xi;h)$ such that $(z - H_h)^{-1} := \mathrm{Op}_h^w(r_z)$. We have seen above that $r_z \in S_\delta^\delta(1)$ for h sufficiently small uniformly in $z \in Z_\delta$, where Z_δ is the intersection of a compact domain with $\{z; |\omega| \geq Ch^\delta\}$. Finally, we define $\sigma_z := z - \sigma$ so that $r_z \sharp_h \sigma_z = \sigma_z \sharp_h r_z = 1$. This equation allows us to obtain an asymptotic expansion for r_z in powers of h.

Since $(e^{i\frac{h}{2}(D_\xi\cdot D_y - D_x \cdot D_\eta)}\sigma_z(x,\xi) r_z(y,\eta))|_{y=x,\eta=\xi} = \sigma_z \sharp_h r_z = 1$ and D commutes with $e^{ihA(D)}$, the above proposition directly yields:

Proposition D.2 *Suppose $\sigma_z \in S(\mathfrak{m})$ and $r_z \in S_\delta^\delta(1)$ for $0 \leq \delta < \frac{1}{2}$. For $N \in \mathbb{N}$, define*

$$b_{z,h,N}(x,\xi) := 1 - \sum_{j \leq N-1} \Big(\frac{(ih(D_\xi \cdot D_y - D_x \cdot D_\eta)/2)^j}{j!} \sigma_z(x,\xi) r_z(y,\eta) \Big)|_{y=x,\eta=\xi}.$$

Then $b_{z,h,N} \in S_\delta^{-\ell_N}(\mathfrak{m})$ uniformly in $z \in Z_\delta$ with $\ell_N := N - 2\delta(N+d+1)$. In particular, $|b_{z,h,N}(x,\xi)| \leq C_N h^{N-2\delta(N+d+1)}\mathfrak{m}(x,\xi)$ uniformly in $(x,\xi) \in \mathbb{R}^{2n}$ and $z \in Z_\delta$.

We now obtain an approximation of r_z to arbitrary order in h.

Proposition D.3 *For $z \in Z_\delta$ and $h \in (0,1]$, define $q_{z,h,N}$ recursively by $q_{z,h,1} := \sigma_z^{-1}$ and*

$$q_{z,h,N} = \sigma_z^{-1}\Big(1 - \sum_{j=1}^{N-1} \Big(\frac{(ih(D_\xi \cdot D_y - D_x \cdot D_\eta)/2)^j}{j!} \sigma_z(x,\xi) q_{z,h,N-j}(y,\eta) \Big)|_{y=x,\eta=\xi}\Big), \quad N \geq 2.$$

Then under the hypotheses of Proposition D.2, we have $r_z - q_{z,h,N} \in S_\delta^{-\ell_N+\delta}(\mathfrak{m})$.

The result is obtained by induction. We finally obtain the following asymptotic result.

Proposition D.4 *Let $\phi \in C_c^\infty(\mathbb{R})$ with $\tilde{\phi}$ an almost analytic extension and $h \in (0,1]$. Let $\sigma \in S(\mathfrak{m})$ for $\mathfrak{m}$ an order function and $H_h = \mathrm{Op}^w(\sigma)$. Let $\nu = \nu(x,\xi;h) \in S(\mathfrak{m}^{-\infty})$ be the symbol such that $\phi(H_h) = \mathrm{Op}^w \nu$. Let $q_{z,h,N}$ be defined as above for $z \in Z = \cup_{\delta>0} Z_\delta$. Assume that for $\mathfrak{m}(x,\xi)$ sufficiently large, then the eigenvalues of σ are all outside of the support of ϕ. Then for all $N \in \mathbb{N}$, we have*

$$\nu + \frac{1}{\pi}\int_Z \bar\partial\tilde{\phi}(z) q_{z,h,N} \mathrm{d}^2 z \in S^{-N+\frac{1}{2}}(\mathfrak{m}^{-\infty}). \tag{D.5}$$

Proof. We know that for an order function $\mathfrak{m}$, then $\nu \in S(\mathfrak{m}^{-\infty})$ is given by the Helffer-Sjöstrand formula

$$\nu = -\frac{1}{\pi}\int_Z \bar\partial\tilde{\phi}(z) r_z d^2 z = -\frac{1}{\pi}\int_{Z_\delta} \bar\partial\tilde{\phi}(z) r_z d^2 z - \frac{1}{\pi}\int_{Z\setminus Z_\delta} \bar\partial\tilde{\phi}(z) r_z d^2 z.$$

We choose $0 < \delta < \frac{1}{2}$. Then using the bound (D.4) and the property of almost analytic extensions $|\bar\partial\tilde{\phi}(z)| \leq C_M |\Im z|^M$, we obtain that the integral over $Z\setminus Z_\delta$ as well as any derivative in (x,ξ) is of order h^p for any $p \geq 0$.

We observe that $z \mapsto q_{z,h,N}$ is analytic on Z_δ and is in fact analytic on $\bar{Z}$ when $\mathfrak{m}(x,\xi)$ is sufficiently large since then by assumption σ_z^{-1} (with $\sigma_z = z - \sigma$) is analytic. This shows that $\int_Z \bar\partial\tilde{\phi}(z) q_{z,h,N}(x,\xi)\mathrm{d}^2 z = 0$ for $\mathfrak{m}(x,\xi)$ sufficiently large. Using (D.4) and $|\bar\partial\tilde{\phi}(z)| \leq C_M|\Im z|^M$ again, we also observe that $\int_{Z\setminus Z_\delta} \bar\partial\tilde{\phi}(z) q_{z,h,N}\mathrm{d}^2 z$ is of order h^p for any $p \geq 0$.

It thus remains to address the integral of $\bar\partial\tilde{\phi}(z)(r_z - q_{z,h,n})$ over Z_δ where $r_z \in S_\delta^\delta(1)$ now, still using (D.4). This is arbitrarily small by Proposition D.3. ∎

Trace-class operators. See [18, Chapter 9] for a trace-class criterion in the semi-classical regime. Suppose $\mathfrak{m} \in L^1(\mathbb{R}^{2d})$, and $|\partial^\alpha a(x,\xi;h)| \le C_\alpha \mathfrak{m}(x,\xi)$ for all $\alpha \in \mathbb{N}^{2d}$ and $h \in (0,1]$ (meaning that $a \in S(\mathfrak{m})$). Then $\mathrm{Op}_h^w(a)$ is trace-class with $\|\mathrm{Op}_h^w(a)\|_1 \le C \max_{|\alpha|\le 2d+1} C_\alpha \|\mathfrak{m}\|_{L^1}$ and

$$\mathrm{Tr}\mathrm{Op}_h^w(a) = \frac{1}{(2\pi h)^d} \int_{\mathbb{R}^{2d}} \mathrm{tr}\, a(x,\xi;h) dx d\xi, \tag{D.6}$$

where C depends only on d and tr is the standard matrix trace [18, Theorem 9.4] .

E Unbounded operators and spectral theory.

The results below are mostly taken from [18, Chapter 4] as well as [41, 43].

We assume $\mathcal{H}$ is a complex separable Hilbert space and $S : \mathfrak{D}(S) \subset \mathcal{H} \to \mathcal{H}$ is an **unbounded** operator with domain $\mathfrak{D}(S)$. The **graph** of S is

$$\mathrm{graph}(S) = \{(x, Sx),\ x \in \mathfrak{D}(S)\} \subset \mathcal{H} \times \mathcal{H} \text{ with norm } \|(u,v)\| = \|u\|^2 + \|v\|^2.$$

S is **closed** if graph(S) is closed.

Proposition. Let S be closed unbounded with dense domain $\mathfrak{D}(S)$. Then there is an **adjoint** unbounded operator S^* given by

$$\begin{aligned} \mathfrak{D}(S^*) &= \{v \in \mathcal{H};\ \exists C(v) \ge 0 \ \text{ s.t. } \ |(Su,v)| \le C(v)\|u\|,\ u \in \mathfrak{D}(S)\} \\ (Su,v) &= (u, S^*v), \ \ \forall u \in \mathfrak{D}(S),\ v \in \mathfrak{D}(S^*). \end{aligned}$$

If $J : \mathcal{H} \times \mathcal{H}$ is given by $J(u,v) = (-v,u)$ then $J^* = -J$ and J is unitary. Moreover, graph$(S^*) = (J(\mathrm{graph}(S)))^\perp$ is closed, and then so is S^*. The closure of graph$(S) = (\mathrm{graph}(S))^{\perp\perp}$ so that if $\mathfrak{D}(S^*)$ is dense, then S^{**} is the closure of S.

If the closure of the graph of S is the graph of an operator, then we say S is *closable* and then $\bar{S}$ corresponding to that closed graph is the closure of S. It is the case when the domains of S and of S^* are dense. We have as for bounded operators

$$\mathrm{Ran}(S)^\perp = \mathrm{Ker}(S^*).$$

Definition: $A \subset B$ if $\mathfrak{D}(A) \subset \mathfrak{D}(B)$ and $Bu = Au$ on $\mathfrak{D}(A)$. If $\mathfrak{D}(A)$ is dense and $A \subset B$, then $B^* \subset A^*$.
Definition: A is symmetric if $A \subset A^*$ and self-adjoint if $A = A*$. A symmetric is called *essentially self-adjoint* if $\bar{A}$ is self-adjoint; then $\bar{A}$ is the only self-adjoint extension of A.

Theorem [Criterion of Self-adjointness]. Let A be symmetric. Then (1) A self-adjoint is equivalent to (2) A closed and $\mathrm{Ker}(A^* \pm i) = \{0\}$ for both sign is equivalent to (3) $\mathrm{Ran}(A \pm i) = \mathcal{H}$ for both signs. Also, (1) A essentially self-adjoint is equivalent to (2) $\mathrm{Ker}(A^* \pm i) = \{0\}$ for both sign is equivalent to (3) $\mathrm{Ran}(A \pm i)$ is dense in $\mathcal{H}$ for both signs.

Above $\pm i$ may be replaced by z_+ and z_- with $\pm \Im z_\pm > 0$.

Assume A densely defined. The resolvent set of A is $\rho(A) = \{z \in \mathbb{C}; (z - A) : \mathfrak{D}(A) \to \mathcal{H}$ bijective with bounded inverse$\}$. This is an open set. Define $R(z) = (z - A)^{-1}$. Then the spectrum of A is $\sigma(A) = \mathbb{C} \backslash \rho(A)$ a closed set. We have

$$R(z) - R(w) = (w - z)R(z)R(w).$$

For A self-adjoint, $\|R(z)\| \le |\Im z|^{-1}$.

Theorem [Friedrichs extension]. Let $q(u,u) \geq \|u\|^2$ a closed and densely defined quadratic form with domain $\mathfrak{D}(q)$. Then there is a unique self-adjoint operator Q with domain $\mathfrak{D}(Q) \subset \mathfrak{D}(q)$ such that $(Qu,u) = q(u,u)$ for $u \in \mathfrak{D}(Q)$.

Let S be symmetric semi-bounded from below, i.e., $(Su,u) \geq -M\|u\|^2$ for $u \in \mathfrak{D}(S)$. Then there is a unique self-adjoint extension A of S with $\mathfrak{D}(A) \subset \mathfrak{D}(q)$ with q quadratic form associated to $S + (M+1)I$. A is the Friedrichs extension of S.

From Simon Chapter 7.

Definition. Let A and B be densely defined on $\mathcal{H}$. We say that B is $A-$bounded (or relatively bounded w.r.t. A) if and only if: (1) $\mathfrak{D}(A) \subset \mathfrak{D}(B)$; (ii) For some $\alpha, \beta > 0$ and all $\phi \in \mathfrak{D}(A)$, we have $\|B\phi\| \leq \alpha\|A\phi\| + \beta\|\phi\|$. The infimum of such α is called the $A-$bound for B.

Theorem [Kato-Rellich]. Let A be an operator on $\mathcal{H}$ and B be $A-$ bounded with bound $\alpha < 1$. Then $A+B$ defined on $\mathfrak{D}(A)$ is closed iif A is closed. Let A be self-adjoint and B Hermitian. Then $A+B$ as an operator on $\mathfrak{D}(A)$ is self-adjoint.

Here is a criterion for the computation of the $A-bound$: Suppose A is self-adjoint and B relatively bounded w.r.t A. The A-bound for B is given by

$$\lim_{\lambda \to +\infty} \|B(\pm i\lambda - A)^{-1}\|.$$

If A is bounded below, we can replace $\pm i\lambda$ by $-\lambda$ above.

Spectral Theorems. The first one is a multiplication theorem:

Theorem [Multiplication spectral theorem]. Let A be SA on separable $\mathcal{H}$. Then there exists a measure space $(M, \mathcal{M}, \mu)$, where μ is a finite measure, a measurable function $a : M \to \mathbb{R}$ and a unitary operator $U : \mathcal{H} \to L^2(d\mu)$ such that
(1) A vector $\psi \in \mathcal{H}$ belongs to $\mathfrak{D}(A)$ iif $aU\psi \in L^2(d\mu)$
(2) If $\psi \in \mathfrak{D}(A)$, then $UA\psi = aU\psi$.
(3) We have $L^2(d\mu) = \oplus_j L^2(\mathbb{R}, d\mu_j)$ with μ_j finite Borel measure on $\mathbb{R}$ so that the above function $a(x) = x$.

We have

$$\sigma(A) = \mathrm{Ran}_{\mathrm{ess}}(a) = \{\lambda \in \mathbb{R}; \mu(a^{-1}[\lambda - \varepsilon, \lambda + \varepsilon])) > 0, \ \forall \varepsilon > 0\}.$$

The second theorem addresses *functional calculus*. If $h : \mathbb{R} \to \mathbb{C}$ is a bounded Borel function, and U and a are as above, we define $h(A) = U^{-1}(h \circ a)U$. We have

Theorem [Functional calculus spectral theorem]. Let A be SA on $\mathcal{H}$ separable. Let $\mathcal{B}_b(\mathbb{R})$ be the set of bounded Borel functions. Then there is a unique map $\mathcal{B}_b(\mathbb{R}) \ni h \to h(A) \in \mathcal{B}(\mathcal{H})$ such that
(1) $h(A)^* = \bar{h}(A)$, $h(A) + g(A) = (h+g)(A)$, $h(A)g(A) = (hg)(A)$.
(2) $\|h(A)\| \leq \sup_{\mathbb{R}} |h_{|\sigma(A)}$.
(3) If $\mathcal{B}_b h_n(x) \to x$ as $n \to \infty$ with $|h_n(x)| \leq |x|$, then $h_n(A) \to A$ strongly on $\mathfrak{D}(A)$.
(4) $h_n \to h$ pointwise and $|h_n(x)| \leq C$, then $h_n(A) \to h(A)$ strongly on $\mathcal{H}$.
(5) If $A\psi = \lambda\psi \in \mathfrak{D}(A)$, then $h(A)\psi = h(\lambda)\psi$.
(6) If $h \geq 0$, then $h(A) \geq 0$.

Spectral measures. For $\phi \in \mathcal{H}$, there is a unique finite Borel measure μ_ϕ (spectral measure) such that for each Borel function g, $(g(A)\phi, \phi) = \int g(\lambda)\mu_\phi(d\lambda)$. By polarization, we construct $\mu_{\phi,\psi}$. We have that $\mu_\phi = a_*(|\phi|^2\mu)$ push-forward by a with a and μ as in the above spectral theorem:

$$\int g(\lambda)(a_*(|\phi|^2\mu)d\lambda = \int g(a(m))|\phi(m)|^2\mu(dm) = (g(A)\phi, \phi) = \int g(\lambda)\mu_\phi(d\lambda).$$

where the first equality is the definition of the push-forward, the second equality is the unitarity of U, and the last equality the definition of μ_ϕ.

Measure decomposition. Every Borel measure on $\mathbb{R}$ has a unique decomposition

$$\mu = \mu_{pp} + \mu_{ac} + \mu_{sc}$$

mutually singular (carried on disjoint sets) where $\mu_{pp} = \sum a_j \delta_{x_j}$ pure point with countable sum; μ_{ac} is absolutely conditions with respect to the Lebesgue measure, i.e., there is a locally integrable function f (Radon-Nikodym derivative) such that $\mu_{ac} = f dx$; μ_{sc} is singular continuous carried by a set of measure zero such that nonetheless $\mu_{sc}(\{x\}) = 0$.

We then have the decomposition

$$\mathcal{H} = \mathcal{H}_{pp} \oplus \mathcal{H}_{ac} \oplus \mathcal{H}_{sc}$$

such that each closed subspace is invariant by $(A+i)^{-1}$ and such that for $\phi \in \mathcal{H}_{xx}$, then μ_ϕ is of the type xx. Every eigenvector of A belongs to $\mathcal{H}_{pp}$, which has an orthogonal basis of vectors.

$\sigma(A) = \overline{\sigma_{pp}(A)} \cup \sigma_{ac}(A) \cup \sigma_{sc}(A)$, the spectra of A restricted to the $\mathcal{H}_{xx}$.

The third form of the spectral theorem is given in terms of *spectral projectors* $P_\Omega = 1_\Omega(A)$ for Ω Borel subset of $\mathbb{R}$ with 1_Ω the characteristic function of Ω. Then P_Ω is a spectral family: (a) P_Ω is an orthogonal projection; (b) $P_\emptyset = 0$; (c) If $\Omega \supset \Omega_j \to \Omega$ when $j \to \infty$, then $P_{\Omega_j} \to P_\Omega$ strongly; (d) $P_\Omega P_X = P_{\Omega \cap X}$.

Define $P_\lambda = P_{(-\infty,\lambda]}$. Then for $\phi \in \mathcal{H}$, $\lambda \to (\phi, P_\lambda \phi)$ is bounded increasing and $d(\phi, P_\lambda\phi) = \mu_\phi$.

For g a complex-valued Borel function on $\mathbb{R}$, we define

$$\mathfrak{D}(g(A)) = \{\phi \in \mathcal{H},\ \int |g(\lambda)|^2 d(\phi, P_\lambda \phi) < \infty\}$$

dense in $\mathcal{H}$ and then $g(A)$ as

$$(g(A)\phi, \psi) = \int g(\lambda) d(P_\lambda \phi, \psi), \quad \forall \phi, \psi \in \mathfrak{D}(g(A)).$$

Formally, we write

$$g(A) = \int g(\lambda) dP_\lambda.$$

In the multiplication spectral theorem, $\phi \in \mathfrak{D}(g(A))$ iif $(g \circ a)\phi \in L^2(d\mu)$ so $g(A)$ is identified with the multiplication by $g \circ a$ in $L^2(d\mu)$. When $g(\lambda) = \lambda$, we get $g(A) = A$ so that

Theorem [Spectral Projectors]. We have

$$A = \int \lambda dP_\lambda, \qquad (\mu - A)^{-1} = \int (\mu - \lambda)^{-1} dP_\lambda.$$

Moreover, if f, g are Borel functions, $\phi \in \mathfrak{D}(g(A))$ and $g(A)\phi \in \mathfrak{D}(f(A))$ then $\phi \in \mathfrak{D}(fg(A))$ and $(fg)(A)\phi = f(A)g(A)\phi$.

Stone Formula. For $\varepsilon > 0$ and $-\infty < a < b < \infty$, we have

$$\frac{1}{2}(P_{[a,b]} + P_{(a,b)}) = \text{stronglimit}_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_a^b ((A - \lambda - i\varepsilon)^{-1} - (A - \lambda + i\varepsilon)^{-1}) d\lambda.$$

This comes from that for $[a,b] \ni \lambda \to g_\lambda \in \mathcal{C}_b(\mathbb{R})$, then by additivity

$$\int_a^b g_\lambda(A)d\lambda = \Big(\int_a^b g_\lambda d\lambda\Big)(A),$$

and then

$$\frac{1}{2\pi i}\int_a^b ((t-\lambda-i\varepsilon)^{-1} - (t-\lambda+i\varepsilon)^{-1})d\lambda = \frac{1}{2\pi}\int_a^b \frac{2\varepsilon d\lambda}{(t-\lambda)^2+\varepsilon^2}.$$

Essential Spectrum. A SA. Then $\lambda \in \sigma(A)$ iif $P_{(\lambda-\varepsilon,\lambda+\varepsilon)} \neq 0$ for all $\varepsilon > 0$. Then **Definition:** $\lambda \in \mathbb{R}$ belongs to the essential spectrum $\sigma_{\rm ess}(A)$ iif $P_{(\lambda-\varepsilon,\lambda+\varepsilon)} \neq 0$ is of infinite rank for each $\varepsilon > 0$. The **discrete spectrum** is $\sigma(A)\backslash\sigma_{\rm ess}(A)$, in other words the union of all eigenvalues of A of finite multiplicity isolated from the rest of the spectrum.

Weyl criterion. Let $\lambda \in \mathbb{C}$. Then
(1) $\lambda \in \sigma(A)$ iif there exists $\phi_n \in \mathfrak{D}(A)$, $\|\phi_n\| = 1$ for $n \geq 1$ such that $(A-\lambda)\phi_n \to 0$;
(2) $\lambda \in \sigma_{\rm ess}(A)$ iif there exists $\phi_n \in \mathfrak{D}(A)$, $\|\phi_n\| = 1$ for $n \geq 1$ *orthonormal* such that $(A-\lambda)\phi_n \to 0$;

Theorem [Weyl]. Let A, B self-adjoint such that $(A+i)^{-1} - (B+i)^{-1}$ is compact. Then $\sigma_{\rm ess}(A) = \sigma_{\rm ess}(B)$.

Definition A SA, C with $\mathfrak{D}(A) \subset \mathfrak{D}(C)$ is **relatively compact** w.r.t. A if $C(A+I)^{-1}$ is compact.

Theorem [Weyl]. Let A self-adjoint and C symmetric and relatively compact w.r.t A. Then $\mathfrak{D}(A+C) = \mathfrak{D}(A)$, $A+C$ is self-adjoint and $\sigma_{\rm ess}(A) = \sigma_{\rm ess}(B)$.

Example. Let $V \in L^2_{\rm loc}(\mathbb{R}^n, \mathbb{R})$ bounded below and $c = \underline{\lim}_{|x|\to\infty} V(x)$. Then $\sigma_{\rm ess}(-\Delta + V) \subset [c, \infty)$. When $c = \infty$, we can thus prove that $-\Delta + V$ has purely discrete spectrum with eigenvalues of finite multiplicity that tend to infinity.

Stability of spectra. We saw that compact perturbations do not modify the essential spectrum. Here are other relevant stability results

Theorem [Kato-Birman]. Let A, B, C bounded with $A = B + C$ and C trace-class. Then $\sigma_{\rm ac}(A) = \sigma_{\rm ac}(B)$. Moreover, A and B restricted to their ac spectrum are unitarily equivalent.
Let A and B unbounded self-adjoint bounded from below. Suppose $\mathbb{R} \ni \lambda \notin \sigma(A) \cup \sigma(B)$ and $(A+\lambda)^{-1} - (B+\lambda)^{-1}$ trace-class. Then $\sigma_{\rm ac}(A) = \sigma_{\rm ac}(B)$.
We say A is subordinate to B if there is a continuous $f(x) \to \infty$ as $|x| \to \infty$ and a function $g(x)$ so $f(A) \leq g(B)$. If A is subordinate to B and vice-versa and if $P_I(A)(A-B)P_I(B)$ is trace-class for all bounded intervals I (with P_I spectral projections), then $\sigma_{\rm ac}(A) = \sigma_{\rm ac}(B)$.

This should be contrasted with

Theorem [Weyl-von Neumann-Kuroda]. Let B be bounded self-adjoint. Then there exists C in every $\mathcal{I}_p$ fpr $p > 1$ so that $A = B + C$ has only point spectrum with an orthonormal basis of eigenvectors. For any $p > 1$ and any $\varepsilon > 0$, we can choose $\|C\|_p < \varepsilon$ so the perturbation to render any bounded operator A with pure point spectrum may be arbitrarily small in any norm by $\mathcal{I}_1$.

Theorem. [Spectral mapping theorem]. (a) Let A be unbounded self-adjoint and F bounded continuous function on $\sigma(A) \cup \{\infty\}$. Then

$$\sigma(F(A)) = F[\sigma(A) \cup \{\infty\}].$$

(b) Suppose there is $k \in \mathbb{N}$ such that for all $\lambda \in \mathbb{R}$, $\sharp[(F^{-1}[|\lambda]) \leq k$. Then

$$\sigma_{\rm ess}(F(A)) = F[\sigma_{\rm ess}(A) \cup \{\infty\}].$$

(c) Suppose F a C^1 function in a neighborhood of $\sigma(A) \sup\{\infty\}$ and suppose $\{x \in \sigma(A); F'(x) = 0\}$ is finite. Then

$$\sigma_{\mathrm{ac}}(F(A)) = F(\sigma_{\mathrm{ac}}(A)).$$

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