Random Correctors.
Lectures 1-3

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PDEs with random potential

We follow the presentation in [B-08]. Consider an equation of the form:

$$\begin{aligned} &P(\mathbf{x},\mathbf{D})u_\varepsilon + q_\varepsilon u_\varepsilon = f, && \mathbf{x}\in D \\ &u_\varepsilon = 0 && \mathbf{x}\in\partial D, \end{aligned} \tag{1}$$

where $P(\mathbf{x},\mathbf{D})$ is a (deterministic) self-adjoint, elliptic, pseudo-differential operator and D an open bounded domain in $\mathbb{R}^d$. We assume that $P(\mathbf{x},\mathbf{D})$ is invertible with symmetric and "more than square integrable" Green's function. More precisely, we assume that the equation

$$\begin{aligned} &P(\mathbf{x},\mathbf{D})u = f, && \mathbf{x}\in D \\ &u = 0 && \mathbf{x}\in\partial D, \end{aligned} \tag{2}$$

admits a unique solution

$$u(\mathbf{x}) = \mathcal{G}f(\mathbf{x}) := \int_D G(\mathbf{x},\mathbf{y})f(\mathbf{y})d\mathbf{y}, \tag{3}$$

and that the real-valued and non-negative (to simplify notation) symmetric kernel $G(\mathbf{x},\mathbf{y}) = G(\mathbf{y},\mathbf{x})$ has more than square integrable singularities so that

$$\mathbf{x} \mapsto \Big(\int_D |G|^{2+\eta}(\mathbf{x},\mathbf{y}) d\mathbf{y} \Big)^{\frac{1}{2+\eta}} \qquad \text{is bounded on } D \text{ for some } \eta > 0. \qquad (4)$$

The assumption is satisfied by operators of the form $P(\mathbf{x}, \mathbf{D}) = -\nabla \cdot a(\mathbf{x})\nabla + \sigma(\mathbf{x})$ for $a(\mathbf{x})$ uniformly bounded and coercive, $\sigma(\mathbf{x}) \geq 0$, and in dimension $d \leq 3$, with $\eta = +\infty$ when $d = 1$ (i.e., the Green's function is bounded), $\eta < \infty$ for $d = 2$, and $\eta < 1$ for $d = 3$.

The assumption is not satisfied for such operators in dimension $d \geq 4$, where deterministic and random correctors are in competition.

Assumptions on potential

Let $q_\varepsilon(\mathbf{x}, \omega) = q(\frac{\mathbf{x}}{\varepsilon}, \omega)$ be a mean zero, (strictly) stationary, process defined on an abstract probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that $q(\mathbf{x}, \omega)$ has an integrable correlation function:

$$R(\mathbf{x}) = \mathbb{E}\{q(\mathbf{y}, \omega)q(\mathbf{y} + \mathbf{x}, \omega)\}, \tag{5}$$

where $\mathbb{E}$ is mathematical expectation associated to $\mathbb{P}$. We assume to simplify that $q_\varepsilon(\mathbf{x}, \omega)$ is sufficiently small so that (1) is well defined. The above expression is independent of $\mathbf{y}$ by stationarity of the process $q(\mathbf{x}, \omega)$.

We also assume that $q(\mathbf{x}, \omega)$ is **strongly mixing** in the following sense. For two Borel sets $A, B \subset \mathbb{R}^d$, we denote by $\mathcal{F}_A$ and $\mathcal{F}_B$ the sub-σ algebras of $\mathcal{F}$ generated by the field $q(\mathbf{x}, \omega)$. Then we assume the existence of a

($\rho-$) mixing coefficient $\varphi(r)$ such that

$$\left|\frac{\mathbb{E}\Big\{(\eta-\mathbb{E}\{\eta\})(\xi-\mathbb{E}\{\xi\})\Big\}}{\Big(\mathbb{E}\{\eta^2\}\mathbb{E}\{\xi^2\}\Big)^{\frac{1}{2}}}\right| \leq \varphi\Big(2\,d(A,B)\Big) \tag{6}$$

for all (real-valued) random variables η on $(\Omega, \mathcal{F}_A, \mathbb{P})$ and ξ on $(\Omega, \mathcal{F}_B, \mathbb{P})$. Here, $d(A,B)$ is the Euclidean distance between the Borel sets A and B.

The multiplicative factor 2 in (6) is here only for convenience. Moreover, we assume that $\varphi(r)$ is bounded and decreasing.

Random integral

We formally recast (1) as

$$u_\varepsilon = \mathcal{G}(f - q_\varepsilon u_\varepsilon), \tag{7}$$

where $\mathcal{G} = P(\mathbf{x}, D)^{-1}$, and after one more iteration as

$$u_\varepsilon = \mathcal{G}f - \mathcal{G}q_\varepsilon\mathcal{G}f + \mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon u_\varepsilon. \tag{8}$$

This is the integral equation we aim to analyze:

$\mathcal{G}f$ is the unperturbed solution

$\mathcal{G}q_\varepsilon\mathcal{G}f$ is the random fluctuation

$\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon u_\varepsilon$ is a lower-order correction

Mixing Lemma

We choose q_ε small so that $(I - \mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon)$ is invertible $\mathbb{P}-$a.s. (this can be significantly relaxed). We then need a few lemmas.

Lemma 1 *Let* $q(\mathbf{x},\omega)$ *be strongly mixing so that* (6) *holds and such that* $\mathbb{E}\{q^6\} < \infty$. *Then, we have:*

$$\begin{aligned} &\Big|\mathbb{E}\{q(\mathbf{x}_1)q(\mathbf{x}_2)q(\mathbf{x}_3)q(\mathbf{x}_4)\}\Big| \\ \lesssim &\sup_{\{\mathbf{y}_k\}_{1\le k\le 4}=\{\mathbf{x}_k\}_{1\le k\le 4}} \varphi^{\frac{1}{2}}(|\mathbf{y}_1-\mathbf{y}_3|)\varphi^{\frac{1}{2}}(|\mathbf{y}_2-\mathbf{y}_4|)\mathbb{E}\{q^6\}^{\frac{2}{3}}. \end{aligned} \tag{9}$$

Here, we use the notation $a \lesssim b$ when there is a positive constant C such that $a \le Cb$.

proof of mixing lemma

Let $\mathbf{y}_1$ and $\mathbf{y}_2$ be two points in $\{\mathbf{x}_k\}_{1\le k\le 4}$ such that $d(\mathbf{y}_1,\mathbf{y}_2) \ge d(\mathbf{x}_i,\mathbf{x}_j)$ for all $1 \le i,j \le 4$ and such that $d(\mathbf{y}_1,\{\mathbf{z}_3,\mathbf{z}_4\}) \le d(\mathbf{y}_2,\{\mathbf{z}_3,\mathbf{z}_4\})$, where $\{\mathbf{y}_1,\mathbf{y}_2,\mathbf{z}_3,\mathbf{z}_4\} = \{\mathbf{x}_k\}_{1\le k\le 4}$.

Let us call $\mathbf{y}_3$ a point in $\{\mathbf{z}_3,\mathbf{z}_4\}$ closest to $\mathbf{y}_1$. We call $\mathbf{y}_4$ the remaining point in $\{\mathbf{x}_k\}_{1\le k\le 4}$. We have, using (6) and $\mathbb{E}\{q\} = 0$, that:

$$\left|\mathbb{E}\{q(\mathbf{x}_1)q(\mathbf{x}_2)q(\mathbf{x}_3)q(\mathbf{x}_4)\}\right| \lesssim \varphi(2|\mathbf{y}_1-\mathbf{y}_3|)(\mathbb{E}\{q^2\})^{\frac{1}{2}}\left(\mathbb{E}\{(q(\mathbf{y}_2)q(\mathbf{y}_3)q(\mathbf{y}_4))^2\}\right)^{\frac{1}{2}}.$$

The last two terms are bounded by $\mathbb{E}\{q^6\}^{\frac{1}{6}}$ and $\mathbb{E}\{q^6\}^{\frac{1}{2}}$, respectively, using Hölder's inequality. Because $\varphi(r)$ is assumed to be decreasing, we deduce that

$$\left|\mathbb{E}\{q(\mathbf{x}_1)q(\mathbf{x}_2)q(\mathbf{x}_3)q(\mathbf{x}_4)\}\right| \lesssim \varphi(|\mathbf{y}_1-\mathbf{y}_3|)\mathbb{E}\{q^6\}^{\frac{2}{3}}. \tag{10}$$

proof of mixing lemma II

If $\mathbf{y}_4$ is (one of) the closest point(s) to $\mathbf{y}_2$, then the same arguments show that

$$\left|\mathbb{E}\{q(\mathbf{x}_1)q(\mathbf{x}_2)q(\mathbf{x}_3)q(\mathbf{x}_4)\}\right| \lesssim \varphi(|\mathbf{y}_2 - \mathbf{y}_4|)\mathbb{E}\{q^6\}^{\frac{2}{3}}. \tag{11}$$

Otherwise, $\mathbf{y}_3$ is the closest point to $\mathbf{y}_2$, and we find that

$$\left|\mathbb{E}\{q(\mathbf{x}_1)q(\mathbf{x}_2)q(\mathbf{x}_3)q(\mathbf{x}_4)\}\right| \lesssim \varphi(2|\mathbf{y}_2 - \mathbf{y}_3|)\mathbb{E}\{q^6\}^{\frac{2}{3}}.$$

However, by construction, $|\mathbf{y}_2 - \mathbf{y}_4| \leq |\mathbf{y}_1 - \mathbf{y}_2| \leq |\mathbf{y}_1 - \mathbf{y}_3| + |\mathbf{y}_3 - \mathbf{y}_2| \leq 2|\mathbf{y}_2 - \mathbf{y}_3|$, so (11) is still valid (this is the only place where the factor 2 in (6) is used).

Combining (10) and (11), the result follows from $a \wedge b \leq (ab)^{\frac{1}{2}}$ for $a, b \geq 0$, where $a \wedge b = \min(a, b)$.

Estimates

Lemma 2 *Let* q_ε *be a stationary process* $q_\varepsilon(\mathbf{x},\omega) = q(\frac{\mathbf{x}}{\varepsilon},\omega)$ *with integrable correlation function in* (5). *Let* f *be a deterministic square integrable function on* D. *Then we have:*

$$\mathbb{E}\{\|\mathcal{G}q_\varepsilon\mathcal{G}f\|^2_{L^2(D)}\} \lesssim \varepsilon^d\|f\|^2_{L^2(D)}. \tag{12}$$

Let q_ε *satisfy one of the following additional hypotheses:*

[H1] $q(\mathbf{x},\omega)$ *is uniformly bounded* $\mathbb{P}$*-a.s.*

[H2] $\mathbb{E}\{q^6\} < \infty$ *and* $q(\mathbf{x},\omega)$ *is strongly mixing with mixing coefficient in* (6) *such that* $\varphi^{\frac{1}{2}}(r)$ *is bounded and* $r^{d-1}\varphi^{\frac{1}{2}}(r)$ *is integrable on* $\mathbb{R}^+$.

Then we find that

$$\mathbb{E}\{\|\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\|^2_{\mathcal{L}(L^2(D))}\} \lesssim \varepsilon^d. \tag{13}$$

Proof

We denote $\|\cdot\| = \|\cdot\|_{L^2(D)}$ and calculate

$$\mathcal{G}q_\varepsilon\mathcal{G}f(\mathbf{x}) = \int_D \Big(\int_D G(\mathbf{x},\mathbf{y})q_\varepsilon(\mathbf{y})G(\mathbf{y},\mathbf{z})d\mathbf{y}\Big) f(\mathbf{z})d\mathbf{z},$$

so that by the Cauchy-Schwarz inequality, we have

$$|\mathcal{G}q_\varepsilon\mathcal{G}f(\mathbf{x})|^2 \le \|f\|^2 \int_D \Big(\int_D G(\mathbf{x},\mathbf{y})q_\varepsilon(\mathbf{y})G(\mathbf{y},\mathbf{z})d\mathbf{y}\Big)^2 d\mathbf{z}.$$

By definition of the correlation function, we thus find that

$$\mathbb{E}\{\|\mathcal{G}q_\varepsilon\mathcal{G}f\|^2\} \lesssim \|f\|^2 \int_{D^4} G(\mathbf{x},\mathbf{y})G(\mathbf{x},\boldsymbol{\zeta})R\Big(\frac{\mathbf{y}-\boldsymbol{\zeta}}{\varepsilon}\Big)G(\mathbf{y},\mathbf{z})G(\boldsymbol{\zeta},\mathbf{z})d\mathbf{x}d\mathbf{y}d\boldsymbol{\zeta}d\mathbf{z}. \tag{14}$$

Extending $G(\mathbf{x},\mathbf{y})$ by 0 outside $D\times D$, we find in the Fourier domain that

$$\mathbb{E}\{\|\mathcal{G}q_\varepsilon\mathcal{G}f\|^2\} \lesssim \|f\|^2 \int_{D^2}\int_{\mathbb{R}^d} |\widehat{G(\mathbf{x},\cdot)G(}\mathbf{z},\cdot)|^2(\mathbf{p})\varepsilon^d \hat{R}(\varepsilon\mathbf{p})d\mathbf{p}d\mathbf{x}d\mathbf{z}.$$

Here $\widehat{f}(\boldsymbol{\xi}) = \int_{\mathbb{R}^d} e^{-i\boldsymbol{\xi}\cdot\mathbf{x}} f(\mathbf{x})d\mathbf{x}$ is the Fourier transform of $f(\mathbf{x})$. Since $R(\mathbf{x})$ is integrable, then $\hat{R}(\varepsilon\mathbf{p})$ (which is always non-negative by e.g. Bochner's theorem) is bounded by a constant we call R_0 so that

$$\mathbb{E}\{\|\mathcal{G}q_\varepsilon\mathcal{G}f\|^2\} \lesssim \|f\|^2\varepsilon^d R_0 \int_{D^3} G^2(\mathbf{x},\mathbf{y})G^2(\mathbf{z},\mathbf{y})d\mathbf{x}d\mathbf{y}d\mathbf{z} \lesssim \|f\|^2\varepsilon^d R_0,$$

by the square-integrability assumption on $G(\mathbf{x},\mathbf{y})$. This yields (12). Let us now consider (13). We denote by $\|\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\|$ the norm $\|\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\|_{\mathcal{L}(L^2(D))}$ and calculate that

$$\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\phi(\mathbf{x}) = \int_D \Big(\int_D G(\mathbf{x},\mathbf{y})q_\varepsilon(\mathbf{y})G(\mathbf{y},\mathbf{z})d\mathbf{y}\Big)q_\varepsilon(\mathbf{z})\phi(\mathbf{z})d\mathbf{z}.$$

Therefore,

$$\Big(\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\phi(\mathbf{x})\Big)^2 \le \int_D \Big(\int_D G(\mathbf{x},\mathbf{y})q_\varepsilon(\mathbf{y})G(\mathbf{y},\mathbf{z})q_\varepsilon(\mathbf{z})d\mathbf{y}\Big)^2 d\mathbf{z} \int_D \phi^2(\mathbf{z})d\mathbf{z},$$

by Cauchy Schwarz. This shows that

$$\|\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\|^2(\omega) \le \int_{D^2} \Big(\int_D G(\mathbf{x},\mathbf{y})q_\varepsilon(\mathbf{y})G(\mathbf{y},\mathbf{z})d\mathbf{y}\Big)^2 q_\varepsilon^2(\mathbf{z})d\mathbf{z}d\mathbf{x}.$$

When $q_\varepsilon(\mathbf{z},\omega)$ is bounded $\mathbb{P}-$a.s., the proof above leading to (12) applies and we obtain (13) under hypothesis [H1].

The hypothesis that q_ε is small or even bounded can be relaxed as the following calculation shows. Using Lemma 1, we obtain that

$$\mathbb{E}\{q_\varepsilon(\mathbf{y})q_\varepsilon(\boldsymbol{\zeta})q_\varepsilon^2(\mathbf{z})\} \lesssim \varphi^{\frac{1}{2}}\Big(\frac{|\mathbf{y}-\boldsymbol{\zeta}|}{\varepsilon}\Big)\varphi^{\frac{1}{2}}(0) + \varphi^{\frac{1}{2}}\Big(\frac{|\mathbf{y}-\mathbf{z}|}{\varepsilon}\Big)\varphi^{\frac{1}{2}}\Big(\frac{|\mathbf{z}-\boldsymbol{\zeta}|}{\varepsilon}\Big).$$

Under hypothesis [H2], we thus obtain that

$$\begin{aligned}\mathbb{E}\{\|\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\|^2\} \;\lesssim\; & \int_{D^4} G(\mathbf{x},\mathbf{y})G(\mathbf{x},\boldsymbol{\zeta})\varphi^{\frac{1}{2}}\Big(\frac{|\mathbf{y}-\boldsymbol{\zeta}|}{\varepsilon}\Big)G(\mathbf{y},\mathbf{z})G(\boldsymbol{\zeta},\mathbf{z})d\mathbf{y}d\boldsymbol{\zeta}d\mathbf{x}d\mathbf{z} \\ & + \int_{D^2}\Big(\int_D G(\mathbf{x},\mathbf{y})\varphi^{\frac{1}{2}}\Big(\frac{|\mathbf{y}-\mathbf{z}|}{\varepsilon}\Big)G(\mathbf{y},\mathbf{z})d\mathbf{y}\Big)^2 d\mathbf{x}d\mathbf{z}.\end{aligned}$$

Because $r^{d-1}\varphi^{\frac{1}{2}}(r)$ is integrable, then $\mathbf{x}\mapsto\varphi^{\frac{1}{2}}(|\mathbf{x}|)$ is integrable as well and the bound of the first term above under hypothesis [H2] is done as in (14) by replacing $R(\mathbf{x})$ by $\varphi^{\frac{1}{2}}(|\mathbf{x}|)$. As for the second term, it is bounded, using the Cauchy Schwarz inequality, by

$$\int_D \Big(\int_D \Big(\int_D G^2(\mathbf{x},\mathbf{y}) d\mathbf{x} \Big) G^2(\mathbf{y},\mathbf{z}) d\mathbf{y} \Big) \Big(\int_D \varphi\Big(\frac{|\mathbf{y}-\mathbf{z}|}{\varepsilon}\Big) d\mathbf{y} \Big) d\mathbf{z} \lesssim \varepsilon^d,$$

since $\mathbf{x} \mapsto \varphi(|\mathbf{x}|)$ is integrable, D is bounded, and (4) holds.

The above lemma may be used to handle cases with q_ε not necessarily bounded. We simply assume here that q_ε is sufficiently small so that the operator $\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon$ is of norm $\rho < 1$ in $\mathcal{L}(L^2(D))$.

Bound on random correctors

Now we can address the behavior of the correctors. We define

$$u_0 = \mathcal{G}f, \tag{15}$$

the solution of the unperturbed problem. We find that

$$(I - \mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon)(u_\varepsilon - u_0) = -\mathcal{G}q_\varepsilon\mathcal{G}f + \mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\mathcal{G}f. \tag{16}$$

Using the results of Lemma 2, we obtain that

Lemma 3 *Let* u_ε *be the solution to the heterogeneous problem* (1) *and* u_0 *the solution to the corresponding homogenized problem. Then we have that*

$$\Big(\mathbb{E}\{\|u_\varepsilon - u_0\|^2\}\Big)^{\frac{1}{2}} \lesssim \varepsilon^{\frac{d}{2}}\|f\|. \tag{17}$$

Bound on "multiple scattering"

$\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon(u_\varepsilon - u_0)$ is bounded by ε^d in $L^1(\Omega; L^2(D))$ by Cauchy-Schwarz:

$$\mathbb{E}\{\|\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon(u_\varepsilon - u_0)\|\} \leq \Big(\mathbb{E}\{\|\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\|^2\}\Big)^{\frac{1}{2}}\Big(\mathbb{E}\{\|u_\varepsilon - u_0\|^2\}\Big)^{\frac{1}{2}} \lesssim \varepsilon^d \ll \varepsilon^{\frac{d}{2}}.$$

This controls the errors coming from multiple scattering. The remaining contributions in $u_\varepsilon - u_0$ are

$$-\mathcal{G}q_\varepsilon\mathcal{G}f + \mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\mathcal{G}f.$$

Estimate for deterministic corrector

We need the following estimate:

Lemma 4 *Under hypothesis* [H2] *of Lemma 2, we find that*

$$\mathbb{E}\{\|\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\mathcal{G}f\|^2\} \lesssim \varepsilon^{2d\frac{1+\eta}{2+\eta}}\|f\|^2 \ll \varepsilon^d\|f\|^2, \tag{18}$$

where η *is such that* $\mathbf{y} \mapsto \Big(\int_D |G|^{2+\eta}(\mathbf{x},\mathbf{y})d\mathbf{x}\Big)^{\frac{1}{2+\eta}}$ *is uniformly bounded on* D*.*

This is where we need that the Green's function be more than square integrable. Otherwise, a deterministic corrector may appear. The estimate in (18) is optimal in powers of ε.

Proof

By Cauchy Schwarz,

$$|\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\mathcal{G}f(\mathbf{x})|^2 \le \|f\|^2 \int_D \Big(\int_{D^2} G(\mathbf{x},\mathbf{y})q_\varepsilon(\mathbf{y})G(\mathbf{y},\mathbf{z})q_\varepsilon(\mathbf{z})G(\mathbf{z},\mathbf{t})d\mathbf{y}d\mathbf{z}\Big)^2 d\mathbf{t}.$$

So we want to estimate

$$A = \mathbb{E}\{\int_{D^6} G(\mathbf{x},\mathbf{y})G(\mathbf{x},\boldsymbol{\zeta})q_\varepsilon(\mathbf{y})q_\varepsilon(\boldsymbol{\zeta})G(\mathbf{y},\mathbf{z})G(\boldsymbol{\zeta},\boldsymbol{\xi})q_\varepsilon(\mathbf{z})q_\varepsilon(\boldsymbol{\xi})G(\mathbf{z},\mathbf{t})G(\boldsymbol{\xi},\mathbf{t})d[\boldsymbol{\xi}\boldsymbol{\zeta}\mathbf{y}\mathbf{z}\mathbf{x}\mathbf{t}]\}.$$

We now use mixing (9) to obtain that $A \lesssim A_1 + A_2 + A_3$, where

$$A_1 = \int_{D^6} G(\mathbf{x},\mathbf{y})G(\mathbf{x},\boldsymbol{\zeta})\varphi^{\frac12}\Big(\frac{|\mathbf{y}-\boldsymbol{\zeta}|}{\varepsilon}\Big)G(\mathbf{y},\mathbf{z})G(\boldsymbol{\zeta},\boldsymbol{\xi})\varphi^{\frac12}\Big(\frac{|\mathbf{z}-\boldsymbol{\xi}|}{\varepsilon}\Big)G(\mathbf{z},\mathbf{t})G(\boldsymbol{\xi},\mathbf{t})d[\boldsymbol{\xi}\boldsymbol{\zeta}\mathbf{y}\mathbf{z}\mathbf{x}\mathbf{t}],$$

$$A_2 = \int_{D^2} \Big(\int_{D^2} G(\mathbf{x},\mathbf{y})G(\mathbf{y},\mathbf{z})\varphi^{\frac12}\Big(\frac{|\mathbf{y}-\mathbf{z}|}{\varepsilon}\Big)G(\mathbf{z},\mathbf{t})d\mathbf{y}d\mathbf{z}\Big)^2 d\mathbf{t}d\mathbf{x},$$

$$A_3 = \int_{D^6} G(\mathbf{x},\mathbf{y})G(\boldsymbol{\xi},\mathbf{t})G(\mathbf{x},\boldsymbol{\zeta})G(\mathbf{z},\mathbf{t})\varphi^{\frac12}\Big(\frac{|\mathbf{y}-\boldsymbol{\xi}|}{\varepsilon}\Big)G(\mathbf{y},\mathbf{z})G(\boldsymbol{\zeta},\boldsymbol{\xi})\varphi^{\frac12}\Big(\frac{|\boldsymbol{\zeta}-\mathbf{z}|}{\varepsilon}\Big)d[\boldsymbol{\xi}\boldsymbol{\zeta}\mathbf{y}\mathbf{z}\mathbf{x}\mathbf{t}].$$

Denote $F_{\mathbf{x},\mathbf{t}}(\mathbf{y},\mathbf{z}) = G(\mathbf{x},\mathbf{y})G(\mathbf{y},\mathbf{z})G(\mathbf{z},\mathbf{t})$. Then in the Fourier domain, we find that

$$A_1 \lesssim \int_{D^2}\int_{\mathbb{R}^{2d}} \varepsilon^{2d}\widehat{\varphi^{\frac{1}{2}}}(\varepsilon\mathbf{p})\widehat{\varphi^{\frac{1}{2}}}(\varepsilon\mathbf{q})|\hat{F}_{\mathbf{x},\mathbf{t}}(\mathbf{p},\mathbf{q})|^2 d\mathbf{p}d\mathbf{q}d\mathbf{x}d\mathbf{t}.$$

Here $\widehat{\varphi^{\frac{1}{2}}}(\mathbf{p})$ is the Fourier transform of $\mathbf{x}\mapsto\varphi^{\frac{1}{2}}(|\mathbf{x}|)$. Since $\widehat{\varphi^{\frac{1}{2}}}(\varepsilon\mathbf{p})$ is bounded because $r^{d-1}\varphi^{\frac{1}{2}}(r)$ is integrable on $\mathbb{R}^+$, we deduce that

$$A_1 \lesssim \varepsilon^{2d}\int_{D^4} G^2(\mathbf{x},\mathbf{y})G^2(\mathbf{y},\mathbf{z})G^2(\mathbf{z},\mathbf{t})d\mathbf{x}d\mathbf{y}d\mathbf{z}d\mathbf{t} \lesssim \varepsilon^{2d},$$

using the integrability condition imposed on $G(\mathbf{x},\mathbf{y})$.

Using $2ab \le a^2+b^2$ for $(a,b) = (G(\mathbf{x},\mathbf{y}),G(\mathbf{x},\boldsymbol{\zeta}))$ and $(a,b) = (G(\boldsymbol{\xi},\mathbf{t}),G(\mathbf{z},\mathbf{t}))$ successively, and integrating in $\mathbf{t}$ and $\mathbf{x}$, we find that

$$A_3 \lesssim \int_{D^4} G(\mathbf{y},\mathbf{z})G(\boldsymbol{\zeta},\boldsymbol{\xi})\varphi^{\frac{1}{2}}\Big(\frac{|\mathbf{y}-\boldsymbol{\xi}|}{\varepsilon}\Big)\varphi^{\frac{1}{2}}\Big(\frac{|\boldsymbol{\zeta}-\mathbf{z}|}{\varepsilon}\Big)d[\mathbf{y}\boldsymbol{\zeta}\mathbf{z}\boldsymbol{\xi}],$$

thanks to the square integrability (4). Now with $(a,b) = (G(\mathbf{y},\mathbf{z}),G(\boldsymbol{\zeta},\boldsymbol{\xi}))$, we find that

$$A_3 \lesssim \int_{D^4} G^2(\mathbf{y},\mathbf{z})\varphi^{\frac{1}{2}}\Big(\frac{|\mathbf{y}-\boldsymbol{\xi}|}{\varepsilon}\Big)\varphi^{\frac{1}{2}}\Big(\frac{|\boldsymbol{\zeta}-\mathbf{z}|}{\varepsilon}\Big)d[\mathbf{y}\boldsymbol{\zeta}\mathbf{z}\boldsymbol{\xi}] \lesssim \varepsilon^{2d},$$

since $\varphi^{\frac{1}{2}}$ is integrable and G is square integrable on D.

Consider the contribution A_2. We write the squared integral as a double integral over the variables $(\mathbf{y},\boldsymbol{\zeta},\mathbf{z},\boldsymbol{\xi})$ and dealing with the integration in $\mathbf{x}$ and $\mathbf{t}$ using $2ab \le a^2+b^2$ as in the A_3 contribution, obtain that

$$A_2 \lesssim \int_{D^4} G(\mathbf{y},\boldsymbol{\zeta})\varphi^{\frac{1}{2}}\Big(\frac{|\mathbf{y}-\boldsymbol{\zeta}|}{\varepsilon}\Big)G(\mathbf{z},\boldsymbol{\xi})\varphi^{\frac{1}{2}}\Big(\frac{|\mathbf{z}-\boldsymbol{\xi}|}{\varepsilon}\Big)d[\mathbf{y}\boldsymbol{\zeta}\mathbf{z}\boldsymbol{\xi}].$$

Using Hölder's inequality, we obtain that

$$A_2 \lesssim \left(\Big(\int_0^\infty \varphi^{\frac{p'}{2}}\Big(\frac{r}{\varepsilon}\Big)r^{d-1}dr\Big)^{\frac{1}{p'}}\Big(\int_{D^2} G^p(\mathbf{y},\mathbf{z})d\mathbf{y}d\mathbf{z}\Big)^{\frac{1}{p}}\right)^2 \lesssim \varepsilon^{2d\frac{1+\eta}{2+\eta}},$$

with $p = 2+\eta$ and $p' = \frac{2+\eta}{1+\eta}$ since $\varphi^{\frac{1}{2}}(r)r^{d-1}$, whence $\varphi^{\frac{p'}{2}}(r)r^{d-1}$, is integrable.

Convergence of multiple scattering

We have therefore obtained that

$$\mathbb{E}\{\|u_\varepsilon - u + \mathcal{G}q_\varepsilon\mathcal{G}f\|\} \lesssim \varepsilon^{d\frac{1+\eta}{2+\eta}} \ll \varepsilon^{\frac{d}{2}}. \tag{19}$$

For what follows, it is useful to recast the above result as:

Proposition 5 *Let $q(\mathbf{x},\omega)$ be constructed so that [H2]-[H3] holds. Let u_ε be the solution to (8) and $u_0 = \mathcal{G}f$. We assume that u_0 is continuous on D. Then we have the following strong convergence result:*

$$\lim_{\varepsilon\to 0} \mathbb{E}\Big\{\Big\|\frac{u_\varepsilon - u_0}{\varepsilon^{\frac{d}{2}}} + \frac{1}{\varepsilon^{\frac{d}{2}}}\mathcal{G}q\Big(\frac{\cdot}{\varepsilon},\omega\Big)u_0\Big\|\Big\} = 0. \tag{20}$$

Oscillatory integral in one space dimension

In dimension $d = 1$, the leading term of the corrector $\varepsilon^{-\frac{1}{2}}(u_\varepsilon - u_0)$ is thus given by:

$$u_{1\varepsilon}(x,\omega) = -\frac{1}{\sqrt{\varepsilon}}\mathcal{G}q_\varepsilon\mathcal{G}f = \int_D -G(x,y)\frac{1}{\sqrt{\varepsilon}}q(\frac{y}{\varepsilon},\omega)u_0(y)dy, \tag{21}$$

where D is an interval (a,b). The convergence is more precise in dimension $d = 1$ than in higher space dimensions. For the Helmholtz equation, the Green function in $d = 1$ is Lipschitz continuous. Then $u_{1\varepsilon}(x,\omega)$ is of class $\mathcal{C}(D)$ $\mathbb{P}$-a.s. and we can seek convergence in that functional class. Since $u_0 = \mathcal{G}f$, it is continuous for $f \in L^2(D)$.

The variance of the random variable $u_{1\varepsilon}(x,\omega)$ is given by

$$\mathbb{E}\{u_{1\varepsilon}^2(x,\omega)\} = \int_{D^2} G(x,y)G(x,z)\frac{1}{\varepsilon}R\Big(\frac{y-z}{\varepsilon}\Big)u_0(y)u_0(z)dydz. \tag{22}$$

Because $R(x)$ is assumed to be integrable, the above integral converges, as $\varepsilon \to 0$, to the following limit:

$$\mathbb{E}\{u_1^2(x,\omega)\} = \int_D G^2(x,y)\widehat{R}(0)u_0^2(y)dy, \tag{23}$$

where

$$\widehat{R}(0) = \sigma^2 := \int_{-\infty}^{\infty} R(r)dr = 2\int_0^{\infty} \mathbb{E}\{q(0)q(r)\}dr. \tag{24}$$

Because (21) is an average of random variables decorrelating sufficiently fast, we expect a central limit-type result to show that $u_{1\varepsilon}(x,\omega)$ converges to a Gaussian random variable. Combined with the variance (24), we expect the limit to be the following stochastic integral:

$$u_1(x,\omega) = -\sigma \int_D G(x,y)u_0(y)dW_y(\omega), \tag{25}$$

where $dW_y(\omega)$ is standard white noise on $(\mathcal{C}(D), \mathcal{B}(\mathcal{C}(D)), \mathbb{P})$. More precisely, we show the following result:

Theorem 6 *Let us assume that $G(x, y)$ is Lipschitz continuous. Then, under the conditions of Proposition 5, the process $u_{1\varepsilon}(x, \omega)$ converges weakly and in distribution in the space of continuous paths $\mathcal{C}(D)$ to the limit $u_1(x, \omega)$ in (25).*

As a consequence, the corrector to homogenization satisfies that

$$\frac{u_\varepsilon - u_0}{\sqrt{\varepsilon}}(x) \xrightarrow{\text{dist.}} -\sigma \int_D G(x, y) u_0(y) dW_y, \qquad \text{as } \varepsilon \to 0, \tag{26}$$

in the space $L^1(\Omega; L^2(D))$.

Weak Convergence and Criterion for Tightness

We recall the classical result on the weak convergence of random variables with values in the space of continuous paths:

Proposition 7 *Suppose* $(Z_n; 1 \le n \le \infty)$ *are random variables with values in the space of continuous functions* $\mathcal{C}(D)$. *Then* Z_n *converges weakly (in distribution) to* Z_∞ *provided that:*

(a) any finite-dimensional joint distribution $(Z_n(x_1), \ldots, Z_n(x_k))$ *converges to the joint distribution* $(Z_\infty(x_1), \ldots, Z_\infty(x_k))$ *as* $n \to \infty$.

(b) (Z_n) *is a tight sequence of random variables. A sufficient condition for tightness of* (Z_n) *is the following Kolmogorov criterion: there exist positive constants* ν, β, *and* δ *such that*

$$\begin{aligned} &(i) \quad \sup_{n \ge 1} \mathbb{E}\{|Z_n(t)|^\nu\} < \infty, \qquad \text{for some } t \in D, \\ &(ii) \quad \mathbb{E}\{|Z_n(s) - Z_n(t)|^\beta\} \lesssim |t - s|^{1+\delta}, \end{aligned} \tag{27}$$

uniformly in $n \ge 1$ *and* $t, s \in D$.

Tightness

Tightness of $u_{1\varepsilon}(x,\omega)$ is obtained with $\nu=\beta=2$ and $\delta=1$. Indeed, we easily obtain that

$$\mathbb{E}\{|u_{1\varepsilon}(x,\omega)|^2\} \lesssim 1,$$

in fact uniformly in $x\in D$. Now by assumption on $G(x,y)$ we obtain that

$$\begin{aligned}
&\mathbb{E}\{|u_{1\varepsilon}(x,\omega)-u_{1\varepsilon}(\xi,\omega)|^2\} = \mathbb{E}\Big(\int_D [G(x,y)-G(\xi,y)]\frac{1}{\sqrt{\varepsilon}}q(\frac{y}{\varepsilon})u_0(y)dy\Big)^2\\
&= \int_{D^2}[G(x,y)-G(\xi,y)][G(x,\zeta)-G(\xi,\zeta)]\frac{1}{\varepsilon}R(\frac{\zeta-y}{\varepsilon})u_0(y)u_0(\zeta)dyd\zeta\\
&\lesssim |x-\xi|^2\int_{D^2}\frac{1}{\varepsilon}|R(\frac{\zeta-y}{\varepsilon})|u_0(y)u_0(\zeta)dyd\zeta \lesssim |x-\xi|^2,
\end{aligned}$$

since the correlation function $R(r)$ is integrable and u_0 is bounded. This proves tightness of the sequence $u_{1\varepsilon}(x,\omega)$, or equivalently weak convergence of the measures $\mathbb{P}_\varepsilon$ generated by $u_{1\varepsilon}(x,\omega)$ on $(\mathcal{C}(D),\mathcal{B}(\mathcal{C}(D)))$.

Finite dimensional distributions

Now any finite-dimensional distribution $(u_{1\varepsilon}(x_j,\omega))_{1\le j\le n}$ has the characteristic function

$$\Phi_\varepsilon(\mathbf{k}) = \mathbb{E}\{e^{ik_j u_{1\varepsilon}(x_j,\omega)}\}, \qquad \mathbf{k} = (k_1,\ldots,k_n).$$

The above characteristic function can be recast as

$$\Phi_\varepsilon(\mathbf{k}) = \mathbb{E}\{e^{i\int_D m(y)\frac{1}{\sqrt{\varepsilon}}q_\varepsilon(y)dy}\}, \qquad m(y) = -\sum_{j=1}^{n} k_j G(x_j,y)u_0(y).$$

As a consequence (Lévi continuity theorem), convergence of the finite dimensional distributions will be proved if we can show convergence of:

$$I_{m\varepsilon} := \int_D m(y)\frac{1}{\sqrt{\varepsilon}}q(\frac{y}{\varepsilon})dy \xrightarrow{\text{dist.}} I_m := \int_D m(y)\sigma dW_y, \qquad \varepsilon\to 0, \qquad (28)$$

for arbitrary continuous moments $m(y)$.

Such integrals have been extensively analyzed in the literature, where the above integral, for $D = (a, b)$ may be seen as the solution $x_\varepsilon(b)$ of the following ordinary differential equation with random coefficients:

$$\dot{x}_\varepsilon = \frac{1}{\sqrt{\varepsilon}} q(\frac{t}{\varepsilon}) m(t), \qquad x_\varepsilon(a) = 0.$$

Since we will use the same methodology in higher space dimensions, we give a short proof of (28) using the central limit theorem for correlated discrete random variables as stated e.g. in [Bo-82].

Approximation by piecewise constant integrand

Note that if we replace $m(y)$ by $m_h(y)$, then

$$\mathbb{E}\{(I_{m\varepsilon} - I_{m_h\varepsilon})^2\} \lesssim \|m - m_h\|_\infty^2, \tag{29}$$

where $\|\cdot\|_\infty$ is the uniform norm on D. It is therefore sufficient to consider (28) for a sequence of functions m_h converging to m in the uniform sense. Since m is (uniformly) continuous, we can approximate it by piecewise constant functions m_h that are constant on M intervals of size $h = \frac{b-a}{M}$. Let m_{hj} be the value of m_h on the j^{th} interval and define the random variables

$$M_{\varepsilon j} = m_{hj} \int_{(j-1)h}^{jh} \frac{1}{\sqrt{\varepsilon}} q(\frac{y}{\varepsilon}) dy.$$

Independence of random variables

We want to show that the variables $M_{\varepsilon j}$ become independent in the limit $\varepsilon \to 0$. This is done by showing that

$$\mathcal{E}(\mathbf{k}) = \left| \mathbb{E}\{e^{i \sum_{j=1}^{M} k_j M_{\varepsilon j}}\} - \prod_{j=1}^{M} \mathbb{E}\{e^{i k_j M_{\varepsilon j}}\} \right| \to 0 \quad \text{as } \varepsilon \to 0,$$

for all $\mathbf{k} = \{k_j\}_{1 \le j \le M} \in \mathbb{R}^M$. Let $\mathbf{k} \in \mathbb{R}^M$ fixed, $0 < \eta < \frac{h}{2}$ and define

$$P_{\varepsilon j}^{\eta} = m_{hj} \int_{(j-1)h+\eta}^{jh-\eta} \frac{1}{\sqrt{\varepsilon}} q(\frac{y}{\varepsilon}) dy, \qquad Q_{\varepsilon j}^{\eta} = M_{\varepsilon j} - P_{\varepsilon j}^{\eta}.$$

Now we write

$$\begin{aligned} \mathbb{E}\{e^{i \sum_{j=1}^{M} k_j M_{\varepsilon j}}\} = \ & \mathbb{E}\{[e^{i k_1 Q_{\varepsilon 1}^{\eta}} - 1] e^{i k_1 P_{\varepsilon 1}^{\eta} + i \sum_{j=2}^{M} k_j M_{\varepsilon j}}\} \\ & + \mathbb{E}\{e^{i k_1 P_{\varepsilon 1}^{\eta} + i \sum_{j=2}^{M} k_j M_{\varepsilon j}}\}. \end{aligned}$$

Using the strong mixing condition (6), we find that

$$\left|\mathbb{E}\{e^{ik_1 P^\eta_{\varepsilon 1}+i\sum_{j=2}^M k_j M_{\varepsilon j}}\} - \mathbb{E}\{e^{ik_1 P^\eta_{\varepsilon 1}}\}\mathbb{E}\{e^{i\sum_{j=2}^M k_j M_{\varepsilon j}}\}\right| \lesssim \varphi(\frac{2\eta}{\varepsilon}).$$

Now we find that $\mathbb{E}\{Q^\eta_{\varepsilon j}\} = 0$ and $\mathbb{E}\{[Q^\eta_{\varepsilon j}]^2\} \lesssim \eta$. The latter result comes from integrating $\varepsilon^{-1}R(\frac{t-s}{\varepsilon})dsdt$ over a cube of size $O(\eta^2)$. Since $|e^{ix}-1| \lesssim |x|$, we deduce that

$$|\mathbb{E}\{[e^{ik_1 Q^\eta_{\varepsilon 1}}-1]e^{ik_1 P^\eta_{\varepsilon 1}+iZ}\}| \le \mathbb{E}\{[e^{ik_1 Q^\eta_{\varepsilon 1}}-1]^2\}^{\frac{1}{2}} \lesssim \eta^{\frac{1}{2}},$$

for an arbitrary random variable Z (equal to 0 or to $\sum_{j=2}^M k_j M_{\varepsilon j}$ here). Thus,

$$\left|\mathbb{E}\{e^{ik_1 M_{\varepsilon 1}+i\sum_{j=2}^M k_j M_{\varepsilon j}}\} - \mathbb{E}\{e^{ik_1 M_{\varepsilon 1}}\}\mathbb{E}\{e^{i\sum_{j=2}^M k_j M_{\varepsilon j}}\}\right| \lesssim \varphi(\frac{2\eta}{\varepsilon}) + \eta^{\frac{1}{2}}.$$

By induction, we thus find that for all $0 < \eta < \frac{h}{2}$,

$$\mathcal{E} \lesssim M\varphi(\frac{2\eta}{\varepsilon}) + \eta^{\frac{1}{2}}.$$

This expression tends to 0 say for $\eta = \varepsilon^{\frac{1}{2}}$.

This shows that the random variables $M_{\varepsilon j}$ become independent as $\varepsilon \to 0$.

We show below that each $M_{\varepsilon j}$ converges to a centered Gaussian variable as $\varepsilon \to 0$.

The sum over j thus yields in the limit a centered Gaussian variable with variance the sum of the M individual variances.

Central Limit Theorem for discrete random variables

By stationarity of the process $q(x,\omega)$, we are thus led to showing that

$$\int_0^h \frac{1}{\sqrt{\varepsilon}} q(\frac{y}{\varepsilon})dy \xrightarrow{\text{dist.}} \int_0^h \sigma dW_y = \sigma W_h = \sigma \mathcal{N}(0,h), \qquad \varepsilon \to 0,$$

where $\mathcal{N}(0,h)$ is the centered Gaussian variable with variance h. We break up h into $N = h/\varepsilon$ (which we assume is an integer) intervals and call

$$q_j = \int_{(j-1)\varepsilon}^{j\varepsilon} \frac{1}{\varepsilon} q(\frac{y}{\varepsilon})dy = \int_{j-1}^{j} q(y)dy, \qquad j \in \mathbb{Z}.$$

The q_j are stationary mixing random variables and we are interested in the limit

$$\sqrt{\varepsilon}\sum_{j=1}^{N} q_j = \frac{\sqrt{h}}{\sqrt{N}}\sum_{j=1}^{N} q_j. \tag{30}$$

Following [Bo-82], we introduce $\mathcal{A}_m$ and $\mathcal{A}^m$ as the $\sigma-$algebras generated by $(q_j)_{j\leq m}$ and $(q_j)_{j\geq m}$, respectively. Let then

$$\rho(n) = \sup\left\{\frac{\mathbb{E}\Big\{(\eta - \mathbb{E}\{\eta\})(\xi - \mathbb{E}\{\xi\})\Big\}}{\Big(\mathbb{E}\{\eta^2\}\mathbb{E}\{\xi^2\}\Big)^{\frac{1}{2}}};\ \eta \in L^2(\mathcal{A}_0), \quad \xi \in L^2(\mathcal{A}^n\}\right\}. \quad (31)$$

Then provided that $\sum_{n\geq 1}\rho(n) < \infty$, we obtain the following central limit theorem

$$\frac{\sqrt{h}}{\sqrt{N}}\sum_{j=1}^{N} q_j \xrightarrow{\text{dist.}} \sqrt{h}\sigma\mathcal{N}(0,1) \equiv \sigma\mathcal{N}(0,h), \quad (32)$$

where $\mathcal{N}(0,1)$ is the standard normal variable, where $\equiv$ is used to mean equality in distribution, and where $\sigma^2 = \sum_{n\in\mathbb{Z}}\mathbb{E}\{q_0 q_n\}$. It remains to verify that the two definitions of σ above and in (24) agree and that

$\sum_{n\geq 1}\rho(n)<\infty$. Note that

$$\sum_{n\in\mathbb{Z}}\mathbb{E}\{q_0 q_n\}=\int_0^1\int_{-\infty}^{\infty}\mathbb{E}\{q(y)q(z)\}dydz$$
$$=\int_0^1\int_{-\infty}^{\infty}\mathbb{E}\{q(y)q(y+z)\}dydz=\int_0^1\widehat{R}(0)dy=\widehat{R}(0),$$

thanks to (24). Now we observe that $\rho(n)\leq\varphi(n-1)$ so that summability of $\rho(n)$ is implied by the integrability of $\varphi(r)$ on $\mathbb{R}^+$. This concludes the proof of the convergence in distribution of $u_{1\varepsilon}$ in the space of continuous paths $\mathcal{C}(D)$.

It now remains to recall the convergence result (20) to obtain (26) in the space $L^1(\Omega;L^2(D))$.

Oscillatory integral in arbitrary space dimensions

In dimension $1 \leq d \leq 3$ for second-order elliptic operators, the leading term in the random corrector $\varepsilon^{-\frac{d}{2}}(u_\varepsilon - u_0)$ is given by:

$$u_{1\varepsilon}(\mathbf{x},\omega) = \int_D -G(\mathbf{x},\mathbf{y})\frac{1}{\varepsilon^{\frac{d}{2}}}q_\varepsilon(\mathbf{y},\omega)u_0(\mathbf{y})d\mathbf{y}. \tag{33}$$

The variance of $u_{1\varepsilon}(\mathbf{x},\omega)$ is given by

$$\mathbb{E}\{u_{1\varepsilon}^2(\mathbf{x},\omega)\} = \int_{D^2} G(\mathbf{x},\mathbf{y})G(\mathbf{x},\mathbf{z})\frac{1}{\varepsilon^d}R\Big(\frac{\mathbf{y}-\mathbf{z}}{\varepsilon}\Big)u_0(\mathbf{y})u_0(\mathbf{z})d\mathbf{y}d\mathbf{z}.$$

As in the one-dimensional case, it converges as $\varepsilon \to 0$ to the limit

$$\mathbb{E}\{u_1^2(\mathbf{x},\omega)\} = \sigma^2 \int_D G^2(\mathbf{x},\mathbf{y})u_0^2(\mathbf{y})d\mathbf{y}, \qquad \sigma^2 = \int_{\mathbb{R}^d} \mathbb{E}\{q(\mathbf{0})q(\mathbf{y})\}d\mathbf{y}. \tag{34}$$

Because of the singularities of the Green's function $G(\mathbf{x},\mathbf{y})$ in dimension $d \geq 2$, we prove here less accurate results than those obtained in dimension $d = 1$.

We want to obtain convergence of the above corrector in distribution on $(\Omega, \mathcal{F}, \mathbb{P})$ and weakly in D. More precisely, let $M_k(\mathbf{x})$ for $1 \leq k \leq K$ be sufficiently smooth functions such that

$$m_k(\mathbf{y}) = -\int_D M_k(\mathbf{x}) G(\mathbf{x}, \mathbf{y}) u_0(\mathbf{y}) d\mathbf{x} = -\mathcal{G} M_k(\mathbf{y}) u_0(\mathbf{y}), \quad 1 \leq k \leq K, \tag{35}$$

are continuous functions (we thus assume that $u_0(\mathbf{x})$ is continuous as well). Let us introduce the random variables

$$I_{k\varepsilon}(\omega) = \int_D m_k(\mathbf{y}) \frac{1}{\varepsilon^{\frac{d}{2}}} q\Big(\frac{\mathbf{y}}{\varepsilon}, \omega\Big) d\mathbf{y}. \tag{36}$$

Because of hypothesis [H3], the accumulation points of the integrals $I_{k\varepsilon}(\omega)$ are not modified if $q(\frac{\mathbf{y}}{\varepsilon}, \omega)$ is replaced by $q_\varepsilon(\mathbf{y}, \omega)$. The main result of this section is the following:

Theorem 8 *Under the above conditions and the hypotheses of Proposition 5, the random variables $I_{k\varepsilon}(\omega)$ converge in distribution to the mean* *zero Gaussian random variables $I_k(\omega)$ as $\varepsilon \to 0$, where the correlation matrix is given by*

$$\Sigma_{jk} = \mathbb{E}\{I_j I_k\} = \sigma^2 \int_D m_j(\mathbf{y}) m_k(\mathbf{y}) d\mathbf{y}, \tag{37}$$

where σ is given by

$$\sigma^2 = \int_{\mathbb{R}^d} \mathbb{E}\{q(\mathbf{0}) q(\mathbf{y})\} d\mathbf{y}. \tag{38}$$

Moreover, we have the stochastic representation

$$I_k(\omega) = \int_D m_k(\mathbf{y}) \sigma dW_{\mathbf{y}}, \tag{39}$$

where $dW_{\mathbf{y}}$ is standard multi-parameter Wiener process.

As a result, for $M(\mathbf{x})$ sufficiently smooth, we obtain that

$$\left(\frac{u_\varepsilon - u_0}{\varepsilon^{\frac{d}{2}}}, M \right) \xrightarrow{\text{dist.}} -\sigma \int_D \mathcal{G}M(\mathbf{y}) \mathcal{G}f(\mathbf{y}) dW_{\mathbf{y}}. \tag{40}$$

Proof

The convergence in (40) is a direct consequence of (39) since

$$\int_{D^2} M(\mathbf{x})G(\mathbf{x},\mathbf{y})u_0(\mathbf{y})dW_{\mathbf{y}}d\mathbf{x} = \int_D \mathcal{G}M(\mathbf{y})\mathcal{G}f(\mathbf{y})dW_{\mathbf{y}},$$

and of the strong convergence (20) in Proposition 5. The equality (39) is directly deduced from (37) since $I_k(\omega)$ is a (multivariate) Gaussian variable. In order to prove (37), we use a methodology similar to that in the proof of Theorem 6.

The characteristic function of the random variables $I_{k\varepsilon}(\omega)$ is given by

$$\Phi_\varepsilon(\mathbf{k}) = \mathbb{E}\{e^{i\sum_{k=1}^K k_j I_{j\varepsilon}(\omega)}\}, \qquad \mathbf{k} = (k_1, \dots, k_K),$$

and may be recast as

$$\Phi_\varepsilon(\mathbf{k}) = \mathbb{E}\{e^{i\int_D m(y)\varepsilon^{\frac{-d}{2}} q(\frac{\mathbf{y}}{\varepsilon},\omega)d\mathbf{y}}\}, \quad m(\mathbf{y}) = \sum_{j=1}^K k_j m_j(\mathbf{y}).$$

So (37) follows from showing that

$$I_\varepsilon(\omega) = \int_D m(\mathbf{y}) \frac{1}{\varepsilon^{\frac{d}{2}}} q\Big(\frac{\mathbf{y}}{\varepsilon}, \omega\Big) d\mathbf{y} \xrightarrow{\text{dist.}} \int_D m(\mathbf{y}) \sigma dW_{\mathbf{y}}, \tag{41}$$

for an arbitrary continuous function $m(\mathbf{y})$. As in the one-dimensional case and for the same reasons, we replace $m(\mathbf{y})$ by $m_h(\mathbf{y})$, which is constant on small hyper-cubes $\mathcal{C}_j$ of size h (and volume h^d) and that there are $M \approx h^{-d}$ of them. Because ∂D is assumed to be sufficiently smooth, it can be covered by $M_S \approx h^{-d+1}$ cubes and we set $m_h(\mathbf{x}) = 0$ on those cubes. The contribution to $I_\varepsilon(\omega)$ is seen to converge to 0 as $h \to 0$ in the mean-square sense as in (29).

We define the random variables

$$M_{\varepsilon j}(\omega) = m_{hj} \int_{\mathcal{C}_j} \frac{1}{\varepsilon^{\frac{d}{2}}} q(\frac{\mathbf{y}}{\varepsilon}, \omega) d\mathbf{y}, \qquad 1 \leq j \leq M,$$

where m_{hj} is the value of m_h on $\mathcal{C}_j$ and are interested in the limiting distribution as $\varepsilon \to 0$ of the random variable

$$I_\varepsilon^h(\omega) = \sum_{j=1}^{M} M_{\varepsilon j}(\omega). \tag{42}$$

We show below that these random variables are again independent in the limit $\varepsilon \to 0$ and each variable converges to a centered Gaussian variable. As a consequence, $I_\varepsilon^h(\omega)$ converges in distribution to a centered Gaussian variable whose variance is the sum of the variances of the variables $M_{\varepsilon j}(\omega)$ in the limit $\varepsilon \to 0$.

That the random variables $M_{\varepsilon j}$ are independent in the limit $\varepsilon \to 0$ is shown using a similar method to that of the one-dimensional case. We want to obtain that

$$\mathcal{E}(\mathbf{k}) = \left| \mathbb{E}\{e^{i\sum_{j=1}^{M} k_j M_{\varepsilon j}}\} - \prod_{j=1}^{M} \mathbb{E}\{e^{ik_j M_{\varepsilon j}}\} \right| \to 0 \text{ as } \varepsilon \to 0,$$

for all $\mathbf{k} = \{k_j\}_j \in \mathbb{R}^M$. Let $0 < \eta < \frac{h}{2}$ and $\mathcal{D}_j^\eta = \{\mathbf{x} \in \mathcal{C}_j; d(\mathbf{x}, \partial\mathcal{C}_j) > \eta\}$. We define

$$P_{\varepsilon j}^\eta = m_{hj} \int_{\mathcal{D}_j^\eta} \frac{1}{\varepsilon^{\frac{d}{2}}} q(\frac{\mathbf{y}}{\varepsilon}, \omega) d\mathbf{y}, \qquad Q_{\varepsilon j}^\eta = M_{\varepsilon j} - P_{\varepsilon j}^\eta.$$

We write again:

$$\begin{aligned}\mathbb{E}\{e^{i\sum_{j=1}^M k_j M_{\varepsilon j}}\} = \ & \mathbb{E}\{[e^{ik_1 Q_{\varepsilon 1}^\eta} - 1] e^{ik_1 P_{\varepsilon 1}^\eta + i\sum_{j=2}^M k_j M_{\varepsilon j}}\} \\ & + \mathbb{E}\{e^{ik_1 P_{\varepsilon 1}^\eta + i\sum_{j=2}^M k_j M_{\varepsilon j}}\}.\end{aligned}$$

Using the strong mixing condition (6), we find that

$$\left| \mathbb{E}\{e^{ik_1 P_{\varepsilon 1}^\eta + i\sum_{j=2}^M k_j M_{\varepsilon j}}\} - \mathbb{E}\{e^{ik_1 P_{\varepsilon 1}^\eta}\} \mathbb{E}\{e^{i\sum_{j=2}^M k_j M_{\varepsilon j}}\} \right| \lesssim \varphi(\frac{2\eta}{\varepsilon}).$$

We find as in the one-dimensional case that $\mathbb{E}\{Q_{\varepsilon j}^\eta\} = 0$ and $\mathbb{E}\{[Q_{\varepsilon j}^\eta]^2\} \lesssim \eta h^{(d-1)} \lesssim \eta$ with a bound independent of ε. This comes from integrating $\varepsilon^{-d} R(\frac{\mathbf{x}-\mathbf{y}}{\varepsilon}) d\mathbf{x} d\mathbf{y}$ on a domain of size $O([\eta h^{d-1}]^2)$. The rest of the proof follows as in the one-dimensional case.

It remains to address the convergence of $M_{\varepsilon j}$ as $\varepsilon \to 0$. By invariance of $q(\mathbf{x})$, it is sufficient to consider integrals on the cube $[\mathbf{0}, \mathbf{h}]$, with $\mathbf{h} = (h, \ldots, h)$. It now remains to show that

$$\int_{[\mathbf{0},\mathbf{h}]} \frac{1}{\varepsilon^{\frac{d}{2}}} q\Big(\frac{\mathbf{y}}{\varepsilon}, \omega\Big) d\mathbf{y} \xrightarrow{\text{dist.}} \sigma \int_{[\mathbf{0},\mathbf{h}]} dW_{\mathbf{y}} = \sigma \mathcal{N}(0, h^d). \tag{43}$$

For a multi-index $\mathbf{j} \in \mathbb{Z}^d$, we define

$$q_{\mathbf{j}}(\omega) = \int_{\mathbf{j}+[\mathbf{0},\mathbf{1}]} q(\mathbf{y}, \omega) d\mathbf{y}.$$

Then (43) will follow by homogeneity if we can show that

$$\frac{1}{\sigma n^{\frac{d}{2}}} \sum_{\mathbf{j} \in [\mathbf{0},\mathbf{n}]} q_{\mathbf{j}} \xrightarrow{\text{dist.}} \mathcal{N}(0, 1). \tag{44}$$

Let A and B be subsets of $\mathbb{Z}^d$ and let $\mathcal{A}$ and $\mathcal{B}$ be the σ algebras generated by $q_{\mathbf{j}}$ on A and B, respectively. Then we define

$$\rho(n) = \sup\left\{\frac{\mathbb{E}\Big\{(\eta - \mathbb{E}\{\eta\})(\xi - \mathbb{E}\{\xi\})\Big\}}{\Big(\mathbb{E}\{\eta^2\}\mathbb{E}\{\xi^2\}\Big)^{\frac{1}{2}}};\ \eta \in L^2(\mathcal{A}), \quad \xi \in L^2(\mathcal{B}\}, \quad d(A,B) \geq n\right\}$$

We then assume that $\mathbb{E}\{q_{\mathbf{j}}^6\} < \infty$ as in hypothesis [H2] and that $\rho(n) = o(n^{-d})$ and that

$$\sum_{n=0}^{\infty} n^{d-1}\rho^{\frac{1}{2}}(n) < \infty. \tag{45}$$

Then we verify that the hypotheses in [Bo-82] are satisfied so that (44) holds with

$$\sigma^2 = \sum_{\mathbf{j}\in\mathbb{Z}^d} \mathbb{E}\{q_{\mathbf{0}}q_{\mathbf{j}}\}.$$

We verify as in the one-dimensional case that the above σ agrees with that in definition (38). Now we verify that (45) is a consequence of the integrability of $r^{d-1}\varphi^{\frac{1}{2}}(r)$. The decay $\rho(n) = o(n^{-d})$ is obtained when $\varphi(r)$ decays faster than $r^{-d-\eta}$ for some $\eta > 0$.

Correctors for one-dimensional elliptic problem

Consider the homogenization of the following one-dimensional elliptic problems:

$$
\begin{aligned}
&-\frac{d}{dx} a_\varepsilon(x,\omega) \frac{d}{dx} u_\varepsilon + (q_0 + q_\varepsilon(x,\omega)) u_\varepsilon = \rho_\varepsilon(x,\omega) f(x), \qquad x \in D = (0,1), \\
&u_\varepsilon(0) = u_\varepsilon(1) = 0.
\end{aligned} \tag{46}
$$

We consider homogeneous Dirichlet conditions to simplify the presentation. The coefficients $a_\varepsilon(x,\omega)$ and $\rho_\varepsilon(x,\omega)$ are uniformly bounded from above and below: $0 < a_0 \le a_\varepsilon(x,\omega), \rho_\varepsilon(x,\omega) \le a_0^{-1}$. The (deterministic) absorption term q_0 is assumed to be a non-negative constant. The generalization to a non-negative smooth function $q_0(x)$ can be done.

Let us introduce the change of variables

$$
z_\varepsilon(x) = a^* \int_0^x \frac{1}{a_\varepsilon(t)} dt, \qquad \frac{dz_\varepsilon}{dx} = \frac{a^*}{a_\varepsilon(x)}, \qquad a^* = \frac{1}{\mathbb{E}\{a^{-1}\}}. \tag{47}
$$

and $\tilde{u}_\varepsilon(z) = u_\varepsilon(x)$. Then we find, with $x = x(z_\varepsilon)$ that

$$\begin{aligned}
&-(a^*)^2 \frac{d^2}{dz^2}\tilde{u}_\varepsilon + a^* q_0 \tilde{u}_\varepsilon + a_\varepsilon[(1 - a_\varepsilon^{-1} a^*) q_0 + q_\varepsilon]\tilde{u}_\varepsilon = a_\varepsilon \rho_\varepsilon f, \qquad 0 < z < z_\varepsilon(1) \\
&\tilde{u}_\varepsilon(0) = \tilde{u}_\varepsilon(z_\varepsilon(1)) = 0.
\end{aligned} \tag{48}$$

Let us introduce the following Green's function

$$\begin{aligned}
&-a^* \frac{d^2}{dx^2} G(x, y; L) + q_0 G(x, y; L) = \delta(x - y) \\
&G(0, y; L) = G(L, y; L) = 0.
\end{aligned} \tag{49}$$

Then, defining

$$\tilde{q}_\varepsilon(x, \omega) = (1 - a_\varepsilon^{-1}(x, \omega) a^*) q_0 + q_\varepsilon(x, \omega), \tag{50}$$

we find that

$$\begin{aligned}
\tilde{u}_\varepsilon(z) &= \int_0^{z_\varepsilon(1)} G(z, y; z_\varepsilon(1))(\rho_\varepsilon f - \tilde{q}_\varepsilon \tilde{u}_\varepsilon)(x(y)) \frac{a_\varepsilon}{a^*}(x(y)) dy, \\
u_\varepsilon(x) &= \int_0^1 G(z_\varepsilon(x), z_\varepsilon(y); z_\varepsilon(1))(\rho_\varepsilon f - \tilde{q}_\varepsilon u_\varepsilon)(y) dy.
\end{aligned}$$

We recast the above equation as

$$u_\varepsilon(x,\omega) = \mathcal{G}_\varepsilon(\rho_\varepsilon f - \tilde{q}_\varepsilon u_\varepsilon), \qquad \mathcal{G}_\varepsilon u(x) = \int_0^1 G(z_\varepsilon(x), z_\varepsilon(y); z_\varepsilon(1)) u(y) dy. \tag{51}$$

After one more iteration, we obtain the following integral equation:

$$u_\varepsilon = \mathcal{G}_\varepsilon \rho_\varepsilon f - \mathcal{G}_\varepsilon \tilde{q}_\varepsilon \mathcal{G}_\varepsilon \rho_\varepsilon f + \mathcal{G}_\varepsilon \tilde{q}_\varepsilon \mathcal{G}_\varepsilon \tilde{q}_\varepsilon u_\varepsilon. \tag{52}$$

A similar convergence result may then be obtained. See [B-08].

Random and periodic homogenization

Let us go back to the problem in the periodic case:

$$\begin{aligned} -\Delta u_\varepsilon + q\Big(\frac{\mathbf{x}}{\varepsilon}\Big) u_\varepsilon = f \quad & D \\ u_\varepsilon = 0 \qquad\qquad\qquad & \partial D, \end{aligned} \tag{53}$$

on a smooth open, bounded, domain $D \subset \mathbb{R}^d$, where $q(\mathbf{y})$ is $[0,1]^d$-periodic. We introduce the fast scale $\mathbf{y} = \frac{\mathbf{x}}{\varepsilon}$ and introduce a function $u_\varepsilon = u_\varepsilon(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})$. Gradients $\nabla_{\mathbf{x}}$ become $\frac{1}{\varepsilon}\nabla_{\mathbf{y}} + \nabla_{\mathbf{x}}$ and (53) becomes formally

$$\Big(-\frac{1}{\varepsilon^2}\Delta_{\mathbf{y}} - \frac{2}{\varepsilon}\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{y}} - \Delta_{\mathbf{x}} + q(\mathbf{y}) \Big) u_\varepsilon(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}).$$

Plugging the expansion $u_\varepsilon = u_0 + \varepsilon u_1 + \varepsilon^2 u_2$ into the above equality and equating like powers of ε yields three equations. The first equation shows that $u_0 = u_0(\mathbf{x})$. The second equation shows that $u_1 = u_1(\mathbf{x})$, which we can choose as $u_1 \equiv 0$. The third equation $-\Delta_{\mathbf{y}} u_2 - \Delta_{\mathbf{x}} u_0 + q(\mathbf{y}) u_0 = f(\mathbf{x})$, admits a solution provided that

$$-\Delta_{\mathbf{x}} u_0 + \langle q \rangle u_0 = f(\mathbf{x}), \qquad D$$

with $u_0 = 0$ on ∂D. Here, $\langle q \rangle$ is the average of q on $[0,1]^d$, which we assume is sufficiently large that the above equation admits a unique solution. We recast the above equation as $u_0 = \mathcal{G}_D f$. The corrector u_2 thus solves

$$-\Delta_{\mathbf{y}} u_2 = \Big(\langle q \rangle - q(\mathbf{y}) \Big) u_0(\mathbf{x}),$$

and is uniquely defined along with the constraint $\langle u_2 \rangle = 0$. We denote the solution operator of the above cell problem as $\mathcal{G}_{\#}$ so that $u_2 = -\mathcal{G}_{\#}(q - \langle q \rangle)\mathcal{G} f$. Thus formally, we have obtained that

$$u_\varepsilon(\mathbf{x}) = \mathcal{G} f(\mathbf{x}) - \varepsilon^2 \mathcal{G}_{\#}(q - \langle q \rangle)\Big(\frac{\mathbf{x}}{\varepsilon}\Big)\mathcal{G} f(\mathbf{x}) + \text{ l.o.t.} \qquad (54)$$

We thus observe that the corrector $u_{2\varepsilon}(\mathbf{x}) := u_2(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})$ is of order $O(\varepsilon^2)$

in the L^2 sense, say. In the sense of distributions, however, the corrector may be of order $o(\varepsilon^m)$ for all integer m in the sense that $\int_D M(\mathbf{x})u_{2\varepsilon}(\mathbf{x})d\mathbf{x} \ll \varepsilon^m$ for all m when $M(\mathbf{x})u_0(\mathbf{x}) \in \mathcal{C}_0^\infty(D)$.

Large deterministic corrector

Consider the equation with random boundary condition:

$$\begin{cases} (-\Delta + \lambda^2) u_\varepsilon(x,\omega) = 0, & x = (x', x_n) \in \mathbb{R}^n_+, \\ \dfrac{\partial}{\partial \nu} u_\varepsilon + (q_0 + q(\dfrac{x'}{\varepsilon}, \omega)) u_\varepsilon = f(x'), & x = (x', 0) \in \partial \mathbb{R}^n_+. \end{cases} \tag{55}$$

We follow the presentation in [BJ-11]

This equation is equivalent to the elliptic pseudo-differential equation:

$$(\sqrt{-\Delta_\perp + \lambda^2} + q_0 + q_\varepsilon(x,\omega)) u_\varepsilon = f, \tag{56}$$

where $\Delta_\perp$ is the Laplacian on $\mathbb{R}^d$, $d = n-1$, obtained from the Laplacian on $\mathbb{R}^n$ with $\partial^2_{x_n}$ eliminated.

The Green's function behaves as $|x|^{1-d}$ for $d = n-1$ and is therefore not square integrable for $d \geq 2$ ($n \geq 3$).

Assumptions on random field

We assume that $q(x,\omega)$ is *stationary* and α-mixing: For any Borel sets $A, B \subset \mathbb{R}^d$, the sub-$\sigma$-algebras $\mathcal{F}_A$ and $\mathcal{F}_B$ generated by the process restricted on A and B respectively decorrelate so rapidly that there exists some function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ with $\alpha(r)$ vanishing to zero as r tends to infinity, and for any $\mathcal{F}_A$ measurable set U and $\mathcal{F}_B$ measurable set V, we have

$$|\mathbb{P}(U)\mathbb{P}(V) - \mathbb{P}(U \cap V)| \leq \alpha\Big(d(A,B)\Big). \tag{57}$$

We further assume that $\alpha(r)$ has the following asymptotic behavior for some real number $\delta > 0$:

$$\alpha(r) \sim \frac{1}{r^{d+\delta}}, \text{ for } r \text{ sufficiently large.} \tag{58}$$

Fourth order cumulants. A further assumption on $q(x,\omega)$ is imposed so that we have an approximate formula for the fourth order cross-moment of the process. To formulate this condition, we need to introduce some terminologies.

Let $F = \{1, 2, 3, 4\}$ and $\mathcal{U}$ be the collections of two pairs of unordered numbers in F, i.e.,

$$\mathcal{U} = \Big\{p = \{\big(p(1), p(2)\big), \big(p(3), p(4)\big)\} \mid p(i) \in F, p(1) \neq p(2), p(3) \neq p(4)\Big\}. \tag{59}$$

As members in a set, the pairs $(p(1), p(2))$ and $(p(3), p(4))$ are required to be distinct; however, they can have one common index. There are three elements in $\mathcal{U}$ whose indices $p(i)$ are all different. They are precisely $\{(1, 2), (3, 4)\}$, $\{(1, 3), (2, 4)\}$ and $\{(1, 4), (2, 3)\}$. Let us denote by $\mathcal{U}_*$ the subset formed by these three elements, and its complement by $\mathcal{U}^*$.

Intuitively, we can visualize $\mathcal{U}$ in the following manner. Draw four points with indices 1 to 4. There are six line segments connecting them. The set $\mathcal{U}$ can be visualized as the collection of all possible ways to choose two line segments among the six. $\mathcal{U}_*$ corresponds to choices so that the two segments have disjoint ends, and $\mathcal{U}^*$ corresponds to choices such that the segments share one common end.

We assume that $q(x,\omega)$ has *controlled fourth order cumulants* in the sense that the following holds: For each $p \in \mathcal{U}^*$, there exists a real valued nonnegative function ϕ_p in $L^1 \cap L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, so that for any four point set $\{x_i\}_{i=1}^4$, $x_i \in \mathbb{R}^d$, we have the following condition on the fourth order cross-moment of $\{q(x_i,\omega)\}$:

$$\begin{aligned}&\left|\mathbb{E}\prod_{i=1}^4 q(x_i) - \sum_{p\in\mathcal{U}_*}\mathbb{E}\{q(x_{p(1)})q(x_{p(2)})\}\mathbb{E}\{q(x_{p(3)})q(x_{p(4)})\}\right|\\ &\le \sum_{p\in\mathcal{U}^*}\phi_p(x_{p(1)}-x_{p(2)},x_{p(3)}-x_{p(4)}).\end{aligned} \tag{60}$$

Deterministic and random correctors in $d = 2$

We decompose the corrector

$$u_\varepsilon - u = (\mathbb{E}\{u_\varepsilon\} - u) + (u_\varepsilon - \mathbb{E}\{u_\varepsilon\}), \tag{61}$$

the *deterministic corrector* and the *stochastic corrector*, respectively. Let us define

$$\tilde{R} := \int_{\mathbb{R}^2} \frac{R(y)}{2\pi|y|} dy, \tag{62}$$

and $\mathcal{G}$ the solution operator to

$$(\sqrt{-\Delta + \lambda^2} + q_0)u = f. \tag{63}$$

Theorem 9 *Let u_ε and u solve (56) and (63) respectively and $d = 2$. Let $q(x, \omega)$ satisfy the same conditions as in the previous theorem. Then* *we have,*

$$\lim_{\varepsilon\to 0}\frac{\mathbb{E}\{u_\varepsilon\}-u}{\varepsilon}=\tilde{R}\mathcal{G}u. \tag{64}$$

Here the limit is taken in the weak sense. That is, for an arbitrary test function $M\in C_c^\infty(\mathbb{R}^2)$, *the real number* $\varepsilon^{-1}\langle M,\mathbb{E}\{\xi_\varepsilon\}\rangle$ *converges to* $\langle \mathcal{G}M,\tilde{R}u\rangle$.

Theorem 10 *Let* u_ε *and* u *solve* (56) *and* (63) *respectively and* $d=2$. *Let* $q(x,\omega)$ *be stationary and mean-zero with strong mixing coefficient* $\alpha(r)$ *satisfying* (58), *and be uniformly bounded. Assume further that the joint fourth order cumulant of* q *satisfies* (60). *Then:*

$$\frac{u_\varepsilon-\mathbb{E}\{u_\varepsilon\}}{\varepsilon}\xrightarrow{\text{distribution}}-\sigma\int_{\mathbb{R}^2}G(x-y)u(y)dW_y, \tag{65}$$

where $\sigma^2=\int_{\mathbb{R}^d}R(x)dx$ *and* W_y *is the standard multi-parameter Wiener process in* $\mathbb{R}^2$. *The convergence here is weakly in* $\mathbb{R}^2$ *and in probability distribution.* *Proofs in [BJ-11].*

Heuristic argument for deterministic corrector

Consider

$$u_\varepsilon = \mathcal{G}f - \mathcal{G}q_\varepsilon\mathcal{G}f + \mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon u_\varepsilon, \tag{66}$$

pushed to

$$u_\varepsilon = \mathcal{G}f - \mathcal{G}q_\varepsilon\mathcal{G}f + \mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\mathcal{G}f - \mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\mathcal{G}u_\varepsilon. \tag{67}$$

Weakly, the third term is of the form

$$(q_\varepsilon\mathcal{G}q_\varepsilon u, \mathcal{G}M).$$

A deterministic contribution thus appears of the form

$$\mathbb{E}q_\varepsilon(x)G(x-y)q_\varepsilon(y) = G(x-y)R\Big(\frac{x-y}{\varepsilon}\Big) \sim \varepsilon^d G(\varepsilon z)R(z) \sim \varepsilon^{d-\alpha}G(z)R(z)$$

This provides a large deterministic corrector when G is singular.

Long Range Potentials

Following [BGMP-08], we are interested in the solution to the following elliptic equation with random coefficients

$$
\begin{aligned}
&-\frac{d}{dx}\Big(a\Big(\frac{x}{\varepsilon},\omega\Big)\frac{d}{dx}u^\varepsilon\Big) = f(x), \qquad 0 \le x \le 1, \quad \omega \in \Omega, \\
&u^\varepsilon(0,\omega) = 0, \qquad u^\varepsilon(1,\omega) = q.
\end{aligned} \tag{68}
$$

Here $a(x,\omega)$ is a stationary ergodic random process such that $0 < a_0 \le a(x,\omega) \le a_0^{-1}$ a.e. for $(x,\omega) \in (0,1)\times\Omega$, where $(\Omega,\mathcal{F},\mathfrak{P})$ is an abstract probability space. The source term $f \in W^{-1,\infty}(0,1)$ and $q \in \mathbb{R}$. Classical theories for elliptic equations then show the existence of a unique solution $u(\cdot,\omega) \in H^1(0,1)$ $\mathfrak{P}-$a.s.

As the scale of the micro-structure ε converges to 0, the solution $u^\varepsilon(x,\omega)$ converges $\mathfrak{P}$-a.s. weakly in $H^1(0,1)$ to the deterministic solution $\bar{u}$ of the homogenized equation

$$-\frac{d}{dx}\Big(a^*\frac{d}{dx}\bar{u}\Big) = f(x), \qquad 0 \le x \le 1, \\ \bar{u}(0) = 0, \qquad \bar{u}(1) = q. \tag{69}$$

The effective diffusion coefficient is given by $a^* = \Big(\mathbb{E}\{a^{-1}(0,\cdot)\}\Big)^{-1}$, where $\mathbb{E}$ is mathematical expectation with respect to $\mathfrak{P}$.

The above one-dimensional boundary value problems admit explicit solutions. Introducing $a^\varepsilon(x) = a(\frac{x}{\varepsilon})$ and $F(x) = \int_0^x f(y)dy$, we have:

$$u^\varepsilon(x,\omega) = c^\varepsilon(\omega)\int_0^x \frac{1}{a^\varepsilon(y,\omega)}dy - \int_0^x \frac{F(y)}{a^\varepsilon(y,\omega)}dy, \; c^\varepsilon(\omega) = \frac{q + \int_0^1 \frac{F(y)}{a^\varepsilon(y,\omega)}dy}{\int_0^1 \frac{1}{a^\varepsilon(y,\omega)}dy}, \tag{70}$$

$$\bar{u}(x) = c^*\frac{x}{a^*} - \int_0^x \frac{F(y)}{a^*}dy, \qquad c^* = a^*q + \int_0^1 F(y)dy. \tag{71}$$

Our aim is to characterize the behavior of $u^\varepsilon - \bar{u}$ as $\varepsilon \to 0$.

Hypothesis on the random process

Let us define the mean zero stationary random process

$$\varphi(x,\omega) = \frac{1}{a(x,\omega)} - \frac{1}{a^*}. \tag{72}$$

Hypothesis 11 *We assume that φ is of the form*

$$\varphi(x) = \Phi(g_x), \tag{73}$$

where Φ is a bounded function such that

$$\int \Phi(g) e^{-\frac{g^2}{2}} dg = 0, \tag{74}$$

and g_x is a stationary Gaussian process with mean zero and variance one. The autocorrelation function of g:

$$R_g(\tau) = \mathbb{E}\Big\{ g_x g_{x+\tau} \Big\},$$

is assumed to have a heavy tail of the form

$$R_g(\tau) \sim \kappa_g \tau^{-\alpha} \textit{ as } \tau \to \infty, \tag{75}$$

where $\kappa_g > 0$ *and* $\alpha \in (0, 1)$*.*

This hypothesis is satisfied by a large class of random coefficients. For instance, if we take $\Phi = \text{sgn}$, then φ models a two-component medium. If we take $\Phi = \tanh$ or arctan, then φ models a continuous medium with bounded variations.

Heavy tail of process

The autocorrelation function of the random process a has a heavy tail, as stated in the following proposition.

Proposition 12 *The process φ defined by (73) is a stationary random process with mean zero and variance V_2. Its autocorrelation function*

$$R(\tau) = \mathbb{E}\{\varphi(x)\varphi(x+\tau)\} \tag{76}$$

has a heavy tail of the form

$$R(\tau) \sim \kappa\tau^{-\alpha} \text{ as } \tau \to \infty, \tag{77}$$

where $\kappa = \kappa_g V_1^2$,

$$V_1 = \mathbb{E}\Big\{g_0\Phi(g_0)\Big\} = \frac{1}{\sqrt{2\pi}}\int g\Phi(g)e^{-\frac{g^2}{2}}dg\,, \tag{78}$$

$$V_2 = \mathbb{E}\Big\{\Phi^2(g_0)\Big\} = \frac{1}{\sqrt{2\pi}}\int \Phi^2(g)e^{-\frac{g^2}{2}}dg\,. \tag{79}$$

Proof. The fact that φ is a stationary random process with mean zero and variance V_2 is straightforward in view of the definition of φ. In particular, Eq. (74) implies that φ has mean zero.

For any x, τ, the vector $(g_x, g_{x+\tau})^T$ is a Gaussian random vector with mean $(0,0)^T$ and 2×2 covariance matrix:

$$C = \begin{pmatrix} 1 & R_g(\tau) \\ R_g(\tau) & 1 \end{pmatrix}.$$

Therefore the autocorrelation function of the process φ is

$$\begin{aligned} R(\tau) &= \mathbb{E}\big\{\Phi(g_x)\Phi(g_{x+\tau})\big\} = \frac{1}{2\pi\sqrt{\det C}} \iint \Phi(g_1)\Phi(g_2) \exp\Big(-\frac{g^T C^{-1} g}{2}\Big) d^2 g \\ &= \frac{1}{2\pi\sqrt{1-R_g^2(\tau)}} \iint \Phi(g_1)\Phi(g_2) \exp\Big(-\frac{g_1^2+g_2^2-2R_g(\tau)g_1 g_2}{2(1-R_g^2(\tau))}\Big) dg_1 dg_2 \,. \end{aligned}$$

For large τ, $R_g(\tau)$ is small and we expand the value of the double integral in powers of $R_g(\tau)$, which gives the autocorrelation function of φ.

Analysis of the corrector

The error term $u^\varepsilon - \bar{u}$ has two different contributions: integrals of random processes with long term memory effects and lower-order terms. We consider the latter. The following lemma provides an estimate for the magnitude of these integrals.

Lemma 13 *Let $\varphi(x)$ be a mean zero stationary random process of the form (73). There exists $K > 0$ such that, for any $F \in L^\infty(0,1)$, we have*

$$\sup_{x\in[0,1]} \mathbb{E}\Big\{ \Big| \int_0^x \varphi^\varepsilon(t) F(t) dt \Big|^2 \Big\} \leq K \|F\|_\infty^2 \varepsilon^\alpha . \tag{80}$$

Proof. We verify that the l.h.s. is bounded by

$$\int_0^1 \int_0^1 F(t)F(s) R\Big(\frac{t-s}{\varepsilon}\Big) dt ds.$$

Since $|R(u)| \le \kappa u^{-\alpha}$, we obtain the bound

$$\varepsilon^\alpha C\|F\|_\infty^2 \int_{[0,1]^2} |z-t|^{-\alpha} dz dt \le \varepsilon^\alpha \frac{2C}{1-\alpha}\|F\|_\infty^2.$$

Corollary 14 *Let $\varphi(x)$ be a mean zero stationary random process of the form (73) and let $f \in W^{-1,\infty}(0,1)$. The solutions u^ε of (70) and $\bar{u}$ of (71) verify that:*

$$u^\varepsilon(x) - \bar{u}(x) = -\int_0^x \varphi^\varepsilon(y)F(y)dy + (c^\varepsilon - c^*)\frac{x}{a^*} + c^* \int_0^x \varphi^\varepsilon(y)dy + r^\varepsilon(x), \quad (81)$$

where

$$\sup_{x\in[0,1]} \mathbb{E}\{|r^\varepsilon(x)|\} \le K\varepsilon^\alpha\,, \quad (82)$$

for some $K > 0$. Similarly, we have that

$$c^\varepsilon - c^* = a^* \int_0^1 \left(F(y) - \int_0^1 F(z)dz - a^* q\right)\varphi^\varepsilon(y)dy + \rho^\varepsilon, \quad (83)$$

where

$$\mathbb{E}\{|\rho^\varepsilon|\} \le K\varepsilon^\alpha\,, \tag{84}$$

for some $K > 0$.

Proof. We first establish the estimate for $c^\varepsilon - c$. We write

$$c^\varepsilon - c^* = \frac{\int_0^1 F(y)\Big(\frac{1}{a^\varepsilon(y)} - \frac{1}{a^*}\Big)dy}{\int_0^1 \frac{1}{a^\varepsilon(y)}dy} + \Big(q + \frac{1}{a^*}\int_0^1 F(y)dy\Big)\Big(\frac{1}{\int_0^1 \frac{1}{a^\varepsilon(y)}dy} - \frac{1}{\frac{1}{a^*}}\Big),$$

which gives (83) with

$$\rho^\varepsilon = \frac{a^*}{\int_0^1 \frac{1}{a^\varepsilon(y)}dy}\left[(a^*q + \int_0^1 F(y)dy)\left(\int_0^1 \varphi^\varepsilon(y)dy\right)^2 - \int_0^1 F(y)\varphi^\varepsilon(y)dy\int_0^1 \varphi^\varepsilon(y)dy\right].$$

Since $\int_0^1 \frac{1}{a^\varepsilon(y)}dy$ is bounded from below a.e. by a positive constant a_0, we deduce from Lemma 13 and the Cauchy-Schwarz estimate that $\mathbb{E}\{|\rho^\varepsilon|\} \le K\varepsilon^\alpha$. The analysis of $u^\varepsilon - \bar{u}$ follows along the same lines. We write

$$u^\varepsilon(x) - \bar{u}(x) = c^\varepsilon \int_0^x \frac{1}{a^\varepsilon(y)} dy - \int_0^x \frac{F(y)}{a^\varepsilon(y)} dy - c^* \frac{x}{a^*} + \int_0^x \frac{F(y)}{a^*} dy,$$

which gives (81) with

$$r^\varepsilon(x) = (c^\varepsilon - c^*) \int_0^x \varphi^\varepsilon(y) dy = r_1^\varepsilon(x) + r_2^\varepsilon(x),$$

where we have defined

$$\begin{aligned} r_1^\varepsilon(x) &= \left[a^* \int_0^1 \left(F(y) - \int_0^1 F(z)dz - a^* q\right) \varphi^\varepsilon(y) dy\right] \left[\int_0^x \varphi^\varepsilon(y) dy\right], \\ r_2^\varepsilon(x) &= \rho^\varepsilon \left[\int_0^x \varphi^\varepsilon(y) dy\right]. \end{aligned}$$

The Cauchy-Schwarz estimate and Lemma 13 give that $\mathbb{E}\{|r_1^\varepsilon(x)|\} \leq K\varepsilon^\alpha$. Besides, φ^ε is bounded by $\|\Phi\|_\infty$, so $|r_2^\varepsilon(x)| \leq \|\Phi\|_\infty |\rho^\varepsilon|$. The estimate on ρ^ε then shows that $\mathbb{E}\{|r_2^\varepsilon(x)|\} \leq K\varepsilon^\alpha$.

Characterization of correctors

Theorem 15 *Let u^ε and $\bar{u}$ be the solutions in* (70) *and* (71)*, respectively, and let $\varphi(x)$ be a mean zero stationary random process of the form (73). Then $u^\varepsilon - \bar{u}$ is a random process in $\mathcal{C}(0,1)$, the space of continuous functions on $[0,1]$. We have the following convergence in distribution in the space of continuous functions $\mathcal{C}(0,1)$:*

$$\frac{u^\varepsilon(x) - \bar{u}(x)}{\varepsilon^{\frac{\alpha}{2}}} \xrightarrow{\text{distribution}} \sqrt{\frac{\kappa}{H(2H-1)}}\,\mathcal{U}^H(x), \tag{85}$$

where

$$\mathcal{U}^H(x) = \int_{\mathbb{R}} K(x,t)\,dW_t^H, \tag{86}$$

$$K(x,t) = \mathbf{1}_{[0,x]}(t)\Big(c^* - F(t)\Big) + x\Big(F(t) - \int_0^1 F(z)dz - a^* q\Big)\mathbf{1}_{[0,1]}(t) \tag{87}$$

Here $\mathbf{1}_{[0,x]}$ is the characteristic function of the set $[0,x]$ and W_t^H is a fractional Brownian motion with Hurst index $H = 1 - \frac{\alpha}{2}$.

The fractional Brownian motion W_t^H is a mean zero Gaussian process with autocorrelation function

$$\mathbb{E}\{W_t^H W_s^H\} = \frac{1}{2}\Big(|t|^{2H} + |s|^{2H} - |s-t|^{2H}\Big). \tag{88}$$

In particular, the variance of W_t^H is $\mathbb{E}\{|W_t^H|^2\} = |t|^{2H}$.

The increments of W_t^H are stationary but not independent for $H \neq \frac{1}{2}$.

Moreover, W_t^H admits the following spectral representation

$$W_t^H = \frac{1}{2\pi C(H)} \int_{\mathbb{R}} \frac{e^{i\xi t} - 1}{i\xi|\xi|^{H-\frac{1}{2}}} d\widehat{W}(\xi), \quad t \in \mathbb{R}, \tag{89}$$

where

$$C(H) = \Big(\frac{1}{2H\sin(\pi H)\Gamma(2H)}\Big)^{1/2}, \tag{90}$$

and $\hat{W}$ is the Fourier transform of a standard Brownian motion W, that is, a complex Gaussian measure such that:

$$\mathbb{E}\Big\{d\hat{W}(\xi)\overline{d\hat{W}(\xi')}\Big\} = 2\pi\delta(\xi-\xi')d\xi d\xi' .$$

Note that the constant $C(H)$ is defined such that $\mathbb{E}\{(W_1^H)^2\} = 1$.

Convergence of random integrals

Theorem 16 *Let* φ *be of the form (73) and let* $F \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$*. We define the mean zero random variable* M_F^ε *by*

$$M_F^\varepsilon = \varepsilon^{-\frac{\alpha}{2}} \int_{\mathbb{R}} \varphi^\varepsilon(t) F(t) dt \,. \tag{91}$$

Then the random variable M_F^ε *converges in distribution as* $\varepsilon \to 0$ *to the mean zero Gaussian random variable* M_F^0 *defined by*

$$M_F^0 = \sqrt{\frac{\kappa}{H(2H-1)}} \int_{\mathbb{R}} F(t) dW_t^H \,, \tag{92}$$

where W_t^H *is a fractional Brownian motion with Hurst index* $H = 1 - \frac{\alpha}{2}$*.*

The limit random variable M_F^0 is a Gaussian random variable with mean zero and variance

$$\mathbb{E}\{|M_F^0|^2\} = \frac{\kappa}{H(2H-1)} \times \frac{1}{2\pi C(H)^2} \int_{\mathbb{R}} \frac{|\widehat{F}(\xi)|^2}{|\xi|^{2H-1}} d\xi \,. \tag{93}$$

We first show that the variance of M_F^ε converges to the variance of M_F^0 as $\varepsilon \to 0$.

We then prove convergence in distribution by using the Gaussian property of the underlying process g_x.

Convergence of the variances

We begin with a key technical lemma.

Lemma 17 *1. There exist $T, K > 0$ such that the autocorrelation function $R(\tau)$ of the process φ satisfies*

$$|R(\tau) - V_1^2 R_g(\tau)| \leq K R_g(\tau)^2, \qquad \textit{for all} \quad |\tau| \geq T.$$

2. There exist T, K such that

$$\Big|\mathbb{E}\{g_x \Phi(g_{x+\tau})\} - V_1 R_g(\tau)\Big| \leq K R_g^2(\tau) \qquad \textit{for all} \quad |\tau| \geq T.$$

Proof. The first point is a refinement of what we proved in Proposition 12: we found that the autocorrelation function of the process φ is

$$R(\tau) = \frac{1}{2\pi\sqrt{1 - R_g^2(\tau)}} \iint \Phi(g_1)\Phi(g_2) \exp\Big(- \frac{g_1^2 + g_2^2 - 2R_g(\tau) g_1 g_2}{2(1 - R_g^2(\tau))}\Big) dg_1 dg_2$$

For large τ, the coefficient $R_g(\tau)$ is small and we can expand the value of the double integral in powers of $R_g(\tau)$, which gives the result of the first item. The proof of the second item follows along the same lines.

We first write

$$\mathbb{E}\{g_x\Phi(g_{x+\tau})\} = \frac{1}{2\pi\sqrt{1-R_g^2(\tau)}}\iint g_1\Phi(g_2)\exp\Big(-\frac{g_1^2+g_2^2-2R_g(\tau)g_1g_2}{2(1-R_g^2(\tau))}\Big)dg_1dg_2\,,$$

and we expand the value of the double integral in powers of $R_g(\tau)$.

Convergence of the variances II

For $F \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, we define the mean zero random variable $M_F^{\varepsilon,g}$ by and recall the definition of M_F^ε:

$$M_F^{\varepsilon,g} = \varepsilon^{-\frac{\alpha}{2}} \int_{\mathbb{R}} g_{\frac{t}{\varepsilon}} F(t)dt \,, \qquad M_F^{\varepsilon} = \varepsilon^{-\frac{\alpha}{2}} \int_{\mathbb{R}} \varphi^\varepsilon(t) F(t)dt \,. \tag{94}$$

Lemma 18 *Let $F \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and let g_x be the Gaussian random process described in Hypothesis 11. Then*

$$\lim_{\varepsilon \to 0} \mathbb{E}\Big\{ \Big| M_F^{\varepsilon,g} \Big|^2 \Big\} = \frac{\kappa_g 2^{-\alpha} \Gamma(\frac{1-\alpha}{2})}{\sqrt{\pi}\Gamma(\frac{\alpha}{2})} \int_{\mathbb{R}} \frac{|\widehat{F}(\xi)|^2}{|\xi|^{1-\alpha}} d\xi \,. \tag{95}$$

Proof. We write the square of the integral as a double integral, which gives

$$\mathbb{E}\Big\{ \Big| \int_{\mathbb{R}} F(y) g_{\frac{y}{\varepsilon}} dy \Big|^2 \Big\} = \int_{\mathbb{R}^2} R_g\Big(\frac{y-z}{\varepsilon}\Big) F(y) F(z) dy dz \,.$$

This implies the estimate

$$\left|\mathbb{E}\Big\{\Big|M_F^{\varepsilon,g}\Big|^2\Big\} - \int_{\mathbb{R}^2} \frac{\kappa_g}{|y-z|^\alpha} F(y)F(z)dydz\right|$$
$$\leq \int_{\mathbb{R}^2} \left|\varepsilon^{-\alpha} R_g\Big(\frac{y-z}{\varepsilon}\Big) - \frac{\kappa_g}{|y-z|^\alpha}\right| |F(y)||F(z)|dydz\,.$$

By (75), for any $\delta > 0$, there exists T_δ such that, for all $|\tau| \geq T_\delta$,

$$\Big|R_g(\tau) - \kappa_g \tau^{-\alpha}\Big| \leq \delta\tau^{-\alpha}\,.$$

We decompose the integration domain into three subdomains D_1, D_2, and D_3:

$$\begin{aligned} D_1 &= \Big\{(y,z)\in\mathbb{R}^2,\, |y-z| \leq T_\delta\varepsilon\Big\}\,, \\ D_2 &= \Big\{(y,z)\in\mathbb{R}^2,\, T_\delta\varepsilon < |y-z| \leq 1\Big\}\,, \\ D_3 &= \Big\{(y,z)\in\mathbb{R}^2,\, 1 < |y-z|\Big\}\,. \end{aligned}$$

First,

$$
\begin{aligned}
&\int_{D_1} \left| \varepsilon^{-\alpha} R_g\Big(\frac{y-z}{\varepsilon}\Big) - \frac{\kappa_g}{|y-z|^{\alpha}} \right| |F(y)||F(z)| dy dz \\
&\quad \leq \int_{D_1} \left| \varepsilon^{-\alpha} R_g\Big(\frac{y-z}{\varepsilon}\Big) \right| |F(y)||F(z)| dy dz + \int_{D_1} \kappa_g |y-z|^{-\alpha} |F(y)||F(z)| dy dz \\
&\quad \leq 2\varepsilon^{-\alpha} \|R_g\|_{\infty} \int_{\mathbb{R}} \int_0^{T_\delta \varepsilon} |F(y+z)| dy |F(z)| dz + 2\kappa_g \int_{\mathbb{R}} \int_0^{T_\delta \varepsilon} y^{-\alpha} |F(y+z)| dy |F(\\
&\quad \leq 2\varepsilon^{-\alpha} \|R_g\|_{\infty} \|F\|_{\infty} \|F\|_1 \int_0^{T_\delta \varepsilon} dy + 2\kappa_g \|F\|_{\infty} \|F\|_1 \int_0^{T_\delta \varepsilon} y^{-\alpha} dy \\
&\quad \leq \|F\|_{\infty} \|F\|_1 \left(2T_\delta R_g(0) + \frac{2\kappa_g T_\delta^{1-\alpha}}{1-\alpha} \right) \varepsilon^{1-\alpha} ,
\end{aligned}
$$

where we have used the fact that $R_g(\tau)$ is maximal at $\tau = 0$, and the value of the maximum is equal to the variance of g. Second,

$$\begin{aligned}\int_{D_2}\left|\varepsilon^{-\alpha}R_g\Big(\frac{y-z}{\varepsilon}\Big)-\frac{\kappa_g}{|y-z|^{\alpha}}\right||F(y)||F(z)|dydz &\leq \delta\int_{D_2}|y-z|^{-\alpha}|F(y)||F(z)|dy\\ &\leq 2\delta\|F\|_{\infty}\|F\|_1\int_{T_\delta\varepsilon}^{1}y^{-\alpha}dy\\ &\leq \frac{2\delta\|F\|_{\infty}\|F\|_1}{1-\alpha},\end{aligned}$$

and finally

$$\begin{aligned}\int_{D_3}\left|\varepsilon^{-\alpha}R_g\Big(\frac{y-z}{\varepsilon}\Big)-\frac{\kappa_g}{|y-z|^{\alpha}}\right||F(y)||F(z)|dydz &\leq \delta\int_{D_3}|y-z|^{-\alpha}|F(y)||F(z)|dy\\ &\leq \delta\int_{D_3}|F(y)||F(z)|dydz\\ &\leq \delta\|F\|_1^2.\end{aligned}$$

Therefore, there exists $K > 0$ such that

$$\limsup_{\varepsilon\to 0}\left|\mathbb{E}\Big\{\Big|M_F^{\varepsilon,g}\Big|^2\Big\}-\int_{\mathbb{R}^2}\frac{\kappa_g}{|y-z|^\alpha}F(y)F(z)dydz\right|\leq K\left(\|F\|_\infty^2+\|F\|_1^2\right)\delta\,.$$

Since this holds true for any $\delta > 0$, we get

$$\lim_{\varepsilon\to 0}\left|\mathbb{E}\Big\{\Big|M_F^{\varepsilon,g}\Big|^2\Big\}-\int_{\mathbb{R}^2}\frac{\kappa_g}{|y-z|^\alpha}F(y)F(z)dydz\right|=0\,.$$

We recall that the Fourier transform of the function $|x|^{-\alpha}$ is

$$\widehat{|x|^{-\alpha}}(\xi)=c_\alpha|\xi|^{\alpha-1}\,,\qquad c_\alpha=\int_{\mathbb{R}}\frac{e^{it}}{|t|^\alpha}dt=\frac{\sqrt{\pi}2^{1-\alpha}\Gamma(\frac{1-\alpha}{2})}{\Gamma(\frac{\alpha}{2})}\,. \tag{96}$$

Using the Parseval equality, we find that

$$\int_{\mathbb{R}^2}\frac{1}{|y-z|^\alpha}F(y)F(z)dydz=\frac{c_\alpha}{2\pi}\int_{\mathbb{R}}\frac{|\hat{F}(\xi)|^2}{|\xi|^{1-\alpha}}d\xi\,.$$

The right-hand side is finite, because (i) $F\in L^1(\mathbb{R})$ so that $\hat{F}(\xi)\in L^\infty(\mathbb{R})$, (ii) $F\in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$ so $F\in L^2(\mathbb{R})$ and $\hat{F}\in L^2(\mathbb{R})$, and (iii) $\alpha\in(0,1)$.

Lemma 19 *Let* $F \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *and let the process* $\varphi(x)$ *be of the form* (73)*. Then we have:*

$$\lim_{\varepsilon \to 0} \mathbb{E}\Big\{(M_F^\varepsilon - V_1 M_F^{\varepsilon,g})^2\Big\} = 0\,.$$

Proof.

We write the square of the integral as a double integral:

$$\mathbb{E}\Big\{(M_F^\varepsilon - V_1 M_F^{\varepsilon,g})^2\Big\} = \varepsilon^{-\alpha} \int_{\mathbb{R}^2} F(y)F(z)Q(\frac{y}{\varepsilon}, \frac{z}{\varepsilon})dydz\,,$$

where

$$Q(y,z) = \mathbb{E}\Big\{\Phi(g_y)\Phi(g_z) - V_1\Phi(g_y)g_z - V_1 g_y \Phi(g_z) + V_1^2 g_y g_z\Big\}\,.$$

By Lemma 17 and (75), there exist K, T such that $|Q(y,z)| \le K|y-z|^{-2\alpha}$ for all $|x-y| \ge T$. Besides, Φ is bounded and g_x is square-integrable, so there exists K such that, for all $y, z \in \mathbb{R}$, $|Q(y,z)| \leq K$. We decompose the integration domain $\mathbb{R}^2$ into three subdomains D_1, D_2, and D_3:

$$\begin{aligned} D_1 &= \left\{(y,z) \in \mathbb{R}^2,\, |y-z| \leq T\varepsilon\right\}, \\ D_2 &= \left\{(y,z) \in \mathbb{R}^2,\, T\varepsilon < |y-z| \leq 1\right\}, \\ D_3 &= \left\{(y,z) \in \mathbb{R}^2,\, 1 < |y-z|\right\}. \end{aligned}$$

We get the estimates

$$\begin{aligned} \left|\int_{D_1} F(y)F(z)Q(\frac{y}{\varepsilon},\frac{z}{\varepsilon})dydz\right| &\leq K\int_{D_1} |F(y)||F(z)|dydz \\ &\leq 2K\int_{\mathbb{R}}\int_0^{T\varepsilon} |F(y+z)|dy|F(z)|dz \\ &\leq 2K\|F\|_\infty\|F\|_1 T\varepsilon\,, \end{aligned}$$

$$
\begin{aligned}
\left|\int_{D_2} F(y)F(z)Q(\frac{y}{\varepsilon},\frac{z}{\varepsilon})dydz\right| &\leq K\int_{D_2}\left|\frac{y}{\varepsilon}-\frac{z}{\varepsilon}\right|^{-2\alpha}|F(y)||F(z)dydz \\
&\leq 2K\varepsilon^{2\alpha}\int_{\mathbb{R}}\int_{T\varepsilon}^{1} y^{-2\alpha}|F(y+z)|dy|F(z)|dz \\
&\leq 2K\|F\|_1\|F\|_\infty\varepsilon^{2\alpha}\int_{T\varepsilon}^{1} y^{-2\alpha}dy \\
&\leq 2K\|F\|_1\|F\|_\infty \begin{cases} \dfrac{1}{1-2\alpha}\varepsilon^{2\alpha} \text{ if } \alpha < \dfrac{1}{2} \\ |\ln(T\varepsilon)|\varepsilon \text{ if } \alpha = \dfrac{1}{2} \\ \dfrac{T^{1-2\alpha}}{2\alpha-1}\varepsilon \text{ if } \alpha > \dfrac{1}{2} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\left|\int_{D_3} F(y)F(z)Q(\frac{y}{\varepsilon},\frac{z}{\varepsilon})dydz\right| &\leq K\int_{D_3}\left|\frac{y}{\varepsilon}-\frac{z}{\varepsilon}\right|^{-2\alpha}|F(y)||F(z)dydz \\
&\leq 2K\varepsilon^{2\alpha}\int_{\mathbb{R}}\int_1^\infty y^{-2\alpha}|F(y+z)|dy|F(z)|dz \\
&\leq 2K\varepsilon^{2\alpha}\int_{\mathbb{R}}\int_1^\infty |F(y+z)|dy|F(z)|dz \\
&\leq 2K\|F\|_1^2\varepsilon^{2\alpha},
\end{aligned}
$$

which gives the desired result:

$$
\lim_{\varepsilon\to 0}\varepsilon^{-\alpha}\left|\int_{\mathbb{R}^2} F(y)F(z)Q(\frac{y}{\varepsilon},\frac{z}{\varepsilon})dydz\right| = 0\,.
$$

The following proposition is now a straightforward corollary of Lemmas 18 and 19 and the fact that $\kappa = \kappa_g V_1^2$.

Proposition 20 *Let* $F \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *and let the process* $\varphi(x)$ *be of* *the form* (73). *Then we find that:*

$$\lim_{\varepsilon\to 0} \mathbb{E}\Big\{\Big|M_F^\varepsilon\Big|^2\Big\} = \frac{\kappa 2^{-\alpha}\Gamma(\frac{1-\alpha}{2})}{\sqrt{\pi}\Gamma(\frac{\alpha}{2})}\int_{\mathbb{R}} \frac{|\hat{F}(\xi)|^2}{|\xi|^{1-\alpha}} d\xi \,. \tag{97}$$

The limit of the variance of M_F^ε is (97) and the variance of M^0 is (93). These two expressions are reconciled by using the identity $1-\alpha = 2H-1$ and standard properties of the Γ function, namely $\Gamma(H)\Gamma(H+\frac{1}{2}) = 2^{1-2H}\sqrt{\pi}\Gamma(2H)$ and $\Gamma(1-H)\Gamma(H) = \pi(\sin(\pi H))^{-1}$. We get

$$\frac{2^{-\alpha}\Gamma(\frac{1-\alpha}{2})}{\sqrt{\pi}\Gamma(\frac{\alpha}{2})} = \frac{2^{-2+2H}\Gamma(H-\frac{1}{2})}{\sqrt{\pi}\Gamma(1-H)} = \frac{2^{-2+2H}\Gamma(H+\frac{1}{2})}{\sqrt{\pi}(H-\frac{1}{2})\Gamma(1-H)} = \frac{\Gamma(2H)\sin(\pi H)}{\pi(2H-1)} \,.$$

By (90) this shows that

$$\frac{2^{-\alpha}\Gamma(\frac{1-\alpha}{2})}{\sqrt{\pi}\Gamma(\frac{\alpha}{2})} 2\pi = \frac{1}{H(2H-1)C(H)^2} \,,$$

and this implies that the variance (93) of M_F^0 is exactly the limit (97) of the variance of M_F^ε:

$$\lim_{\varepsilon \to 0} \mathbb{E}\left\{ \left| M_F^\varepsilon \right|^2 \right\} = \mathbb{E}\left\{ \left| M_F^0 \right|^2 \right\}.$$

Convergence of random integrals

We can now give the proof of Theorem 16.

Step 1. The sequence of random variables $M_F^{\varepsilon,g}$ *defined by (94) converges in distribution as* $\varepsilon \to 0$ *to*

$$M_F^{0,g} = \sqrt{\frac{\kappa_g}{H(2H-1)}} \int_{\mathbb{R}} F(t) dW_t^H \,.$$

Since the random variable $M_F^{\varepsilon,g}$ is a linear transform of a Gaussian process, it has Gaussian distribution. Moreover, its mean is zero. The same statements hold true for $M_F^{0,g}$. Therefore, the characteristic functions of $M_F^{\varepsilon,g}$ and $M_F^{0,g}$ are

$$\mathbb{E}\Big\{ e^{i\lambda M_F^{\varepsilon,g}} \Big\} = \exp\left(-\frac{\lambda^2}{2} \mathbb{E}\Big\{ (M_F^{\varepsilon,g})^2 \Big\} \right) , \quad \mathbb{E}\Big\{ e^{i\lambda M_F^{0,g}} \Big\} = \exp\left(-\frac{\lambda^2}{2} \mathbb{E}\Big\{ (M_F^{0,g})^2 \Big\} \right) ,$$

where $\lambda \in \mathbb{R}$. Convergence of the characteristic functions implies that of the distributions [?]. Therefore, it is sufficient to show that the variance of $M_F^{\varepsilon,g}$ converges to the variance of $M_F^{0,g}$ as $\varepsilon \to 0$. This follows from Lemma 18.

Step 2: M_F^{ε} converges in distribution to M_F^0 as $\varepsilon \to 0$.

Let $\lambda \in \mathbb{R}$. Since $M_F^0 = V_1 M_F^{0,g}$, we have

$$\left|\mathbb{E}\left\{e^{i\lambda M_F^{\varepsilon}}\right\} - \mathbb{E}\left\{e^{i\lambda M_F^{0}}\right\}\right| \leq \left|\mathbb{E}\left\{e^{i\lambda M_F^{\varepsilon}}\right\} - \mathbb{E}\left\{e^{i\lambda V_1 M_F^{\varepsilon,g}}\right\}\right| + \left|\mathbb{E}\left\{e^{i\lambda V_1 M_F^{\varepsilon,g}}\right\} - \mathbb{E}\left\{e^{i\lambda V_1 M_F^{0,g}}\right\}\right| . \quad (98)$$

Since $|e^{ix} - 1| \leq |x|$ we can write

$$\left|\mathbb{E}\left\{e^{i\lambda M_F^{\varepsilon}}\right\} - \mathbb{E}\left\{e^{i\lambda V_1 M_F^{\varepsilon,g}}\right\}\right| \leq |\lambda| \mathbb{E}\left\{|M_F^{\varepsilon} - V_1 M_F^{\varepsilon,g}|\right\} \leq |\lambda| \mathbb{E}\left\{(M_F^{\varepsilon} - V_1 M_F^{\varepsilon,g})^2\right\}^{1/2},$$

which goes to zero by the result of Lemma 19. This shows that the first term of the right-hand side of (98) converges to 0 as $\varepsilon \to 0$. The second term of the right-hand side of (98) also converges to zero by the result of Step 1. This completes the proof of Theorem 16.

Convergence of random integral processes

Let F_1, F_2 be two functions in $L^\infty(0,1)$. We consider the random process $M^\varepsilon(x)$ defined for any $x \in [0,1]$ by

$$M^\varepsilon(x) = \varepsilon^{-\frac{\alpha}{2}} \left(\int_0^x F_1(t)\varphi^\varepsilon(t)dt + x \int_0^1 F_2(t)\varphi^\varepsilon(t)dt \right) . \tag{99}$$

With the notation (91) of the previous section, we have

$$M^\varepsilon(x) = M^\varepsilon_{F_x} = \varepsilon^{-\frac{\alpha}{2}} \int_{\mathbb{R}} F_x(t)\varphi^\varepsilon(t)dt \, ,$$

where

$$F_x(t) = F_1(t)\mathbf{1}_{[0,x]}(t) + xF_2(t)\mathbf{1}_{[0,1]}(t) \tag{100}$$

is indeed a function in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

Theorem 21 *Let φ be a random process of the form (73) and let $F_1, F_2 \in L^\infty(0,1)$. Then the random process $M^\varepsilon(x)$ defined by (99) converges in* *distribution as $\varepsilon \to 0$ in the space of the continuous functions $\mathcal{C}(0,1)$ to the continuous Gaussian process*

$$M^0(x) = \sqrt{\frac{\kappa}{H(2H-1)}} \int_{\mathbb{R}} F_x(t) dW_t^H \,, \tag{101}$$

where F_x is defined by (100) and W_t^H is a fractional Brownian motion with Hurst index $H = 1 - \frac{\alpha}{2}$.

The limit random process M^0 is a Gaussian process with mean zero and autocorrelation function given by

$$\mathbb{E}\Big\{M^0(x)M^0(y)\Big\} = \frac{\kappa}{H(2H-1)} \times \frac{1}{2\pi C(H)^2} \int_{\mathbb{R}} \frac{\widehat{F}_x(\xi)\overline{\widehat{F}_y(\xi)}}{|\xi|^{2H-1}} d\xi \,. \tag{102}$$

The proof of Theorem 21 is based on a classical result on the weak convergence of continuous random processes [Billingsley]:

Proposition 22 *Suppose* $(M^\varepsilon)_{\varepsilon\in(0,1)}$ *are random processes with values in the space of continuous functions* $\mathcal{C}(0,1)$ *with* $M^\varepsilon(0) = 0$*. Then* M^ε *converges in distribution to* M^0 *provided that:*

(i) for any $0 \le x_1 \le \ldots \le x_k \le 1$*, the finite-dimensional distribution* $(M^\varepsilon(x_1), \cdots, M^\varepsilon(x_k))$ *converges to the distribution* $(M^0(x_1), \ldots, M^0(x_k))$ *as* $\varepsilon \to 0$*.*

(ii) $(M^\varepsilon)_{\varepsilon\in(0,1)}$ *is a tight sequence of random processes in* $\mathcal{C}(0,1)$*. A sufficient condition for tightness of* $(M^\varepsilon)_{\varepsilon\in(0,1)}$ *is the Kolmogorov criterion:* $\exists\delta, \beta, C > 0$ *such that*

$$\mathbb{E}\Big\{\Big|M^\varepsilon(s) - M^\varepsilon(t)\Big|^\beta\Big\} \le C|t-s|^{1+\delta}, \qquad (103)$$

uniformly in $\varepsilon, t, s \in (0,1)$*.*

Convergence of finite-dimensional distributions

For the proof of convergence of the finite-dimensional distributions, we want to show that for each set of points $0 \leq x_1 \leq \cdots \leq x_k \leq 1$ and each $\Lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k$, we have the following convergence result for the characteristic functions:

$$\mathbb{E}\Big\{ \exp\Big(i \sum_{j=1}^{k} \lambda_j M^\varepsilon(x_j)\Big)\Big\} \xrightarrow{\varepsilon \to 0} \mathbb{E}\Big\{ \exp\Big(i \sum_{j=1}^{k} \lambda_j M^0(x_j)\Big)\Big\}. \qquad (104)$$

Convergence of the characteristic functions implies that of the joint distributions. Now the above characteristic function may be recast as

$$\mathbb{E}\Big\{ \exp\Big(i \sum_{j=1}^{k} \lambda_j M^\varepsilon(x_j)\Big)\Big\} = \mathbb{E}\Big\{ \exp i\Big(\varepsilon^{-\frac{\alpha}{2}} \int_{\mathbb{R}} \varphi^\varepsilon(t) F_\Lambda(t) dt\Big)\Big\}, \qquad (105)$$

where

$$F_\Lambda(t) = \Big(\sum_{j=1}^{k} \lambda_j \mathbf{1}_{[0,x_j]}(t)\Big) F_1(t) + \Big(\sum_{j=1}^{k} \lambda_j x_j\Big) \mathbf{1}_{[0,1]}(t) F_2(t)\,.$$

Since $F_\Lambda \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ when $F_1, F_2 \in L^\infty(0,1)$, we can apply Theorem 16 to obtain that:

$$\mathbb{E}\Big\{\exp\Big(i\sum_{j=1}^{k} \lambda_j M^\varepsilon(x_j)\Big)\Big\} \overset{\varepsilon\to 0}{\longrightarrow} \mathbb{E}\Big\{\exp i\Big(\sqrt{\frac{\kappa}{H(2H-1)}}\int_{\mathbb{R}} F_\Lambda(t)\,dW_t^H\Big)\Big\}\,,$$

which in turn establishes (104).

Tightness

It is possible to control the increments of the process M^ε, as shown by the following proposition.

Proposition 23 *There exists K such that, for any $F_1, F_2 \in L^\infty(0,1)$ and for any $x, y \in [0,1]$,*

$$\sup_{\varepsilon\in(0,1)} \mathbb{E}\Big\{\Big|M^\varepsilon(y) - M^\varepsilon(x)\Big|^2\Big\} \leq K\Big(\|F_1\|_\infty^2 |y-x|^{2-\alpha} + \|F_2\|_\infty^2 |y-x|^2\Big), \quad (106)$$

where M^ε is defined by (99).

Proof.

The proof is a refinement of the ones of Lemmas 18 and 19. We can split the random process M^ε into two components: $M^\varepsilon(x) = M^{\varepsilon,1}(x) +$

$M^{\varepsilon,2}(x)$, with

$$M^{\varepsilon,1}(x) = \varepsilon^{-\frac{\alpha}{2}} \int_0^x F_1(t)\varphi^\varepsilon(t)dt\,, \qquad M^{\varepsilon,2}(x) = x\varepsilon^{-\frac{\alpha}{2}} \int_0^1 F_2(t)\varphi^\varepsilon(t)dt\,.$$

We have

$$\mathbb{E}\Big\{\Big|M^\varepsilon(y)-M^\varepsilon(x)\Big|^2\Big\} \le 2\mathbb{E}\Big\{\Big|M^{\varepsilon,1}(y)-M^{\varepsilon,1}(x)\Big|^2\Big\}+2\mathbb{E}\Big\{\Big|M^{\varepsilon,2}(y)-M^{\varepsilon,2}(x)\Big|^2\Big\}\,.$$

The second moment of the increment of $M^{\varepsilon,2}$ is given by

$$\mathbb{E}\Big\{\Big|M^{\varepsilon,2}(y) - M^{\varepsilon,2}(x)\Big|^2\Big\} = |x-y|^2\varepsilon^{-\alpha} \int_{[0,1]^2} R\Big(\frac{z-t}{\varepsilon}\Big)F_2(z)F_2(t)dzdt\,.$$

Since there exists $K > 0$ such that $|R(\tau)| \le K\tau^{-\alpha}$ for all τ, we have

$$\begin{aligned}
\varepsilon^{-\alpha} \int_{[0,1]^2} R\Big(\frac{z-t}{\varepsilon}\Big)F_2(z)F_2(t)dzdt \;&\le\; K \int_{[0,1]^2} |z-t|^{-\alpha}|F_2(z)||F_2(t)|dzdt \\
&\le\; K\|F_2\|_\infty^2 \int_{-1}^1 |z|^{-\alpha}dz = \frac{2K}{1-\alpha}\|F_2\|_\infty^2\,,
\end{aligned}$$

which gives the following estimate

$$\mathbb{E}\Big\{\Big|M^{\varepsilon,2}(y)-M^{\varepsilon,2}(x)\Big|^2\Big\}\leq\frac{2K}{1-\alpha}\|F_2\|_\infty^2|x-y|^2\,.$$

The second moment of the increment of $M^{\varepsilon,1}$ for $x<y$ is given by

$$\mathbb{E}\Big\{\Big|M^{\varepsilon,1}(y)-M^{\varepsilon,1}(x)\Big|^2\Big\}=\varepsilon^{-\alpha}\int_{[x,y]^2}R\Big(\frac{z-t}{\varepsilon}\Big)F_1(z)F_1(t)dzdt\,.$$

We distinguish the cases $|y-x|\leq\varepsilon$ and $|y-x|\geq\varepsilon$.

First case. Let us assume that $|y-x|\leq\varepsilon$. Since R is bounded by V_2, we have

$$\mathbb{E}\Big\{\Big|M^{\varepsilon,1}(y)-M^{\varepsilon,1}(x)\Big|^2\Big\}\leq V_2\|F_1\|_\infty^2\varepsilon^{-\alpha}|y-x|^2\,.$$

Since $|y-x|\leq\varepsilon$, this implies

$$\mathbb{E}\Big\{\Big|M^{\varepsilon,1}(y)-M^{\varepsilon,1}(x)\Big|^2\Big\}\leq V_2\|F_1\|_\infty^2|y-x|^{2-\alpha}\,.$$

Second case. Let us assume that $|y-x| \geq \varepsilon$. Since R can be bounded by a power-law function $|R(\tau)| \leq K\tau^{-\alpha}$ we have

$$\begin{aligned}
\mathbb{E}\Big\{\Big|M^{\varepsilon,1}(y) - M^{\varepsilon,1}(x)\Big|^2\Big\} &\leq K\|F_1\|_\infty^2 \int_{[x,y]^2} |z-t|^{-\alpha} dz dt \\
&\leq 2K\|F_1\|_\infty^2 \int_x^y \int_0^{y-x} t^{-\alpha} dt dz \\
&\leq \frac{2K}{1-\alpha}\|F_1\|_\infty^2 |y-x|^{2-\alpha},
\end{aligned}$$

which completes the proof.

This Proposition allows us to get two results.

1) Applying Prop. 23 with $F_2 = 0$ and $y = 0$, we re-prove Lemma 13.

2) By applying Proposition 23, we obtain that the increments of the process M^ε satisfy the Kolmogorov criterion (103) with $\beta = 2$ and $\delta = 1 - \alpha > 0$. This gives the tightness of the family of processes M^ε in the space $\mathcal{C}(0,1)$.

Proof of convergence theorem

We can now give the proof of Theorem 15. The error term can be written in the form

$$\varepsilon^{-\frac{\alpha}{2}}(u^\varepsilon(x)-\bar{u}(x)) = \varepsilon^{-\frac{\alpha}{2}}\left(\int_0^x F_1(t)\varphi^\varepsilon(t)dt + x\int_0^1 F_2(t)\varphi^\varepsilon(t)dt\right) + \tilde{r}^\varepsilon(x)\,,$$

where $F_1(t) = c^* - F(t)$, $F_2(t) = F(t) - \int_0^1 F(z)dz - a^*q$, and $\tilde{r}^\varepsilon(x) = \varepsilon^{-\alpha/2}[r^\varepsilon(x) + \rho^\varepsilon a^{*-1}x]$. The first term of the right-hand side is of the form (99). Therefore, by applying Theorem 21, we get that this process converges in distribution in $\mathcal{C}(0,1)$ to the limit process (86). It remains to show that the random process $\tilde{r}^\varepsilon(x)$ converges as $\varepsilon \to 0$ to zero in $\mathcal{C}(0,1)$ in probability.

We have

$$\mathbb{E}\{|\tilde{r}^\varepsilon(x)-\tilde{r}^\varepsilon(y)|^2\} \le 2\varepsilon^{-\alpha}\mathbb{E}\{|r^\varepsilon(x)-r^\varepsilon(y)|^2\} + 2a^{*-2}\varepsilon^{-\alpha}\mathbb{E}\{|\rho^\varepsilon|^2\}|x-y|^2\,,$$

From the expression (85) of r^ε, and the fact that c^ε can be bounded uniformly in ε by a constant c_0, we get

$$\varepsilon^{-\alpha}\mathbb{E}\{|r^\varepsilon(x)-r^\varepsilon(y)|^2\} \leq 2\varepsilon^{-\alpha}c_0\mathbb{E}\left\{\left|\int_x^y \varphi^\varepsilon(t)dt\right|^2\right\}.$$

Upon applying Proposition 23, we obtain that there exists $K>0$ such that

$$\varepsilon^{-\alpha}\mathbb{E}\{|r^\varepsilon(x)-r^\varepsilon(y)|^2\} \leq K|x-y|^{2-\alpha}.$$

Besides, since ρ^ε can be bounded uniformly in ε by a constant ρ_0, we have $\mathbb{E}\{|\rho^\varepsilon|^2\} \leq \rho_0\mathbb{E}\{|\rho^\varepsilon|\} \leq K\varepsilon^\alpha$ for some $K>0$. Therefore, we have established that there exists $K>0$ such that

$$\mathbb{E}\{|\tilde{r}^\varepsilon(x)-\tilde{r}^\varepsilon(y)|^2\} \leq K|x-y|^{2-\alpha},$$

uniformly in ε, x, y. This shows that $\tilde{r}^\varepsilon(x)$ is a tight sequence in the space $\mathcal{C}(0,1)$ by the Kolmogorov criterion (103). Furthermore, the finite-dimensional distributions of $\widetilde{r}^{\varepsilon}(x)$ converges to zero because

$$\sup_{x\in[0,1]} \mathbb{E}\Big\{\big|\widetilde{r}_{\varepsilon}(x)\big|\Big\} \stackrel{\varepsilon\to 0}{\longrightarrow} 0$$

by (82) and (84). Proposition 22 then shows that $\widetilde{r}^{\varepsilon}(x)$ converges to zero in distribution in $\mathcal{C}(0,1)$. Since the limit is deterministic, the convergence actually holds true in probability.

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