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On the attenuated Radon transform with full and partial measurements

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Abstract
This paper deals with the reconstruction and the redundancy of the two-dimensional attenuated Radon transform (AtRT). When full measurements are available, we characterize the information on a spatially and angularly dependent function that can be retrieved from its AtRT. In particular we consider the reconstruction of isotropic scalar and vector-valued source terms. Next we consider the reconstruction of a spatially dependent function from its AtRT with half angular measurements. We show that the inversion is feasible provided that the spatial variations of the absorption coefficient are not too large. The reconstruction is based on the decomposition of the reconstruction operator in the Novikov formula into three components bounded in the L^2 sense. The first component involves the measured partial data. The second component is a skew-symmetric operator. The third component is a symmetric and compact contribution whose spectral radius, which depends on the attenuation, needs to be smaller than unity in our reconstruction. In a paper (Bal and Moireau, *Preprint*) on the numerical implementation of the reconstruction, we show that the reconstruction algorithm can be successfully applied in situations of practical interest.

1. Introduction

The reconstruction of a function from its attenuated Radon transform (AtRT) finds many applications in medical imaging, for instance image reconstruction in single-photon emission computed tomography (SPECT) [13, 16], or in Doppler tomography [7, 8, 24, 27], with non-uniform attenuation. Whereas reconstruction formulae in the absence of absorption [22] and for constant absorption [28] have been known for several decades now, the inversion of the general AtRT in the case of arbitrary absorption has been obtained only recently; see [15, 18, 19] for the derivation of the Novikov formula and [4, 7, 8] for a different approach. Direct discretizations of the Novikov formula have successfully been used to invert the AtRT numerically from synthetic and real data [10, 12]. Other contributions to the analysis of the attenuated Radon transform include [6, 9, 11, 21, 25].

Denoting by $g(s, \theta)$ with $s \in \mathbb{R}$ and $\theta \in (0, 2\pi)$ the (non-attenuated) Radon transform of a source term $f(\mathbf{x})$ defined over $\mathbb{R}^2$, it is well-known that $g(-s, \theta + \pi) = g(s, \theta)$. There is therefore a redundancy of order 2 in the measured data when the source term is isotropic. This redundancy can be used to reduce the effect of noise in the data. In many practical situations, however, we would like to reconstruct the source term $f(\mathbf{x})$ from half measurements, for instance for $\theta \in (0, \pi)$ only.

Whereas the case of vanishing absorption is easily handled since $g(-s, \theta+\pi) = g(s, \theta)$, no such simplification arises in the case of arbitrary absorption. In the case of constant absorption, a method was recently derived in [17] to iteratively reconstruct the source term $f(\mathbf{x})$ from its exponential Radon transform (ERT: the AtRT in the case of constant absorption) with half measurements. The method was generalized to more arbitrary domains of data acquisition in [20]. Some work also exists on the analysis of the Range conditions for full measurements, i.e., the constraints that the measured data must satisfy to be the ERT of a source term of the form $f(\mathbf{x})$ [2].

These reconstruction techniques are based on a judicious change of contours of integration in the complex plane and on the Fourier slice theorem. As such they do not adapt to the case of spatially varying attenuation. In the latter case, the redundancy is expressed in terms of Range conditions for the measured data [13, 19]. Such Range conditions can be used to determine more information than the source term as in [14] but do not provide any algorithm to reconstruct the source term from partial measurements. Let us mention [26] for an alternative perspective on the analysis of redundant information in the AtRT and [3, 23] for results on the simultaneous detection of the source term and the absorption parameter.

In this paper we analyse the redundancy in the AtRT by characterizing the information that can be reconstructed from a possibly angularly dependent function. We show that two spatially independent scalar functions can be reconstructed from the AtRT as in the case of the Radon transform and propose explicit reconstructions in simple cases. The reconstructions are based on an extension of the technique developed in [19] involving the solution of a Riemann–Hilbert problem. We consider two applications: reconstruction of a mildly angularly dependent source term in SPECT and reconstruction of a vector-valued source term of the form $\boldsymbol{\theta} \cdot \mathbf{F}(\mathbf{x})$ in Doppler tomography. In particular we show that both components of $\mathbf{F}(\mathbf{x})$ can be reconstructed provided that the support of $\mathbf{F}(\mathbf{x})$ is included in the support of $a(\mathbf{x})$.

In the case where half of the angular measurements are available we propose an algorithm that allows us to reconstruct compactly supported (in the unit ball B to simplify) source terms from the AtRT $g(s, \theta)$ for $s \in \mathbb{R}$ and $\theta \in M$, where M is such that $(0, 2\pi)\backslash M \subset \overline{M + \pi}$. This implies that either $g(s, \theta)$ or $g(s, \theta + \pi)$ is measured for all θ and all s. The simplest example is $M = [0, \pi)$ as in [17]. More general situations such as those in [20] are also included. The algorithm is based on a decomposition of the Novikov formula [18] into three components. The first component involves only the measured data on $\mathbb{R} \times M$. The second component involves an operator bounded in $L^2(B)$ and skew-symmetric. It is similar to the skew-symmetric operator introduced in [17] in the case of constant absorption. The key observation to obtain a skew-symmetric operator is a symmetry of the AtRT (see (61) below). The third operator is a compact and self-adjoint operator in $L^2(B)$. It vanishes in the case of constant absorption. Our reconstruction algorithm is iterative and requires that the latter operator be of spectral radius less than unity. This constraints corresponds to assuming that the absorption parameter does not have too large spatial variations. The case of large variations of the absorption parameter remains open. However, we show in [5], devoted to the numerical implementation of the method, that in many situations of practical interest the method of reconstruction can be successfully applied.

The rest of the paper is organized as follows. In section 2 we present an extension of the Novikov formula in the case where the source term $f(\mathbf{x}, \theta)$ may depend on the angle of incidence θ. This derivation gives some insight on the redundancy of the measured data and the type of information on $f(\mathbf{x}, \theta)$ that can be obtained from its AtRT. It also allows us to reconstruct $\mathbf{F}(\mathbf{x})$ in a source term of the form $f(\mathbf{x}, \theta) = \boldsymbol{\theta} \cdot \mathbf{F}(\mathbf{x})$ in Doppler tomography. The iterative reconstruction algorithm is derived in section 3. The proof of convergence of the reconstruction algorithm is based on regularity estimates for the various components that appear in the Novikov reconstruction formula. Such estimates are given in section 4. Concluding remarks are presented in section 5.

2. Reconstruction from full measurements

We derive in this section a generalization of the Novikov formula [18, 19], which allows us to reconstruct a function from its attenuated Radon transform (AtRT). We consider here functions that may depend on the direction of propagation and obtain the information on the source term that can be reconstructed from the AtRT. This generalization of the original Novikov reconstruction formula displays quite explicitly the redundancy in two-dimensional AtRT and provides useful reconstruction formulae in SPECT and Doppler tomography applications.

The rest of the section is organized as follows. Section 2.1 introduces the AtRT and the main notation used in the paper. Section 2.2 recasts the reconstruction problem as a Riemann–Hilbert problem. This Riemann–Hilbert problem is solved in sections 2.3–2.5, where our main reconstruction formulae are proposed. Section 2.6 shows that the reconstruction of two spatial functions from given 'measured data' is optimal in the sense that the AtRT of the reconstructed source term is equal to the measured data. Finally, section 2.7 considers reconstruction formulae in Doppler tomography.

2.1. Transport equation and AtRT

The transport equation with possibly anisotropic source term is given by

$$\boldsymbol{\theta} \cdot \nabla\psi(\mathbf{x}, \theta) + a(\mathbf{x})\psi(\mathbf{x}, \theta) = f(\mathbf{x}, \theta), \qquad \mathbf{x} \in \mathbb{R}^2, \quad \boldsymbol{\theta} \in S^1. \tag{1}$$

Throughout we identify $\boldsymbol{\theta} = (\cos\theta, \sin\theta) \in S^1$ and $\theta \in (0, 2\pi)$. The source term $f(\mathbf{x}, \theta)$ cannot be completely arbitrary in θ and we make here the simplifying assumption that

$$f(\mathbf{x}, \theta) = \sum_{k=-N}^{N} f_k(\mathbf{x})\mathrm{e}^{\mathrm{i}k\theta} \tag{2}$$

for some $N \in \mathbb{N}$ with $f_{-k} = \bar{f}_k$ so that $f(\mathbf{x}, \theta)$ is real-valued. We assume that the $f_k(\mathbf{x})$ are compactly supported and that for all $\mathbf{x} \in \mathbb{R}^2$,

$$\lim_{s\to+\infty} \psi(\mathbf{x} - s\boldsymbol{\theta}, \theta) = 0, \tag{3}$$

so that (1) can uniquely be solved by the method of characteristics. Since the absorption parameter $a(\mathbf{x})$ is important only on the support of the source term in practice, we assume that it is sufficiently smooth and decays exponentially at infinity to simplify. This allows us to define

$$D_\theta a(\mathbf{x}) = \frac{1}{2}\int_0^\infty [a(\mathbf{x} - t\boldsymbol{\theta}) - a(\mathbf{x} + t\boldsymbol{\theta})]\,\mathrm{d}t. \tag{4}$$

This is a *symmetrized* beam transform. It is different from the beam transform defined in [15, 19]. This symmetrized notation is convenient in the sequel. We observe that $\boldsymbol{\theta} \cdot \nabla D_\theta a(\mathbf{x}) = a(\mathbf{x})$ so that the solution to the transport equation (1) is given by

$$\mathrm{e}^{D_\theta a(\mathbf{x})} \psi(\mathbf{x}, \theta) = \int_0^\infty (\mathrm{e}^{D_\theta a} f)(\mathbf{x} - t\boldsymbol{\theta}, \theta)\, \mathrm{d}t. \tag{5}$$

Let us define $\boldsymbol{\theta}^\perp = (-\sin\theta, \cos\theta)$ and decompose $\mathbf{x} = s\boldsymbol{\theta}^\perp + t\boldsymbol{\theta}$. We find that

$$\lim_{t\to+\infty} \mathrm{e}^{D_\theta a(s\boldsymbol{\theta}^\perp + t\boldsymbol{\theta})} \psi(s\boldsymbol{\theta}^\perp + t\boldsymbol{\theta}, \theta) = \int_{\mathbb{R}} (\mathrm{e}^{D_\theta a} f)(s\boldsymbol{\theta}^\perp + t\boldsymbol{\theta}, \theta)\, \mathrm{d}t.$$

This can be recast as

$$\lim_{t\to+\infty} \psi(s\boldsymbol{\theta}^\perp + t\boldsymbol{\theta}, \theta) = \mathrm{e}^{-(P_\theta a)(s)/2} (R_{a,\theta} f)(s), \tag{6}$$

where we have defined the Radon transform and the (symmetrized) attenuated Radon transform as follows:

$$\begin{aligned} P_\theta f(s) &= \int_{\mathbb{R}} f(s\boldsymbol{\theta}^\perp + t\boldsymbol{\theta}, \theta)\, \mathrm{d}t = \int_{\mathbb{R}^2} f(\mathbf{x}, \theta) \delta(\mathbf{x} \cdot \boldsymbol{\theta}^\perp - s)\, \mathrm{d}\mathbf{x} \\ (R_{a,\theta} f)(s) &= (P_\theta(\mathrm{e}^{D_\theta a} f))(s). \end{aligned} \tag{7}$$

We deduce from (6) that up to a factor $\mathrm{e}^{-(P_\theta a)(s)/2}$, the (symmetrized) attenuated Radon transform $R_{a,\theta} f(s)$ is what can be measured physically. Up to this known multiplicative factor we call $R_{a,\theta} f(s)$ the 'measured data'.

We now wish to understand how much of $f(\mathbf{x}, \theta)$ can be reconstructed from the complete or partial knowledge of $R_{a,\theta} f(s)$ for $s \in \mathbb{R}$ and $\theta \in [0, 2\pi)$. In the case of full measurements and $f(\mathbf{x}, \theta) = f(\mathbf{x})$, there exists an exact inversion method [4] and an explicit inversion formula [18] to uniquely reconstruct $f(\mathbf{x})$ from the AtRT; see [4, 7, 15, 18, 19] for additional details.

2.2. Riemann–Hilbert problem

To obtain reconstruction formulae we essentially follow the technique of the original derivation in [18]. We aim to recast the reconstruction problem as a Riemann–Hilbert problem. To do so we parameterize the unit circle in the complex plane and define

$$\lambda = \mathrm{e}^{\mathrm{i}\theta}, \qquad \theta \in (0, 2\pi). \tag{8}$$

Using $\mathbf{x} = (x, y)$, the transport equation (1) may thus be recast as

$$\left(\frac{\lambda + \lambda^{-1}}{2}\frac{\partial}{\partial x} + \frac{\lambda - \lambda^{-1}}{2\mathrm{i}}\frac{\partial}{\partial y} + a(\mathbf{x})\right)\psi(\mathbf{x}, \lambda) = f(\mathbf{x}, \lambda) = \sum_{k=-N}^{N} \lambda^k f_k(\mathbf{x}).$$

Identifying $\mathbf{x}$ with $z = x + \mathrm{i}y$ we can simplify the above equation as

$$\left(\lambda\frac{\partial}{\partial z} + \lambda^{-1}\frac{\partial}{\partial \bar{z}} + a(z)\right)\psi(z, \lambda) = f(z, \lambda), \tag{9}$$

where we have used

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right).$$

The above equation may also be considered for arbitrary complex values of λ. As we shall now see, $\psi(z, \lambda)$ is analytic for $\lambda \in \mathbb{C}\backslash(T \cup \{0\})$, where $T = \{\lambda \in \mathbb{C}, |\lambda| = 1\}$ is the unit circle. The justification is based on the analysis of the Green's function solution of

$$\left(\lambda\frac{\partial}{\partial z} + \lambda^{-1}\frac{\partial}{\partial \bar{z}}\right)G(z, \lambda) = \delta(z), \qquad |G(z, \lambda)| \to 0 \qquad \text{as } |z| \to \infty, \tag{10}$$

for $\lambda \in \mathbb{C}\backslash(T \cup \{0\})$. Such a spatial decay as infinity is not possible for $\lambda \in T$. However, when $|\lambda| \neq 1$, the above equation is elliptic and its solution is given by

$$G(z, \lambda) = \frac{\operatorname{sgn}(|\lambda| - 1)}{\pi(\lambda\bar{z} - \lambda^{-1}z)}. \tag{11}$$

Obviously the Green's function is analytic in λ on $\mathbb{C}\backslash(T \cup \{0\})$. Let us now define

$$h(z, \lambda) = \int_{\mathbb{C}} G(z - \zeta, \lambda)a(\zeta)\,\mathrm{d}m(\zeta), \tag{12}$$

where $\mathrm{d}m(\zeta)$ is the Lebesgue measure on $\mathbb{C} \simeq \mathbb{R}^2$, the solution of

$$\left(\lambda\frac{\partial}{\partial z} + \lambda^{-1}\frac{\partial}{\partial \bar{z}}\right)h(z, \lambda) = a(z), \qquad |h(z, \lambda)| \to 0 \qquad \text{as } |z| \to \infty. \tag{13}$$

The function $h(z, \lambda)$ can be seen as a complex extension of $D_\theta a(\mathbf{x}, \theta)$. The solution to the transport equation (9) with vanishing conditions at $z \to \infty$ is then given by

$$\psi(z, \lambda) = \mathrm{e}^{-h(z,\lambda)} \int_{\mathbb{C}} G(z - \zeta, \lambda)\mathrm{e}^{h(\zeta,\lambda)} f(\zeta, \lambda)\,\mathrm{d}m(\zeta). \tag{14}$$

We verify that $\psi(z, \lambda)$ is sectionally analytic for $|\lambda| > 1$ and $0 < |\lambda| < 1$. As it stands ψ is not necessarily analytic at $\lambda = 0$ unless f is independent of λ. Moreover ψ is not necessarily of order $\mathrm{O}(\lambda^{-1})$ at infinity. We shall subtract from ψ the terms that are responsible for this behaviour. Once this is done we shall observe that the jump of $\psi(z, \lambda)$ across the unit circle $|\lambda| = 1$ is actually a function of the measured data $R_{a,\theta} f(s)$. This is the setting for the Riemann–Hilbert problem.

Let us denote $D^+ = \{\lambda \in \mathbb{C}, |\lambda| < 1\}$ and $D^- = \{\lambda \in \mathbb{C}, |\lambda| > 1\}$. Let $\phi(\lambda)$ be analytic on D^+ and on D^- and such that $\lambda\phi$ is bounded at infinity. Then we have the Cauchy type integral

$$\phi(\lambda) = \frac{1}{2\pi\mathrm{i}} \int_T \frac{\varphi(t)}{t - \lambda}\,\mathrm{d}t, \qquad \lambda \in \mathbb{C}\backslash T, \tag{15}$$

where

$$\varphi(t) = \lim_{0<\varepsilon\to 0} (\phi((1 - \varepsilon)t) - \phi((1 + \varepsilon)t)) \equiv \phi^+(t) - \phi^-(t). \tag{16}$$

Thus $\phi^\pm(t)$ is the limit of $\phi(\lambda)$ as $\lambda \in D^\pm$ reaches $t \in T$. The Riemann–Hilbert problem aims to find the sectionally analytic function on $D^\pm$, which is of order $\mathrm{O}(\lambda^{-1})$ at infinity and is such that (16) holds. Its unique solution is given by (15). We refer to [1] for details of the Riemann–Hilbert problem. For any function $\psi(\lambda)$, we denote in the sequel by $\psi^\pm(t)$ the limit of $\psi(\lambda)$ as $\lambda \in D^\pm$ reaches $t \in T$.

The reconstruction formulae for the source term from the AtRT are obtained as follows.

(i) We first replace $\psi(z, \lambda)$ given in (14) by a function $\phi(z, \lambda)$ that is sectionally analytic on $D^\pm$ and of order $\mathrm{O}(\lambda^{-1})$ at infinity. This is done by subtracting from $\psi(z, \lambda)$ a finite number of analytic functions on $\mathbb{C}\backslash\{0\}$. Notice that $\psi(z, \lambda)$ in (14) already satisfies such conditions when $f(z, \lambda) = f(z)$ independent of λ as in [19].

(ii) Second we verify that $\varphi(z, t) = \phi^+(z, t) - \phi^-(z, t)$, which is equal to $\psi^+(z, t) - \psi^-(z, t)$ since $(\phi - \psi)(z, \lambda)$ is analytic, can be written as a function of the measured data $R_{a,\theta} f(s)$. Thus $\phi(z, \lambda)$ is the solution to the Riemann–Hilbert problem and is given by (15).

(iii) Finally we use the expression for $\phi(z, \lambda)$ in the vicinity of $\lambda = 0$ (or equivalently $|\lambda| = \infty$) to obtain reconstruction formulae for the source term $f(z, \lambda)$.

2.3. Taylor series expansions (Step 1)

We first analyse $\psi(z, \lambda)$ in the vicinity of 0 and ∞. Let us consider λ on D^+. We observe that on this domain,

$$G(z, \lambda) = \frac{\lambda}{\pi} \frac{1}{z - \lambda^2 \bar{z}} = \frac{1}{\pi z} \sum_{m=0}^{\infty} \left(\frac{\bar{z}}{z}\right)^m \lambda^{2m+1} \equiv \sum_{m=0}^{\infty} G_m(z) \lambda^{2m+1}. \tag{17}$$

We denote by $\mathcal{G}_m$ the integral operator of kernel G_m and thus obtain that

$$h(z, \lambda) = \sum_{m=0}^{\infty} (\mathcal{G}_m a)(z) \lambda^{2m+1}, \qquad |\lambda| < 1. \tag{18}$$

This allows us to obtain the asymptotic expansion of $\psi(z, \lambda)$ around the pole of finite multiplicity $\lambda = 0$. We therefore recast (14) as

$$\psi(z, \lambda) = \sum_{m=1}^{\infty} (\mathcal{H}_m f)(z, \lambda) \lambda^m, \qquad |\lambda| < 1, \tag{19}$$

for some spatial operators $\mathcal{H}_m$ that are explicitly computable. We deduce from (17) and (18) that $\mathcal{H}_1 = \mathcal{G}_0$, and more explicitly that

$$\begin{aligned} \mathcal{H}_1 f(z) &= \frac{1}{\pi} \int_{\mathbb{C}} \frac{f(\zeta)}{z - \zeta} \mathrm{d}m(\zeta), \\ \mathcal{H}_2 f(z) &= \frac{1}{\pi} \int_{\mathbb{C}} \frac{f(\zeta)}{z - \zeta} ((\mathcal{H}_1 a)(\zeta) - (\mathcal{H}_1 a)(z))\, \mathrm{d}m(\zeta). \end{aligned} \tag{20}$$

From the equation that $\psi(z, \lambda)$ satisfies with $f(z) = \delta(z)$, we deduce that

$$\sum_{m=1}^{\infty} \lambda^m \left(\lambda \frac{\partial}{\partial z} + \lambda^{-1} \frac{\partial}{\partial \bar{z}} + a\right) \mathcal{H}_m(z) = I. \tag{21}$$

Equating like powers of λ we obtain

$$\begin{aligned} &\frac{\partial}{\partial \bar{z}} \mathcal{H}_1 = I, \\ &\frac{\partial}{\partial \bar{z}} \mathcal{H}_2 + a\mathcal{H}_1 = 0, \\ &\frac{\partial}{\partial \bar{z}} \mathcal{H}_{k+2} + a\mathcal{H}_{k+1} + \frac{\partial}{\partial z} \mathcal{H}_k = 0, \qquad k \geqslant 1. \end{aligned} \tag{22}$$

This is a triangular system of equations that can be solved by induction. We verify that $\mathcal{H}_1$ is indeed given by (20) and that $\mathcal{H}_2 = -\mathcal{H}_1 a \mathcal{H}_1$ from (22) is indeed equivalent to (20).

Notice that $G(z, \lambda^{-1}) = \bar{G}(z, \bar{\lambda})$ and $h(z, \lambda^{-1}) = \bar{h}(z, \bar{\lambda})$, so that when the source is real-valued on the unit circle, $\psi(z, \lambda^{-1}) = \bar{\psi}(z, \bar{\lambda})$. So we get in the vicinity of $\lambda^{-1} = 0$ by linearity of the transport equation that

$$\psi(z, \lambda) = \sum_{m=1}^{\infty} (\overline{\mathcal{H}_m}\, f)(z, \lambda) \lambda^{-m}, \qquad |\lambda| > 1. \tag{23}$$

Using the structure of the source term (2) we obtain that

$$\psi(z, \lambda) = \begin{cases} \displaystyle\sum_{k=-N}^{N} \sum_{m=1}^{\infty} (\mathcal{H}_m f_k)(z) \lambda^{m+k}, & |\lambda| < 1, \\ \displaystyle\sum_{n=-\infty}^{\infty} \lambda^n \sum_{m=1}^{\infty} (\mathcal{H}_m f_{n-m})(z), & |\lambda| < 1, \\ \displaystyle\sum_{n=-\infty}^{\infty} \lambda^{-n} \sum_{m=1}^{\infty} (\overline{\mathcal{H}_m} f_{m-n})(z), & |\lambda| > 1. \end{cases} \tag{24}$$

Here we have defined $f_n(z) \equiv 0$ for $|n| > N$. To obtain an analytic function at $\lambda = 0$ and a function of order λ^{-1} at infinity, we define

$$\phi(z,\lambda) = \psi(z,\lambda) - \sum_{n=-\infty}^{-1} \lambda^n \sum_{m=1}^{\infty} (\mathcal{H}_m f_{n-m})(z) - \sum_{n=-\infty}^{0} \lambda^{-n} \sum_{m=1}^{\infty} (\overline{\mathcal{H}_m} f_{m-n})(z). \tag{25}$$

Notice that the difference $\phi(z,\lambda) - \psi(z,\lambda)$ is the sum of a finite number of analytic functions on $\mathbb{C}\backslash\{0\}$ according to (2). We verify that on D^+, we have

$$\phi(z,\lambda) = \sum_{n=0}^{\infty} \lambda^n \sum_{m=1}^{\infty} (\mathcal{H}_m f_{n-m} - \overline{\mathcal{H}_m} f_{n+m})(z). \tag{26}$$

The function $\phi(z,\lambda)$ is now sectionally analytic on $D^\pm$ and of order $O(\lambda^{-1})$ at infinity. It is therefore a good candidate to solve a Riemann–Hilbert problem. We now show that its jump across T can be expressed as a function of the boundary measurements.

2.4. *Jump conditions (Step 2)*

The jump conditions for $\phi(z,\lambda)$ across T are the same as those for $\psi(z,\lambda)$ since the difference is analytic on $\mathbb{C}\backslash\{0\}$. The latter jump conditions are obtained as follows. First writing $\lambda = re^{i\theta}$ and sending $r - 1$ to ± 0, we obtain as in [18] that

$$G_\pm(\mathbf{x},\theta) = \frac{\pm 1}{2\pi i(\boldsymbol{\theta}^\perp \cdot \mathbf{x} \mp i0\,\mathrm{sgn}(\boldsymbol{\theta}\cdot\mathbf{x}))}. \tag{27}$$

This implies that $h_\pm(\mathbf{x},\theta)$ obtained from (12) is defined as

$$h_\pm(\mathbf{x},\theta) = \pm\frac{1}{2i}(HP_\theta a)(\mathbf{x}\cdot\boldsymbol{\theta}^\perp) + (D_\theta a)(\mathbf{x}). \tag{28}$$

Here H is the Hilbert transform defined as

$$Hu(t) = \frac{1}{\pi} \int\!\!\!\!\!-_{\mathbb{R}} \frac{u(t-s)}{s}\,ds = \frac{1}{\pi} \int\!\!\!\!\!-_{\mathbb{R}} \frac{u(s)}{t-s}\,ds, \tag{29}$$

where the above integrals need be taken in the principal value sense. Since P_θ and D_θ involve integrations in the direction $\boldsymbol{\theta}$ only, we easily verify that

$$P_\theta[u(\mathbf{x})v(\mathbf{x}\cdot\boldsymbol{\theta}^\perp)](s) = v(s)P_\theta[u](s), \qquad D_\theta[u(\mathbf{x})v(\mathbf{x}\cdot\boldsymbol{\theta}^\perp)](\mathbf{x}) = v(\mathbf{x}\cdot\boldsymbol{\theta}^\perp)D_\theta[u](\mathbf{x}).$$

Using the above results we obtain that

$$\psi^\pm(\mathbf{x},\theta) = e^{-D_\theta a} e^{\frac{\mp 1}{2i}(HP_\theta a)(\mathbf{x}\cdot\boldsymbol{\theta}^\perp)} \frac{\mp 1}{2i} H(e^{\frac{\pm 1}{2i}(HP_\theta a)(s)} P_\theta(e^{D_\theta a} f))(\mathbf{x}\cdot\boldsymbol{\theta}^\perp) + e^{-D_\theta a} D_\theta(e^{D_\theta a} f)(\mathbf{x}). \tag{30}$$

Notice that the difference of the above terms is a function of $R_{a,\theta}f(s) = P_\theta(e^{D_\theta a}f)(s)$, i.e., of the measurements, whereas $\psi_\pm$ individually cannot be written in terms of the sole measurements. Let us define

$$\varphi(\mathbf{x},\theta) = (\psi^+ - \psi^-)(\mathbf{x},\theta). \tag{31}$$

We can show using (30) that

$$i\varphi(\mathbf{x},\theta) = R^*_{-a,\theta}(2H_a)R_{a,\theta}f(\mathbf{x},\theta), \tag{32}$$

where we have defined the following operators:

$$\begin{aligned} R^*_{a,\theta}g(\mathbf{x}) &= e^{D_\theta a(\mathbf{x})} g(\mathbf{x}\cdot\boldsymbol{\theta}^\perp), & H_a &= \tfrac{1}{2}(C_c H C_c + C_s H C_s), \\ C_c g(s,\theta) &= g(s,\theta)\cos\left(\frac{HP_\theta a(s)}{2}\right), & C_s g(s,\theta) &= g(s,\theta)\sin\left(\frac{HP_\theta a(s)}{2}\right). \end{aligned} \tag{33}$$

Here $R^*_{a,\theta}$ is the formal adjoint operator to $R_{a,\theta}$. This shows that the jump of ψ, hence of ϕ, across T can be written in terms of the boundary measurements. Notice that $i\varphi(\mathbf{x},\theta)$ is *real-valued* and that $\varphi(\mathbf{x},\theta)$ is of the form $e^{-D_\theta a(\mathbf{x})}M(\mathbf{x}\cdot\boldsymbol{\theta}^\perp,\theta)$ so that φ satisfies that $\boldsymbol{\theta}\cdot\nabla\varphi + a\varphi = 0$.

2.5. Reconstruction formulae (Step 3)

We still denote $\varphi(\mathbf{x}, t) = \varphi(\mathbf{x}, \theta)$ for $t = \mathrm{e}^{\mathrm{i}\theta}$. We observe that $\phi(z, \lambda)$ defined in (25) is sectionally analytic on D^+ and D^-, of order $|\lambda|^{-1}$ at infinity, and such that

$$\varphi(z, \theta) = \phi^+(z, \theta) - \phi^-(z, \theta) \quad \text{on } S^1.$$

Thus it solves the Riemann–Hilbert problem and we have that

$$\phi(z, \lambda) = \frac{1}{2\pi \mathrm{i}} \int_T \frac{\varphi(z, t)}{t - \lambda} \,\mathrm{d}t, \qquad \lambda \in \mathbb{C}\backslash T. \tag{34}$$

Using the Taylor expansion of the right-hand side we obtain that

$$\phi(z, \lambda) = \sum_{n=0}^{\infty} \lambda^n \frac{1}{2\pi \mathrm{i}} \int_T \frac{\varphi(z, t)\,\mathrm{d}t}{t^{n+1}}, \qquad \lambda \in D^+.$$

Comparing with (26) we obtain that for all $n \geqslant 0$,

$$\sum_{m=1}^{\infty} (\mathcal{H}_m f_{n-m} - \overline{\mathcal{H}_m} f_{n+m})(z) = \frac{1}{2\pi \mathrm{i}} \int_T \frac{\varphi(z, t)\,\mathrm{d}t}{t^{n+1}} \equiv \varphi_n(z). \tag{35}$$

There are actually only two independent equations in the above system, corresponding to $n = 0$ and 1. Indeed, after change of variables, we obtain that

$$\varphi_n(z) = \frac{1}{2\pi \mathrm{i}} \int_T \frac{\varphi(z, t)\,\mathrm{d}t}{t^{n+1}} = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}n\theta} \varphi(z, \theta)\,\mathrm{d}\theta.$$

Since $\boldsymbol{\theta} \cdot \nabla\varphi + a\varphi = 0$ as noted below (31), we have that

$$\left(\mathrm{e}^{\mathrm{i}\theta} \frac{\partial}{\partial z} + \mathrm{e}^{-\mathrm{i}\theta} \frac{\partial}{\partial \bar{z}} + a(z)\right) \varphi(z, \theta) = 0,$$

which implies that

$$\frac{\partial \varphi_{n-1}(z)}{\partial z} + \frac{\partial \varphi_{n+1}(z)}{\partial \bar{z}} + a(z)\varphi_n(z) = 0, \qquad n \geqslant 1. \tag{36}$$

We thus deduce equivalently that

$$\varphi_n(z) = -\mathcal{H}_1 a \varphi_{n-1}(z) - \mathcal{H}_1 \frac{\partial}{\partial z} \varphi_{n-2}(z) \qquad \text{for all } n \geqslant 2. \tag{37}$$

Since these constraints are independent of the source term $f(\mathbf{x})$, this implies that the relations (35) for $n \geqslant 2$ do not bring any additional information.

Reconstruction in a simplified setting. The above relations (35) for $n = 0$ and 1 are all that we can obtain from the measurements. The measurements $R_{a,\theta} f(s)$ are not sufficiently rich to allow us to reconstruct all the terms $f_k(\mathbf{x})$. However, assuming that $N = 1$ and that only two terms are present in (2), they can be reconstructed uniquely. We obtain from (35) that

$$\begin{aligned} \mathcal{H}_1 f_{-1}(z) - \overline{\mathcal{H}_1} f_1(z) &= \frac{1}{2\pi \mathrm{i}} \int_T \frac{\varphi(z, t)\,\mathrm{d}t}{t} = \varphi_0(z), \\ \mathcal{H}_2 f_{-1}(z) + \mathcal{H}_1 f_0(z) &= \frac{1}{2\pi \mathrm{i}} \int_T \frac{\varphi(z, t)\,\mathrm{d}t}{t^2} = \varphi_1(z). \end{aligned} \tag{38}$$

Some algebra (identifying t with $\mathrm{e}^{\mathrm{i}\theta}$) shows that

$$\begin{aligned} \varphi_0(z) &= \frac{1}{2\pi} \int_0^{2\pi} \varphi(z, \theta)\,\mathrm{d}\theta, \\ \varphi_1(z) &= \frac{1}{2\pi} \int_0^{2\pi} (\cos\theta - \mathrm{i}\sin\theta)\varphi(z, \theta)\,\mathrm{d}\theta. \end{aligned} \tag{39}$$

We observe that $\varphi_0(z) \in \mathrm{i}\mathbb{R}$, which is consistent with (38). The latter system cannot be solved with three unknowns. We need to specify the relationship between the real and imaginary parts of f_1. Let us define $\boldsymbol{\omega} = (\cos\omega, \sin\omega)$ and assume that

$$\begin{aligned} &f_1(z) = \mathrm{e}^{\mathrm{i}\omega}\rho_1(z), \qquad f_{-1}(z) = \mathrm{e}^{-\mathrm{i}\omega}\rho_1(z), \\ &\text{so that } f_1(z)\mathrm{e}^{\mathrm{i}\theta} + f_{-1}(z)\mathrm{e}^{-\mathrm{i}\theta} = 2\cos(\theta+\omega)\rho_1(z), \end{aligned} \tag{40}$$

with $\rho_1(z)$ real-valued. We can now solve (38) uniquely. We have seen in (20) that $\mathcal{H}_1$ is a convolution operator of kernel $1/(\pi z) = 1/(\pi(x+\mathrm{i}y))$. In the Fourier domain we thus verify that

$$\widehat{\mathcal{H}_1 f}(\boldsymbol{\xi}) = \frac{-2(\xi_y + \mathrm{i}\xi_x)}{|\boldsymbol{\xi}|^2}\hat{f}(\boldsymbol{\xi}), \qquad \widehat{\overline{\mathcal{H}_1} f}(\boldsymbol{\xi}) = \frac{2(\xi_y - \mathrm{i}\xi_x)}{|\boldsymbol{\xi}|^2}\hat{f}(\boldsymbol{\xi}). \tag{41}$$

Denoting by $\boldsymbol{\omega}_{\mathrm{s}} = (\sin\omega, \cos\omega)$, we therefore obtain that

$$\hat{\rho}_1(\boldsymbol{\xi}) = \frac{-|\boldsymbol{\xi}|^2}{4\boldsymbol{\xi}\cdot\boldsymbol{\omega}_{\mathrm{s}}}\hat{\varphi}_0(\boldsymbol{\xi}), \qquad \text{or equivalently} \qquad f_1(\mathbf{x}) = \frac{1}{4}D_{\boldsymbol{\omega}_{\mathrm{s}}}\Delta(\mathrm{i}\varphi_0)(\mathbf{x}), \tag{42}$$

where Δ is the usual two-dimensional Laplace operator and $D_{\boldsymbol{\omega}_{\mathrm{s}}}$ is the symmetrized beam (4) in the direction $\boldsymbol{\omega}_{\mathrm{s}}$. We recall that $\boldsymbol{\theta}\cdot\nabla D_\theta a = a$.

Once $f_1(\mathbf{x})$ is reconstructed, we use the relations

$$\frac{\partial}{\partial\bar{z}}\mathcal{H}_1 = I, \qquad \frac{\partial}{\partial\bar{z}}\mathcal{H}_2 = -a(z)\mathcal{H}_1, \tag{43}$$

to obtain

$$\begin{aligned} f_0(z) &= \frac{\partial}{\partial\bar{z}}\varphi_1 + a(z)\mathcal{H}_1 f_{-1}(z) \\ &= \frac{1}{4\pi}\int_0^{2\pi}\boldsymbol{\theta}^\perp\cdot\nabla(\mathrm{i}\varphi)\,\mathrm{d}\theta + \frac{1}{4\pi}\int_0^{2\pi}\boldsymbol{\theta}\cdot\nabla\varphi\,\mathrm{d}\theta + a(z)\mathcal{H}_1 f_{-1}(z) \\ &= \frac{1}{4\pi}\int_0^{2\pi}\boldsymbol{\theta}^\perp\cdot\nabla(\mathrm{i}\varphi)\,\mathrm{d}\theta + a(z)\left(\mathcal{H}_1 f_{-1} - \frac{\varphi_0}{2}\right). \end{aligned} \tag{44}$$

Here we have used that $\boldsymbol{\theta}\cdot\nabla\varphi + a\varphi = 0$ as was noticed below (31). We deduce from (38) that the last term in the above equation is real-valued so that $f_0(z)$ is real-valued as expected. When f_1 is given by (40) some algebra shows that

$$\widehat{H_1 f_{-1}}(\boldsymbol{\xi}) - \frac{1}{2}\hat{\varphi}_0(\boldsymbol{\xi}) = \frac{1}{2}\frac{\mathrm{i}\boldsymbol{\xi}\cdot\boldsymbol{\omega}_{\mathrm{s}}^\perp}{\mathrm{i}\boldsymbol{\xi}\cdot\boldsymbol{\omega}_{\mathrm{s}}}(\mathrm{i}\hat{\varphi}_0)(\boldsymbol{\xi}).$$

This implies the reconstruction formula

$$f_0(\mathbf{x}) = \frac{1}{4\pi}\int_0^{2\pi}\boldsymbol{\theta}^\perp\cdot\nabla(\mathrm{i}\varphi)(\mathbf{x},\theta)\,\mathrm{d}\theta + \frac{1}{2}D_{\boldsymbol{\omega}_{\mathrm{s}}}\boldsymbol{\omega}_{\mathrm{s}}^\perp\cdot\nabla(\mathrm{i}\varphi_0)(\mathbf{x}). \tag{45}$$

This is the generalization of the Novikov formula as it appears in [15, 19]. Notice that our reconstruction is parameterized by the rotation factor $\boldsymbol{\omega}$. Unless $\varphi_0(\mathbf{x})$ uniformly vanishes, the reconstruction of $f_0(\mathbf{x})$ will depend on the choice of $\boldsymbol{\omega}$.

Compatibility condition. In the case where $f(z,\lambda) = f(z)$ we deduce that

$$\begin{aligned} 0 &= \frac{1}{2\pi\mathrm{i}}\int_T\frac{\varphi(z,t)}{t}\,\mathrm{d}t = \frac{1}{2\pi}\int_0^{2\pi}\varphi(z,\theta)\,\mathrm{d}\theta, \\ f(z) &= \frac{1}{2\pi\mathrm{i}}\int_T\frac{\partial\varphi}{\partial\bar{z}}(z,t)\frac{\mathrm{d}t}{t^2} = \frac{1}{4\pi}\int_0^{2\pi}\boldsymbol{\theta}^\perp\cdot\nabla(\mathrm{i}\varphi)(z,\theta)\,\mathrm{d}\theta. \end{aligned} \tag{46}$$

The latter equation is the usual Novikov formula [15, 19]. The former equation is a *compatibility condition* (the Range condition in [18, 19]) ensuring that measured data indeed correspond to a source term of the form $f(z,\lambda) = f(z)$. When the compatibility condition is not satisfied, for instance because the measured data are noisy, we may be able remove part of this noise from the reconstruction by reconstructing f_1 and f_0 as was done in (45).

2.6. Compatible reconstructions

We have seen in the preceding section that full measurements allowed us to reconstruct two spatially dependent functions f_0 and f_1 (assuming that (40) holds). We claim that the above reconstruction is in some sense optimal as the AtRT of the reconstructed source yields back the data we started with. This is in some sense a Range condition for the AtRT reminiscent of the one in [19]. Not surprisingly the derivation is very similar to that in [19]. The main argument in the proof is the uniqueness of the solution to the Riemann–Hilbert problem we introduced earlier. To avoid technical details that are well explained in [19] we assume that all our functions are sufficiently smooth, and refer to [19] for additional details.

The uniqueness of the reconstruction is demonstrated as follows. Let $g(s,\theta)$ be given 'measured data' on $\mathbb{R}\times(0,2\pi)$. We define $\mathrm{i}\varphi(\mathbf{x},\theta) = R^*_{-a,\theta}H_a g(\mathbf{x})$ following (32). Notice that $\mathrm{i}\varphi(\mathbf{x},\theta)$ is real-valued. We now solve the Riemann–Hilbert problem for $\phi(z,\lambda)$. Thus $\phi(z,\lambda)$ is the unique function analytic (in λ) on D^+ and D^- such that $\phi(z,\lambda)$ is $\mathrm{O}(\lambda^{-1})$ as $|\lambda|\to\infty$, and such that $\phi^+ - \phi^- = \varphi$ on T. We may now define the functions

$$\phi(z,\lambda) = \begin{cases} \phi_{-1}(z)\lambda^{-1} + \mathrm{O}(\lambda^{-2}) & \lambda\to\infty, \\ \phi_0(z) + \phi_1(z)\lambda + \mathrm{O}(\lambda^2) & \lambda\to 0. \end{cases} \tag{47}$$

We shall show in step 1 below that $\phi_{-1}(z) = \overline{\phi_1}(z)$. We also define $f_0(z)$ and $f_1(z)$ as the functions given by (45) and (42), respectively, for an arbitrary fixed value of $\boldsymbol{\omega}\in S^1$.

Let us now define $\psi(z,\lambda) = \phi(z,\lambda) + \overline{\mathcal{H}_1} f_1$. We shall prove in step 2 below that

$$T_a\psi(z,\lambda) \equiv \left(\lambda\frac{\partial}{\partial z} + \lambda^{-1}\frac{\partial}{\partial\bar{z}} + a\right)\psi(z,\lambda) = f(z,\lambda), \tag{48}$$

with $f(z,\lambda) = f_0(z) + \lambda f_1(z) + \lambda^{-1} f_{-1}(z)$ and $f_{-1}(z) = \bar{f}_1(z)$. We recognize the transport equation (9) with appropriate source term. We can thus calculate the new AtRT and apply $R^*_{-a,\theta}H_a$ to obtain

$$g_r(s,\theta) = R_{a,\theta}(f_0 + \mathrm{e}^{\mathrm{i}\theta} f_1 + \mathrm{e}^{-\mathrm{i}\theta} f_{-1})(s), \qquad \mathrm{i}\varphi_r(\mathbf{x},\theta) = R^*_{-a,\theta}H_a g_r(\mathbf{x}). \tag{49}$$

We next verify that

$$\varphi_r = \psi^+ - \psi^- = \phi^+ - \phi^- = \varphi.$$

It remains to verify that $R^*_{-a,\theta}H_a h = 0$ implies that $h = 0$ (see step 3 below) to conclude that $g_r(s,\theta) = g(s,\theta)$.

We have thus shown that for any (sufficiently smooth) data $g(s,\theta)$, we can construct source terms f_0, f_1, and $f_{-1} = \bar{f}_1$ such that $g(s,\theta)$ is the AtRT of the source term $f(z,\theta) = f_0(z) + \mathrm{e}^{\mathrm{i}\theta} f_1(z) + \mathrm{e}^{-\mathrm{i}\theta} f_{-1}(z)$. The source term is by no means unique since a whole family parameterized by $\boldsymbol{\omega}\in S^1$ can be obtained.

Verification of the three steps. Step 1. Let $\lambda = \mathrm{e}^{\mathrm{i}\theta}$ on T and define the limiting points $\lambda^\pm = (1\mp 0)\mathrm{e}^{\mathrm{i}\theta}$. We observe that $(\bar{\lambda}^\pm)^{-1} = \lambda^\mp$. Thus $\phi(\bar{\lambda}^{-1})$ satisfies the same Riemann–Hilbert problem as ϕ but with φ replaced by $-\varphi$. Now since $\mathrm{i}\varphi$ is real, $-\varphi = \bar{\varphi}$. By uniqueness of the solution to the Riemann–Hilbert problem (including the constraint that the solution be of order $O(\lambda^{-1})$ at infinity), we deduce that $\phi(z, \bar{\lambda}^{-1}) - \phi_0(z) = \bar{\phi}(z, \lambda)$. This implies that $\phi_{-1}(z) = \overline{\phi_1}(z)$.

Step 2. Using (34) and (35), we deduce that $\phi_k = \varphi_k$ for $k = 0, 1$. We now remark that $T_a\phi$, where T_a is defined in (48), is analytic on $D^+\backslash\{0\}$ and D^-. From the asymptotic expansion of ϕ, we deduce that

$$u(z, \lambda) = T_a\phi(z, \lambda) - \frac{\partial}{\partial z}\bar{\varphi}_1(z) - \lambda^{-1}\frac{\partial}{\partial \bar{z}}\varphi_0(z)$$

is sectionally analytic on D^+ and D^- and is of order $O(\lambda^{-1})$ at infinity. It is thus the unique solution of the Riemann–Hilbert problem with jump condition $\varphi_u = u^+ - u^-$. Since, however, $u^+ - u^- = T_a\phi^+ - T_a\phi^- = T_a\varphi = 0$ as was remarked below (33), we deduce that $\varphi_u = 0$ and, by uniqueness of the solution to the Riemann–Hilbert problem, that $u = 0$. By construction of ψ, this implies that

$$T_a\psi = T_a\overline{\mathcal{H}_1} f_1 + \frac{\partial}{\partial z}\bar{\varphi}_1 + \lambda^{-1}\frac{\partial}{\partial \bar{z}}\varphi_0(z).$$

Upon equating like powers of λ, we obtain that $T_a\psi = f(z, \lambda)$ is equivalent to the system of equations:

$$\begin{aligned}
\lambda^{-1}&: \bar{f}_1 = \frac{\partial}{\partial \bar{z}}\varphi_0 + \frac{\partial}{\partial \bar{z}}\overline{\mathcal{H}_1 f_1},\\
\lambda^{0}&: f_0 \;= \frac{\partial}{\partial z}\bar{\varphi}_1 + a\overline{\mathcal{H}_1} f_1,\\
\lambda^{1}&: f_1 \;= \frac{\partial}{\partial z}\overline{\mathcal{H}_1} f_1.
\end{aligned}$$

Applying $\mathcal{H}_1$ to the first equation yields the first equation in (38). The third equation is nothing but (43). Applying $\overline{\mathcal{H}_1}$ to the second equation and recalling that $\overline{\mathcal{H}_2} = -\mathcal{H}_1 a\overline{\mathcal{H}_1}$, the second equation is equivalent to

$$\overline{\mathcal{H}_1} f_0 + \overline{\mathcal{H}_2} f_1 = \bar{\varphi}_1.$$

This is equivalent to the second equation in (38) because f_0 is real-valued by construction.

Step 3. This is a corollary of statement 3.5 in [19]. This concludes our derivation of the reconstruction of a source term of the form $f(z, \lambda) = f_0(z) + \lambda f_1(z) + \lambda^{-1} f_{-1}(z)$ from arbitrary (smooth) AtRT data $g(s, \theta)$.

2.7. Doppler tomography

The attenuated Doppler transform can be seen as an attenuated Radon transform with source term of the form

$$f(\mathbf{x}, \theta) = \boldsymbol{\theta} \cdot \mathbf{F}(\mathbf{x}), \qquad \mathbf{F}(\mathbf{x}) = (F_1(\mathbf{x}), F_2(\mathbf{x})). \tag{50}$$

In the absence of attenuation, it is known that only $\nabla \times \mathbf{F}(\mathbf{x}) = \frac{\partial}{\partial x}F_2(\mathbf{x}) - \frac{\partial}{\partial y}F_1(\mathbf{x})$ can be reconstructed. In the presence of absorption, we now show that both components of $\mathbf{F}(\mathbf{x})$ can be reconstructed on the support of $a(\mathbf{x})$. We first verify that

$$f(\mathbf{x}, \theta) = \lambda\left(\frac{F_1(\mathbf{x})}{2} - \mathrm{i}\frac{F_2(\mathbf{x})}{2}\right) + \lambda^{-1}\left(\frac{F_1(\mathbf{x})}{2} + \mathrm{i}\frac{F_2(\mathbf{x})}{2}\right).$$

So we can identify the above source term with (2) choosing $f_1(\mathbf{x}) = \frac{1}{2}(F_1(\mathbf{x}) - \mathrm{i}F_2(\mathbf{x}))$ and $f_k(\mathbf{x}) \equiv 0$ for $|k| \neq 1$. Therefore, (38) holds with $f_0(z) \equiv 0$. From the first equation we deduce that

$$\mathrm{Im}(\mathcal{H}_1 f_{-1}(z)) = \mathrm{i}\varphi_0(z), \tag{51}$$

which upon using (41) yields

$$\nabla \times \mathbf{F}(\mathbf{x}) = \frac{\partial F_2(\mathbf{x})}{\partial x} - \frac{\partial F_1(\mathbf{x})}{\partial y} = \frac{1}{2}\Delta(\mathrm{i}\varphi_0)(\mathbf{x}). \tag{52}$$

This is the usual reconstruction that also holds in the absence of absorption. We can also use the second constraint in (38) to obtain that

$$\mathcal{H}_2 f_{-1}(z) = \varphi_1(z).$$

Since $\mathcal{H}_2 = -\mathcal{H}_1 a \mathcal{H}_1$, we deduce that

$$\mathcal{H}_1 f_{-1}(z) = \frac{1}{a(z)}\frac{\partial \varphi_1(z)}{\partial \bar{z}}, \qquad \text{whence } \frac{1}{2}(F_1(z) + \mathrm{i}F_2(z)) = -\frac{\partial}{\partial \bar{z}}\frac{1}{a(z)}\frac{\partial \varphi_1(z)}{\partial \bar{z}}. \tag{53}$$

The imaginary part of the first relation yields (51) since $\mathrm{Im}(\partial_{\bar{z}}\varphi_1) + a\varphi_0 = 0$, which follows from $\boldsymbol{\theta}\cdot\nabla\varphi + a\varphi = 0$. The latter relation holds on the support of $a(\mathbf{x})$ and is equivalent in the spatial variables to

$$\begin{aligned} F_1(\mathbf{x}) &= -\left(\frac{\partial}{\partial x}\frac{f_R(\mathbf{x})}{a(\mathbf{x})} - \frac{\partial}{\partial y}\frac{f_I(\mathbf{x})}{a(\mathbf{x})}\right), \qquad & F_2(\mathbf{x}) &= -\left(\frac{\partial}{\partial x}\frac{f_I(\mathbf{x})}{a(\mathbf{x})} + \frac{\partial}{\partial y}\frac{f_R(\mathbf{x})}{a(\mathbf{x})}\right), \\ f_R(\mathbf{x}) &= \frac{1}{4\pi}\int_0^{2\pi}\boldsymbol{\theta}^\perp\cdot\nabla(\mathrm{i}\varphi)(\mathbf{x},\theta)\,\mathrm{d}\theta, \qquad & f_I(\mathbf{x}) &= \frac{a(\mathbf{x})}{4\pi}\int_0^{2\pi}(\mathrm{i}\varphi)(\mathbf{x},\theta)\,\mathrm{d}\theta. \end{aligned} \tag{54}$$

Similar (equivalent) formulae were first proposed in [7, 8].

3. Reconstruction from partial measurements

We have seen in the preceding section that full measurements $R_{a,\theta} f(s,\theta)$ for $s \in \mathbb{R}$ and $\theta \in (0, 2\pi)$ allow us to reconstruct two spatial functions. It is therefore natural to try and reconstruct one spatial function from half of the boundary measurements. In this section we assume that $f(\mathbf{x},\theta) = f(\mathbf{x})$ and consider reconstruction algorithms from partial measurements in the angular variable θ. We assume that $f(\mathbf{x})$ has compact support and to simplify assume that the support is included in the unit ball B. Let us denote the measured data by

$$g(s,\theta) = R_{a,\theta} f(s). \tag{55}$$

We assume that $g(s,\theta)$ is available for all values of $s \in \mathbb{R}$ and for $\theta \in M \subset [0, 2\pi)$. The assumption on M is that $M^c = [0, 2\pi)\backslash M \subset \overline{M+\pi}$. This implies that measurements are available for at least one of the angles θ or $\theta+\pi$. For simplicity of analysis, we assume that M is composed of a finite number of intervals in $[0, 2\pi)$ closed on the left and open on the right. The simplest example is $M = [0, \pi)$ and $M^c = [\pi, 2\pi)$. Notice that uniqueness of the reconstruction is also ensured when fewer data are available (see [18] for the AtRT and [13] for the Radon transform). However, the result is based on analytic continuation and thus results in a much more severely ill-posed problem than what is proposed below.

Let us recast (31) as

$$\frac{\mathrm{i}\varphi(\mathbf{x},\theta)}{2} = R^*_{-a,\theta} H_a R_{a,\theta} f(\mathbf{x}) \equiv \Phi_{a,\theta} f(\mathbf{x}) \tag{56}$$

and define the operators

$$\begin{aligned} F_\theta &= \boldsymbol{\theta}^\perp\cdot\nabla\Phi_{a,\theta} = F_{1,\theta} + F_{2,\theta}, \\ F_{1,\theta} &= R^*_{-a,\theta}\frac{\partial}{\partial s}H_a R_{a,\theta}, \\ F_{2,\theta} &= \left(\boldsymbol{\theta}^\perp\cdot\nabla R^*_{-a,\theta} - R^*_{-a,\theta}\frac{\partial}{\partial s}\right)H_a R_{a,\theta}. \end{aligned} \tag{57}$$

Formally, the reconstruction formulae obtained in earlier sections show that

$$2\pi I = \int_0^{2\pi} F_\theta \, \mathrm{d}\theta. \tag{58}$$

We shall see in the next section that this equality holds in the L^2 sense for compactly supported functions. Since information is available on M only we would like to obtain reconstruction formulae involving an integration over M. We propose the following decomposition:

$$2\pi I = \int_M F_\theta \, \mathrm{d}\theta + \int_{M^c} F_{1,\theta}^* \, \mathrm{d}\theta + \int_{M^c} (F_{1,\theta} - F_{1,\theta}^*) \, \mathrm{d}\theta + \int_{M^c} F_{2,\theta} \, \mathrm{d}\theta. \tag{59}$$

The main reason for introducing this decomposition is that

$$F_{1,\theta}^* = R_{a,\theta}^* H_a \frac{\partial}{\partial s} R_{-a,\theta}, \tag{60}$$

so that $F_{1,\theta}^*$ on M^c involves

$$R_{-a,\theta} f(s) = R_{a,\theta+\pi} f(-s), \tag{61}$$

where now $\theta + \pi \in M$ by construction. So $F_{1,\theta}^*$ is an operator that involves the measured data only. The third operator on the right-hand side of (59) is now formally skew-symmetric. It remains the operator $F_{2,\theta}$, which unfortunately does not seem to have any useful symmetry properties. We can always recast it as

$$F_{2,\theta} = F_{2,\theta}^s + F_{2,\theta}^a, \qquad F_{2,\theta}^s = \tfrac{1}{2}(F_{2,\theta} + F_{2,\theta}^*). \tag{62}$$

By doing so we recast (59) as

$$I = F^d + F^a + F^s, \tag{63}$$

where

$$\begin{aligned} F^d &= \frac{1}{2\pi} \int_M F_\theta \, \mathrm{d}\theta + \frac{1}{2\pi} \int_{M^c} F_{1,\theta}^* \, \mathrm{d}\theta, \\ F^a &= \frac{1}{2\pi} \int_{M^c} (F_{1,\theta} - F_{1,\theta}^* + F_{2,\theta}^a) \, \mathrm{d}\theta, \qquad F^s = \frac{1}{2\pi} \int_{M^c} F_{2,\theta}^s \, \mathrm{d}\theta. \end{aligned} \tag{64}$$

The operator F^d involves only the measured data $g(s,\theta)$ on M, whereas the operators F^a and F^s are formally skew-symmetric and symmetric, respectively.

We show in theorem 4.1 in the next section that all operators are bounded in $\mathcal{L}(L^2(B))$, where B is the unit ball (or any arbitrary ball by rescaling). Moreover, the operator F^s is compact in the same sense.

We define $d(\mathbf{x}) = F^d f(\mathbf{x})$ obtained from the measured data. The problem we aim to solve is thus to find $f(\mathbf{x})$ such that

$$f(\mathbf{x}) = \mathrm{d}(\mathbf{x}) + F^a f(\mathbf{x}) + F^s f(\mathbf{x}). \tag{65}$$

Our main result is the following.

Theorem 3.1. *Let us assume that F^s as an operator in $\mathcal{L}(L^2(B))$ has spectral radius $\rho(F^s) < 1$. Then we can reconstruct $f(\mathbf{x})$ uniquely from the measurements $g(s,\theta)$ on M. The reconstruction is obtained as follows. We have that*

$$f(\mathbf{x}) = (I - F^s)^{-1/2} h(\mathbf{x}), \tag{66}$$

where $h(\mathbf{x})$ is the unique solution to the following equation:

$$h(\mathbf{x}) = (I - F^s)^{-1/2} d(\mathbf{x}) + (I - F^s)^{-1/2} F^a (I - F^s)^{-1/2} h(\mathbf{x}). \tag{67}$$

The above equation admits a unique solution that can be computed explicitly by an iterative method; see (69) below.

Proof. Since F^s is self-adjoint and has spectral radius less than 1, $I - F^s$ is also a self-adjoint operator with non-negative spectrum, and we can define $(I - F^s)^{-1/2}$, which is also a self-adjoint and bounded operator. We obtain then that $G^a = (I - F^s)^{-1/2} F^a (I - F^s)^{-1/2}$ is a bounded and skew-symmetric operator in $\mathcal{L}(L^2(B))$. This implies that $\mathrm{i}G^a$ is Hermitian with real-valued spectrum. Hence i is in the resolvent of $\mathrm{i}G^a$ and $\mathrm{i}I - \mathrm{i}G^a$ is invertible with bounded inverse in $\mathcal{L}(L^2(B))$. This ensures the existence of a unique solution to the Fredholm equation (67). We can also obtain an explicit iterative algorithm as, for example, in [17]. Indeed we recast (67) as

$$h(\mathbf{x}) = \gamma (I - F^s)^{-1/2} d(\mathbf{x}) + ((1-\gamma) I + \gamma G^a) h(\mathbf{x}), \tag{68}$$

and choose $\gamma = (1 + \|G^a\|_2^2)^{-1}$. Because G^a is skew-symmetric, one observes that

$$\|(1-\gamma)I + \gamma G^a\|_2 = \frac{\|G^a\|_2}{(1 + \|G^a\|_2^2)^{1/2}} < 1,$$

so that (68) can be solved iteratively:

$$h^{k+1}(\mathbf{x}) = \gamma (I - F^s)^{-1/2} d(\mathbf{x}) + ((1-\gamma) I + \gamma G^a) h^k(\mathbf{x}) \tag{69}$$

with $h^k \to h$ in $L^2(B)$ as $k \to \infty$. □

The case where F^s is of spectral radius greater than 1 remains open. In such situation it is unclear whether (65) admits any solution. However we can use the regularization properties of F^s to obtain some information about $f(\mathbf{x})$. For instance let us denote by f^a the solution to

$$f^a(\mathbf{x}) = d(\mathbf{x}) + F^a f^a(\mathbf{x}). \tag{70}$$

Then we obtain that the source term $f(\mathbf{x})$ that generated the data $d(\mathbf{x})$ is such that

$$f(\mathbf{x}) - f^a(\mathbf{x}) \in \mathrm{Range}(F^s). \tag{71}$$

So $f^a(\mathbf{x})$ captures the most singular part of $f(\mathbf{x})$ since F^s is compact.

Remark in the case of constant absorption

In the case where a is constant on the unit ball, we can show that $F_{2,\theta} = 0$ so that the reconstruction is always possible. Indeed, let us assume that f is compactly supported in the unit ball and that the absorption parameter $a(\mathbf{x}) = \mu$ for $|\mathbf{x}| < 1$ and $a(\mathbf{x}) = 0$ otherwise. We verify that

$$\mathrm{e}^{D_\theta a(\mathbf{x})} = \mathrm{e}^{\mu \mathbf{x}\cdot\boldsymbol{\theta}}, \qquad |\mathbf{x}| < 1,$$

i.e., on the support of the source term f. We thus deduce in this case

$$\boldsymbol{\theta}^\perp \cdot \nabla(\mathrm{e}^{D_\theta a(\mathbf{x})} g(\mathbf{x}\cdot\boldsymbol{\theta}^\perp, \theta)) = \mathrm{e}^{D_\theta a(\mathbf{x})} \left(\frac{\partial g(s,\theta)}{\partial s}\right)(\mathbf{x}\cdot\boldsymbol{\theta}^\perp),$$

which implies that $F_{2,\theta} \equiv 0$. We thus recover the results of [17, 20] that a complete reconstruction is possible when half of the measurements are available. More specifically, assuming that $M^c = M + \pi$ (so that M and M^c both have measure π), we can verify in the case of a constant on the unit ball that

$$H_a \frac{\partial}{\partial s} = \frac{\partial}{\partial s} H_a,$$

since both operators H_a and $\frac{\partial}{\partial s}$ are diagonal in the Fourier domain. More specifically, we obtain that [12]

$$\widehat{H_a u}(\sigma) = -\mathrm{i}\,\mathrm{sgn}_\mu(\sigma)\hat{u}(\sigma), \qquad \mathrm{sgn}_\mu(\sigma) = \begin{cases} \mathrm{sgn}(\sigma), & |\sigma| \geqslant \mu, \\ 0, & |\sigma| < \mu. \end{cases}$$

The product $H_a \frac{\partial}{\partial s} = \frac{\partial}{\partial s} H_a$ is thus the generalized Riesz operator with symbol $r_\mu(\sigma) = |\sigma|$ for $|\sigma| \geqslant \mu$ and $r_\mu(\sigma) = 0$ for $|\sigma| < \mu$, as it appears in the inversion formula proposed in [28]. These observations imply that

$$F_\theta^* = F_{1,\theta}^* = F_{\theta+\pi}.$$

We have thus in this case the following decomposition of identity:

$$I = \frac{2}{2\pi} \int_M F_\theta \, \mathrm{d}\theta + \frac{1}{2\pi} \int_{M+\pi} (F_\theta - F_{\theta+\pi}) \, \mathrm{d}\theta = F^d + F^a, \tag{72}$$

where F^d is the operator that depends on the measured data on M and F^a is a skew-symmetric operator. So in practice (72) implies that the measured data are multiplied by a factor 2 to obtain a first guess of the reconstruction. The error between the guess and the exact solution is modeled by a skew-symmetric operator and (72) can therefore be solved iteratively as in (69). This is the procedure first suggested in [17] for $M = [0, \pi)$ and extended to more general domains of data acquisition in [20].

4. Regularity results

We have seen in the preceding section that the reconstruction from partial measurements involves operators of the form

$$H_\alpha^\beta = \frac{1}{2\pi} \int_\alpha^\beta \boldsymbol{\theta}^\perp \cdot \nabla \Phi_{a,\theta} \, \mathrm{d}\theta, \tag{73}$$

where $0 \leqslant \alpha < \beta \leqslant 2\pi$ and $\Phi_{a,\theta}$ is defined in (56). Up to rescaling we assume that the source term $f(\mathbf{x})$ has support in the unit ball B. We still denote by $f(\mathbf{x})$ the function extended to $\mathbb{R}^2$ by 0 outside the unit ball B. We want to show that the operators H_α^β are uniformly bounded in $\mathcal{L}(L^2(B))$.

Let us denote $h(\mathbf{x}) = H_\alpha^\beta f(\mathbf{x})$ and consider first the case where $a \equiv 0$. We deduce that

$$(\Phi_{a,\theta} f)(\mathbf{x}) = \frac{H P_\theta f(\mathbf{x} \cdot \boldsymbol{\theta}^\perp)}{2}.$$

Some algebra, see [13], shows that the operators P_θ in (7) and H in (29) are given in the Fourier domain by

$$\widehat{P_\theta[f(\mathbf{x})]}(\sigma) = \hat{f}(\sigma\boldsymbol{\theta}^\perp), \qquad \widehat{Hu}(\sigma) = -\mathrm{i}\,\mathrm{sgn}(\sigma)\hat{u}(\sigma). \tag{74}$$

We recognize in the first equality the Fourier slice theorem. We deduce that

$$\widehat{\Phi_{a,\theta} f}(\boldsymbol{\xi}) = -\mathrm{i}\,\mathrm{sgn}(\boldsymbol{\xi}\cdot\boldsymbol{\theta}^\perp)\hat{f}(\boldsymbol{\xi}\cdot\boldsymbol{\theta}^\perp\boldsymbol{\theta}^\perp)\pi\delta(\boldsymbol{\xi}\cdot\boldsymbol{\theta}) = -\mathrm{i}\,\mathrm{sgn}(\boldsymbol{\xi}\cdot\boldsymbol{\theta}^\perp)\hat{f}(\boldsymbol{\xi})\pi\delta(\boldsymbol{\xi}\cdot\boldsymbol{\theta}).$$

Denoting $\boldsymbol{\xi} = |\boldsymbol{\xi}|\hat{\boldsymbol{\xi}}$ and $\hat{\boldsymbol{\xi}} = (\cos\xi, \sin\xi)$, we verify that

$$\widehat{\Phi_{a,\theta} f}(\boldsymbol{\xi}) = -\mathrm{i}\,\mathrm{sgn}(\boldsymbol{\xi}\cdot\boldsymbol{\theta}^\perp)\hat{f}(\boldsymbol{\xi})\pi \frac{1}{|\boldsymbol{\xi}\cdot\boldsymbol{\theta}^\perp|}\delta(\hat{\boldsymbol{\xi}}\cdot\boldsymbol{\theta}) = \frac{-\mathrm{i}}{\boldsymbol{\xi}\cdot\boldsymbol{\theta}^\perp}\pi\hat{f}(\boldsymbol{\xi})\delta(\hat{\boldsymbol{\xi}}\cdot\boldsymbol{\theta}).$$

The measure $\delta(\hat{\boldsymbol{\xi}}\cdot\boldsymbol{\theta})$ is supported on the values

$$\theta = \xi - \frac{\pi}{2} \equiv \xi_B \qquad \text{for } \hat{\boldsymbol{\xi}} = \boldsymbol{\theta}^\perp \qquad \text{and} \qquad \theta = \xi + \frac{\pi}{2} \equiv \xi_F \qquad \text{for } \hat{\boldsymbol{\xi}} = -\boldsymbol{\theta}^\perp. \tag{75}$$

This implies that for $h(\mathbf{x}) = H_\alpha^\beta f(\mathbf{x})$, we have

$$\hat{h}(\boldsymbol{\xi}) = \tfrac{1}{2}(\chi_{(\alpha,\beta)}(\xi_B) + \chi_{(\alpha,\beta)}(\xi_F))\hat{f}(\boldsymbol{\xi}). \tag{76}$$

Here, $\chi_{(\alpha,\beta)}(\theta) = 1$ for $\theta \in (\alpha, \beta)$ and $\chi_{(\alpha,\beta)}(\theta) = 0$ otherwise. This shows that $h(\mathbf{x}) \in L^2(\mathbb{R}^2)$ independently of the interval (α, β) and that $h(\mathbf{x}) = f(\mathbf{x})$ when $(\alpha, \beta) = (0, 2\pi)$. Thus the restriction of $h(\mathbf{x})$ to B belongs to $L^2(B)$ and H_α^β is bounded in $\mathcal{L}(L^2(B))$ independently of the interval (α, β).

We aim to extend these results to the case $a \neq 0$. To do so we recast $H_\alpha^\beta f(\mathbf{x})$ using (33) and (57) as a sum of terms of the form

$$h(\mathbf{x}) = \frac{1}{2\pi} \int_\alpha^\beta \boldsymbol{\theta}^\perp \cdot \nabla (u(\mathbf{x}, \theta) H[v(s, \theta) P_\theta(w(\mathbf{x}, \theta))(s)](\mathbf{x} \cdot \boldsymbol{\theta}^\perp))\, \mathrm{d}\theta, \tag{77}$$

where the functions $u(\mathbf{x}, \theta)$ and $v(s, \theta)$ are smooth since the absorption map $a(\mathbf{x})$ is smooth and $w(\mathbf{x}, \theta) = \mathrm{e}^{D_\theta a(\mathbf{x})} f(\mathbf{x}) \in L^2(B; \mathcal{C}^0(0, 2\pi))$ since $f(\mathbf{x}) \in L^2(B)$.

Following the decomposition $F_\theta = F_{1,\theta} + F_{2,\theta}$ in the preceding section we split h into two contributions $h_a + h_b$ as follows:

$$\begin{aligned} h_a(\mathbf{x}) &= \frac{1}{2\pi} \int_\alpha^\beta (\boldsymbol{\theta}^\perp \cdot \nabla u(\mathbf{x}, \theta)) H[v(s, \theta) P_\theta(w(\mathbf{x}, \theta))(s)](\mathbf{x} \cdot \boldsymbol{\theta}^\perp)\, \mathrm{d}\theta, \\ h_b(\mathbf{x}) &= \frac{1}{2\pi} \int_\alpha^\beta u(\mathbf{x}, \theta) \boldsymbol{\theta}^\perp \cdot \nabla (H[v(s, \theta) P_\theta(w(\mathbf{x}, \theta))(s)](\mathbf{x} \cdot \boldsymbol{\theta}^\perp))\, \mathrm{d}\theta. \end{aligned} \tag{78}$$

The first term h_a involves two one-dimensional integrations of $f(\mathbf{x})$ and no differentiation, so it is natural to assume that it corresponds to a compact operator. The second term h_b involves a derivation applied to the function $f(\mathbf{x})$ as in the case $a = 0$, and so we expect that it corresponds to a bounded operator in $L^2(B)$.

Compactness of the first contribution

Let us consider the bound for h_a first. Since $w(\mathbf{x}, \theta) \in L^2(B; \mathcal{C}^0(0, 2\pi))$, we obtain that $P_\theta(w(\mathbf{x}, \theta))(s) \in H^{1/2}(Z)$ with $Z = \mathbb{R} \times S^1$. We follow here the notation in [13] and define $H^\alpha(Z)$ as the Sobolev space of functions $g(s, \theta)$ bounded for the norm

$$\left(\int_0^{2\pi} \int_{\mathbb{R}} (1 + \sigma^2)^\alpha |\hat{g}(\sigma, \theta)|^2\, \mathrm{d}\sigma\, \mathrm{d}\theta \right)^{1/2},$$

where $\hat{g}(\sigma, \theta)$ is the Fourier transform of $g(s, \theta)$ in the first variable only.

For v sufficiently smooth, we deduce that $H[v(s, \theta) P_\theta(w(\mathbf{x}, \theta))(s)](s)$ belongs to $H^{1/2}(Z)$ since the Hilbert transform preserves functions in the Hilbert scale as can be seen in (74). This implies that

$$H[v(s, \theta) P_\theta(w(\mathbf{x}, \theta))(s)](\mathbf{x} \cdot \boldsymbol{\theta}^\perp) \varphi(|\mathbf{x}|) \in H^{1/2}(\mathbb{R}^2 \times S^1),$$

where $H^{1/2}(\mathbb{R}^2 \times S^1)$ is defined as above with $\mathbb{R}$ replaced by $\mathbb{R}^2$, and where $\varphi(r)$ is a smooth function over $\mathbb{R}^+$ such that $\varphi(r) = 1$ on $[0, 1]$ and $\varphi(r) = 0$ for $r > 2$. For u sufficiently smooth, we obtain that

$$(\boldsymbol{\theta}^\perp \cdot \nabla u(\mathbf{x}, \theta)) H[v(s, \theta) P_\theta(w(\mathbf{x}, \theta))(s)](\mathbf{x} \cdot \boldsymbol{\theta}^\perp) \in L^2((0, 2\pi); H^{1/2}(B)),$$

since $\varphi(|\mathbf{x}|) = 1$ on B. Since the interval (α, β) is bounded, we deduce from the Cauchy–Schwarz inequality that $h_a(\mathbf{x}) \in H^{1/2}(B)$. We thus obtain that the operator

$$f(\mathbf{x}) \to h_a(\mathbf{x}) \qquad \text{is compact in } \mathcal{L}(L^2(B)). \tag{79}$$

L^2 bound for the second contribution

We now obtain a bound for the $L^2(B)$ norm of $h_b(\mathbf{x})$. We want to use (76) to obtain a bound in the $L^2(B)$ sense. There are three issues to consider: $u(\mathbf{x}, \theta)$ depends on θ, $v(s, \theta)$ is not 1, and $w(\mathbf{x}, \theta)$ depends on θ.

We first decompose

$$u(\mathbf{x}, \theta) = \sum_{n=1}^{\infty} u_n(\theta)\phi_n(\mathbf{x}), \tag{80}$$

where $\phi_n(\mathbf{x})$ are uniformly bounded functions that form an orthonormal (in the $L^2(B)$ sense) basis of the unit ball B (the eigenvectors of the Laplacian for instance). We thus have that

$$u_n(\theta) = \int_\Omega u(\mathbf{x}, \theta)\phi_n(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

Let us now define

$$h_b(\mathbf{x}) = \sum_{n=1}^{\infty} h_n(\mathbf{x}), \qquad h_n(\mathbf{x}) = \phi_n(\mathbf{x})\frac{1}{4\pi}\int_\alpha^\beta u_n(\theta)\tau(\mathbf{x}, \theta)\,\mathrm{d}\theta,$$

where

$$\tau(\mathbf{x}, \theta) = \boldsymbol{\theta}^\perp \cdot \nabla(H[v(s, \theta)P_\theta(w(\mathbf{x}, \theta))(s)](\mathbf{x}\cdot\boldsymbol{\theta}^\perp)).$$

We thus have

$$\|h_b(\mathbf{x})\|_2 \leqslant \sum_{n=1}^{\infty}\|h_n(\mathbf{x})\|_2 \leqslant \sum_{n=1}^{\infty}\|\phi_n(\mathbf{x})\|_\infty \left\| \frac{1}{4\pi}\int_\alpha^\beta u_n(\theta)\tau(\mathbf{x}, \theta)\,\mathrm{d}\theta \right\|_2.$$

Here $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are the $L^2(B)$ and $L^\infty(B)$ norms, respectively. Therefore we have to estimate terms of the form

$$p_n(\mathbf{x}) = \frac{1}{4\pi}\int_\alpha^\beta u_n(\theta)\boldsymbol{\theta}^\perp \cdot \nabla(H[\nu(s, \theta)](\mathbf{x}\cdot\boldsymbol{\theta}^\perp))\,\mathrm{d}\theta,$$

where we have defined $\nu(s, \theta) = v(s, \theta)P_\theta(w(\mathbf{x}, \theta))(s)$. We observe that

$$\begin{aligned}\hat{p}_n(\boldsymbol{\xi}) &= \tfrac{1}{2}\int_\alpha^\beta u_n(\theta)|\boldsymbol{\xi}\cdot\boldsymbol{\theta}^\perp|\delta(\boldsymbol{\xi}\cdot\boldsymbol{\theta})\hat{\nu}(\boldsymbol{\xi}\cdot\boldsymbol{\theta}^\perp, \theta)\,\mathrm{d}\theta \\ &= \tfrac{1}{2}\chi_{(\alpha,\beta)}(\xi_B)u_n(\xi_B)\hat{\nu}(-|\boldsymbol{\xi}|, \xi_B) + \tfrac{1}{2}\chi_{(\alpha,\beta)}(\xi_F)u_n(\xi_F)\hat{\nu}(|\boldsymbol{\xi}|, \xi_F).\end{aligned}$$

The notation ξ_B and ξ_F is defined in (75). Thus independently of the interval (α, β), we have

$$2\pi\|p_n(\mathbf{x})\|_2 = \|\hat{p}_n(\boldsymbol{\xi})\|_2 \leqslant \sup_\theta |u_n(\theta)|\mathcal{N},$$

where $\mathcal{N} = \sup(\|\hat{\nu}(|\boldsymbol{\xi}|, \xi_F)\|_2, \|\hat{\nu}(-|\boldsymbol{\xi}|, \xi_B)\|_2)$. This implies that

$$\|h_b(\mathbf{x})\|_2 \leqslant \left(\sum_{n=1}^{\infty}\sup_\theta |u_n(\theta)|\right)\mathcal{N}. \tag{81}$$

The infinite sum in n converges provided that $u(\mathbf{x}, \theta)$ is sufficiently regular. It thus remains to show that $\boldsymbol{\xi} \mapsto \hat{\nu}(|\boldsymbol{\xi}|, \xi_F)$ is bounded in $L^2(\mathbb{R}^2)$. The term involving ξ_B is treated similarly. From the definition of ν and the Fourier slice theorem (74), we deduce that

$$\hat{\nu}(|\boldsymbol{\xi}|, \xi_F) = \hat{v}(|\boldsymbol{\xi}|, \xi_F) * \hat{w}(\boldsymbol{\xi}, \xi_F) = \int_{\mathbb{R}} \hat{v}(|\boldsymbol{\xi}| - t, \xi_F)\hat{w}(t\hat{\boldsymbol{\xi}}, \xi_F)\,\mathrm{d}t. \tag{82}$$

The L^2 norm squared of the above quantity in polar coordinates is thus bounded by

$$\int |\hat{v}(|\boldsymbol{\xi}| - t, \xi_F)\hat{w}(t\hat{\boldsymbol{\xi}}, \xi_F)\hat{v}^*(|\boldsymbol{\xi}| - s, \xi_F)\hat{w}^*(s\hat{\boldsymbol{\xi}}, \xi_F)|\,\mathrm{d}s\,\mathrm{d}t|\boldsymbol{\xi}|\mathrm{d}|\boldsymbol{\xi}|\mathrm{d}\hat{\boldsymbol{\xi}}.$$

We assume that v is sufficiently regular so that

$$|\hat{v}(t, \theta)| \leqslant \frac{C}{1 + |t|^m}, \tag{83}$$

with $m > 2$. Let us first integrate over $t > 0$ and $s > 0$. The integration in $r = |\boldsymbol{\xi}|$ above yields then a term bounded by

$$\int_0^\infty \frac{r}{(1+|t-r|^m)(1+|s-r|^m)} \mathrm{d}r \leqslant C_m \frac{\sup(t,s)}{1+|t-s|^m}$$

when $m > 2$. We thus obtain that the integration for $s, t > 0$ is bounded by

$$\int_{S^1}\int_0^\infty\int_0^\infty |\hat{w}(t\hat{\boldsymbol{\xi}}, \xi_F)\hat{w}^*(s\hat{\boldsymbol{\xi}}, \xi_F)| \frac{C_m \sup(t,s)}{1+|t-s|^m} \,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}\hat{\boldsymbol{\xi}}$$

$$\leqslant \frac{1}{2}\int_{S^1}\left(\int_0^\infty |\hat{w}(t\hat{\boldsymbol{\xi}}, \xi_F)|^2 \left(\int_0^\infty \frac{C_m \sup(t,s)}{1+|t-s|^m}\,\mathrm{d}s\right)\mathrm{d}t\right)\mathrm{d}\hat{\boldsymbol{\xi}},$$

using $2|ab^*| \leqslant |a|^2 + |b|^2$ and the symmetry $(t,s) \to (s,t)$. So we have that

$$\int_{S^1}\int_0^\infty t|\hat{w}(t\hat{\boldsymbol{\xi}}, \xi_F)|^2 \,\mathrm{d}t\,\mathrm{d}\hat{\boldsymbol{\xi}} \leqslant \|\hat{w}\|^2_{L^2(B;\mathcal{C}^0(0,2\pi))}.$$

The latter term is bounded by a multiple of $\|f\|_2$ provided that a is sufficiently smooth. The integral over $s < 0$ or $t < 0$ is dealt with in a similar (and slightly simpler) fashion.

This shows that the operator that maps $f(\mathbf{x})$ to $h(\mathbf{x})$ defined in (12) is bounded in $\mathcal{L}(L^2(B))$ independently of the interval (α, β). We summarize the above results in the following theorem.

Theorem 4.1. *Let us assume that M is a finite union of intervals in $[0, 2\pi)$ of the form $[\alpha_k, \beta_k)$ and that M^c is $[0, 2\pi)\backslash M$. Then for the domains $D = M$ and $D = M^c$, the operators*

$$\int_D F_{k,\theta}\,\mathrm{d}\theta \in \mathcal{L}(L^2(B)), \qquad \text{for } k = 1, 2,$$

and moreover the operators

$$\int_D F_{2,\theta}\,\mathrm{d}\theta \text{ are compact in } \mathcal{L}(L^2(B)) \qquad \text{with range in } H^{1/2}(B).$$

The operators $F_{k,\theta}$ are defined in (57).

5. Conclusions

Using a variation of the Novikov formula derived in [18, 19], we have shown that a source term of the form $f(\mathbf{x})+2\cos(\theta+\omega)f_1(\mathbf{x})$ can be reconstructed from the attenuated Radon transform $g(s,\theta)$ for all $s \in \mathbb{R}$ and all $\theta \in (0, 2\pi)$. The reconstruction is based on recasting the inverse problem as a Riemann–Hilbert problem. The same procedure allows us to reconstruct source terms of the form $f(\mathbf{x}) = F_1(\mathbf{x})\cos\theta + F_2(\mathbf{x})\sin\theta$ on the support of $a(\mathbf{x})$, which is of interest in Doppler tomography.

When only partial angular measurements are available for $\theta \in M$, with M such that $(0, 2\pi)\backslash M \subset \overline{M+\pi}$, we have shown that a compactly supported source term of the form $f(\mathbf{x})$ can be reconstructed provided that the variations of the absorption term $a(\mathbf{x})$ are not too large. The theory is based on showing that the Novikov reconstruction formula can be decomposed as the sum of three bounded operators in the L^2 sense. The first operator is a function of the measured data, whereas the second operator is of arbitrary size and is skew symmetric, and the third operator is symmetric and must be of spectral radius strictly bounded by 1. For constant absorption a on the support of $f(\mathbf{x})$ we recover results obtained earlier in [17, 20].

The case of arbitrary large spatial variations of the absorption parameter $a(\mathbf{x})$ is still open. However, we show in [5], devoted to the implementation of the fast slant stack algorithm in the context of the attenuated Radon transform, that the algorithms proposed in section 3 provide accurate reconstructions in situations of practical interest.

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