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Reconstructions in impedance and optical tomography with singular interfaces

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Abstract

Singular layers modelled by a tangential diffusion process supported on an embedded closed surface (of co-dimension 1) have found applications in tomography problems. In optical tomography they may model the propagation of photons in thin clear layers, which are known to hamper the use of classical diffusion approximations. In impedance tomography they may be used to model thin regions of very high conductivity profile. In this paper we show that such surfaces can be reconstructed from boundary measurements (more precisely, from a local Neumann-to-Dirichlet operator) provided that the material properties between the measurement surface and the embedded surface are known. The method is based on the *factorization* technique introduced by Kirsch. Once the location of the surface is reconstructed, we show under appropriate assumptions that the full tangential diffusion process and the material properties in the region enclosed by the surface can also uniquely be determined.

1. Introduction

Many applications require the reconstruction or imaging of physical coefficients with high contrasts. When large differences occur over thin domains, one may not so much be interested in separately reconstructing both the contrast and the support of the domain as in characterizing the global (non-local) effect the whole inclusion has on the rest of the domain. In such cases, it may be valuable to model such variations as a singular term supported on a surface. This paper considers the reconstruction from boundary measurements of such surfaces, the singular term supported on the surface and, when possible, the rest of the parameters of interest.

We have two primary applications in mind. The first application is the modelling of clear layers in optical tomography. Optical tomography consists of probing human tissues with near-infrared photons [3]. Although photons are best modelled with radiative transfer equations [2, 8], diffusion models are usually preferred because of their much lower computational cost. The presence of thin clear layers filled with optically thin (non-scattering) cerebro-spinal fluid hampers the use of classical diffusion so that more careful modelling is required [4, 13, 26]. Following the works in [6, 7] we consider here the modelling of the thin clear layer by a tangential diffusion process supported on a co-dimension one closed surface. The second application is the modelling of highly conducting cracks in impedance tomography. Cracks of thickness ε and conductivity of order ε^{-1} can also be modelled in the limit $\varepsilon \to 0$ as tangential diffusion processes supported on a surface [15].

The setting of the results presented here is as follows. In both cases the physical quantity of interest satisfies a second-order elliptic equation. We assume that we have access to the Neumann-to-Dirichlet (NtD) data at the boundary of a bounded domain and that the coefficients in the elliptic equation are *known* between the surface of the domain and the singular interface. Then using the factorization method introduced in [21] in the context of scattering theory and extended in [11, 16] to elliptic problems, we propose a method for locating the singular interface from the boundary measurements. The factorization method (or similarly the linear sampling method) is based on estimating the range conditions of an infinite-dimensional operator and as such has similarities with the MUSIC algorithm used to locate localized scatterers from scattering data [12, 22]. Once the interface is reconstructed we show under appropriate assumptions that the tangential diffusion process supported on the interface and the parameters of the elliptic equation are uniquely determined by the boundary measurements. We also show that the above results still hold when only partial measurements modelled by knowledge of a local Neumann-to-Dirichlet operator are available. We refer the reader to [1] for a similar result on the Schrödinger equation. We also mention that the probe method developed in [17, 18] may be used to reconstruct inclusions from partial boundary measurements.

The rest of the paper is organized as follows. The theory for the forward models in impedance and optical tomography is introduced in section 2. The reconstruction and uniqueness results are shown in section 3 in the case corresponding to impedance tomography. The reconstruction from partial measurements is addressed in section 3.4. The theory is generalized to the optical tomography problem in section 4 and concluding remarks are presented in section 5.

2. Forward models

We consider two types of inversions. The first problem consists of reconstructing the conductivity tensor $\gamma(\mathbf{x})$ from boundary measurements of potentials and currents. The potential $u(\mathbf{x})$ solves the following equation:

$$\begin{aligned}
&\nabla\cdot\gamma\nabla u = 0 && \text{in} \quad \Omega\backslash\Sigma\\
&[u] = 0 && \text{on} \quad \Sigma\\
&[\mathbf{n}\cdot\gamma\nabla u] = -\nabla_\perp\cdot \mathrm{d}\nabla_\perp u && \text{on} \quad \Sigma\\
&\mathbf{n}\cdot\gamma\nabla u = g && \text{on} \quad \partial\Omega\\
&\int_{\partial\Omega} u\,\mathrm{d}\sigma = 0.
\end{aligned} \tag{1}$$

Our assumptions are as follows. The domain Ω is a connected, open bounded subset in $\mathbb{R}^n$ for $n = 2$ or $n = 3$ with Lipschitz boundary $\partial\Omega$. The layer Σ is a closed surface of class C^2 strictly included in Ω. We define the open bounded domain D such that $\Sigma = \partial D$ and such that its complementary $D^c = \Omega\backslash\overline{D}$ in Ω has boundary $\partial D^c = \partial\Omega\cup\Sigma$. This means that D is the domain 'inside' the surface Σ. We assume that D^c is connected and that Σ is the union of N Lipschitz connected components Σ_j for $1 \leqslant j \leqslant N$, defining N open domains D_j such that $\Sigma_j = \partial D_j$ and thus $\partial D^c = \partial\Omega \cup \left(\cup_{j=1}^N \Sigma_j\right)$.

The $n \times n$ symmetric tensor $\gamma(\mathbf{x})$ is of class $C^2(\overline{\Omega})$ and positive definite such that $\xi_i\xi_j\gamma_{ij}(\mathbf{x}) \geqslant \alpha_0 > 0$ uniformly in $\mathbf{x} \in \Omega$ and in $\{\xi_i\}_{i=1}^n = \boldsymbol{\xi} \in S^{n-1}$, the unit sphere in $\mathbb{R}^n$. The $(n-1) \times (n-1)$ symmetric tensor $d(\mathbf{x})$ is uniformly bounded and positive definite such that $\xi_i\xi_j d_{ij}(\mathbf{x}) \geqslant \alpha_0 > 0$ uniformly in $\boldsymbol{\xi} \in T_{\mathbf{x}}\Sigma$ such that $|\boldsymbol{\xi}| = 1$. The notation $[\cdot]$ stands for the jump across the surface Σ in the direction $\mathbf{n}(\mathbf{x})$, the outward unit normal at $\mathbf{x} \in \Sigma$ to the domain D. We also denote by $\mathbf{n}(\mathbf{x})$ the outward unit normal to Ω at $\mathbf{x} \in \partial\Omega$. The operator $\nabla_\perp$ is the restriction of ∇ to Σ, so that for a sufficiently smooth function $\phi(\mathbf{x})$ defined on Ω, we have $\nabla_\perp\phi(\mathbf{x}) = \nabla\phi(\mathbf{x}) - (\mathbf{n}(\mathbf{x}) \cdot \nabla\phi(\mathbf{x}))\mathbf{n}(\mathbf{x})$ for $\mathbf{x} \in \Sigma$. The (Lebesgue) surface measure on Σ and $\partial\Omega$ is denoted by $\mathrm{d}\sigma(\mathbf{x})$. Finally $g(\mathbf{x})$ is a mean zero current imposed at the boundary of the domain.

Let us introduce the following Hilbert spaces:

$$H_0^s(S) = \left\{\phi \in H^s(S) \text{ such that } \int_S \phi \,\mathrm{d}\sigma = 0\right\}, \tag{2}$$

where S is a connected closed Lipschitz surface in $\mathbb{R}^n$ and $H^s(S)$ is the usual Sobolev space [29]. We denote as usual $L_0^2(S) = H_0^0(S)$. For $s \geqslant 1/2$ we also define

$$H_{0,\Sigma}^s(\Omega) = \left\{u \in H^s(\Omega) \text{ s.t. } \int_{\partial\Omega} u \,\mathrm{d}\sigma = 0 \text{ and } \int_\Sigma |\nabla_\perp u|^2 \,\mathrm{d}\sigma < \infty\right\}, \tag{3}$$

equipped with its natural norm $\|\cdot\|_{H_{0,\Sigma}^s(\Omega)}$. We verify that the latter space is a Hilbert space. We then have the following result.

Proposition 2.1. *Assume that the above hypotheses are satisfied and that* $g \in H_0^{-1/2}(\partial\Omega)$. *Then the system (1) admits a unique solution* $u \in H_{0,\Sigma}^1(\Omega)$ *with trace* $u_{|\partial\Omega} \in H_0^{1/2}(\partial\Omega)$.

Proof. Let ψ be a smooth test function on D and D^c. Upon multiplying the first equation in (1) by ψ and integrating by parts, we obtain that

$$\begin{aligned} &\int_D \gamma\nabla u \cdot \nabla\psi \,\mathrm{d}\mathbf{x} - \int_\Sigma \mathbf{n} \cdot \gamma\nabla u^- \psi^- \,\mathrm{d}\sigma = 0 \\ &\int_{D^c} \gamma\nabla u \cdot \nabla\psi \,\mathrm{d}\mathbf{x} + \int_\Sigma \mathbf{n} \cdot \gamma\nabla u^+ \psi^+ \,\mathrm{d}\sigma = \int_{\partial\Omega} g\psi \,\mathrm{d}\sigma. \end{aligned} \tag{4}$$

Here, we denote by $\psi^\pm(\mathbf{y})$ for $\mathbf{y} \in \Sigma$ the limits of $\psi(\mathbf{x})$ as $\mathbf{x} \to \mathbf{y}$ for $\mathbf{x} \in D^c$ and $\mathbf{x} \in D$, respectively. Upon summing the above contributions, choosing $\psi \in H_{0,\Sigma}^1(\Omega)$, using the jump conditions in (1), and integrating by parts on Σ, we find that

$$\int_\Omega \gamma\nabla u \cdot \nabla\psi \,\mathrm{d}\mathbf{x} + \int_\Sigma \mathrm{d}\nabla_\perp u \cdot \nabla_\perp\psi \,\mathrm{d}\sigma = \int_{\partial\Omega} g\psi \,\mathrm{d}\sigma. \tag{5}$$

Upon choosing $\psi = u$ in the above expression, we deduce that the above left-hand side is a coercive bilinear form, i.e., is bounded from below by $C\|u\|^2_{H_{0,\Sigma}^1(\Omega)}$ thanks to a standard Poincaré inequality (since the average of u on $\partial\Omega$ is assumed to vanish). Moreover, the above right-hand side is bounded from above by $C\|g\|_{H_0^{-1/2}(\partial\Omega)}\|u\|_{H_{0,\Sigma}^1(\Omega)}$ by a standard trace estimate. Existence and uniqueness of a solution to (1) then follows from the Lax–Milgram theory. The same trace estimate as above shows that $u_{|\partial\Omega} \in H_0^{1/2}(\partial\Omega)$. This concludes the proof of the proposition. □

We define the Neumann-to-Dirichlet operator Λ_Σ as

$$\Lambda_\Sigma : H_0^{-1/2}(\partial\Omega) \longrightarrow H_0^{1/2}(\partial\Omega), \qquad g \longmapsto u_{|\partial\Omega}, \tag{6}$$

where $u(\mathbf{x})$ is the solution to (1) with boundary normal current $g(\mathbf{x})$. We also define the 'background' Neumann-to-Dirichlet operator Λ_0 as above, where $\gamma(\mathbf{x})$ is replaced by a known background $\gamma_0(\mathbf{x})$ satisfying the same regularity constraints as $\gamma(\mathbf{x})$ and where $d(\mathbf{x})$ is replaced by 0. It is well known that Λ_0 is an isomorphism from $H_0^{-1/2}(\partial\Omega)$ onto $H_0^{1/2}(\partial\Omega)$. Our assumptions on the background $\gamma_0(\mathbf{x})$ are that it is the true conductivity tensor on D^c and a lower bound to the true conductivity tensor on D:

$$\gamma_0(\mathbf{x}) \leqslant \gamma(\mathbf{x}) \quad \text{on} \quad D, \qquad \gamma_0(\mathbf{x}) = \gamma(\mathbf{x}) \quad \text{on} \quad D^c. \tag{7}$$

The tensor inequality $\gamma_1 \geqslant \gamma_2$ is meant in the sense that $\xi_i\xi_j(\gamma_{1,ij} - \gamma_{2,ij}) \geqslant 0$ for all $\boldsymbol{\xi} \in \mathbb{R}^n$. The main assumption is thus that we *assume* that the physical parameters in (1) are known in D^c. Since Σ, whence D and D^c, is not known, this means in practice that the coefficients are known in a vicinity of the boundary and that the singular interface Σ lies within that vicinity.

The second problem models the propagation of photons in tissues in the diffusive regime except within a thin clear layer where a tangential diffusion process needs to be introduced [6, 7]. The density of photons $u(\mathbf{x})$ then solves the following system of equations:

$$\begin{aligned}
&Au \equiv -\nabla\cdot\gamma\nabla u + au = 0 && \text{in} \quad \Omega\backslash\Sigma \\
&[u] = 0 && \text{on} \quad \Sigma \\
&[\mathbf{n}\cdot\gamma\nabla u] = -\nabla_\perp\cdot \mathrm{d}\nabla_\perp u + \alpha u \equiv Bu && \text{on} \quad \Sigma \\
&\mathbf{n}\cdot\gamma\nabla u = g && \text{on} \quad \partial\Omega.
\end{aligned} \tag{8}$$

Here $\gamma(\mathbf{x})$, $d(\mathbf{x})$ and $g(\mathbf{x})$ satisfy the same constraints as before (except that $g(\mathbf{x})$ need no longer be mean zero), $a(\mathbf{x})$ is a uniformly positive and bounded absorption parameter and α is a non-negative parameter modelling absorption within the clear layer. Let us define the following Hilbert spaces for $s \geqslant 1/2$:

$$H^s_\Sigma(\Omega) = \left\{ u \in H^s(\Omega) \text{ s.t. } \int_\Sigma |\nabla_\perp u|^2 \, \mathrm{d}\sigma < \infty \right\}. \tag{9}$$

Then we have the following result.

Proposition 2.2. *Assume that the above hypotheses are satisfied and that* $g \in H^{-1/2}(\partial\Omega)$. *Then the system (8) admits a unique solution* $u \in H^1_\Sigma(\Omega)$ *with trace* $u_{|\partial\Omega} \in H^{1/2}(\partial\Omega)$.

Proof. The variational formulation of (8) is given for every test function $\psi \in H^1_\Sigma(\Omega)$ by

$$\begin{aligned}
&b(u,\psi) + b_\Sigma(u,\psi) = l(\psi), \\
&b(u,\psi) = \int_\Omega [\gamma\nabla u\cdot\nabla\psi + au\psi]\,\mathrm{d}\mathbf{x} \\
&b_\Sigma(u,\psi) = \int_\Sigma [\mathrm{d}\nabla_\perp u\cdot\nabla_\perp\psi + \alpha u\psi]\,\mathrm{d}\sigma \\
&l(\psi) = \int_{\partial\Omega} g\psi\,\mathrm{d}\sigma.
\end{aligned} \tag{10}$$

The rest of the proof goes as for proposition 2.1. □

As before we define the Neumann-to-Dirichlet operator Λ_Σ as

$$\Lambda_\Sigma : H^{-1/2}(\partial\Omega) \longrightarrow H^{1/2}(\partial\Omega), \qquad g \longmapsto u_{|\partial\Omega}, \tag{11}$$

where $u(\mathbf{x})$ is the solution to (8) with boundary normal current $g(\mathbf{x})$. The background diffusion Neumann-to-Dirichlet operator Λ_0 is defined as above with $\gamma(\mathbf{x})$ and $a(\mathbf{x})$ replaced by $\gamma_0(\mathbf{x})$ and $a_0(\mathbf{x})$ and $d(\mathbf{x})$ set to 0. We assume that the background satisfies

$$\begin{aligned}
&\gamma_0(\mathbf{x}) \leqslant \gamma(\mathbf{x}) \quad \text{on} \quad D, \qquad \gamma_0(\mathbf{x}) = \gamma(\mathbf{x}) \quad \text{on} \quad D^c, \\
&a_0(\mathbf{x}) \leqslant a(\mathbf{x}) \quad \text{on} \quad D, \qquad a_0(\mathbf{x}) = a(\mathbf{x}) \quad \text{on} \quad D^c.
\end{aligned} \tag{12}$$

Some of these assumptions can be slightly relaxed as we shall see in section 4 though they are not very constraining physically: they simply mean that lower bounds for the tensor $\gamma(\mathbf{x})$ and the absorption $a(\mathbf{x})$ are known *a priori* on the inner domain D. As before we assume that the parameters in (8) are known in D^c.

In both equations (1) and (8), the matching conditions across the singular interface Σ have been modelled as a tangential diffusion process along Σ. The analysis in the following sections may be extended to more general models so long as the operator B in (8) say, generates enough 'coercivity'. The theory applies for instance when B is a positive fourth-order differential operator defined on a sufficiently regular Σ; see theorem 4.1, corollary 4.2 and the following paragraph.

3. Reconstructions in impedance tomography

In this section, we consider the reconstruction of the surface Σ, the tangential diffusion process $d(\mathbf{x})$ and partial information about the tensor $\gamma(\mathbf{x})$ in D, from knowledge of the Neumann-to-Dirichlet (NtD) operator Λ_Σ associated with (1). We have to assume that the diffusion tensor $\gamma(\mathbf{x})$ is known *a priori* on D^c and that a lower bound is known on D. We thus assume the existence of a known tensor $\gamma_0(\mathbf{x})$ such that (7) holds.

A typical result we show is as follows:

Theorem 3.1. *Let us assume that the tensor $\gamma(\mathbf{x})$ is of class $C^2(\overline{\Omega})$ for $n = 2, 3$, is known on D^c and is proportional to identity (i.e., $\gamma(\mathbf{x}) = \frac{1}{n}\mathrm{Tr}(\gamma(\mathbf{x}))I$) on $\overline{D}$. Then the surface $\Sigma = \partial D$, the symmetric tangential diffusion tensor $d(\mathbf{x})$ and the conductivity tensor $\gamma(\mathbf{x})$ are uniquely determined by knowledge of the Neumann-to-Dirichlet operator Λ_Σ in $\mathcal{L}\big(H_0^{-1/2}(\partial\Omega), H_0^{1/2}(\partial\Omega)\big)$.*

The method of reconstruction is based on the factorization technique introduced in [21] in scattering theory and adapted to impedance tomography in [11]. The idea is to factor the difference of NtD operators as follows:

$$\Lambda_0 - \Lambda_\Sigma = L^* F L, \tag{13}$$

where L maps Neumann data on $\partial\Omega$ to Dirichlet data on Σ, F maps Dirichlet data on Σ to Neumann data on Σ, and L^*, which is in duality with L for the L^2 inner products on Σ and $\partial\Omega$, maps Neumann data on Σ back to Dirichlet data on $\partial\Omega$. We derive the factorization and show that F generates a coercive form in appropriate spaces in section 3.1.

From the above factorization and the properties on F, we next show that

$$\mathcal{R}((\Lambda_0 - \Lambda_\Sigma)^{1/2}) = \mathcal{R}(L^*). \tag{14}$$

This implies that the range of the operator L^* can be obtained from the measured data. We finally construct functions $\mathbf{y} \mapsto g_{\mathbf{y}}(\cdot)$ from the measured data that are in the range of L^* when $\mathbf{y} \in D$ and not in the range of L^* when $\mathbf{y} \in D^c$. This allows us to image the interface Σ from the boundary measurements.

Once Σ is reconstructed we use the uniqueness of the solution to the Cauchy problem to show the injectivity of the operator L^* and construct the Dirichlet-to-Neumann map at the boundary of the domain D. This implies the uniqueness of the reconstruction of $d(\mathbf{x})$ and $\gamma(\mathbf{x})$.

3.1. Factorization technique

In this section, we derive the factorization (13) and some properties satisfied by the operator F. We first need to introduce the functional spaces

$$Y_0^s(\Sigma) = \bigotimes_{j=1}^{N} H_0^s(\Sigma_j) = \left\{ v \in H^s(\Sigma), \text{ such that } \int_{\Sigma_j} v_{|\Sigma_j}\, d\sigma = 0, 1 \leqslant j \leqslant N \right\}, \tag{15}$$

for $s \in \mathbb{R}$ and verify that $Y_0^{1/2}(\Sigma)$ and $Y_0^{-1/2}(\Sigma)$ (the only spaces of interest here with $Y_0^0(\Sigma)$) are in duality. Note also that $Y_0^s(\Sigma) = H_0^s(\Sigma)$ when Σ is connected. The only reason why we introduce these product spaces is that we need functions whose averages over each connected component of the surface vanish. The space $Y_0^s(\Sigma)$ is equipped with the natural norm of $H^s(\Sigma)$.

We now define the operator L, which maps $\phi \in H_0^{-1/2}(\partial\Omega)$ to $\Pi v_{|\Sigma} \in Y_0^{1/2}(\Sigma)$, and its adjoint operator L^*, which maps $\phi \in Y_0^{-1/2}(\Sigma)$ to $w_{|\partial\Omega} \in H_0^{1/2}(\partial\Omega)$, where v and w are the unique solutions to the following problems:

$$\begin{aligned}
&\nabla\cdot\gamma\nabla v = 0 \quad &&\text{in } D^c \qquad &&\nabla\cdot\gamma\nabla w = 0 \quad &&\text{in } D^c,\\
&\mathbf{n}\cdot\gamma\nabla v = \phi &&\text{on } \partial\Omega &&\mathbf{n}\cdot\gamma\nabla w = 0 &&\text{on } \partial\Omega,\\
&\mathbf{n}\cdot\gamma\nabla v = 0 &&\text{on } \Sigma &&\mathbf{n}\cdot\gamma\nabla w = -\phi &&\text{on } \Sigma,\\
&\int_{\partial\Omega} v\, d\sigma = 0 && &&\int_{\partial\Omega} w\, d\sigma = 0,
\end{aligned} \tag{16}$$

and $I - \Pi$ is the L^2 projection of functions defined on Σ onto their average on each component Σ_j of Σ. More precisely, for each $v \in L^2(\Sigma)$, we define

$$(\Pi v)_{|\Sigma_j} = v_{|\Sigma_j} - \frac{1}{|\Sigma_j|}\int_{\Sigma_j} v\, d\sigma, \tag{17}$$

where $|\Sigma_j|$ is the surface measure of Σ_j. We verify that the above equations admit unique solutions and that the operators L and L^* are in duality in the sense that

$$(L\phi, \psi)_\Sigma \equiv \int_\Sigma \psi L\phi\, d\sigma = \int_{\partial\Omega} \phi L^*\psi\, d\sigma \equiv (\phi, L^*\psi)_{\partial\Omega}.$$

Indeed, we deduce from integrations by parts in (16) that

$$\int_{\partial\Omega} \mathbf{n}\cdot\gamma\nabla v w\, d\sigma = -\int_\Sigma v\mathbf{n}\cdot\gamma\nabla w\, d\sigma = -\int_\Sigma \Pi v\mathbf{n}\cdot\gamma\nabla w\, d\sigma,$$

since $\int_{\Sigma_j} \mathbf{n}\cdot\gamma\nabla w\, d\sigma = 0$.

Note that the operator L^* is injective. This follows from the uniqueness of the Cauchy problem in (16) on the connected domain D^c (see [20, chapter 3] for instance). Indeed, knowledge of $L^*\phi = w_{|\partial\Omega}$ and $\mathbf{n}\cdot\gamma\nabla w$ on $\partial\Omega$ uniquely determines $\phi = -\mathbf{n}\cdot\gamma\nabla w_{|\Sigma}$, whence the injectivity of L^*. Note, however, that L as constructed above is not injective.

We now come to the definition of the operator F and introduce two operators, G_Σ which maps $g \in H_0^{-1/2}(\partial\Omega)$ to $G_\Sigma g = \mathbf{n}\cdot\gamma\nabla v^+_{|\Sigma} \in Y_0^{-1/2}(\Sigma)$ and G_Σ^* which maps $\phi \in Y_0^{1/2}(\Sigma)$ to $G_\Sigma^*\phi = w_{|\partial\Omega} \in H_0^{1/2}(\partial\Omega)$, where v and w are the unique solutions to the following problems:

$$\begin{aligned}
&\nabla\cdot\gamma\nabla v = 0 \quad &&\text{in } \Omega\backslash\Sigma \qquad &&\nabla\cdot\gamma\nabla w = 0 \quad &&\text{in } \Omega\backslash\Sigma\\
&[v] = 0 &&\text{on } \Sigma &&[w] = \phi &&\text{on } \Sigma\\
&[\mathbf{n}\cdot\gamma\nabla v] = -\nabla_\perp d\nabla_\perp v &&\text{on } \Sigma &&[\mathbf{n}\cdot\gamma\nabla w] = -\nabla_\perp d\nabla_\perp w^- &&\text{on } \Sigma\\
&\mathbf{n}\cdot\gamma\nabla v = g &&\text{on } \partial\Omega &&\mathbf{n}\cdot\gamma\nabla w = 0 &&\text{on } \partial\Omega\\
&\int_{\partial\Omega} v\, d\sigma = 0 && &&\int_{\partial\Omega} w\, d\sigma = 0.
\end{aligned} \tag{18}$$

The equation for v is the same as (1) so that proposition 2.1 shows the existence of a unique solution $v \in H^1_{0,\Sigma}(\Omega)$. This also implies the well posedness of the operator G_Σ as one verifies that

$$\int_{\Sigma_j} \mathbf{n} \cdot \gamma \nabla v^+_{|\Sigma} \, d\sigma = \int_{\Sigma_j} \mathbf{n} \cdot \gamma \nabla v^-_{|\Sigma} \, d\sigma - \int_{\Sigma_j} \nabla_\perp \cdot (d \nabla_\perp v_{|\Sigma}) \, d\sigma = 0,$$

since $\nabla \cdot \gamma \nabla v = 0$ on D_j with boundary $\partial D_j = \Sigma_j$ and the second term is in divergence form (and thus we can use Stokes' theorem). The operator G^*_Σ is more delicate as both the potential and the current jump across Σ. We state the following result:

Proposition 3.2. *Let $\phi \in Y_0^{1/2}(\Sigma)$. Then the problem stated in (18) for $w(\mathbf{x})$ admits a unique solution in $X_0^1(\Omega)$, where the latter Hilbert space is defined by*

$$X_0^1(\Omega) = \left\{ w \in H^1(D) \otimes H^1(D^c); \int_{\partial\Omega} w \, d\sigma = 0, w^-_{|\Sigma} \in H_0^1(\Sigma) \text{ and } [w] \in Y_0^{1/2}(\Sigma) \right\}, \tag{19}$$

equipped with its natural norm $\| \cdot \|_{X_0^1(\Omega)}$ defined for $v \in X_0^1(\Omega)$ by

$$\|v\|^2_{X_0^1(\Omega)} = \|v_{|D}\|^2_{H^1(D)} + \|v_{|D^c}\|^2_{H^1(D^c)} + \int_\Sigma |\nabla_\perp v^-_{|\Sigma}|^2 \, d\sigma. \tag{20}$$

Note that the last constraint in (19) may be replaced by the constraint that the average of $[w]$ on each Σ_j vanishes. The regularity of $[w]$ on Σ is a consequence of the first constraint in (19) and a standard trace theorem.

Proof. By integrations by parts we obtain first that

$$\int_{D^c} \gamma \nabla w \cdot \nabla \psi \, d\mathbf{x} + \int_\Sigma \mathbf{n} \cdot \gamma \nabla w^+ \psi^+ \, d\sigma = 0$$
$$\int_D \gamma \nabla w \cdot \nabla \psi \, d\mathbf{x} - \int_\Sigma \mathbf{n} \cdot \gamma \nabla w^- \psi^- \, d\sigma = 0.$$

This implies that

$$\int_\Omega \gamma \nabla w \cdot \nabla \psi \, d\mathbf{x} + \int_\Sigma d \nabla_\perp w^- \cdot \nabla_\perp \psi^- = \int_\Sigma (-\mathbf{n} \cdot \gamma \nabla w^+)[\psi] \, d\sigma. \tag{21}$$

Choosing $\psi = w$ yields

$$I(w) = \int_\Omega \gamma \nabla w \cdot \nabla w \, d\mathbf{x} + \int_\Sigma d \nabla_\perp w^- \cdot \nabla_\perp w^- \, d\sigma = \int_\Sigma (-\mathbf{n} \cdot \gamma \nabla w^+) \phi \, d\sigma. \tag{22}$$

We then use the fact that $\|\mathbf{n} \cdot \gamma \nabla w^+\|_{Y_0^{-1/2}(\Sigma)}$ is bounded by the norm of $\gamma \nabla w$ in $H(\text{div}, D^c)$ (see [11, 14]) to obtain that the above right-hand side is bounded *a priori* by $C \|\phi\|_{Y_0^{1/2}(\Sigma)} \|w\|_{X_0^1(\Omega)}$.

We now want to show that the above left-hand side $I(w)$ defines a norm on $X_0^1(\Omega)$ equivalent to the natural one. Let us first show that $I(w)$ defines a norm on $X_0^1(\Omega)$ and assume that $w \in X_0^1(\Omega)$ and that $I(w) = 0$. Then, w is constant and equal to w_0 on D^c by connectedness of the latter domain. By connectedness of D_j, we also deduce that $w = w_j$ on D_j, where w_j is a constant. This implies that $\phi_j = w_0 - w_j$ is constant, hence uniformly vanishes since it has zero average on Σ_j. Now $w_0 = 0$ from the constraint on $\partial\Omega$ so that $w \equiv 0$. Note that the result applies to the case of multiple-component surfaces because the average of ϕ is forced to vanish on each connected component of the interface Σ.

We observe that $I(w) \leqslant \|w\|^2_{X^1_0(\Omega)}$. The reverse inequality follows from a classical proof, which we briefly recall here. Indeed let us assume otherwise and deduce the existence of a series of functions $\tilde{w}_m$ such that

$$\frac{1}{m^2}\|\tilde{w}_m\|^2_{X^1_0(\Omega)} > I(\tilde{w}_m).$$

Then set $w_m = \tilde{w}_m/\|\tilde{w}_m\|_{X^1_0(\Omega)}$ so that $\|w_m\|_{X^1_0(\Omega)} = 1$ and $I(w_m) \to 0$ as $m \to \infty$. Since the canonical embedding from $H^1(D) \otimes H^1(D^c)$ to $L^2(\Omega)$ is compact, we deduce the existence of $w \in L^2(\Omega)$ such that $\|w_m - w\|_{L^2(\Omega)} \to 0$. Now since $I(w_m) \to 0$, we deduce that $\|\nabla w_m\|_{L^2(\Omega)} + \|\nabla_\perp w^-_{m|\Sigma}\|_{L^2(\Sigma)} \to 0$. This implies that $\nabla w = 0$ on D and on D^c, that $\nabla_\perp w^-_{|\Sigma} = 0$ and that $w \in X^1_0(\Omega)$ since its average over $\partial\Omega$ and the average of its jump across each component Σ_j of Σ vanish, whence $w_m \to w$ in $X^1_0(\Omega)$ by completeness. We have seen that $I(w) = 0$ implies that $w \equiv 0$, which is impossible since $\|w\|_{X^1_0(\Omega)} = \lim_{m\to\infty}\|w_m\|_{X^1_0(\Omega)} = 1$. This yields the existence of a constant C such that for all $w \in X^1_0(\Omega)$,

$$\|w\|^2_{X^1_0(\Omega)} < CI(w),$$

and the equivalence of $I(w)^{1/2}$ with the natural norm on $X^1_0(\Omega)$.

We deduce from the Lax–Milgram theory the existence of a unique solution w to (18) in $X^1_0(\Omega)$. □

The usual trace theorems and the above proposition show that G^*_Σ described above is a bounded operator. Note that G_Σ and G^*_Σ are in duality in the sense that

$$\int_\Sigma G_\Sigma g\phi\,\mathrm{d}\sigma = \int_{\partial\Omega} gG^*_\Sigma\phi\,\mathrm{d}\sigma. \tag{23}$$

Indeed, we deduce from the definition of G_Σ and the equalities in (4) that for any smooth test function φ, we have

$$\int_\Omega \gamma\nabla v\cdot\nabla\varphi + \int_\Sigma \mathrm{d}\nabla_\perp v\cdot\nabla_\perp\varphi^- = \int_{\partial\Omega} g\varphi - \int_\Sigma (G_\Sigma g)[\varphi].$$

It now suffices to choose $\psi = v$ in (21) and $\varphi = w$ in the above equation to obtain (23) since $[v] = 0$.

We define F_Σ as the operator that maps $\phi \in Y^{1/2}_0(\Sigma)$ to $F_\Sigma\phi = -\mathbf{n}\cdot\gamma\nabla w^+ \in Y^{-1/2}_0(\Sigma)$, where w is the solution to (18). We deduce from (21) that

$$(F_\Sigma[w], [\psi])_\Sigma = \int_\Omega \gamma\nabla w\cdot\nabla\psi\,\mathrm{d}\mathbf{x} + \int_\Sigma \mathrm{d}\nabla_\perp w^-\cdot\nabla_\perp\psi^- = ([w], F_\Sigma[\psi])_\Sigma, \tag{24}$$

so that $F_\Sigma = F^*_\Sigma$, and by choosing $[w] = [\psi]$ that F_Σ is coercive on $Y^{1/2}_0(\Sigma)$ (and actually coercive on $Y^1_0(\Sigma)$). We next verify that $G^*_\Sigma = L^*F_\Sigma$ so that $G_\Sigma = F_\Sigma L$. We define G_0 and F_0 using the same procedure as G_Σ and F_Σ except that $d \equiv 0$ and that γ is replaced by γ_0. We thus also obtain that F_0 is a coercive form on $Y^{1/2}_0(\Sigma)$.

Let us define the operator M, which maps $g \in H^{-1/2}_0(\partial\Omega)$ to $u_{|\partial\Omega} \in H^{1/2}_0(\partial\Omega)$, where u is the solution to

$$\begin{aligned}
&\nabla\cdot\gamma\nabla u = 0, && \text{in} \quad D^c\\
&\mathbf{n}\cdot\gamma\nabla u = 0, && \text{on} \quad \Sigma\\
&\mathbf{n}\cdot\gamma\nabla u = g, && \text{on} \quad \partial\Omega\\
&\int_{\partial\Omega} u\,\mathrm{d}\sigma = 0.
\end{aligned} \tag{25}$$

We now verify the following decomposition for the Neumann-to-Dirichlet operators:

$$\Lambda_\Sigma = M - L^* G_\Sigma = M - L^* F_\Sigma L, \qquad \Lambda_0 = M - L^* G_0 = M - L^* F_0 L. \tag{26}$$

This is a direct application of the principle of superposition, which for instance implies that $u_{|\partial\Omega} = Mg + L^* \mathbf{n} \cdot \gamma \nabla u^+_{|\Sigma}$, where u is the solution to (1). Upon subtracting the two relations, we obtain that

$$\Lambda_0 - \Lambda_\Sigma = L^* F L, \qquad F = F_\Sigma - F_0. \tag{27}$$

The reason why this relation is useful is because we will show that F generates a coercive form on $Y_0^{1/2}(\Sigma)$ and thus can be decomposed as B^*B with B^* *surjective*. Note that $F^* = F$ since this property holds for F_Σ and F_0. We now show the following result:

Proposition 3.3. *The bilinear form F is coercive on* $Y_0^{1/2}(\Sigma)$*, which implies the existence of a constant C independent of* $\phi \in Y_0^{1/2}(\Sigma)$ *such that*

$$(F\phi, \phi)_\Sigma^{1/2} \geqslant C \|\phi\|_{Y_0^{1/2}(\Sigma)}. \tag{28}$$

Proof. Let us denote by w_Σ the solution to (18) and by w_0 the same solution corresponding to $d = 0$ and γ replaced by γ_0. Note that the latter is an element in $H_0^1(\Omega)$. We also introduce $\delta w = w_0 - w_\Sigma$ and verify that it belongs to the Hilbert space

$$Z_0^1(\Omega) = \left\{ w \in H^1(D) \otimes H^1(D^c), \text{ such that } \int_{\partial\Omega} w \, \mathrm{d}\sigma = 0 \text{ and } [w] \in Y_0^{1/2}(\Sigma) \right\}, \tag{29}$$

equipped with its natural product-space norm

$$\|v\|^2_{Z_0^1(\Omega)} = \|v_{|D}\|^2_{H^1(D)} + \|v_{|D^c}\|^2_{H^1(D^c)}. \tag{30}$$

Note that $Z_0^1(\Omega) = X_0^1(\Omega) \cup H_0^1(\Omega)$ so that $\delta w \in Z_0^1(\Omega)$.

We recall that $\gamma = \gamma_0$ on D^c. Upon multiplying $\nabla \cdot \gamma_0 \nabla \delta w + \nabla \cdot (\gamma_0 - \gamma) \nabla w_\Sigma = 0$ by δw and integrating by parts on D^c and D we have (we do not write the integration measures explicitly)

$$\int_\Omega \gamma_0 \nabla \delta w \cdot \nabla \delta w + \int_D (\gamma_0 - \gamma) \nabla w_\Sigma \cdot \nabla \delta w = \int_\Sigma \mathbf{n} \cdot \gamma \nabla \delta w^- \delta w^- - \int_\Sigma \mathbf{n} \cdot \gamma \nabla \delta w^+ \delta w^+.$$

Since $[\delta w] = 0$ we deduce that

$$\int_\Omega \gamma_0 \nabla \delta w \cdot \nabla \delta w + \int_D (\gamma - \gamma_0) \nabla w_\Sigma \cdot \nabla w_\Sigma = \int_D (\gamma - \gamma_0) \nabla w_0 \cdot \nabla w_\Sigma - \int_\Sigma [\mathbf{n} \cdot \gamma \nabla \delta w] \delta w^-.$$

Integrating the equation satisfied by w_Σ multiplied by w_0 on D and subtracting the equation satisfied by w_0 multiplied by w_Σ, we obtain that

$$\int_D (\gamma - \gamma_0) \nabla w_0 \cdot \nabla w_\Sigma = \int_\Sigma (\mathbf{n} \cdot \gamma \nabla w_\Sigma^- w_0^- - \mathbf{n} \cdot \gamma \nabla w_0 w_\Sigma^-).$$

Here we have used that $[\mathbf{n} \cdot \gamma \nabla w_0] = 0$ across Σ. The same integrations on D^c yield

$$\int_\Sigma \mathbf{n} \cdot \gamma \nabla w_0^+ w_\Sigma^+ - \mathbf{n} \cdot \gamma \nabla w_\Sigma^+ w_0^+ = 0.$$

The three preceding equalities and the jump conditions for w_0 and w_Σ imply that

$$\int_D (\gamma - \gamma_0) \nabla w_0 \cdot \nabla w_\Sigma - \int_\Sigma [\mathbf{n} \cdot \gamma \nabla \delta w] \delta w^- = \int_\Sigma -[\mathbf{n} \cdot \gamma \nabla w_\Sigma] w_\Sigma^- + (F_\Sigma - F_0)\phi\phi.$$

The above results then show that

$$\int_\Sigma F\phi\phi = \int_\Omega \gamma_0 \nabla \delta w \cdot \nabla \delta w + \int_D (\gamma - \gamma_0) \nabla w_\Sigma \cdot \nabla w_\Sigma + \int_\Sigma \mathrm{d} \nabla_\perp w_\Sigma^- \cdot \nabla_\perp w_\Sigma^-. \tag{31}$$

We want to show that the above bilinear form is coercive on $Y_0^{1/2}(\Sigma)$. Note that the middle term on the above right-hand side is not very helpful for this purpose as we may have $\gamma = \gamma_0$. Both other terms are crucial in our estimate as each one alone is not sufficient to grant coercivity.

Let us assume that $(F\phi, \phi)_\Sigma$ is bounded so that by (31), $|\nabla \delta w| \in L_0^2(\Omega)$ and $(w_\Sigma^-)_{|\Sigma} \in H^1(\Sigma)$. The same proof as that of proposition 3.2 shows that $J(w) = \int_\Omega \gamma_0 \nabla w \cdot \nabla w$ is a norm on $Z_0^1(\Omega)$ defined in (29) equivalent to the natural norm defined in (30). This implies that δw is bounded in $Z_0^1(\Omega)$. Next, we obtain from the equation satisfied by w_Σ that $(w_\Sigma)_{|D} \in H^1(D)$. We thus deduce that $(w_0)_{|D} \in H^1(D)$ since $\delta w_{|D} \in H^1(D)$, so that $(w_0^-)_{|\Sigma} \in H^{1/2}(\Sigma)$ and $(\mathbf{n} \cdot \gamma \nabla w_0)_{|\Sigma} \in H^{-1/2}(\Sigma)$. Since the latter term does *not* jump across Σ, we also obtain that $(w_0)_{|D^c} \in H^1(D^c)$ and that $\left(w_0^+\right)_{|\Sigma} \in H^{1/2}(\Sigma)$. This implies that the jump $\phi = [w_0] \in H^{1/2}(\Sigma)$. Since by construction, $\int_{\Sigma_j} \phi \, d\sigma = 0$ for $1 \leqslant j \leqslant N$, and all the preceding embeddings are continuous, we have just shown the existence of a positive constant C such that

$$C^{-1} \|\phi\|_{Y_0^{1/2}(\Sigma)} \leqslant \|\nabla \delta w\|_{L_0^2(\Omega)} + \|(w_\Sigma^-)_{|\Sigma}\|_{H^1(\Sigma)} \leqslant C(F\phi, \phi)_\Sigma^{1/2}. \tag{32}$$

We see that the role of the first constraint is to provide an estimate for w_Σ on D^c, which the constraint on $w_{\Sigma|\Sigma}^-$ alone cannot grant. □

The factorization method allows us to derive the main result of this subsection, namely, that the range of L^* is characterized by the boundary measurements.

Theorem 3.4. *The following range characterization holds:*

$$\mathcal{R}((\Lambda_0 - \Lambda_\Sigma)^{1/2}) = \mathcal{R}(L^*). \tag{33}$$

Proof. Let $\mathcal{I}$ be the canonical isomorphism between $Y_0^{-1/2}(\Sigma)$ and $Y_0^{1/2}(\Sigma)$. It can be defined as the square root of the Laplace–Beltrami operator $\Delta_\perp$ on each connected component Σ_j of Σ [29]. We can similarly define the square root of $\mathcal{I}$ and decompose the latter as

$$\mathcal{I} = \mathcal{J}^* \mathcal{J}, \qquad \mathcal{J} : Y_0^{-1/2}(\Sigma) \to Y_0^0(\Sigma), \quad \mathcal{J}^* : Y_0^0(\Sigma) \to Y_0^{1/2}(\Sigma).$$

We recall that when Σ is connected, the above spaces read $Y_0^s(\Sigma) = H_0^s(\Sigma)$ and thus $Y_0^0(\Sigma) = L_0^2(\Sigma)$. Both $\mathcal{J}$ and $\mathcal{J}^*$ are isometries as defined above. We can thus recast the coercivity of F as

$$(F\phi, \phi) = (F\mathcal{J}^* u, \mathcal{J}^* u) = (\mathcal{J} F \mathcal{J}^* u, u) \geqslant \alpha \|\phi\|^2_{Y_0^{1/2}(\Sigma)} = \alpha \|u\|^2_{Y_0^0(\Sigma)}.$$

So $\mathcal{J} F \mathcal{J}^*$ as a self-adjoint positive definite operator on $Y_0^0(\Sigma)$ can be written as C^*C, where C and C^* are bounded operators from $Y_0^0(\Sigma)$ to $Y_0^0(\Sigma)$. Since

$$\|Cu\|^2_{Y_0^0(\Sigma)} \geqslant \alpha \|u\|^2_{Y_0^0(\Sigma)},$$

we deduce that C^* is surjective. We thus obtain that $F = B^*B$ where $B = C(\mathcal{J}^*)^{-1}$ maps $Y_0^{1/2}(\Sigma)$ to $Y_0^0(\Sigma)$ and its adjoint operator $B^* = \mathcal{J}^{-1} C^*$ maps $Y_0^0(\Sigma)$ to $Y_0^{-1/2}(\Sigma)$. Since $\mathcal{J}$ is an isomorphism, we deduce that B^* is surjective.

From the above calculations we obtain that

$$\Lambda_0 - \Lambda_\Sigma = L^* F L = L^* B^* (L^* B^*)^* = A^* A,$$

for $A = BL$. Since the range of $(A^*A)^{1/2}$ for A acting on Hilbert spaces is equal to the range of A^*, we deduce that

$$\mathcal{R}((\Lambda_0 - \Lambda_\Sigma)^{1/2}) = \mathcal{R}(L^* B^*) = \mathcal{R}(L^*) \tag{34}$$

since B^* is surjective. Indeed we always have that $\mathcal{R}(L^* B^*) \subset \mathcal{R}(L^*)$. Now for $y \in \mathcal{R}(L^*)$ there is x such that $y = L^* x$ and since B^* is surjective u such that $y = L^* B^* x$ so that $y \in \mathcal{R}(L^* B^*)$; whence $\mathcal{R}(L^*) \subset \mathcal{R}(L^* B^*)$. □

3.2. Reconstruction of Σ

In this section we use the characterization (33) to reconstruct the surface Σ from boundary measurements. Since $\Lambda_0 - \Lambda_\Sigma$ is assumed to be known, then so is $\mathcal{R}(L^*)$. Our objective is to explicitly construct a family of functions $g_{\mathbf{y}}(\mathbf{x})$ for $\mathbf{x} \in \partial\Omega$ parametrized by the points $\mathbf{y} \in \Omega$ such that $g_{\mathbf{y}}(\mathbf{x}) \in \mathcal{R}(L^*)$ when $\mathbf{y} \in D$ and such that this does not hold when $\mathbf{y} \in D^c$.

For each $\mathbf{y} \in \Omega$, we define $N(\mathbf{x}; \mathbf{y})$ as the unique (fundamental) solution to

$$\begin{aligned} &\nabla \cdot \gamma_0 \nabla N(\cdot; \mathbf{y}) = \delta(\cdot - \mathbf{y}), && \text{in} \quad \Omega \\ &\mathbf{n} \cdot \gamma_0 \nabla N(\cdot; \mathbf{y}) = 0, && \text{on} \quad \partial\Omega \\ &\int_{\partial\Omega} N(\cdot; \mathbf{y})\, \mathrm{d}\sigma = 0, && \end{aligned} \tag{35}$$

and construct the family of boundary conditions $g_{\mathbf{y}}(\mathbf{x}) = N(\mathbf{x}; \mathbf{y})_{|\partial\Omega}$ on $\partial\Omega$. Since γ_0 is a known tensor, the family $g_{\mathbf{y}}(\mathbf{x})$ does not depend on the unknown quantities Σ, $d(\mathbf{x})$ and $\gamma(\mathbf{x})$. Then we have the following result.

Proposition 3.5. *The functions* $g_{\mathbf{y}}(\mathbf{x})$ *on* $\partial\Omega$ *constructed above belong to* $\mathcal{R}(L^*)$ *when* $\mathbf{y} \in D$ *and do not belong to* $\mathcal{R}(L^*)$ *when* $\mathbf{y} \in D^c$. *This provides us with an explicit method for imaging* D *and hence* $\Sigma = \partial D$ *from the boundary measurements via the range characterization (33).*

Proof. When $\mathbf{y} \in D$, there is an integer $1 \leqslant J \leqslant N$ such that $\mathbf{y} \in D_J$. We verify that $\nabla \cdot \gamma_0 \nabla N(\cdot; \mathbf{y}) = 0$ in $\Omega \backslash \overline{D_J}$ so that the average of $\mathbf{n} \cdot \gamma \nabla N(\mathbf{x}; \mathbf{y})^+_{|\Sigma}$ on each Σ_j for $j \neq J$ vanishes. Integrations by parts on $\Omega \backslash \overline{D_J}$ also show that the average of $\mathbf{n} \cdot \gamma \nabla N(\mathbf{x}; \mathbf{y})^+_{|\Sigma}$ on Σ_J vanishes thanks to the Neumann conditions verified on $\partial\Omega$. This implies that $\mathbf{n} \cdot \gamma \nabla N(\mathbf{x}; \mathbf{y})^+_{|\Sigma} \in Y_0^{-1/2}(\Sigma)$ and that $g_{\mathbf{y}} \in \mathcal{R}(L^*)$.

Let us now consider a point $\mathbf{y} \in D^c$ and assume that $g_{\mathbf{y}}(\mathbf{x})$ belongs to the range of L^*. This implies the existence of $\phi \in Y_0^{-1/2}(\Sigma)$ such that $g_{\mathbf{y}} = L^*\phi = w_{|\partial\Omega}$, where w is the solution to (16). Let $B_\varepsilon(\mathbf{y}) \subset D^c$ be the ball of radius ε centred at $\mathbf{y}$ for ε sufficiently small. On $D^c \backslash \overline{B_\varepsilon}$, both w and $N(\mathbf{x}; \mathbf{y})$ satisfy the same equation. By uniqueness of the solution to the Cauchy problem imposing Dirichlet data and vanishing Neumann data on $\partial\Omega$, we deduce that $w = N(\cdot; \mathbf{y})$ on $D^c \backslash \overline{B_\varepsilon}$ [20]. Let $\varepsilon_0 > \varepsilon$ be such that $B_\varepsilon \subset B_{\varepsilon_0} \subset D^c$. The $H^1(B_{\varepsilon_0} \backslash B_\varepsilon)$ norm of w on $B_{\varepsilon_0} \backslash B_\varepsilon$ remains bounded by $\|w\|_{H^1(B_{\varepsilon_0})} < \infty$ since w solves (16). However, the $H^1(B_{\varepsilon_0} \backslash B_\varepsilon)$ norm of the fundamental solution $N(\cdot; \mathbf{y})$ is not bounded independently of ε since $|\nabla N(\mathbf{x}; \mathbf{y})|$ can be locally bounded from below by a function of the form $c(\partial\Omega)|\mathbf{x} - \mathbf{y}|^{1-n}$ with a constant $c(\partial\Omega) > 0$ for $n \geqslant 2$, which is not square integrable [29]. This gives a contradiction with our assumption that $w = N(\cdot; \mathbf{y})$ on $D^c \backslash \overline{B_\varepsilon}$ for all $\varepsilon > 0$ and implies that $g_{\mathbf{y}}$ cannot belong to the range of L^* when $\mathbf{y} \in D^c$. □

3.3. Reconstruction of $d(\mathbf{x})$ *and* $\gamma(\mathbf{x})$

Once the surface Σ is reconstructed as shown in the preceding section, we can have access to the operator F from density properties of the operators L and L^*.

Indeed, L^* is surjective from $Y_0^{-1/2}(\Sigma)$ to $\mathcal{R}(L^*)$. Moreover, we have recalled in section 3.1 that L^* was injective by uniqueness of the Cauchy problem in (16). We can thus define $(L^*)^{-1}$ from $\mathcal{R}(\Lambda_0 - \Lambda_\Sigma) \subset \mathcal{R}(L^*)$ (since (27) holds) to its image $(L^*)^{-1}(\mathcal{R}(\Lambda_0 - \Lambda_\Sigma)) \subset Y_0^{-1/2}(\Sigma)$. We thus have the operator equality

$$(L^*)^{-1}(\Lambda_0 - \Lambda_\Sigma) = FL,$$

defined in $\mathcal{L}\big(Y_0^{-1/2}(\partial\Omega), Y_0^{-1/2}(\Sigma)\big)$. Note that $(L^*)^{-1}$ depends only on $\gamma = \gamma_0$ on D^c and on Σ, which are known by assumption. Moreover, still from the injectivity of L^*, we obtain that the range of L is dense in $Y_0^{1/2}(\Sigma)$ since $\overline{\mathcal{R}(L)} = \mathcal{N}(L^*)^\perp = \{0\}^\perp = Y_0^{1/2}(\Sigma)$. Therefore, for each function $\phi \in Y_0^{1/2}(\Sigma)$, there exists a sequence of functions $g_n \in Y_0^{-1/2}(\partial\Omega)$ such that $Lg_n \to \phi$ in $Y_0^{1/2}(\Sigma)$ as $n \to \infty$. Consequently the sequence of functions

$$u_n = (L^*)^{-1}(\Lambda_0 - \Lambda_\Sigma)g_n \to F\phi \qquad \text{as} \quad n \to \infty.$$

Since the functions u_n can be uniquely constructed from the boundary measurements, we have access to the full mapping F in $\mathcal{L}\big(Y_0^{1/2}(\Sigma), Y_0^{-1/2}(\Sigma)\big)$.

We have the following result:

Proposition 3.6. *The symmetric tensor $d(\mathbf{x})$ is uniquely determined by the Neumann-to-Dirichlet operator Λ_Σ.*

Proof. Since F and F_0 are known, then so is F_Σ and $G_\Sigma = F_\Sigma L$. We obtain that the range of G_Σ is dense in $Y_0^{-1/2}(\Sigma)$ since F_Σ and L^*, hence $G_\Sigma^* = F_\Sigma L^*$, are injective. So for $h^+ \in Y_0^{-1/2}(\Sigma)$ we find a converging sequence (in the same sense) $h_n^+ = G_\Sigma g_n \in Y_0^{-1/2}(\Sigma) \to h^+$ as $n \to \infty$. Let $1 \leqslant J \leqslant N$ be fixed. Upon solving the equation for w in (16) on D^c and renormalizing w such that $\int_{\Sigma_J} w^+ \,\mathrm{d}\sigma = 0$ (this problem also admits a unique solution), we can thus construct the Neumann-to-Dirichlet operator Λ_{NJ} on D_J, which to $h^+ \in Y_0^{-1/2}(\Sigma)$, hence to its restriction $h^+_{|\Sigma_J} \in H_0^{-1/2}(\Sigma_J)$ associates $w^+_{|\Sigma_J} \in H_0^{1/2}(\Sigma_J)$.

We recast the operator Λ_{NJ} as mapping $h^+ \in H_0^{-1}(\Sigma_J)$ to $\Lambda_{NJ}h^+ = u_{|\Sigma_J} \in H_0^1(\Sigma_J)$, where u solves the following equation:

$$\begin{aligned} &\nabla\cdot\gamma\nabla u = 0, && \text{in} \quad D_J \\ &\mathbf{n}\cdot\gamma\nabla u - \nabla_\perp\cdot \mathrm{d}\nabla_\perp u = h^+, && \text{on} \quad \Sigma_J \\ &\int_{\Sigma_J} u \,\mathrm{d}\sigma = 0. && \end{aligned} \tag{36}$$

Indeed, $H_0^{-1/2}(\Sigma_J)$ is dense in $H_0^{-1}(\Sigma_J)$ so that the incoming current $h^+ \in H_0^{-1}(\Sigma)$ can be constructed from boundary measurements by density. The variational formulation for the above equation is

$$\int_{D_J} \gamma\nabla u\cdot\nabla\phi \,\mathrm{d}\mathbf{x} + \int_{\Sigma_J} \mathrm{d}\nabla_\perp u\cdot\nabla_\perp\phi \,\mathrm{d}\sigma = \int_{\Sigma_J} h^+\phi \,\mathrm{d}\sigma. \tag{37}$$

Upon choosing $\phi = u$, we deduce as in the proof of proposition 2.1 the existence of a unique solution $u \in H^1_{0,\Sigma_J}(D_J)$. Moreover, upon choosing $\phi_{|\Sigma_J} = \Lambda_{NJ} f^+$, we deduce that

$$\int_{D_J} \gamma\nabla u\cdot\nabla\phi \,\mathrm{d}\mathbf{x} + \int_{\Sigma_J} \mathrm{d}\nabla_\perp u\cdot\nabla_\perp\phi \,\mathrm{d}\sigma = \int_{\Sigma_J} h^+\Lambda_N f^+ \,\mathrm{d}\sigma = \int_{\Sigma_J} \Lambda_{NJ}h^+ f^+ \,\mathrm{d}\sigma,$$

since the left-hand side is symmetric in u and ϕ, so that $\Lambda_{NJ} = \Lambda^*_{NJ}$. Since Λ^*_{NJ} is injective, we deduce that $\overline{\mathcal{R}(\Lambda_{NJ})} = H_0^1(\Sigma_J)$. Thus by density we can construct the inverse operator $\Lambda_{NJ}^{-1} = \Lambda_{DJ}$, the Dirichlet-to-Neumann operator on D_J, which maps $u_{|\Sigma_J} \in H_0^1(\Sigma_J)$ to $h^+ = \mathbf{n}\cdot\gamma\nabla u_{|\Sigma_J} - \nabla_\perp\cdot\mathrm{d}\nabla_\perp u_{|\Sigma_J} \in H_0^{-1}(\Sigma_J)$. This means that we can construct from boundary measurements the operator

$$\Lambda_{DJ} = -\nabla_\perp\cdot\mathrm{d}\nabla_\perp + \tilde{\Lambda}_{DJ},$$

where $\tilde{\Lambda}_{DJ}$ is the usual Dirichlet-to-Neumann operator for the domain D_J. The latter contribution is a bounded operator from $H_0^1(\Sigma_J)$ to $L_0^2(\Sigma_J)$ and is thus less singular than the first contribution in Λ_{DJ}. Let Σ_J be given locally by $x^n = 0$ in the coordinates $(\mathbf{x}', x^n)$. We then verify that

$$\boldsymbol{\omega}'\cdot d(\mathbf{x}')\boldsymbol{\omega}' = \lim_{s\to\infty}\frac{1}{s^2}\,\mathrm{e}^{-\mathrm{i}s\boldsymbol{\omega}'\cdot\mathbf{x}'}\Lambda_{DJ}\,\mathrm{e}^{\mathrm{i}s\boldsymbol{\omega}'\cdot\mathbf{x}'}, \qquad \text{for all} \quad \boldsymbol{\omega}'\in S^{n-2}. \tag{38}$$

This fully characterizes the symmetric tensor $d(\mathbf{x}')$ on Σ_J for $1 \leqslant J \leqslant N$. □

Once $d(\mathbf{x})$ is known, then so is the Dirichlet-to-Neumann operator Λ_{DJ} on the domain D_J. Known results [25, 28, 30] on the uniqueness of the reconstruction of the conductivity from boundary measurements allow us to conclude the proof of theorem 3.1.

The hypotheses on $\gamma(\mathbf{x})$ in D can be refined and theorem 3.1 modified accordingly. In the case of isotropic conductivities, a uniqueness result in dimension $n \geqslant 3$ exists for conductivities of class $C^{3/2+\varepsilon}(D)$ [9]. Uniqueness was also established for piecewise analytic conductivities [23] and for piecewise $C^2(D)$ conductivities [19]. In dimension $n = 2$ uniqueness is established for conductivities with gradient in $L^p(D)$ for $p > 2$ [10].

For anisotropic tensors the results are in dimension $n = 2$ [27] that two tensors γ_1 and γ_2 in $C^{2,\alpha}(D)$, $0 < \alpha < 1$, with boundary ∂D of class $C^{3,\alpha}$ with same Neumann-to-Dirichlet data Λ_γ are such that there exists a $C^{3,\alpha}(D)$ diffeomorphism Φ with $\Phi_{|\partial\Omega} = I_{\partial\Omega}$, the identity operator on $\partial\Omega$, and

$$\gamma_2(x) = \frac{(D\Phi)^T \gamma_1 (D\Phi)}{|D\Phi|} \circ \Phi^{-1}(x).$$

In dimension $n \geqslant 3$ the same results hold [24] provided that γ_1, γ_2 and ∂D are real-analytic (in which case Φ is also real-analytic).

This concludes the reconstruction of the singular interface Σ and the physical parameters $d(\mathbf{x})$ and $\gamma(\mathbf{x})$ on D.

Remark 3.7. Let us conclude this section by a remark on the reconstruction of embedded inclusions characterized by conductivity tensors that differ from the background γ_0. This is the problem treated in [11]. In this context, where the above calculations apply with $d \equiv 0$, we verify that

$$\begin{aligned}\int_\Sigma F\phi\,\phi\,\mathrm{d}\sigma &= \int_\Omega \gamma_0 \nabla\delta w \cdot \nabla\delta w\,\mathrm{d}\mathbf{x} + \int_D (\gamma - \gamma_0)\nabla w_\Sigma \cdot \nabla w_\Sigma\,\mathrm{d}\mathbf{x} \\ \int_\Sigma -F\phi\,\phi\,\mathrm{d}\sigma &= \int_\Omega \gamma \nabla\delta w \cdot \nabla\delta w\,\mathrm{d}\mathbf{x} + \int_D (\gamma_0 - \gamma)\nabla w_0 \cdot \nabla w_0\,\mathrm{d}\mathbf{x}.\end{aligned} \tag{39}$$

Let us assume that either one of the following hypotheses holds:

$$\gamma(\mathbf{x}) - \gamma_0(\mathbf{x}) \geqslant \alpha_1 > 0 \quad \text{on} \quad D, \qquad \gamma_0(\mathbf{x}) = \gamma(\mathbf{x}), \quad \text{on} \quad D^c, \tag{40}$$

$$\gamma_0(\mathbf{x}) - \gamma(\mathbf{x}) \geqslant \alpha_1 > 0 \quad \text{on} \quad D, \qquad \gamma_0(\mathbf{x}) = \gamma(\mathbf{x}), \quad \text{on} \quad D^c, \tag{41}$$

for some constant positive definite tensor α_1. The reasoning presented in this section allows us to deduce from (39) that F is coercive on $Y_0^{1/2}(\Sigma)$ provided that (40) holds and that $-F$ is coercive in the same sense when (41) is satisfied. We thus recover in a marginally more general context the result obtained in [11] that the interface Σ can be imaged from knowledge of the operator Λ_Σ (more precisely, we have shown that theorem 3.4 applies in this context; see [11] for details on how the range condition is used to image Σ). Note that we do not need to show that F is an isomorphism to obtain the range relation (33). The results stated in propositions 3.5 and 3.6 also hold in this context.

3.4. Local Neumann-to-Dirichlet measurements

So far, we have assumed that we had access to the full Neumann-to-Dirichlet measurements Λ_Σ. A similar factorization method can be used to show that local measurements on an arbitrary small portion of positive measure of the boundary $\partial\Omega$ are actually sufficient. Let Γ

be a smooth open subset of $\partial\Omega$ and Γ^c the complementary open subset such that $\overline{\Gamma^c} \cup \Gamma = \partial\Omega$. Let us assume that u is the unique solution to

$$\begin{aligned}
&\nabla\cdot\gamma\nabla u = 0 && \text{in} \quad \Omega\backslash\Sigma \\
&[u] = 0 && \text{on} \quad \Sigma \\
&[\mathbf{n}\cdot\gamma\nabla u] = -\nabla_\perp\cdot \mathrm{d}\nabla_\perp u && \text{on} \quad \Sigma \\
&\mathbf{n}\cdot\gamma\nabla u = g && \text{on} \quad \Gamma \\
&\mathbf{n}\cdot\gamma\nabla u = 0 && \text{on} \quad \Gamma^c \\
&\int_\Gamma u\,\mathrm{d}\sigma = 0,
\end{aligned} \tag{42}$$

where $g \in H_0^{-1/2}(\Gamma)$. We denote by Λ_Γ the local Neumann-to-Dirichlet operator mapping $g \in H_0^{-1/2}(\Gamma)$ to $u_{|\Gamma} \in H_0^{1/2}(\Gamma)$, where u is the solution to the above problem. We claim that knowledge of $\Lambda_\Gamma \in \mathcal{L}\big(H_0^{-1/2}(\Gamma), H_0^{1/2}(\Gamma)\big)$ is sufficient to obtain the results stated in theorem 3.1.

Let Θ be a smooth connected surface embedded in D^c, D^c_Θ the subset of D^c with boundary $\partial D^c_\Theta = \Theta \cup \partial\Omega$ and D_Θ its complement in Ω with boundary $\partial D_\Theta = \Theta$. All the results that follow in this section are independent of the presence of the singular interface. We will show that knowledge of Λ_Γ implies knowledge of the full Neumann-to-Dirichlet operator on D_Θ. This is not influenced by the presence of the interface Σ and so we set $d \equiv 0$ to simplify notation.

We define the operators M_Γ, L_Γ, L^*_Γ, G_Γ, G^*_Γ and F_Γ as follows. Let u, v and w be the unique solutions to the following equations:

$$\begin{aligned}
&\nabla\cdot\gamma\nabla u = 0, && \nabla\cdot\gamma\nabla v = 0, && \nabla\cdot\gamma\nabla w = 0, && \text{in} \quad D^c_\Theta \\
&\mathbf{n}\cdot\gamma\nabla u = g, && \mathbf{n}\cdot\gamma\nabla v = \phi, && \mathbf{n}\cdot\gamma\nabla w = 0, && \text{on} \quad \Gamma \\
&\mathbf{n}\cdot\gamma\nabla u = 0, && \mathbf{n}\cdot\gamma\nabla v = 0, && \mathbf{n}\cdot\gamma\nabla w = 0, && \text{on} \quad \Gamma^c \\
&\mathbf{n}\cdot\gamma\nabla u = 0, && \mathbf{n}\cdot\gamma\nabla v = 0, && \mathbf{n}\cdot\gamma\nabla w = -\phi, && \text{on} \quad \Theta \\
&\int_\Gamma u\,\mathrm{d}\sigma = 0, && \int_\Theta v\,\mathrm{d}\sigma = 0, && \int_\Gamma w\,\mathrm{d}\sigma = 0.
\end{aligned} \tag{43}$$

Then, M_Γ maps $g \in H_0^{-1/2}(\Gamma)$ to $M_\Gamma g = u_{|\Gamma} \in H_0^{1/2}(\Gamma)$, L_Γ maps $\phi \in H_0^{-1/2}(\Gamma)$ to $L_\Gamma\phi = v|_\Theta \in H_0^{1/2}(\Theta)$ and L^*_Γ maps $\phi \in H_0^{-1/2}(\Theta)$ to $L^*_\Gamma\phi = w_{|\Gamma} \in H_0^{1/2}(\Gamma)$.

We now define the operator G_Γ mapping $g \in H_0^{-1/2}(\Gamma)$ to $G_\Sigma g = \mathbf{n}\cdot\gamma\nabla v_{|\Theta} \in H_0^{-1/2}(\Theta)$, and its adjoint G^*_Γ mapping $\phi \in H_0^{1/2}(\Theta)$ to $G^*_\Gamma\phi = w|_\Gamma \in H_0^{1/2}(\Gamma)$, where v and w are the solutions to

$$\begin{aligned}
&\nabla\cdot\gamma\nabla v = 0 && \text{in} \quad \Omega\backslash\Theta && \nabla\cdot\gamma\nabla w = 0 && \text{in} \quad \Omega\backslash\Theta \\
&\mathbf{n}\cdot\gamma\nabla v = g && \text{on} \quad \Gamma && \mathbf{n}\cdot\gamma\nabla w = 0 && \text{on} \quad \Gamma \\
&\mathbf{n}\cdot\gamma\nabla v = 0 && \text{on} \quad \Gamma^c && \mathbf{n}\cdot\gamma\nabla w = 0 && \text{on} \quad \Gamma^c \\
&[v] = 0 && \text{on} \quad \Theta && [w] = \phi && \text{on} \quad \Theta \\
&[\mathbf{n}\cdot\gamma\nabla v] = 0 && \text{on} \quad \Theta && [\mathbf{n}\cdot\gamma\nabla w] = 0 && \text{on} \quad \Theta \\
&\int_\Theta v\,\mathrm{d}\sigma = 0 && && \int_\Gamma w\,\mathrm{d}\sigma = 0.
\end{aligned} \tag{44}$$

We finally define F_Σ as the operator mapping $\phi \in H_0^{1/2}(\Theta)$ to $F_\Sigma\phi = \mathbf{n}\cdot\gamma\nabla w|_\Theta \in H_0^{-1/2}(\Theta)$.

We verify that

$$\Lambda_\Gamma = M_\Gamma - L^*_\Gamma G_\Gamma = M_\Gamma - L^*_\Gamma F_\Gamma L_\Gamma,$$

as before. Since Λ_Γ is known by measurements and M_Γ can be computed since γ is known on D^c_Θ, we have access to $L^*_\Gamma G_\Gamma$ from $H_0^{-1/2}(\Gamma)$ to $H_0^{1/2}(\Gamma)$. By uniqueness of the solution to the Cauchy problem, we also obtain as before that L^*_Γ is injective. We can then invert it on the range of $\Lambda_\Gamma - M_\Gamma$ and thus construct G_Γ. It remains to show that $\overline{\mathcal{R}(G_\Gamma)} = H_0^{-1/2}(\Theta)$ to deduce that knowledge of G_Γ implies that of the Neumann-to-Dirichlet operator Λ_Θ on D_Θ. Indeed, we find that G^*_Γ is injective since $G^*_\Gamma = L^*_\Gamma F_\Gamma$, where we know that L^*_Γ is injective, and where we verify that $F_\Gamma = F^*_\Gamma$ generates a coercive form on $H_0^{-1/2}(\Gamma)$ according to (24), hence F_Γ is injective.

We have thus been able to construct the Neumann-to-Dirichlet operator Λ_Θ on D_Θ from knowledge of the local Neumann-to-Dirichlet operator Λ_Γ. We easily verify that the results are not modified by the presence of a singular interface Σ embedded in D_Θ. It remains to apply theorem 3.1 to the domain D_Θ instead of Ω to conclude that the results of theorem 3.1 hold when Λ_Γ is known.

4. Reconstructions in optical tomography

The setting for the reconstruction of clear layers and the absorbing and diffusing properties of human tissues inside the region enclosed by the clear layer is very similar to the setting of the preceding section. We mainly outline the differences here. The generalization to local boundary measurements can be carried out as was done in section 3.4. We do not consider it here.

Let us first mention that since the absorption parameter $a(\mathbf{x})$ is uniformly strictly positive, we can safely replace $Y_0^s(\Sigma)$ by $H^s(\Sigma) \equiv \otimes_{j=1}^N H^s(\Sigma_J)$ in this section. All the norm equivalences encountered in the preceding section become trivial in this context thanks to the presence of the term $\int_\Omega au\psi\,d\mathbf{x}$ in $b(u,\psi)$.

The operator L maps $\phi \in H^{-1/2}(\partial\Omega)$ to $v_{|\Sigma} \in H^{1/2}(\Sigma)$, and its adjoint operator L^* maps $\phi \in H^{-1/2}(\Sigma)$ to $w_{|\partial\Omega} \in H^{1/2}(\partial\Omega)$, where v and w are the unique solutions to the following problems:

$$\begin{aligned} Av &= 0 && \text{in } D^c & Aw &= 0 && \text{in } D^c \\ \mathbf{n}\cdot\gamma\nabla v &= \phi && \text{on } \partial\Omega & \mathbf{n}\cdot\gamma\nabla w &= 0 && \text{on } \partial\Omega \\ \mathbf{n}\cdot\gamma\nabla v &= 0 && \text{on } \Sigma & \mathbf{n}\cdot\gamma\nabla w &= -\phi && \text{on } \Sigma. \end{aligned} \tag{45}$$

The operator G_Σ maps $g \in H^{-1/2}(\partial\Omega)$ to $G_\Sigma g = \mathbf{n}\cdot\gamma\nabla v^+_{|\Sigma} \in H^{-1/2}(\Sigma)$ and G^*_Σ maps $\phi \in H_0^{1/2}(\Sigma)$ to $G^*_\Sigma\phi = w_{|\partial\Omega} \in H^{1/2}(\partial\Omega)$, where v and w are the unique solutions to the following problems:

$$\begin{aligned} Av &= 0, && \text{in } \Omega\backslash\Sigma & Aw &= 0, && \text{in } \Omega\backslash\Sigma \\ [v] &= 0, && \text{on } \Sigma & [w] &= \phi, && \text{on } \Sigma \\ [\mathbf{n}\cdot\gamma\nabla v] &= Bv, && \text{on } \Sigma & [\mathbf{n}\cdot\gamma\nabla w] &= Bw^-, && \text{on } \Sigma \\ \mathbf{n}\cdot\gamma\nabla v &= g, && \text{on } \partial\Omega & \mathbf{n}\cdot\gamma\nabla w &= 0, && \text{on } \partial\Omega. \end{aligned} \tag{46}$$

As before, the operators G_Σ and G^*_Σ are bounded and in duality with each other, i.e., (23) holds. We define F_Σ as the operator that maps $\phi \in H^{1/2}(\Sigma)$ to $-\mathbf{n}\cdot\gamma\nabla w^+ \in H^{-1/2}(\Sigma)$, where w is the solution to (46). We verify that

$$(F_\Sigma[w],[\psi])_\Sigma = b(w,\psi) + b_\Sigma(w^-,\psi^-) = ([w], F_\Sigma[\psi])_\Sigma, \tag{47}$$

where the bilinear forms b and b_Σ are defined in (10), so that $F_\Sigma = F^*_\Sigma$. We still verify that $G^*_\Sigma = L^*F_\Sigma$ so that $G_\Sigma = F_\Sigma L$ and define G_0 and F_0 using the same procedure as G_Σ and

F_Σ except that $d \equiv 0$ and $\alpha \equiv 0$ and that γ and a are replaced by γ_0 and a_0. We thus obtain as in (27) that

$$\Lambda_0 - \Lambda_\Sigma = L^* F L, \qquad F = F_\Sigma - F_0. \tag{48}$$

It remains to show that F generates a coercive form on $H^{1/2}(\Sigma)$. This is done as follows. We still denote $\delta w = w_0 - w_\Sigma$ and define $\delta b = b - b_0$, where b_0 is the form defined as in (10) with γ and a replaced by γ_0 and a_0. We similarly define the operator

$$A_0 u = -\nabla \cdot \gamma_0 \nabla u + a_0 u. \tag{49}$$

We also introduce the bilinear forms b_D, b_{0D} and $b_{D^c} = b - b_D$, where b_D is the same as the form b defined in (10) except that the integration is performed over D only, and b_{0D} is constructed from b_0 similarly.

Some integrations by parts show that

$$\begin{aligned} b_D(w_0, w_\Sigma) &= \int_\Sigma \mathbf{n} \cdot \gamma \nabla w_\Sigma^- w_0^- \,\mathrm{d}\sigma \\ b_{0D}(w_0, w_\Sigma) &= \int_\Sigma \mathbf{n} \cdot \gamma \nabla w_0^- w_\Sigma^- \,\mathrm{d}\sigma \\ b_{D^c}(w_0, w_\Sigma) &= -\int_\Sigma \mathbf{n} \cdot \gamma \nabla w_\Sigma^+ w_0^+ \,\mathrm{d}\sigma = -\int_\Sigma \mathbf{n} \cdot \gamma \nabla w_0^+ w_\Sigma^+ \,\mathrm{d}\sigma. \end{aligned}$$

Now we multiply $A_0 \delta w + (A_0 - A) w_\Sigma = 0$ by δw and integrate to obtain

$$b_0(\delta w, \delta w) - \delta b(w_\Sigma, \delta w) = -\int_\Sigma [\mathbf{n} \cdot \gamma \nabla \delta w \delta w] \,\mathrm{d}\sigma,$$

so that

$$\begin{aligned} b_0(\delta w, \delta w) + \delta b(w_\Sigma, w_\Sigma) &= \delta b(w_0, w_\Sigma) + \int_\Sigma [\mathbf{n} \cdot \gamma \nabla w_\Sigma] \delta w^- \,\mathrm{d}\sigma \\ &= \int_\Sigma F\phi\, \phi \,\mathrm{d}\sigma - \int_\Sigma B w_\Sigma^- w_\Sigma^- \,\mathrm{d}\sigma. \end{aligned}$$

This implies that

$$(F\phi, \phi)_\Sigma = b_0(\delta w, \delta w) + \delta b(w_\Sigma, w_\Sigma) + b_\Sigma(w_\Sigma^-, w_\Sigma^-). \tag{50}$$

We are now in a position to state the following result.

Theorem 4.1. *Let us assume that the operator F generates a coercive form on $H^{1/2}(\Sigma)$ in the sense that there exists a positive constant C such that*

$$(F\phi, \phi)_\Sigma \geqslant C \|\phi\|^2_{H^{1/2}(\Sigma)}. \tag{51}$$

Then the surface Σ can uniquely be reconstructed from the Cauchy data at the boundary of the domain.

The proof of this theorem follows the same lines as the derivation in section 3. There exists a canonical isomorphism $\mathcal{I}$ from $H^{-1/2}(\Sigma)$ to $H^{1/2}(\Sigma)$ (for instance the inverse of the square root of $\Delta_\perp - 1$) allowing us to recast F as $F = B^* B$ with B surjective so that

$$\mathcal{R}((\Lambda_0 - \Lambda_\Sigma)^{1/2}) = \mathcal{R}(L^*). \tag{52}$$

It suffices to consider the solutions to

$$\begin{aligned} A_0 N(\cdot; \mathbf{y}) &= \delta(\cdot - \mathbf{y}), && \text{in} \quad \Omega \\ \mathbf{n} \cdot \gamma_0 \nabla N(\cdot; \mathbf{y}) &= 0 && \text{on} \quad \partial\Omega, \end{aligned} \tag{53}$$

to construct a family of functions $g_{\mathbf{y}}(\mathbf{x}) = N(\mathbf{x}; \mathbf{y})_{|\partial\Omega}$ such that $g_{\mathbf{y}}$ is in the range of L^* when $\mathbf{y} \in D$ and not in its range when $\mathbf{y} \in D^c$.

Corollary 4.2. *Let us assume that $\gamma \geqslant \gamma_0$ and that $a \geqslant a_0$ on D. Then the results stated in theorem 4.1 hold.*

Indeed, these hypotheses are sufficient to show (51) as was done in section 3 before (32). Note that all that is required from the bilinear form $b_\Sigma(w_\Sigma^-, w_\Sigma^-)$ is that it enforces that $w_\Sigma \in H^1(D)$. The above result thus generalizes to a large class of operators B modelling the jump of fluxes across the interface Σ. We now state the following result.

Proposition 4.3. *The symmetric tensor $d(\mathbf{x})$ and absorption parameter $\alpha(\mathbf{x})$ are uniquely determined by knowledge of Λ_Σ.*

Similar to the proof of proposition 3.6, we have access to $\Lambda_D = B + \tilde{\Lambda}_D$. The coefficient $d(\mathbf{x})$ is still given by formula (38). The coefficient α can be reconstructed as follows. Let $\mathbf{x}_0 \in \Sigma$ such that Σ is locally given by $\mathbf{x}_n = 0$ in coordinates $(\mathbf{x}', x_n)$. We know [30] that $\tilde{\Lambda}_D$ is a classical pseudo-differential operator whose symbol in coordinates $(\mathbf{x}', x_n)$ is given by

$$\tilde{\lambda}(\mathbf{x}', \boldsymbol{\xi}') = \gamma(\mathbf{x}', 0)|\boldsymbol{\xi}'| + \mu_0(\mathbf{x}', \boldsymbol{\xi}') + r(\mathbf{x}', \boldsymbol{\xi}'), \tag{54}$$

where μ_0 depends only on γ and a in the vicinity of Σ and is thus known, and r is a symbol of order -1. At the same time, the symbol $b(\mathbf{x}', \boldsymbol{\xi}')$ of B in the same coordinates is given by

$$b(\mathbf{x}', \boldsymbol{\xi}') = d(\mathbf{x}')|\boldsymbol{\xi}'|^2 - \mathrm{i}\nabla_\perp d(\mathbf{x}') \cdot \boldsymbol{\xi}' + \alpha(\mathbf{x}'). \tag{55}$$

We can thus reconstruct α from knowledge of Λ_D by constructing the zeroth order term of its symbol. Once B is known, then so is the Neumann-to-Dirichlet operator $\tilde{\Lambda}_D$ on the domain D.

Let us now consider the reconstruction of the unknown parameters on D in spatial dimension $n \geqslant 3$ and assume that the diffusion tensor γ is scalar, i.e., of the form $\gamma(\mathbf{x})I$. It is known that only one of the coefficients γ and a can be reconstructed from such data [5]. Indeed we verify that

$$\gamma^{-1/2} A \gamma^{-1/2} = -\Delta + q, \qquad q = \frac{\Delta\gamma^{1/2}}{\gamma^{1/2}} + \gamma^{-1} a. \tag{56}$$

Thus q can be reconstructed from boundary measurements when the spatial dimension $n \geqslant 3$ [28, 30]. In order to reconstruct both a and γ, more information is necessary at the boundary. Such information can be obtained when modulated source terms are used. In this case, the operator A modelling photon propagation is given by

$$A = -\nabla \cdot \gamma \nabla + (\mathrm{i}\omega + a), \tag{57}$$

where $\omega \in \mathbb{R}$. The operator B is not modified. Formally all the calculations performed in this section hold with a replaced by $a + \mathrm{i}\omega$. We can still define the operator L as above. Because the solutions are now complex valued, we can express L as $L = L_R + \mathrm{i}L_I$, where L_R and L_I are real-valued operators. We can then decompose the Neumann-to-Dirichlet operators as

$$\Lambda_0 - \Lambda_\Sigma = L' F L, \qquad L' = L_R^* + \mathrm{i}L_I^*.$$

Note, however, that L' is different from $L^* = L_R - \mathrm{i}L_I$. The factorization technique developed in this paper thus does not apply directly. It is unknown to the author how the range of L and that of the measurement operator are related. Let us assume that data are available at the modulation $\omega = 0$ and at a second modulation $\omega \neq 0$. From the measurements at $\omega = 0$, we reconstruct the interface Σ and the coefficient d and α as above. For $\omega \neq 0$, we can then invert

L' from $\mathcal{R}(L')$ to $H^{-1/2}(\Sigma)$ and have access to the Dirichlet-to-Neumann operator $\tilde{\Lambda}_D(\omega)$. Such data are sufficient to reconstruct both the absorption parameter a and the diffusion coefficient γ. Indeed we observe that

$$\gamma^{-1/2} A \gamma^{-1/2} = -\Delta + q, \qquad q = \frac{\Delta\gamma^{1/2}}{\gamma^{1/2}} + \gamma^{-1}(a + \mathrm{i}\omega). \tag{58}$$

We can thus retrieve γ from the imaginary part of the potential q and a from its real part once γ is known.

5. Conclusions

We have presented a method to explicitly reconstruct the location of a singular interface from global and local boundary measurements in two problems of practical interest, namely, impedance tomography and optical tomography. The technique is based on the factorization method. We have shown that the constitutive parameters of the considered elliptic equations in the region enclosed by the singular interface could also be uniquely reconstructed from the same boundary measurements. We have considered two problems of singular interfaces: clear layers arising in optical tomography and highly conducting thin inclusions in impedance tomography.

In the optical tomography application, the clear layer is in practice relatively close to the surface of the domain. The assumption that the properties of the human tissues are known between the boundary of the domain and the clear layer is therefore not totally unrealistic. Nevertheless the following mathematically more challenging question remains open, namely, whether complete boundary measurements allow us to uniquely reconstruct the singular interface and the coefficients on the whole domain *without* any *a priori* assumption on what the coefficients should be between the boundary of the domain and the singular interface.

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