ANALYSIS OF THE DOUBLE SCATTERING SCINTILLATION OF WAVES IN RANDOM MEDIA

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Abstract. High frequency waves propagating in highly oscillatory media are often modeled by radiative transfer equations that describes the propagation of the energy density of the waves. When the medium is statistically homogeneous, averaging effects occur in such a way that in the limit of vanishing wavelength, the wave energy density solves a deterministic radiative transfer equation. In this paper, we are interested in the remaining stochasticity of the energy density. More precisely, we wish to understand how such stochasticity depends on the statistics of the random medium and on the initial phase-space structure of the propagating wave packets.

The analysis of stochasticity is a formidable task involving complicated analytical calculations. In this paper, we consider the propagation of waves modeled by a scalar Schrödinger equation and limit the interaction of the waves with the underlying structure to second order. We calculate the scintillation function (second statistical moment) for such signals, which thus involve fourth-order moments of the random fluctuations, which we assume have Gaussian statistics. Our main result is a detailed analysis of the scintillation function in that setting. This requires the analysis of non-trivial oscillatory integrals, which is carried out in detail.

1. Introduction

In this paper, wave propagation is modeled by the following Schrödinger equation:

$$\left(i\varepsilon\frac{\partial}{\partial t}+\frac{\varepsilon^2}{2}\Delta-\sqrt{\varepsilon}V\left(\frac{x}{\varepsilon}\right)\right)u_\varepsilon(t,x)=0,\quad t>0,\quad x\in\mathbb{R}^d,$$

augmented with a deterministic initial condition $u_\varepsilon(0,\cdot)$ uniformly bounded in $L^2(\mathbb{R}^d)$ with respect to ε, for $d\geq 1$. Here, V is a mean-zero Gaussian stationary random field with autocorrelation $R(x):=\mathbb{E}V(x+y)V(y)$ and is time-independent. The symbol $\mathbb{E}$ denotes the ensemble average with respect to a given probability space $(\Omega,\mathcal{F},\mathbb{P})$ on which V is defined. The Wigner transform of u_ε is defined as [8]:

$$W_\varepsilon(t,x,k):=\frac{1}{(2\pi)^d}\int_{\mathbb{R}}e^{ik\cdot y}u_\varepsilon\left(t,x-\frac{\varepsilon y}{2}\right)\bar{u}_\varepsilon\left(t,x+\frac{\varepsilon y}{2}\right)dy,$$

where $\bar{u}_\varepsilon$ is the complex conjugate of u_ε and W_ε solves the Wigner equation

$$\frac{\partial}{\partial t}W_\varepsilon+k\cdot\nabla_x W_\varepsilon=A_\varepsilon W_\varepsilon,\tag{1.1}$$

with

$$\begin{aligned}(A_\varepsilon W_\varepsilon)(x,k) &:= \int_{\mathbb{R}^d} f_\varepsilon(x,k-\eta)W_\varepsilon(x,\eta)d\eta,\\ f_\varepsilon(x,\xi) &:= \frac{i}{\sqrt{\varepsilon}\pi^d}\left[\hat{V}(-2\xi)e^{-i2\xi\cdot x/\varepsilon} - \hat{V}(2\xi)e^{i2\xi\cdot x/\varepsilon}\right],\end{aligned}$$

where $\hat{V}$ denotes the Fourier transform of V with the convention

$$\hat{V}(k) = \int_{\mathbb{R}^d} e^{-ik\cdot x}V(x)dx.$$

The initial condition of (1.1), denoted by $W_\varepsilon^0(x,k)$, is the Wigner transform of $u_\varepsilon(0,\cdot)$. We denote by $a_\varepsilon := \mathbb{E}W_\varepsilon$ the ensemble average of W_ε. For sufficiently rapidly decaying correlation function R and Gaussian potentials, a_ε is known to converge in a proper functional setting to the solution a_0 of a radiative transfer equation, see [11, 5]:

$$\frac{\partial}{\partial t}a_0 + k\cdot\nabla_x a_0 = \int_{\mathbb{R}^d}\sigma(p,k)[a_0(t,x,p) - a_0(t,x,k)]dp, \tag{1.2}$$

with a scattering cross section $\sigma(p,k) = \hat{R}(p-k)\delta(|k|^2 - |p|^2)$, where δ is the Dirac distribution and the power spectrum $\hat{R}(k)$ is the Fourier transform of the correlation function $R(x)$. The above radiative transfer equation is known to hold for other frameworks involving the Schrödinger equation or classical wave equations [1, 3, 9, 10]. Although we are interested in classical as well as quantum waves, we restrict ourselves to the Schrödinger model because of its simpler mathematical structure.

Long-range correlations. We are interested in random fields with possibly long range interactions, which can be modeled with slowly decaying autocorrelations that are not integrable. Assuming $R(x) \sim_{|x|\to\infty} x^{\delta-d}$, with $0<\delta<d$, some standard rescaling arguments show that $\hat{R}$ is singular in the origin and behaves like $|k|^{-\delta}$. This leads us to consider correlation functions with singular Fourier transforms near the origin of the form

$$\hat{R}(k) = \frac{S(k)}{|k|^\delta}, \quad 0<\delta<d, \tag{1.3}$$

with $S\in L^1(\mathbb{R}^d)\cap\mathcal{C}_b(\mathbb{R}^d)$, $\mathcal{C}_b(\mathbb{R}^d)$ denoting the space of bounded continuous fonctions. Since, $0<\delta<d$, $\hat{R}$ is locally integrable. Physically realizable media impose that $\int\hat{R}(k)dk = (2\pi)^dR(0)<\infty$. Short-range correlations correspond to integrable R. In this case $\hat{R}$ is bounded so we may take $\delta=0$ in (1.3).

Scintillation function. An appropriate tool in the analysis is the scintillation function J_ε (or covariance function), defined as

$$J_\varepsilon(t,x,k,y,p) = \mathbb{E}W_\varepsilon(t,x,k)W_\varepsilon(t,y,p) - \mathbb{E}W_\varepsilon(t,x,k)\,\mathbb{E}W_\varepsilon(t,y,p), \tag{1.4}$$

whose weak convergence to zero implies the convergence in probability thanks to the Chebyshev inequality

$$\mathbb{P}\Big(|\langle W_\varepsilon(t),\varphi\rangle - \langle a_\varepsilon(z),\varphi\rangle| \geq \epsilon\Big) \leq \frac{1}{\epsilon^2}\langle J_\varepsilon(t),\varphi\otimes\varphi\rangle.$$

Convergence in probability shows that W_ε is self-averaging as $\varepsilon \to 0$. Introducing first the free transport semigroup J, $Jh(t,x,k) := h(x-tk,k)$, and the operator

$$D^{-1}h(t,x,k) := \int_0^t h(t-s,x-sk,k)ds,$$

then (1.1) can be recast as the integral equation

$$(I - D^{-1}A_\varepsilon)W_\varepsilon = JW_\varepsilon^0,$$

whose solution can be decomposed formally as the multiple scattering expansion:

$$W_\varepsilon = \sum_{j=0}^{\infty}(D^{-1}A_\varepsilon)^j JW_\varepsilon^0. \tag{1.5}$$

We cannot obtain closed form equations for statistical moments of W_ε as can be done in the Itô-Schrödinger regime. Nevertheless, it is shown in the Itô-Schrödinger regime that the single and double scattering contributions give the leading terms of the scintillation function. We expect such a property to still hold in our regime of interest. Retaining only the terms $j \leq 2$ in the latter decomposition, and writing

$$W_\varepsilon \approx B + SS + DS,$$

where $B = JW_\varepsilon^0$ is the ballistic part, $SS = D^{-1}A_\varepsilon JW_\varepsilon^0$ the single scattering contribution and $DS = D^{-1}A_\varepsilon JW_\varepsilon^0 D^{-1}A_\varepsilon JW_\varepsilon^0$ the double scattering one, we have:

$$\begin{aligned} J_\varepsilon &\approx \mathbb{E}\left((B+SS+DS)(B+SS+DS)\right) - \mathbb{E}(B+SS+DS)\mathbb{E}(B+SS+DS), \\ &= J_{SS} + \mathbb{E}\left(DS(B+SS+DS)\right) + \mathbb{E}\left((B+SS)DS\right) \\ &\quad -\mathbb{E}(DS)\mathbb{E}(B+SS+DS) - \mathbb{E}(B+SS)\mathbb{E}(DS). \end{aligned}$$

Since the potential is mean-zero and Gaussian, $\mathbb{E}(DSSS) = 0$, and we have

$$J_\varepsilon = J_{SS} + J_{DS} := J_{SS} + \mathbb{E}\left(DSDS\right) - \mathbb{E}(DS)\mathbb{E}(DS).$$

Here, J_{SS} is the scintillation function of single scattering. It was analyzed in [2] and the main results of that study are recalled below. The purpose of this work is to carefully analyze the convergence properties of J_{DS} to complement that of J_{SS}.

Initial conditions. The scintillation function is known to strongly depend on the structure of the initial conditions, see [2, 4]. We consider initial conditions $u_\varepsilon(0,\cdot)$ oscillating at frequencies of order ε^{-1} and with a spatial support of size ε^α for $0 \leq \alpha \leq 1$. The parameter α quantifies the macroscopic concentration of the initial condition. The simplest example is a modulated plane wave of the form (or a pure state):

$$u_\varepsilon(0,x) = \frac{1}{\varepsilon^{\frac{d\alpha}{2}}}\chi\Big(\frac{x}{\varepsilon^\alpha}\Big)e^{i\frac{x\cdot q_0}{\varepsilon}}, \tag{1.6}$$

where $\chi \in \mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}$ denotes the Schwarz class of functions. The direction of propagation is given by q_0, and we suppose for simplicity that $|q_0| = 1$. Note that the above sequence of initial conditions is uniformly bounded in $L^2(\mathbb{R}^d)$, and that the corresponding Wigner transform reads

$$W_\varepsilon^0(x,q) = \frac{1}{\varepsilon^d} W_0\left(\frac{x}{\varepsilon^\alpha}, \frac{q-q_0}{\varepsilon^{1-\alpha}}\right), \tag{1.7}$$

where $W_0(x,k)$ is the Wigner transform of the rescaled initial condition $u_{\varepsilon=1}$ and is real-valued. We restrict α to be less than 1 to ensure that ε^{-1} is the highest frequency in the problem. Such an initial condition allows for a precise characterization of the convergence of J_{DS}.

Some notation. We denote by $\mathcal{F}f$ the Fourier transform of $f(x,q)$ with respect to both variables x and q. For a function $f(z^1,\cdots,z^n) \in \mathcal{C}^m(\mathbb{R}^{nd})$, $z^j \in \mathbb{R}^d$, $j=1,\cdots,n$ and a multi-index $i=(i_1,\cdots,i_{nd}) \in \mathbb{N}^{nd}$ with $|i| = i_1+\cdots+i_{nd} \le m$, we introduce

$$\partial^i_{z^1,\cdots,z^n} f := \frac{\partial^{i_1}}{\partial_{z^1_1}} \cdots \frac{\partial^{i_{nd}}}{\partial_{z^n_d}} f.$$

Let $\langle x\rangle := (1+|x|^2)^{1/2}$ for $x \in \mathbb{R}^d$, $|x|$ being the Euclidian norm of the vector x and let $a \wedge b$ (resp. $a \vee b$) be the minimum (resp. maximum) of a and b. All along the paper C denotes a universal constant that might differ from line to line.

2. Results

This section summarizes our main results on the scintillation function. For a real test function $\varphi \in \mathcal{S}(\mathbb{R}^{2d})$, we define

$$w_\varepsilon(\tau) = \int_{\mathbb{R}^{4d}} dxdydpdq\varphi(x,p)\varphi(y,q)J_{DS}(\tau,x,p,y,q). \tag{2.1}$$

Simple but lengthy calculations show that w_ε admits the expression

$$\begin{aligned} w_\varepsilon(\tau) \quad = \quad & \frac{1}{\varepsilon^2(2\pi)^{4d}} \int_{\mathbb{R}^{4d}} d\xi_1 d\eta_1 d\xi_2 d\eta_2 \; \mathbb{E}\{\hat V(\eta_1)\hat V(\xi_1)\hat V(\eta_2)\hat V(\xi_2)\} \\ & \qquad\qquad \times F^\varepsilon(\tau,\xi_1+\eta_1,\eta_1)F^\varepsilon(\tau,\xi_2+\eta_2,\eta_2) \\ & - \int_{\mathbb{R}^{4d}} dxdydpdq\varphi(x,p)\varphi(y,q)\mathbb{E}(DS)(\tau,x,p)\mathbb{E}(DS)(\tau,y,q), \end{aligned} \tag{2.2}$$

where we have defined

$$\begin{aligned} F^\varepsilon(\tau,\xi,\eta) = & \sum_{\sigma_1,\sigma_2=\pm1} \sigma_1\sigma_2 \int_0^\tau \int_0^t dtds \exp\Big\{ -\frac{i}{\varepsilon}\Big[\frac{1}{2}\sigma_2 s(\xi-\eta)\cdot\eta + q_0\cdot(t\xi - s\eta)\Big]\Big\} \\ & \times \int_{\mathbb{R}^{2d}} dxdp \, \exp\Big\{-\frac{i}{\varepsilon^{1-\alpha}} x\cdot\xi\Big\} \exp\Big\{-\frac{i}{\varepsilon^\alpha} p\cdot(t\xi - s\eta)\Big\} W_0(x,p)\psi^\varepsilon(x,p,[z]), \end{aligned}$$

$$\begin{aligned} \psi^\varepsilon(x,p,[z]) = & \sum_{\sigma_1=\pm1} \sigma_1 \varphi\Big(\varepsilon^\alpha x + \frac{1}{2}(\tau-t)\sigma_1(\xi-\eta) + \tau(q_0+\varepsilon^{1-\alpha}p) \\ & + \frac{1}{2}\sigma_2(\tau-t+s)\eta, q_0+\varepsilon^{1-\alpha}p + \frac{1}{2}\sigma_1(\xi-\eta)+\frac{1}{2}\sigma_2\eta\Big). \end{aligned}$$

Above (and also in the sequel), we used the shorthand notation $[z] = (\tau, t, s, \xi, \eta, \sigma_2)$ to denote the variables (after possible rescaling) related to the function ψ. Since the potential is Gaussian, we have the property

$$(2\pi)^{-2d}\mathbb{E}(\hat{V}(\eta_1)\hat{V}(\xi_1)\hat{V}(\eta_2)\hat{V}(\xi_2)) = \hat{R}(\eta_1)\delta(\eta_1+\xi_1)\hat{R}(\eta_2)\delta(\eta_2+\xi_2)$$
$$+\hat{R}(\eta_1)\delta(\eta_1+\eta_2)\hat{R}(\xi_1)\delta(\xi_1+\xi_2)+\hat{R}(\eta_1)\delta(\eta_1+\xi_2)\hat{R}(\xi_1)\delta(\xi_1+\eta_2),$$

where δ denotes the Dirac measure. The first term on the right of the latter equation generates a scintillation that is equal to the second term on the right of (2.2), so that only two terms remain in w_ε:

$$w_\varepsilon(\tau) = w_\varepsilon^1(\tau) + w_\varepsilon^2(\tau), \tag{2.3}$$

$$w_\varepsilon^1(\tau) = \frac{1}{\varepsilon^2(2\pi)^{2d}}\int_{\mathbb{R}^{2d}} d\xi d\eta \hat{R}(\xi-\eta)\hat{R}(\eta)|F^\varepsilon(\tau,\xi,\eta)|^2, \tag{2.4}$$

$$w_\varepsilon^2(\tau) = \frac{1}{\varepsilon^2(2\pi)^{2d}}\int_{\mathbb{R}^{2d}} d\xi d\eta \hat{R}(\xi-\eta)\hat{R}(\eta)F^\varepsilon(\tau,\xi,\eta)F^\varepsilon(\tau,-\xi,-\xi+\eta). \tag{2.5}$$

As will be explained at the end of section 3.2, the two terms above are actually equal in the limit $\varepsilon \to 0$ and we will therefore only consider w_ε^1 in our analysis. When $d \geq 2$, we decompose the variables ξ and η according to the line defined by the vector q_0 (recall that $|q_0| = 1$) as

$$\eta = (\eta_\|, \eta_\perp) \in \mathbb{R}^d \qquad ; \qquad \xi = (\xi_\|, \xi_\perp) \in \mathbb{R}^d, \tag{2.6}$$

where $(\xi_\|, \eta_\|) \in \mathbb{R}^2$, $(\xi_\perp, \eta_\perp) \in \mathbb{R}^{2(d-1)}$, $\xi_\| = \xi \cdot q_0$, $\eta_\| = \eta \cdot q_0$, $\xi^0 \cdot q_0 = \eta^0 \cdot q_0 = 0$, with ξ^0 and η^0 denoting the vectors $(0, \xi_\perp)$ and $(0, \eta_\perp)$. When $d = 1$, such a transformation is not necessary as ξ and η are always aligned with q_0. Let B_1 be the unit ball of $\mathbb{R}^{d-1}$. When $|\eta_\perp| \leq 1$, we define $\eta_\pm^\star(\eta_\perp) = 1 \pm \sqrt{1-|\eta_\perp|^2}$ and $\eta^\pm = (\eta_\pm^\star, \eta_\perp)$. The theorem below characterizes the limit of w_ε^1 (and therefore that of w_ε^2) in the limit $\varepsilon \to 0$ according to the physically relevant parameters α and δ. When $d \geq 2$, we do not address the case $\alpha = 1$ in detail as it is of lesser interest since the double scattering contribution is of higher order in ε than the single scattering contribution and is therefore asymptotically negligible. When $d = 1$, single and double scattering contributions have the same order when $\alpha = 1$ and the corresponding case is treated in theorem 2.2. All convergences below are pointwise in time and can be shown to be uniform provided an initial layer is avoided. More precisely, we have the following result when $d \geq 2$:

Theorem 2.1. *We have, pointwise in τ, $\forall \delta \in (0, d)$, $\forall \alpha \in [0, 1]$:*

$$w_\varepsilon^1(\tau) \to 0 \qquad \text{as} \qquad \varepsilon \to 0,$$

and when $\alpha = \frac{1}{2}$, there exists a positive non-identically vanishing function f such that, pointwise in τ,

$$f(\tau) \leq \liminf_{\varepsilon \to 0} \varepsilon^{\delta - d} w_\varepsilon^1(\tau). \tag{2.7}$$

Besides, when $\delta = 0$ and $d \geq 3$:

$$\varepsilon^{-d(1-\alpha)-(2\alpha-1)-(2\alpha-1)\vee 0} w_\varepsilon^1(\tau) \to w(\tau). \tag{2.8}$$

The limits depend on several parameters. When $0<\alpha<1$, we have:

$$w(\tau) = C_d \sum_{\pm} \int_0^{\tau_\alpha} \int_0^\infty \int_{\mathbb{R}^{d-1}} \int_{B_1} dt ds d\xi_\perp d\eta_\perp \frac{\hat{R}^2(\eta^\pm)}{(1-|\eta_\perp|^2)^{\frac12}} \left|\psi_\alpha \mathcal{F}(W_0)\left(\xi^0, t\xi_\alpha - s\eta^\pm\right)\right|^2,$$

$$C_d = \frac{2}{(2\pi)^{2(d-1)}}, \quad \psi_\alpha = \sum_{\sigma_1=\pm1} \sigma_1 \varphi\left(\tau q_0 - \frac12(1+\sigma_1)(\tau - t_\alpha)\eta^\pm, q_0 - \frac12(1+\sigma_1)\eta^\pm\right),$$

with $\tau_\alpha=\tau$ when $\alpha\le\frac12$, $\tau_\alpha=\infty$ when $\alpha>\frac12$, $\xi_\alpha=0$ when $\alpha<\frac12$, $\xi_\alpha=\xi^0$ when $\alpha\ge\frac12$, $t_\alpha=t$ when $\alpha\le\frac12$, and $t_\alpha=0$ when $\alpha>\frac12$.

When $\alpha=0$, then we find that:

$$w(\tau) = C_d \sum_{\pm} \int_{\mathbb{R}^d} \int_{B_1} \int_0^\tau d\xi d\eta_\perp ds \frac{\hat{R}^2(\eta^\pm)}{(1-|\eta_\perp|^2)^{1/2}} \left| \int_s^\tau dt e^{-it\xi_\parallel} \mathcal{F}(W_0\psi)\left(\xi, s\eta^\pm\right)\right|^2,$$

$$\psi = \sum_{\sigma_1=\pm1} \sigma_1\varphi\Big(x + \frac12(\tau-t)\sigma_1(\xi-\eta^\pm) + \tau q_0 - \frac12(\tau - t + s)\eta^\pm, q_0 + \frac12\sigma_1(\xi-\eta^\pm) - \frac12\eta^\pm\Big).$$

When $d=2$, $\delta=0$ and $\alpha\in[0,\frac12]$, (2.8) still holds while when $\alpha\in(\frac12,1)$:

$$\varepsilon^{-2(1-\alpha)-(2\alpha-1)-(2\alpha-1)\vee 0} |\log \tau\varepsilon^{1-2\alpha}|^{-1} w^1_\varepsilon(\tau) \to w(\tau),$$

$$w(\tau) = \sum_{\pm} \int_{\mathbb{R}} \int_{B_1} \int_0^\infty \frac{d\xi_\perp d\eta_\perp ds}{4\pi^2} \frac{\hat{R}^2(\eta^\pm)}{(1-|\eta_\perp|^2)^{1/2}} |\psi_\alpha \mathcal{F}(W_0)\left(0, \xi^0 - s\eta^\pm\right)|^2,$$

for the ψ_α previously defined.

In the one-dimensional case, we have the following result:

Theorem 2.2. *$\forall\delta\in(0,d)$, $\forall\alpha\in(0,1]$, pointwise in τ:*

$$w^1_\varepsilon(\tau) \to 0 \qquad as \qquad \varepsilon\to 0.$$

When $\delta=0$, we have $\lim\limits_{\varepsilon\to0}\varepsilon^{-\alpha}w_\varepsilon(\tau) = w^\alpha(\tau)$, where

$$\begin{aligned}
w^\alpha(\tau) &= \frac{\hat{R}^2(2q_0)}{2} \int_0^\tau \int_0^\infty dt ds \left|\mathcal{F}(W_0\psi_\alpha)(0,s)\right|^2, \qquad \alpha\in(0,1),\\
w^0(\tau) &= \frac{\hat{R}^2(2q_0)}{4\pi} \int_0^\tau \int_{\mathbb{R}} ds d\xi \left| \int_s^\tau dt \exp\{-it\xi\} \mathcal{F}(W_0\psi_\alpha)(\xi,s)\right|^2, \qquad \alpha=0,\\
w^1(\tau) &= 4\int_{\mathbb{R}} \int_0^\tau \frac{d\eta ds}{2\pi|\eta|} \hat{R}^2(\eta) \Big| \sum_{\sigma_2=\pm1} \sigma_2 \int_0^\infty ds \exp\{is\Psi_{\sigma_2}(\eta)\} \mathcal{F}(W_0\psi)(0,s\eta)\Big|^2, \ \alpha=1,
\end{aligned}$$

and

$$\psi_\alpha(x,p) = \sum_{\sigma_1=\pm1} \sigma_1\varphi\left(x_\alpha - (\tau-t)(1+\sigma_1)q_0 + \tau q_0 + (\tau - s_\alpha)q_0, q_0 - (1+\sigma_1)q_0\right),$$

$$x_\alpha = x \textit{ if } \alpha=0, \quad x_\alpha = 0 \textit{ otherwise}, \qquad s_\alpha = s \textit{ if } \alpha=0, \quad s_\alpha=0 \textit{ otherwise},$$

$$\psi(x,p) = \sum_{\sigma_1=\pm1} \sigma_1\varphi\left(\frac12(\tau-t)(\sigma_2-\sigma_1)\eta + \tau(p+q_0), p + q_0 + \frac12(\sigma_2-\sigma_1)\eta\right).$$

All the expressions that appear in the above two theorems can be shown to be finite since W_0 and φ belong to $\mathcal{S}(\mathbb{R}^{2d})$; see also section 6. When $d \geq 2$, theorem 2.1 shows that the scintillation function of the double scattering converges to zero, even for long-range correlations. A similar result was obtained in [2] for the single scattering contribution. This means that the truncated Wigner function (including only the ballistic part and the single and double scattering parts) is statistically stable in the limit $\varepsilon \to 0$ in media with spatial long-range correlations. There is nevertheless a striking difference with the single scattering contribution. The latter was shown in [2] to be (approximately) of order $\sqrt{\varepsilon}$ when $\delta \sim d$ and not of lower order. Here, the estimate from below (2.7) shows that the limiting scintillation of the second scattering is greater than a term of order $\varepsilon^{d-\delta}$ when $\alpha = \frac{1}{2}$, which is therefore close to one as $\delta \sim d$. This means that long-range effects are stronger on double scattering (and likely on any higher order scattering events) than on single scattering. This agrees with the physical intuition since long-range effects are supposed to be seen at large distances and double scattering events take place at a larger distance from the source than single scattering events do. Whether or not higher order scattering are statistically stable in media with long-range correlations remains an open problem for the random Schrödinger equation with time-independent potentials. Getting such a result may require an analysis of the whole series (1.5), which is already very difficult for short-range correlations [5].

We treated in the theorems the cases that it seemed to use were most relevant. Obtaining all the limiting expressions of w^1_ε when $\delta > 0$ is a fairly long task. Indeed, as explained in section 3.2, three different scales have to be defined in order to handle the singularity of $\hat{R}$ around the origin. This leads to three different expressions in the limit $\varepsilon \to 0$. The leading term thus depends on the value of δ. We computed the exact limit for the smallest of the three scales in section 4.3, and this allowed to obtain the bound from below (2.7). We presented the result only for $\alpha = \frac{1}{2}$ as it is the most interesting case. Indeed, it shows the scintillation is increasing as $\delta \to d$, a result that no longer holds when $\alpha \neq \frac{1}{2}$.

Theorem 2.1 also shows that when $d \geq 3$ and $\delta = 0$, scintillation of double scattering is of order $\varepsilon^{d(1-\alpha)+(2\alpha-1)+(2\alpha-1)\vee 0}$. It reaches a minimum of order ε^d when $\alpha = 0$, and a maximum of order $\varepsilon^{\frac{3}{2}}$ when $\alpha = \frac{1}{2}$ and $d = 3$ for instance. The scintillation of single scattering was shown to be of order $\varepsilon^{d(1-\alpha)+1-\alpha\wedge(1-\alpha)}$. It is then interesting to notice that single scattering dominates when $\alpha > \frac{2}{3}$, which is precisely the same threshold observed for the Itô-Schrödinger regime in [4]. The case $d = 2$ is very similar except that there is an additional logarithmic correction when $\alpha \in (\frac{1}{2}, 1)$. Also, the optimal estimates of our theorem together with those of [2] allow us to quantify the (self-)averaging effects of the fast oscillations in time of the random potential in the Itô-Schrödinger case. For the time independent potential considered here, the highest order of single scattering is ε^{d+1} (obtained for $\alpha = 0$), while it is ε^{d+2} for the Itô-Schrödinger case. For double scattering, we have ε^{d-1} when $\alpha = 0$, while the order is ε^d for Itô-Schrödinger. The fast oscillations then provide an additional order in ε when $\alpha = 0$. It is interesting to compare with the case of initial conditions that do not localize in the momentum variables, i.e., when

$\alpha = 1$. We find for both the stationary potential case and the Itô-Schrödinger case that single scattering is of order ε while double scattering is of order ε^2. This means that the fast oscillations in time have no longer a self-averaging effect when $\alpha = 1$ and therefore that spatial self-averaging becomes dominant as the initial condition gets unlocalized in momentum.

Application to precursors. When $\delta = 0$, the optimal estimates obtained in theorem 2.1 provide the dynamics of the statistical instabilities. The test function φ appears as

$$\varphi\Big(\tau q_0 - \frac{1}{2}(1+\sigma_1)(\tau - t_\alpha)\eta^\pm, q_0 - \frac{1}{2}(1+\sigma_1)\eta^\pm\Big),$$

with $t_\alpha = t$ when $\alpha \le \frac{1}{2}$ and $t_\alpha = 0$ if $\alpha > \frac{1}{2}$. When $\alpha > \frac{1}{2}$, this means that the instabilities of double scattering propagate freely in the random medium with momentum $q_0 + \frac{1}{2}(1+\sigma_1)\eta^\pm$. The instabilities are therefore created by an initial condition as was already observed in [4] for the Itô-Schrödinger regime. When $\alpha \le \frac{1}{2}$, they are generated by a source term as the position is now determined by $\tau q_0 + \frac{1}{2}(1+\sigma_1)(\tau - t)\eta^\pm$. It is interesting to notice that instabilities propagate not only with the initial momentum q_0, but also with a momentum $q_0 + \eta^\pm$, whose distribution admits the following fairly complex expression when $\alpha \le \frac{1}{2}$:

$$\sigma(\eta_\perp) = \frac{\hat{R}^2(\eta^\pm)}{(1-|\eta_\perp|^2)^{\frac{1}{2}}} \int_0^\infty \int_{\mathbb{R}^{d-1}} ds d\xi_\perp \left|\mathcal{F}(W_0)\left(\xi^0, s\eta^\pm\right)\right|^2.$$

The integrated term on the right can be shown to be uniformly bounded with respect to $\eta_\perp$ so that the main characteristics of the distribution are that of

$$\frac{\hat{R}^2(\eta^\pm)}{(1-|\eta_\perp|^2)^{\frac{1}{2}}}.$$

This is in contrast with the dynamics of the limiting Wigner transform, which is known to be the solution of a transport equation with a conservative collision operator, see (1.2) and [5]. This means that if the initial condition for the transport equation has only one frequency content, the same holds for the solution at all times. Statistical instabilities thus propagate with a larger range of frequencies than the average Wigner function, and this property can be used for imaging purposes as explained below. See [7] for an exposition of precursors in a one-dimensional setting. Recall that $|q_0| = 1$. We have $|\eta^\pm| = (2 \pm 2\sqrt{1-|\eta_\perp|^2})^{1/2}$, so that since $|\eta_\perp| \le 1$, we have $|\eta^\pm| \in [0,2]$. Therefore, instabilities propagate with both lower and higher frequencies than $|q_0|$. The distribution will be maximum for momenta k^m with $|k^m_\perp| = 1$. The above cross-section therefore mainly generates momenta with norm $\sqrt{2}|q_0|$. This corresponds to high frequency waves that are not suited for precursors. Nevertheless, the cross-section also creates low frequencies k^l (provided $\hat{R}$ does not vanish around the origin) whose amplitude decays like $r^{-\frac{1}{2}}$ if $|k^l_\perp|^2 = 1 - r$. If the related low-frequency waves can be measured, which could be a difficult experimental task since: (i) the amplitude decreases as the frequency does; and (ii) the amplitude is of order $\varepsilon^{d(1-\alpha)+(2\alpha-1)}$ (when $\delta = 0$ and $\alpha \le \frac{1}{2}$) and therefore small, they can be of interest for imaging purposes as they weakly interact with the random fluctuations of the medium and thus approximately propagate in a homogeneous medium. It is interesting to perform a comparison with the single scattering contribution. According to the results of [2], when $\alpha < 1$, the instabilities only propagate in the direction q_0, so that no low frequencies are created during the propagation and no precursors are generated.

Comments on the one dimensional case. A first observation of theorem 2.2 is that, as for single scattering, double scattering is stable when $\alpha > 0$ in the presence of long-range correlations. There is no contradiction with the well-known localization property of waves propagating in one-dimensional random media, see for instance [6]. Our result shows that scattering events of order at least three are responsible for localization when $\alpha > 0$. The case $\alpha > 0$ corresponds to initial conditions localized in the spatial variables, while $\alpha = 0$ corresponds to unlocalized initial conditions. When $\alpha = 0$, we find that the scintillation is of order one and therefore is compatible with localization. The results seems to indicate that waves need to spread spatially first in order to localize: when $\alpha = 0$, waves have a wide spatial support and statistical instability occurs for the double scattering; when $\alpha > 0$, waves needs to disperse first and then double scattering is stable. Also, double scattering is dominant whenever $\alpha < 1$ and is of same order as single scattering when $\alpha = 1$.

3. **Outline of the proof**

3.1. **Preliminary calculations.** We need to perform additional computations before describing the outline: when $\alpha \in [0, \frac{1}{2}]$, we make in (2.4) the change of variables $s \to \varepsilon^\alpha s$ and $\xi \to \varepsilon^{1-\alpha}\xi$, $\xi_\| \to \varepsilon^\alpha \xi_\|$, where we used the notation (2.6). When $d = 1$, by convention $\xi_\perp \equiv \eta_\perp \equiv 0$. Let $\xi^\varepsilon = (\varepsilon^\alpha \xi_\|, \xi_\perp)$. Still using the notation F^ε for the rescaled version of F^ε, as well as $d\xi = d\xi_\| d\xi_\perp$ and $d\eta = d\eta_\| d\eta_\perp$, and defining

$$\hat{R}_\varepsilon(\xi, \eta) = \hat{R}(\varepsilon^{1-\alpha}\xi - \eta), \qquad \Psi_{\sigma_2}(\eta) = \eta_\| + \frac{\sigma_2}{2}\left(|\eta_\perp|^2 + \eta_\|^2\right) \tag{3.1}$$

$$\begin{aligned} a^\varepsilon(u, v, [z^\varepsilon]) &= \mathcal{F}(W_0(\cdot,\cdot)\psi^\varepsilon(\cdot,\cdot,[z^\varepsilon]))(u, v), &(3.2)\\ F^\varepsilon(\tau, \xi^\varepsilon, \eta) &= \sum_{\sigma_2=\pm 1} \sigma_2 F^\varepsilon_{\sigma_2}(\tau, \xi^\varepsilon, \eta), &(3.3)\end{aligned}$$

with $[z^\varepsilon] = (\tau, t, \varepsilon^\alpha s, \varepsilon^{1-\alpha}\xi^\varepsilon, \eta, \sigma_2)$, and

$$\begin{aligned} F^\varepsilon_{\sigma_2}(\tau, \xi^\varepsilon, \eta) = \int_0^\tau \int_0^{\varepsilon^{-\alpha} t} dt ds \exp\left\{-i\sigma_2 s \xi^\varepsilon \cdot \eta/2\right\} \exp\left\{-it\xi_\|\right\} \exp\left\{\frac{i}{\varepsilon^{1-\alpha}} s \Psi_{\sigma_2}(\eta)\right\}\\ \times a^\varepsilon\left(\xi^\varepsilon, \varepsilon^{1-2\alpha} t \xi^\varepsilon - s\eta, [z^\varepsilon]\right), \qquad (3.4)\end{aligned}$$

we find the expression for the first scintillation w^1_ε

$$w^1_\varepsilon(\tau) = \varepsilon^{d(1-\alpha)+3\alpha-2} \int_{\mathbb{R}^{2d}} \frac{d\xi d\eta}{(2\pi)^{2d}} \hat{R}_\varepsilon(\xi^\varepsilon, \eta)\hat{R}(\eta)|F^\varepsilon(\tau, \xi^\varepsilon, \eta)|^2. \tag{3.5}$$

When $\alpha \in (\frac{1}{2}, 1]$, we make in addition the change of variables $t \to \varepsilon^{2\alpha-1}t$ and $\xi_\parallel \to \varepsilon^{1-2\alpha}\xi_\parallel$. This yields

$$w^1_\varepsilon(\tau) = \varepsilon^{d(1-\alpha)+5\alpha-3} \int_{\mathbb{R}^{2d}} \frac{d\xi d\eta}{(2\pi)^{2d}} \hat{R}_\varepsilon(\xi^\varepsilon, \eta)\hat{R}(\eta)|F^\varepsilon(\tau, \xi^\varepsilon, \eta)|^2, \tag{3.6}$$

with now $\xi^\varepsilon = (\varepsilon^{1-\alpha}\xi_\parallel, \xi_\perp)$ and

$$F^\varepsilon_{\sigma_2}(\tau, \xi^\varepsilon, \eta) = \int_0^{\tau\varepsilon^{1-2\alpha}} \int_0^{\varepsilon^{\alpha-1}t} dtds \exp\left\{-i\sigma_2 s\xi^\varepsilon \cdot \eta/2\right\} \exp\left\{-it\xi_\parallel\right\}$$
$$\times \exp\left\{\frac{i}{\varepsilon^{1-\alpha}} s\Psi_{\sigma_2}(\eta)\right\} a^\varepsilon\left(\xi^\varepsilon, t\xi^\varepsilon - s\eta, [z^\varepsilon]\right), \tag{3.7}$$

$$[z^\varepsilon] = (\tau, \varepsilon^{2\alpha-1}t, \varepsilon^\alpha s, \varepsilon^{1-\alpha}\xi^\varepsilon, \eta, \sigma_2).$$

The second scintillation contribution w^2_ε is discussed below.

3.2. **Outline.** Assume $d \geq 2$. The case $d = 1$ is simpler and treated in section 7. Let us start with a formal analysis for w^ε_1: assume $\alpha \in (0, \frac{1}{2})$ and decompose F^ε as

$$|F^\varepsilon|^2 \quad = \sum_{\sigma_2, \sigma_2' = \pm 1} \sigma_2\sigma_2' F^\varepsilon_{\sigma_2} \overline{F}^\varepsilon_{\sigma_2'}.$$

which leads for w^ε_1 (before the change of variables $\xi_\parallel \to \varepsilon^\alpha\xi_\parallel$) to oscillatory integrals of the form

$$I = \int_{\mathbb{R}^{2d}} \int_0^\tau \int_0^\tau dt_1\, dt_2 d\xi\, d\eta\, \hat{R}_\varepsilon(\xi, \eta)\, \hat{R}(\eta)$$
$$\exp\left\{-\frac{i}{\varepsilon^\alpha}(t_1 - t_2)\xi_\parallel\right\} \mathfrak{F}(t_1, \xi, \eta, \sigma_2)\overline{\mathfrak{F}}(t_2, \xi, \eta, \sigma_2'), \tag{3.8}$$

with

$$\mathfrak{F}(t, \xi, \eta, \sigma_2) \;=\; \int_0^{\varepsilon^{-\alpha}t} ds \exp\left\{-i\sigma_2 s\xi \cdot \eta/2\right\} \exp\left\{\frac{i}{\varepsilon^{1-\alpha}} s\Psi^{\sigma_2}(\eta)\right\} a^\varepsilon\Big(\xi, \varepsilon^{1-2\alpha}t\xi - s\eta\Big).$$

We drop the dependence of a^ε in $[z^\varepsilon]$ to simplify. The product $\mathfrak{F}\overline{\mathfrak{F}}$ can be written as

$$\int_0^{\varepsilon^{-\alpha}t_1} \int_0^{\varepsilon^{-\alpha}t_2} ds_1 ds_2 \exp\left\{-i(\sigma_2 s_1 - s_2\sigma_2')\xi \cdot \eta/2\right\} \exp\left\{\frac{i}{\varepsilon^{1-\alpha}}(s_1\Psi^{\sigma_2} - s_2\Psi^{\sigma_2'})\right\}$$
$$a^\varepsilon\left(\xi, \varepsilon^{1-2\alpha}t_1\xi - s_1\eta\right) \bar{a}^\varepsilon\left(\xi, \varepsilon^{1-2\alpha}t_2\xi - s_2\eta\right).$$

The first exponential term in the integral above plays no role. When $\alpha < 1$, the second oscillatory phase localizes η on the (hyper)surface on which the phase factor $s_1\Psi^{\sigma_2} - s_2\Psi^{\sigma_2'}$ vanishes. The phase is equal to

$$s_1\Psi^{\sigma_2} - s_2\Psi^{\sigma_2'} = (s_1 - s_2)\eta_\parallel + \frac{s_1\sigma_2 - s_2\sigma_2'}{2}\left(|\eta_\perp|^2 + \eta_\parallel^2\right).$$

Assume first $\sigma_2' = \sigma_2$. Then the phase reads

$$s_1\Psi^{\sigma_2} - s_2\Psi^{\sigma_2'} = (s_1 - s_2)\left(\eta_\parallel + \frac{\sigma_2}{2}\left(|\eta_\perp|^2 + \eta_\parallel^2\right)\right),$$

which vanishes on the surfaces S_{σ_2} given by

$$\begin{aligned} S_{\sigma_2} &= \{(\eta_\parallel,\eta_\perp)\in\mathbb{R}\times\mathbb{R}^{d-1},\quad 2\eta_\parallel+\sigma_2|\eta_\perp|^2+\sigma_2\eta_\parallel^2=0\},\\ &= \Big\{(\eta_\parallel,\eta_\perp)\in\mathbb{R}\times\mathbb{R}^{d-1},\quad |\eta_\perp|\le 1,\quad \eta_\parallel=-\sigma_2\pm\sqrt{1-|\eta_\perp|^2}\Big\}. \end{aligned}$$

Setting $s_1 = s_2 + \varepsilon^{1-\alpha}s_1$ then yields formally:

$$\begin{aligned} &\mathfrak{F}(t_1,\xi,\eta,\sigma_2)\overline{\mathfrak{F}}(t_2\xi,\eta,\sigma_2)\sim\\ &\qquad \varepsilon^{1-\alpha}\delta_{S_{\sigma_2}}(\eta)\int_0^{\varepsilon^{-\alpha}t_2} ds_2\, a^\varepsilon\left(\xi,\varepsilon^{1-2\alpha}t_1\xi-s_2\eta\right)\bar{a}^\varepsilon\left(\xi,\varepsilon^{1-2\alpha}t_2\xi-s_2\eta\right), \end{aligned}$$

where $\delta_{S_{\sigma_2}}$ denotes here the Dirac measure on the surface S_{σ_2}. When $\sigma_2'=-\sigma_2$, we have

$$\begin{aligned} (s_2+\varepsilon^{1-\alpha}s_1)\Psi^{\sigma_2}-s_2\Psi^{-\sigma_2} &= \varepsilon^{1-\alpha}s_1\eta_\parallel+\sigma_2\frac{2s_2+\varepsilon^{1-\alpha}s_1}{2}\left(|\eta_\perp|^2+\eta_\parallel^2\right),\\ &\sim \sigma_2 s_2\left(|\eta_\perp|^2+\eta_\parallel^2\right), \end{aligned}$$

which therefore does not vanish except at the origin. Hence, it is expected that the contribution of the terms corresponding to $\sigma_2'=-\sigma_2$ will be negligible compared to that of $\sigma_2'=\sigma_2$ since it oscillates like $\exp\{\sigma_2\varepsilon^{\alpha-1}s_2(|\eta_\perp|^2+\eta_\parallel^2)\}$. When $\alpha=1$, the situation is different since the integral in s_1 no longer displays fast oscillations and all terms are of the same order whether $\sigma_2'=\sigma_2$ or not. Let us now go back to (3.8) and study integration with respect to t. When $\alpha>0$, the oscillatory integral localizes $\xi_\parallel$ around zero, so that, after the change of variables $t_1=t_2+\varepsilon^\alpha t_1$, we find formally

$$\begin{aligned} I &\sim \varepsilon\int_{\mathbb{R}^{2d}} d\xi\, d\eta\, \hat{R}_\varepsilon(\xi,\eta))\,\hat{R}(\eta)\delta_{S_{\sigma_2}}(\eta)\delta_0(\xi_\parallel)\int_0^\tau\int_0^\infty dt_2 ds_2\left|a^\varepsilon\left(\xi,\varepsilon^{1-2\alpha}t_2\xi-s_2\eta\right)\right|^2,\\ &\sim \varepsilon\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}^d} d\xi_\perp\, d\eta\,\hat{R}^2(\eta)\delta_{S_{\sigma_2}}(\eta)\int_0^\tau\int_0^\infty dt_2ds_2\left|a^\varepsilon\left(\xi^0,\varepsilon^{1-2\alpha}t_2\xi^0-s_2\eta\right)\right|^2, \end{aligned} \tag{3.9}$$

where following the notation (2.6), $\xi^0=(0,\xi_\perp)$. A close look at the surface S_{σ_2} shows that they include the origin. This is problematic since $\hat{R}^2(\eta)$ is singular near zero when $\delta>0$ and behaves like $|\eta|^{-2\delta}$. Even if $a^\varepsilon\sim|\eta|$ around the origin when $\alpha\in(0,1)$, as can be seen from (3.2) and the definition of ψ^ε, the singularity is not integrable when δ becomes too large (but less than d according to our assumptions). One has therefore to be careful when the correlations are very long distance to justify the formal computations. A possibility is to precisely control the rate at which η gets closer to the origin: the term $\hat{R}_\varepsilon(\xi,\eta)\,\hat{R}(\eta)$ behaves like $|\varepsilon^{1-\alpha}\xi-\eta|^{-\delta}|\eta|^{-\delta}$ which generates three natural scales:

- $|\eta|\gg|\varepsilon^{1-\alpha}\xi^\varepsilon|$, so that $|\varepsilon^{1-\alpha}\xi^\varepsilon-\eta|^{-\delta}|\eta|^{-\delta}\sim|\eta|^{-2\delta}$ and we will prevent in this case η from approaching the origin because of the singularity,
- when $|\eta|\sim|\varepsilon^{1-\alpha}\xi^\varepsilon|$, $|\varepsilon^{1-\alpha}\xi-\eta|^{-\delta}|\eta|^{-\delta}$ is integrable since $\delta<d$,
- and finally when $|\eta|\ll|\varepsilon^{1-\alpha}\xi|$, so that $|\varepsilon^{1-\alpha}\xi-\eta|^{-\delta}|\eta|^{-\delta}\sim|\varepsilon^{1-\alpha}\xi|^{-\delta}|\eta|^{-\delta}$ which is also integrable.

The last scale allows us to obtain bounds from below that show that scintillation grows in some cases as the correlation distance gets longer. The latter analysis leads us to decompose the domain of integration in η over $\mathbb{R}^d$, as follows: let B_a be the closed ball of $\mathbb{R}^d$ centered at the origin of radius a and C_a^b be the corona of radii a and b with $b > a$ also centered at the origin. Let

$$\begin{aligned} D^0_\pm &= \left\{(\eta_\|, \eta_\perp) \in \mathbb{R} \times \mathbb{R}^{d-1}, \quad |\eta_\perp| \leq 1, \quad 0 \leq \pm\eta_\| \leq 1\right\}, \\ D^1_\pm &= \left\{(\eta_\|, \eta_\perp) \in \mathbb{R} \times \mathbb{R}^{d-1}, \quad |\eta_\perp| \leq 1, \quad 1 \leq \pm\eta_\|\right\}, \\ C_\varepsilon^\pm &= C_{\varepsilon^{\gamma_2}}^{\varepsilon^{\gamma_1}} \cap \{\pm\eta_\| \geq 0\}, \\ \mathcal{D}^0_\pm &= D^0_\pm \backslash \left\{D^0_\pm \cap B_{\varepsilon^{\gamma_1}}\right\}, \\ \mathcal{C} &= \mathbb{R}^d \backslash \left\{D^0_+ \cup D^0_- \cup D^1_+ \cup D^1_-\right\}. \end{aligned}$$

The latter decomposition is depicted in figure 1 when $d = 2$. A fair amount of calculations is redundant and we will focus our analysis only on the subdomains $\mathcal{D}^0_+$, C_ε^+ and $B_{\varepsilon^{\gamma_2}}$ that contain all the relevant difficulties. The techniques used to treat these domains can then easily be transported to the other subdomains. When $\alpha < 1$, following the latter discussion, the contribution of the domain $\mathcal{C}$ can be shown to be asymptotically negligible since it does not include the surfaces S_1 and S_{-1}. It will therefore not be treated in detail. The contributions of D^1_+ and D^1_- will not be discussed further either as they are simpler to analyze than those of D^0_+ and D^0_- since the domains D^1_+ and D^1_- do not include the origin. The domain $D^0_- \cup D^1_-$ is similar to $D^0_+ \cup D^1_+$ and symmetry arguments show they yield the same limit. The related results will thus be given without proof. We will focus our attention on the case $\alpha \leq \frac{1}{2}$. The rigorous treatment is fairly lengthy and technical. When $\alpha \leq \frac{1}{2}$, the proofs can be performed in a relatively simple systematic manner and we will present them in details. When $\alpha > \frac{1}{2}$, the technicalities are heavier due to the extra infinite domain integration with respect to t_2 (τ has to be replaced by ∞ in (3.9)) and do not bring much novelty compared to the case $\alpha \leq \frac{1}{2}$. For the sake of conciseness, we decided therefore to remain at a less formal level when $\alpha > \frac{1}{2}$.

We set in the sequel $\gamma_1 = \frac{1-\alpha}{3}$, $\gamma_2 = 1 - \frac{\alpha}{2}$ and decompose w^1_ε according to the various subdomains (and omit the dependence in ε for simplicity):

$$w^1_\varepsilon(\tau) = \sum_{D_i \in \mathcal{D}} w_i(\tau), \; \mathcal{D} = \left\{\mathcal{D}^0_+, C_\varepsilon^+, B_{\varepsilon^{\gamma_2}}, \mathcal{D}^0_-, C_\varepsilon^-, D^1_+, D^1_-, \mathcal{C}\right\}, \; w_i(\tau) = \int_{D_i} (\cdots) d\eta.$$

The value of γ_2 is the scale that allows us to capture the corrector (the exact limiting term) around the origin, while γ_1 defines a (non-optimal) scale at which the long-range correlations have a weaker effect than around the origin. As annouced earlier, we will focus only on the terms w_1, w_2 and w_3. Our main technique to prove theorems 2.1 and 2.2 is a careful estimation of the dependence of the function F^ε on ξ and η. This amounts to analyzing the different oscillatory integrals involved in the definition of F^ε so as to obtain optimal estimates. Part of this task is carried out in the appendix in Lemma 8.2 where we study parametrized oscillatory integrals of the form

$$\int_0^\tau \int_0^{\varepsilon^{-a}t} dtds \exp\{-isA\} \exp\{-itB\} \exp\{is\Psi\} f(t,s),$$

and obtain accurate estimates of their behavior as $|B|$ and $|\Psi|$ become large. In sections 4.1, 4.2, 4.3, we show the terms w_1, w_2 and w_3 tend to zero for $\alpha \leq \frac{1}{2}$ and $d \geq 2$ while the optimal estimates for $\delta = 0$ are obtained in section 5. The case $\alpha > \frac{1}{2}$ is adressed in section 6 and the one-dimensional setting in section 7.

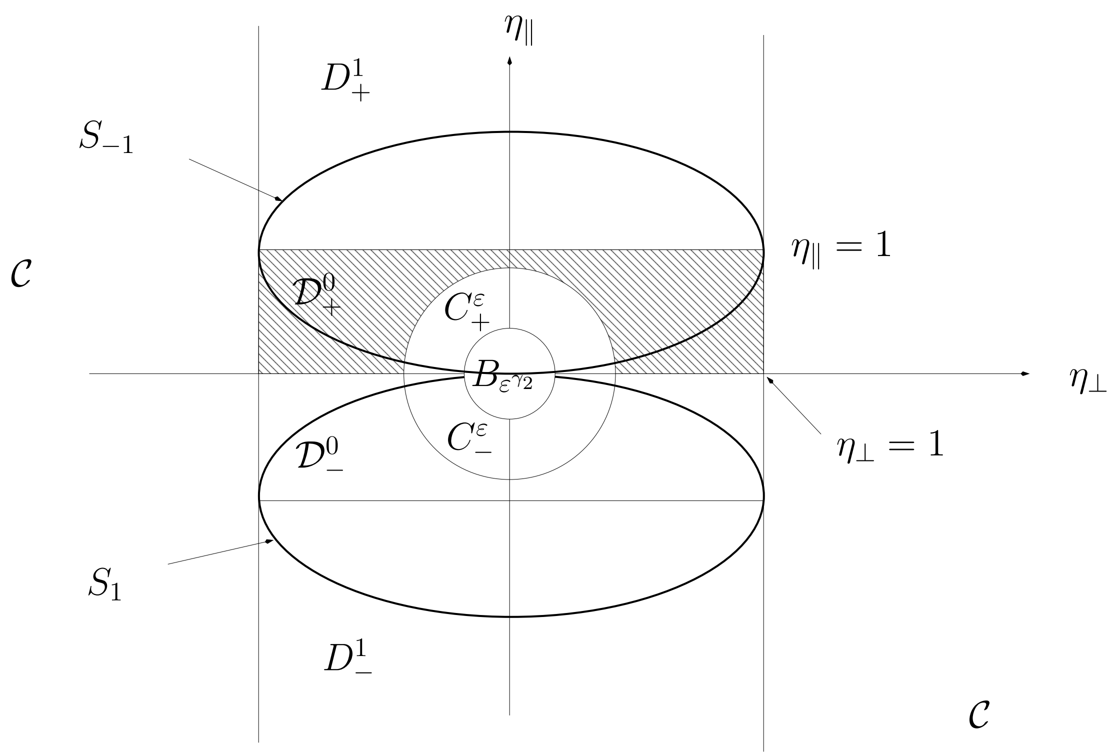

Figure 1. Decomposition of the integration domain in η when $d = 2$. The subdomain $\mathcal{D}^0_+$ corresponds to the shaded zone.

To conclude this outline, recall that the total scintillation function w_ε is the sum of w^1_ε and w^2_ε; see (2.3). Starting from (2.5), the expression of w^2_ε may be recast as

$$w^2_\varepsilon(\tau) = \frac{1}{\varepsilon^2 \pi^{2d}} \int_{\mathbb{R}^{2d}} d\xi d\eta \hat{R}(\xi - \eta)\hat{R}(\eta) F^\varepsilon(\tau,\xi,\eta) G^\varepsilon(\tau,\xi,\eta),$$

where

$$G^\varepsilon(\tau,\xi,\eta) =$$
$$\sum_{\sigma_1,\sigma_2=\pm1}\sigma_1\sigma_2\int_0^\tau\int_0^t dtds\exp\left\{-\frac{i}{\varepsilon}\left[\frac{1}{2}\sigma_2 s(\xi-\eta)\cdot\eta - q_0\cdot((t-s)\xi+s\eta)\right]\right\}$$
$$\int_{\mathbb{R}^{2d}}dxdp\,\exp\left\{\frac{i}{\varepsilon^{1-\alpha}}x\cdot\xi\right\}\exp\left\{\frac{i}{\varepsilon^{\alpha}}p\cdot((t-s)\xi+s\eta)\right\}W_0(x,p)\tilde{\psi}^\varepsilon(x,p,[z]),$$
$$\tilde{\psi}^\varepsilon(x,p,[z]) = \sum_{\sigma_1=\pm1}\sigma_1\varphi\left(\varepsilon^\alpha x - \frac{1}{2}(\tau-t)\sigma_1\eta + \tau(q_0+\varepsilon^{1-\alpha}p) - \frac{1}{2}\sigma_2(\tau-t+s)(\xi-\eta),\right.$$
$$\left. q_0+\varepsilon^{1-\alpha}p - \frac{1}{2}\sigma_1\eta - \frac{1}{2}\sigma_2(\xi-\eta)\right),$$

and $[z]=(\tau,t,s,\xi,\eta,\sigma_2)$. After the change of variables $s\to\varepsilon^\alpha s$, G^ε reads (dropping the multiplicative ε factors):

$$G^\varepsilon(\tau,\xi,\eta) := \sum_{\sigma_2=\pm1}\sigma_2 G^\varepsilon_{\sigma_2}(\tau,\xi,\eta),$$
$$G^\varepsilon_{\sigma_2}(\tau,\xi,\eta) = \int_0^\tau\int_0^{\varepsilon^{-\alpha}t}dtds\exp\left\{-i\sigma_2 s\xi\cdot\eta/2\right\}\exp\left\{\frac{i}{\varepsilon^\alpha}(t-\varepsilon^\alpha s)\xi_\parallel\right\}$$
$$\times\exp\left\{\frac{i}{\varepsilon^{1-\alpha}}s\Psi_{\sigma_2}(\eta)\right\}a^\varepsilon\left(-\xi,-\varepsilon^{1-2\alpha}(t-\varepsilon^\alpha s)\xi - s\eta,[z^\varepsilon]\right).$$

A simple inspection then shows that the expression of w^2_ε is extremely close to that of w^1_ε (actually equal up to high order terms in ε after appropriate rescaling): the variable η is also localized on the surfaces S_1 and S_{-1} in the limit as well as $\xi_\parallel$ at the origin. The case $\alpha=0$ seems to yield a different result than w^1_ε because of the extra $\varepsilon^\alpha s$ factors. They actually have no influence at the limit as explained further in the following sections. Hence, all the methods used for w^1_ε can be applied to w^2_ε with very few modifications and show that w^1_ε and w^2_ε share the same limit. We will therefore not analyze w^2_ε in detail and focus mainly on w^1_ε.

4. The Case $\alpha\in[0,\frac{1}{2}]$

4.1. **Contribution of $D_1 := \mathcal{D}^0_+$.** Our starting point is (3.5) with the related definitions. Our goal is to estimate F^ε and to prove that w_1 tends to zero. For this, we first need an estimate for the function a^ε: it stems from applying Lemma 8.1 of the appendix with $\gamma=1-2\alpha$, $r=r'=0$, $h=\alpha$ and we find $\forall n\geq 0$, $k,l=0,1$, pointwise in all variables:

$$\begin{aligned}&|\partial_t^k\partial_s^l a^\varepsilon\left(\xi^\varepsilon,\varepsilon^{1-2\alpha}t\xi^\varepsilon - s\eta, z^\varepsilon\right)|\\ &\qquad\leq C_n\frac{(|\varepsilon^{1-\alpha}\xi^\varepsilon-\eta|^k+|\eta|^k)|\varepsilon^\alpha\eta|^l + |\varepsilon^{1-2\alpha}\xi^\varepsilon|^k|\eta|^l}{(1+|\xi^\varepsilon|^2+|\varepsilon^{1-2\alpha}t\xi^\varepsilon - s\eta|^2)^n}|\varepsilon^{1-\alpha}\xi^\varepsilon-\eta|. \qquad (4.1)\end{aligned}$$

As announced in the outline, we decompose F^ε as

$$|F^\varepsilon|^2 \;=\; |F^\varepsilon_{-1}|^2 + R^\varepsilon, \qquad R^\varepsilon \;=\; |F^\varepsilon_1|^2 - \sum_{\sigma_2=\pm 1} F^\varepsilon_{\sigma_2}\overline{F}^\varepsilon_{-\sigma_2}, \tag{4.2}$$

and split $w_1 := w_1^L + w_1^N$ accordingly as leading and negligible parts. We treat these two terms separately. The first step of the analysis is to control F^ε using (4.1). Recall for this that the characteristic surface in $\mathcal{D}^0_+$ is given by

$$\left\{(\eta_\|, \eta_\perp) \in \mathbb{R}\times\mathbb{R}^{d-1}, \quad |\eta_\perp| \le 1, \quad \eta_\| = \eta^\star(\eta_\perp) := 1 - \sqrt{1-|\eta_\perp|^2}\right\}.$$

We then perform the change of variables $\eta_\| \to \eta^\star(\eta_\perp) + \varepsilon^{1-\alpha}\eta_\|$ and as in (2.6), introduce the notation $\eta^\varepsilon = (\eta^\star(\eta_\perp) + \varepsilon^{1-\alpha}\eta_\|, \eta_\perp)$. The phase Ψ_{-1} reads after this transformation

$$\Psi_{-1}(\eta^\varepsilon) = \varepsilon^{1-\alpha}\eta_\| \left(1 - \eta^\star(\eta_\perp) - \frac{1}{2}\varepsilon^{1-\alpha}\eta_\|\right).$$

In order to control F^ε_{-1}, we need to bound Ψ_{-1} from below. This stems from the following geometrical constraints in the domain $\mathcal{D}^0_+$:

$$\varepsilon^{\frac{2}{3}(1-\alpha)} \le |\eta^\varepsilon|^2 = |\eta^\star(\eta_\perp) + \eta^{1-\alpha}\eta_\||^2 + |\eta_\perp|^2, \qquad 0 \le \eta^\star(\eta_\perp) + \eta^{1-\alpha}\eta_\| \le 1, \tag{4.3}$$

along with $|\eta_\perp| \le 1$. These relations yield

$$1 - \frac{1}{2}\eta^\star(\eta_\perp) - \frac{1}{2}(\eta^\star(\eta_\perp) + \varepsilon^{1-\alpha}\eta_\|) \ge \frac{1}{2}\left(1 - \eta^\star(\eta_\perp)\right) = \frac{1}{2}\sqrt{1-|\eta_\perp|^2},$$

so that

$$\begin{aligned} |\Psi_{-1}| \;&=\; \left|\varepsilon^{1-\alpha}\eta_\| \left(1 - \frac{1}{2}\eta^\star(\eta_\perp) - \frac{1}{2}(\eta^\star(\eta_\perp) + \varepsilon^{1-\alpha}\eta_\|)\right)\right|, \\ &\ge\; \frac{1}{2}\varepsilon^{1-\alpha}|\eta_\||\sqrt{1-|\eta_\perp|^2}. \end{aligned} \tag{4.4}$$

Using (4.1), we apply Lemma 8.2 of the appendix with $a = h = \alpha$, $\gamma' = 1-2\alpha$, $A = \xi^\varepsilon \cdot \eta^\varepsilon$, $B = \xi_\|$, $\Psi = \Psi_{-1}\varepsilon^{\alpha-1}$, and $r = 0$. Depending on the values of $\xi_\|$ and $\eta_\|$, we use four different estimates. Lemma 8.2 gives, pointwise in $\eta^\varepsilon \in \mathcal{D}^0_+$ and $\xi^\varepsilon \in \mathbb{R}^d$, $\forall n \ge 0$:

$$|F^\varepsilon_{-1}(\tau, \xi^\varepsilon, \eta^\varepsilon)| \le C_n \langle\xi^\varepsilon\rangle^{-n}\, \Psi^1 \wedge \Psi^2 \wedge \Psi^3 \wedge \Psi^4,$$

$$\begin{aligned} &\Psi^1 = 1, \qquad\qquad \Psi^2 = |\eta_\||^{-1}(1-|\eta_\perp|^2)^{-\frac{1}{2}}(1 + |\xi^\varepsilon\cdot\eta^\varepsilon| + |\eta^\varepsilon|), \\ &\Psi^3 = |\xi_\||^{-1}(1 + |\xi^\varepsilon| + |\eta^\varepsilon|), \\ &\Psi^4 = |\xi_\||^{-1}|\eta_\||^{-1}(1-|\eta_\perp|^2)^{-\frac{1}{2}}(1 + |\xi^\varepsilon\cdot\eta^\varepsilon|^2 + |\eta^\varepsilon|^2 + |\xi^\varepsilon|^2 + \varepsilon^\alpha|\xi_\||). \end{aligned}$$

Using the fact that $|\eta^\varepsilon| \le 2$ in the domain $\mathcal{D}^0_+$, together with, for $n \ge 2$,

$$\langle\xi^\varepsilon\rangle^{-n}(|\xi^\varepsilon|^2 + \varepsilon^\alpha|\xi_\||) \le \langle\xi^\varepsilon\rangle^{-(n-2)},$$

we arrive at

$$|F^\varepsilon_{-1}(\tau, \xi^\varepsilon, \eta^\varepsilon)| \le C\langle\xi_\perp\rangle^{-n}\, \Psi^1 \wedge \Psi^2 \wedge \Psi^3 \wedge \Psi^4, \tag{4.5}$$

$$\Psi^1 = 1, \quad \Psi^2 = |\eta_\||^{-1}(1-|\eta_\perp|^2)^{-\frac{1}{2}}, \quad \Psi^3 = |\xi_\||^{-1}, \quad \Psi^4 = |\xi_\||^{-1}|\eta_\||^{-1}(1-|\eta_\perp|^2)^{-\frac{1}{2}}.$$

Defining the functions f and g by

$$f(\eta_\parallel,\eta_\perp) = 1 \wedge \Big(|\eta_\parallel|^{-3/2}(1-|\eta_\perp|^2)^{-\frac{3}{4}}\Big), \qquad g(\xi_\parallel) = 1\wedge |\xi_\parallel|^{-3/2},$$

with by construction $g\in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$, $f\in L^1(\mathbb{R}\times B_1)\cap L^\infty(\mathbb{R}\times B_1)$, B_1 being the $d-1$ dimensional unit ball, we can thus control $|F^\varepsilon_{-1}|^2$ by

$$|F^\varepsilon_{-1}|^2 \le C\langle\xi^\varepsilon\rangle^{-n} g(\xi_\parallel) f(\eta_\parallel,\eta_\perp). \tag{4.6}$$

This provides us with the desired bound. In order to estimate w_1, it remains to treat the integration with respect to ξ and η and thus the multiplicative (and singular when $\delta>0$) terms $\hat R(\eta^\varepsilon)$ and $\hat R_\varepsilon(\xi^\varepsilon,\eta^\varepsilon) = \hat R(\varepsilon^{1-\alpha}\xi^\varepsilon-\eta^\varepsilon)$ in (3.5). For the first term, we have using (4.3), so that $|\eta^\varepsilon|\ge \varepsilon^{\frac{1-\alpha}{3}}$ when $\eta^\varepsilon\in\mathcal{D}^0_+$ (recall the S below was defined in (1.3)):

$$\mathbb{1}_{\mathcal{D}^0_+}(\eta^\varepsilon)S(\eta^\varepsilon)|\eta^\varepsilon|^{-\delta} \le \|S\|_{L^\infty}\left(\mathbb{1}_{|\eta_\perp|\ge\varepsilon^{\frac{1-\alpha}{3}}}(\eta_\perp)|\eta_\perp|^{-\delta} + \mathbb{1}_{|\eta_\perp|\le\varepsilon^{\frac{1-\alpha}{3}}}(\eta_\perp)\varepsilon^{-\frac{1-\alpha}{3}\delta}\right). \tag{4.7}$$

For the second term $\hat R_\varepsilon(\xi^\varepsilon,\eta^\varepsilon)$, assume first that $|\xi^\varepsilon|\le\varepsilon^{-\gamma_0}$, with $0<\gamma_0<\frac{2(1-\alpha)}{3}$. Since $|\eta^\varepsilon|\ge\varepsilon^{\frac{1-\alpha}{3}}$ according to (4.3), there exists, for $\varepsilon\le\varepsilon_0$ small enough, a constant C_{ε_0} such that $|\varepsilon^{1-\alpha}\xi^\varepsilon-\eta^\varepsilon|\ge C_{\varepsilon_0}\varepsilon^{\frac{1-\alpha}{3}}$, and therefore, for $\eta^\varepsilon\in\mathcal{D}^0_+$:

$$\hat R(\varepsilon^{1-\alpha}\xi^\varepsilon-\eta^\varepsilon)\le C'_{\varepsilon_0}\|S\|_{L^\infty}\,\varepsilon^{-\frac{\delta(1-\alpha)}{3}}, \qquad \text{when } |\xi^\varepsilon|\le\varepsilon^{-\gamma_0}. \tag{4.8}$$

The contribution of the set $\{|\xi^\varepsilon|>\varepsilon^{-\gamma_0}\}$ is of higher order thanks to the arbitrary decay of $\langle\xi^\varepsilon\rangle^{-n}$. More precisely, we have, introducing $D_{1,\varepsilon}=\{\eta\in\mathbb{R}^d, \eta^\varepsilon\in D_1\}$:

$$\begin{aligned} w_1^L(\tau) &= \varepsilon^{d(1-\alpha)+2\alpha-1}\int_{\mathbb{R}^d}\int_{\mathcal{D}_{1,\varepsilon}}\frac{d\xi d\eta}{(2\pi)^{2d}}\hat R_\varepsilon(\xi^\varepsilon,\eta^\varepsilon)\hat R(\eta^\varepsilon)|F^\varepsilon_{-1}(\tau,\xi^\varepsilon,\eta^\varepsilon)|^2,\\ &= \int_{|\xi^\varepsilon|\le\varepsilon^{-\gamma_0}}(\cdots)d\xi + \int_{|\xi^\varepsilon|>\varepsilon^{-\gamma_0}}(\cdots)d\xi \;:=\; T_1+T_2. \end{aligned}$$

Using (4.8) and (4.6), T_1 is controlled by

$$\begin{aligned} T_1 &\le C\varepsilon^{d(1-\alpha)+2\alpha-1-\frac{\delta(1-\alpha)}{3}}\|S\|^2_{L^\infty}\int_{|\xi^\varepsilon|\le\varepsilon^{-\gamma_0}}\int_{\mathcal{D}_{1,\varepsilon}} d\xi d\eta\,\langle\xi_\perp\rangle^{-n}\,|\eta^\varepsilon|^{-\delta}g(\xi_\parallel)f(\eta_\parallel,\eta_\perp),\\ &\le C\varepsilon^{d(1-\alpha)+2\alpha-1-\frac{\delta(1-\alpha)}{3}}\|S\|^2_{L^\infty}\int_{\mathcal{D}_{1,\varepsilon}} d\eta\,|\eta^\varepsilon|^{-\delta}f(\eta_\parallel,\eta_\perp), \end{aligned}$$

since $g\in L^1\cap L^\infty$. Moreover, thanks to (4.7),

$$\begin{aligned} C^{-1}\int_{\mathcal{D}_{1,\varepsilon}} d\eta\,|\eta^\varepsilon|^{-\delta}f(\eta_\parallel,\eta_\perp) \;\le\; &\int_{\mathbb{R}}\int_{\varepsilon^{\frac{1-\alpha}{3}}\le|\eta_\perp|\le1} d\eta_\parallel d\eta_\perp|\eta_\perp|^{-\delta}f(\eta_\parallel,\eta_\perp)\\ &+\varepsilon^{-\frac{\delta(1-\alpha)}{3}}\int_{\mathbb{R}}\int_{|\eta_\perp|\le\varepsilon^{\frac{1-\alpha}{3}}} d\eta_\parallel d\eta_\perp f(\eta_\parallel,\eta_\perp). \end{aligned}$$

Treating the domains $|\eta_\parallel|\le1$ and $|\eta_\parallel|>1$ separately, and using that when $|\eta_\perp|\le\frac12$,

$$(1-|\eta_\perp|^2)^{-\frac{3}{4}}\le 2^{3/4},$$

we find

$$
\begin{aligned}
C^{-1}\int_{\mathcal{D}_{1,\varepsilon}} d\eta\,|\eta^\varepsilon|^{-\delta} f(\eta_\|,\eta_\perp) &\le \int_{\varepsilon^{\frac{1-\alpha}{3}}}^{1/2} r^{d-2-\delta} + \varepsilon^{(d-1-\delta)(1-\alpha)/3} + 1,\\
&= \mathcal{O}(1+\varepsilon^{(d-1-\delta)(1-\alpha)/3}), \qquad (4.9)
\end{aligned}
$$

so that

$$
T_1 \le C\varepsilon^{d(1-\alpha)+2\alpha-1-\frac{\delta(1-\alpha)}{3}}(1+\varepsilon^{(d-1-\delta)(1-\alpha)/3}). \qquad (4.10)
$$

Regarding T_2, we denote by B the ball centered at $\varepsilon^{\alpha-1}\eta^\varepsilon$ of radius one and by B^c its complementary in $\mathbb{R}^d$. We have

$$
\begin{aligned}
T_2 &\le C\varepsilon^{d(1-\alpha)+2\alpha-1}\|S\|_{L^\infty}\int_{|\xi^\varepsilon|>\varepsilon^{-\gamma_0}}\int_{\mathcal{D}_{1,\varepsilon}} \frac{d\xi d\eta\,\hat{R}(\eta^\varepsilon)}{|\varepsilon^{1-\alpha}\xi^\varepsilon-\eta^\varepsilon|^\delta}\langle\xi^\varepsilon\rangle^{-n} g(\xi_\|) f(\eta_\|,\eta_\perp),\\
&:= C\varepsilon^{d(1-\alpha)+2\alpha-1}\|S\|_{L^\infty}\left[\int_{\mathcal{D}_{1,\varepsilon}}\int_{\{|\xi^\varepsilon|>\varepsilon^{-\gamma_0}\}\cap B} + \int_{\mathcal{D}_{1,\varepsilon}}\int_{\{|\xi^\varepsilon|>\varepsilon^{-\gamma_0}\}\cap B^c}\right].
\end{aligned}
$$

In the first term, we perform the change of variable $\xi_\| \to \varepsilon^{-\alpha}\xi_\|$, which yields an integration of ξ on the domain $\{|\xi|>\varepsilon^{-\gamma_0}\}\cap B$ and a loss of a factor $\varepsilon^{-\alpha}$. Since $\delta<d$, the function $|\xi-\varepsilon^{\alpha-1}\eta^\varepsilon|^{-\delta}$ is integrable on B. This implies that the first term is bounded by, $\forall n\ge 0$:

$$
C\varepsilon^{(d-\delta)(1-\alpha)+2\alpha-1-\alpha+\gamma_0 n}\int_{\mathcal{D}_{1,\varepsilon}} d\eta\hat{R}(\eta^\varepsilon) f(\eta_\|,\eta_\perp),
$$

which, thanks to (4.9) is controlled by

$$
h_\varepsilon = C\varepsilon^{(d-\delta)(1-\alpha)+2\alpha-1-\alpha+\gamma_0 n}(1+\varepsilon^{(d-1-\delta)(1-\alpha)/3}).
$$

The second term controlling T_2 is also readily bounded by h_ε. It finally suffices to choose n large enough so that

$$
h_\varepsilon \ll \varepsilon^{d(1-\alpha)+2\alpha-1-\frac{\delta(1-\alpha)}{3}}(1+\varepsilon^{(d-1-\delta)(1-\alpha)/3})
$$

to obtain that T_2 is higher order than T_1. The main result of this section is therefore, that, pointwise in τ:

$$
w_1^L(\tau) = \mathcal{O}\left(\varepsilon^{d(1-\alpha)+2\alpha-1-\frac{\delta(1-\alpha)}{3}}(1+\varepsilon^{(d-1-\delta)(1-\alpha)/3})\right). \qquad (4.11)
$$

A close inspection then shows that $\forall\delta\in(0,d)$, $\forall\alpha\in[0,\frac{1}{2}]$, for $d\ge 2$, we have, pointwise in τ:

$$
w_1^L(\tau)\to 0 \qquad \text{as} \qquad \varepsilon\to 0.
$$

Regarding the remainder R^ε defined in (4.2), there are two contributions. They both involve the phase factor Ψ_1, that reads

$$
\begin{aligned}
\Psi_1(\eta^\varepsilon) &= 2\eta^\star(\eta_\perp) + \varepsilon^{1-\alpha}\eta_\|\left(1+\eta^\star(\eta_\perp)+\frac{1}{2}\varepsilon^{1-\alpha}\eta_\|\right),\\
&\ge \eta^\star(\eta_\perp)+\eta^\star(\eta_\perp)+\varepsilon^{1-\alpha}\eta_\|+\varepsilon^{1-\alpha}\eta_\|\eta^\star(\eta_\perp),\\
&\ge \eta^\star(\eta_\perp)(1+\varepsilon^{1-\alpha}\eta_\|)\ge\eta^\star(\eta_\perp)\sqrt{1-|\eta_\perp|^2},
\end{aligned}
$$

since

$$1-\sqrt{1-|\eta_\perp|^2}+\varepsilon^{1-\alpha}\eta_\parallel=\eta^\star(\eta_\perp)+\varepsilon^{1-\alpha}\eta_\parallel\geq 0$$

according to (4.3). Contrary to Ψ_{-1}, Ψ_1 is bounded from below independently of ε, so that the oscillatory term $\exp\{is\varepsilon^{\alpha-1}\Psi_1\}$ provides an additional averaging effect that renders R^ε of higher order. Proceeding as for F^ε_{-1} and using Lemma 8.2, we find the estimate

$$|F_1^\varepsilon|\leq C\langle\xi_\perp\rangle^{-n}\,\Psi^1\wedge\Psi^2\wedge\Psi^3\wedge\Psi^4,$$

with

$$\Psi^1=1,\quad \Psi^2=\varepsilon^{1-\alpha}|\eta^\star|^{-1}(1-|\eta_\perp|^2)^{-\frac12},$$
$$\Psi^3=|\xi_\parallel|^{-1},\quad \Psi^4=\varepsilon^{1-\alpha}|\xi_\parallel|^{-1}|\eta^\star|^{-1}(1-|\eta_\perp|^2)^{-\frac12}.$$

Therefore, using (4.5), we have

$$C^{-1}|F^\varepsilon_{-1}F^\varepsilon_1|\leq \qquad (4.12)$$
$$\langle\xi^\varepsilon\rangle^{-n}g(\xi_\parallel)\left(\varepsilon^{(1-\alpha)}|\eta^\star|(1-|\eta_\perp|^2)^{-\frac12}\right)\wedge\left(\varepsilon^{(1-\alpha)/2}|\eta^\star|^{-1/2}|\eta_\parallel|^{-3/2}(1-|\eta_\perp|^2)^{-\frac34}\right)$$

where g is as in (4.6). We then proceed as for $|F^\varepsilon_{-1}|$ and underline here only the main differences. Consider first the contribution for $|\eta_\parallel|\leq 1$. We control $|F^\varepsilon_{-1}F^\varepsilon_1|$ using the first term on the right hand side of (4.12). Since the term $\eta^\star$ behaves likes $|\eta_\perp|^2$ near the origin, $r^{d-2-\delta}$ in (4.9) has to be replaced by $r^{d-4-\delta}$ and we lose a factor $\varepsilon^{\frac{2(1-\alpha)}{3}}$ compared to w_1^L. This is compensated by the multiplicative $\varepsilon^{1-\alpha}$ and yields an overall gain of a factor $\varepsilon^{\frac{(1-\alpha)}{3}}$ compared to w_1^L. When $\varepsilon^{\alpha-1}\geq|\eta_\parallel|\geq 1$, we use the second term of (4.12). The term $|\eta^\star|^{-1/2}$ creating a loss of $\varepsilon^{\frac{(1-\alpha)}{3}}$, the overall gain is a factor $\varepsilon^{\frac{(1-\alpha)}{6}}$ compared to w_1^L. We proceed exactly the same way for $|F_1^\varepsilon|^2$ and obtain finally that w_1^N is negligible compared to w_1^L when $\varepsilon\to 0$.

As claimed in the outline, the results of this section can be directly generalized to the domain $\mathcal{D}^0_-$ and to w^2_ε. This ends this section about the domain D_1.

4.2. **Contribution of $D_2:=C_1^+$.** The method is the same as the one for the domain D_1: we first find an estimate for F^ε and then show that w_2 goes to zero. We have the following geometrical constraint in the domain D_2:

$$\varepsilon^{2(1-\frac{\alpha}{2})}\leq|\eta|^2\leq\varepsilon^{\frac{2(1-\alpha)}{3}},\qquad 0\leq\eta_\parallel. \qquad (4.13)$$

Starting from (3.5), we perform the change of variables $\eta\to\eta\varepsilon^{(1-\alpha/2)}$, $s\to s^{-\alpha}s$. This yields:

$$w_2(\tau)=\varepsilon^{d(1-\alpha)+d(1-\frac{\alpha}{2})+\alpha-2}\int_{\mathbb{R}^d\times D_{2,\varepsilon}}\hat R_\varepsilon(\xi^\varepsilon,\varepsilon^{1-\alpha/2}\eta)\hat R(\varepsilon^{1-\alpha/2}\eta)|F^\varepsilon(\tau,\xi^\varepsilon,\eta)|^2\frac{d\xi d\eta}{(2\pi)^{2d}}, \qquad (4.14)$$

where $D_{2,\varepsilon}=\{\eta\in\mathbb{R}^d,\varepsilon^{(1-\alpha/2)}\eta\in D_2\}$. As in the previous section, F^ε is decomposed as, with same notations,

$$|F^\varepsilon|=|F^\varepsilon_{-1}|^2+R^\varepsilon. \qquad (4.15)$$

We split $w_2 := w_2^L + w_2^N$ accordingly into leading and negligible parts. F_{-1}^ε reads:

$$F_{-1}^\varepsilon(\tau,\xi^\varepsilon,\eta) = \int_0^\tau \int_0^t dtds \exp\left\{i\varepsilon^{(1-3\alpha/2)} s\xi^\varepsilon\cdot\eta/2\right\}\exp\left\{-it\xi_\|\right\}$$
$$\times \exp\left\{\frac{i}{\varepsilon^{\alpha/2}} s\Psi_{-1}(\eta)\right\} a^\varepsilon\left(\xi^\varepsilon, \varepsilon^{1-2\alpha} t\xi^\varepsilon - \varepsilon^{(1-3\alpha/2)} s\eta, [z^\varepsilon]\right),$$

$$\Psi_{-1}(\eta) = \eta_\| - \frac{\varepsilon^{1-\alpha/2}}{2}\left(|\eta_\perp|^2 + \eta_\|^2\right),$$

with $[z^\varepsilon] = (\tau, t, s, \varepsilon^{1-\alpha}\xi^\varepsilon, \varepsilon^{(1-\alpha/2)}\eta, -1)$. The phase Ψ_{-1} vanishes for

$$\eta_\| = \eta^\star(\eta_\perp) = \varepsilon^{-(1-\alpha/2)}\left(1 - \sqrt{1 - \varepsilon^{2(1-\alpha/2)}|\eta_\perp|^2}\right).$$

We then set $\eta_\| \to \eta^\star(\eta_\perp) + \varepsilon^{\alpha/2}\eta_\|$ and introduce the notation $\eta^\varepsilon = (\eta^\star(\eta_\perp) + \varepsilon^{\alpha/2}\eta_\|, \eta_\perp)$. The phase reads after the latter change of variables

$$\Psi_{-1} = \varepsilon^{\alpha/2}\eta_\|\left(1 - \varepsilon^{1-\alpha/2}\eta^\star(\eta_\perp) - \frac{1}{2}\varepsilon\eta_\|\right),$$

and the constraint (4.13) becomes

$$1 \le |\eta^\varepsilon|^2 = |\eta^\star(\eta_\perp) + \eta^{\alpha/2}\eta_\||^2 + |\eta_\perp|^2 \le \varepsilon^{-\frac{2(4-\alpha)}{6}}, \qquad 0 \le \eta^\star(\eta_\perp) + \eta^{\alpha/2}\eta_\|.$$

The definition of $\eta^\star$ gives $\varepsilon^{1-\alpha/2}\eta^\star \le 1$. This, together with the constraints above yield the following lower bound when $\eta^\varepsilon \in D_{2,\varepsilon}$:

$$\begin{aligned} 1 - \varepsilon^{1-\alpha/2}\eta^\star(\eta_\perp) - \frac{1}{2}\varepsilon\eta_\| &= 1 - \frac{1}{2}\varepsilon^{1-\alpha/2}(\eta^\star(\eta_\perp) + \varepsilon^{\alpha/2}\eta_\|) - \frac{1}{2}\varepsilon^{1-\alpha/2}\eta^\star(\eta_\perp) \\ &\ge 1 - \frac{1}{2}\varepsilon^{\frac{1-\alpha}{3}} - \frac{1}{2} = \frac{1}{2}(1 - \varepsilon^{\frac{(1-\alpha)}{3}}) \ge C > 0, \end{aligned} \qquad (4.16)$$

for ε small enough. This implies

$$|\Psi_{-1}| \ge C\,\varepsilon^{\alpha/2}\,|\eta_\||, \qquad \forall \eta^\varepsilon \in D_{2,\varepsilon}. \qquad (4.17)$$

In order to apply Lemma 8.2, we use first Lemma 8.1 with $h = 0$, $r = 1 - \alpha/2$, $r' = 1 - 3\alpha/2$ and obtain the estimate

$$\begin{aligned} &|\partial_t^k \partial_s^l a^\varepsilon\left(\xi^\varepsilon, \varepsilon^{1-2\alpha} t\xi^\varepsilon - s\varepsilon^{1-3\alpha/2}\eta^\varepsilon, z^\varepsilon\right)| \\ &\le C\frac{(|\varepsilon^{1-\alpha}\xi^\varepsilon - \varepsilon^{1-\alpha/2}\eta^\varepsilon|^k + |\varepsilon^{1-\alpha/2}\eta^\varepsilon|^k)|\varepsilon^{1-3\alpha/2}\eta^\varepsilon|^l + |\varepsilon^{1-2\alpha}\xi^\varepsilon|^k|\varepsilon^{1-3\alpha/2}\eta^\varepsilon|^l}{(1 + |\xi^\varepsilon|^2 + |\varepsilon^{1-2\alpha} t\xi^\varepsilon - s\varepsilon^{1-3\alpha/2}\eta^\varepsilon|^2)^n} \\ &\quad \times |\varepsilon^{1-\alpha}\xi^\varepsilon - \varepsilon^{1-\alpha/2}\eta^\varepsilon|. \end{aligned}$$

We next apply lemma 8.2 with $a = h = 0$, $\gamma' = 1 - 2\alpha$, $A = \varepsilon^{1-3\alpha/2}\xi^\varepsilon\cdot\eta^\varepsilon$, $B = \xi_\|$, $\Psi = \Psi_{-1}\varepsilon^{-\alpha/2}$, $r = 1 - \alpha/2$, $r' = 1 - 3\alpha/2$. We find, using (4.17) and the notation of the Lemma:

$$\begin{aligned} I_1 &= \varepsilon^{1-\alpha}|\xi^\varepsilon - \varepsilon^{\alpha/2}\eta^\varepsilon|, \\ I_2 &\le \varepsilon^{1-\alpha}|\eta_\||^{-1}(1 + \varepsilon^{1-3\alpha/2}|\xi^\varepsilon||\eta^\varepsilon| + \varepsilon^{1-3\alpha/2}|\eta^\varepsilon|)|\xi^\varepsilon - \varepsilon^{\alpha/2}\eta^\varepsilon|, \\ I_3 &\le \varepsilon^{1-\alpha}|\xi_\||^{-1}(1 + |\xi^\varepsilon| + \varepsilon^{1-\alpha/2}|\eta^\varepsilon|)|\xi^\varepsilon - \varepsilon^{\alpha/2}\eta^\varepsilon|. \end{aligned}$$

Since $|\eta^\varepsilon| \leq \varepsilon^{-(4-\alpha)/6}$, $\forall n \geq 1$, we have:

$$\begin{aligned}\langle \xi^\varepsilon\rangle^{-n} I_2 &\leq \varepsilon^{1-\alpha}\langle \xi^\varepsilon\rangle^{-(n-1)}|\eta_\parallel|^{-1}(1+\varepsilon^{(1-4\alpha)/3})|\xi^\varepsilon - \varepsilon^{\alpha/2}\eta^\varepsilon|,\\ \langle \xi^\varepsilon\rangle^{-n} I_3 &\leq \varepsilon^{1-\alpha}\langle \xi^\varepsilon\rangle^{-(n-1)}|\xi_\parallel|^{-1}|\xi^\varepsilon - \varepsilon^{\alpha/2}\eta^\varepsilon|,\end{aligned}$$

which implies that

$$|F_{-1}^\varepsilon|^2 \leq C\varepsilon^{2(1-\alpha)}\langle \xi^\varepsilon\rangle^{-n}(1+\varepsilon^{(1-4\alpha)/3})H(\xi_\parallel,\eta_\parallel)|\xi^\varepsilon - \varepsilon^{\alpha/2}\eta^\varepsilon|^2, \tag{4.18}$$

where

$$H(\xi_\parallel,\eta_\parallel) = 1 \wedge \left(|\xi_\parallel|^{-3/2}\right) \wedge \left(|\eta_\parallel|^{-1}|\xi_\parallel|^{-1}\right).$$

From (4.14), the scintillation w_2^L is then controlled by

$$\begin{aligned} w_2^L(\tau) &\leq C\varepsilon^{d(1-\alpha)+d(1-\alpha/2)-\alpha/2}\int_{\mathbb{R}^d}\int_{D_{2,\varepsilon}} d\xi d\eta \hat{R}_\varepsilon(\xi^\varepsilon,\varepsilon^{1-\alpha/2}\eta^\varepsilon)\hat{R}(\varepsilon^{(1-\alpha/2)}\eta^\varepsilon)\\ &\qquad\qquad \times\langle \xi^\varepsilon\rangle^{-n}(1+\varepsilon^{(1-4\alpha)/3})H(\xi_\parallel,\eta_\parallel)|\xi^\varepsilon - \varepsilon^{\alpha/2}\eta^\varepsilon|^2,\\ &\leq C\varepsilon^{(d-\delta)(1-\alpha)+(d-\delta)(1-\alpha/2)-\alpha/2}\|S\|^2_{L^\infty}\int_{\mathbb{R}^d}\int_{D_{2,\varepsilon}} d\xi d\eta |\xi^\varepsilon - \varepsilon^{\alpha/2}\eta^\varepsilon|^{2-\delta}|\eta^\varepsilon|^{-\delta},\\ &\qquad\qquad \times\langle \xi^\varepsilon\rangle^{-n}(1+\varepsilon^{(1-4\alpha)/3})H(\xi_\parallel,\eta_\parallel).\end{aligned}$$

We control the latter integral for $|\xi_\parallel| \geq 1$ and $|\eta_\parallel| \geq 1$ only treating the most technical part in detail. The corresponding integral is denoted by I. The remaining part is simpler to tackle. We have, $\forall n \geq 1$, for some $\gamma > 0$:

$$\langle \xi^\varepsilon\rangle^{-n} H(\xi_\parallel,\eta_\parallel) \leq \varepsilon^{-\gamma}|\eta_\parallel|^{-1}|\xi_\parallel|^{-1-\gamma}\langle \xi^\varepsilon\rangle^{-(n-1)}.$$

Owing to the fact that $|\eta^\varepsilon| \geq 1$ in $D_{2,\varepsilon}$, this yields

$$|\eta^\varepsilon|^{-\delta} \leq \mathbb{1}_{\varepsilon^{-\frac{(4-\alpha)}{6}} \geq |\eta_\perp| \geq 1}(\eta_\perp)|\eta_\perp|^{-\delta} + \mathbb{1}_{|\eta_\perp|\leq 1}(\eta_\perp), \qquad \forall \eta^\varepsilon \in D_{2,\varepsilon}. \tag{4.19}$$

Besides, since $\varepsilon^{1-\alpha/2}\eta^\star \leq 1$ and $0 \leq \eta^\star(\eta_\perp) + \eta^{\alpha/2}\eta_\parallel \leq \varepsilon^{-\frac{(4-\alpha)}{6}}$, we find that $|\eta_\parallel| \leq \varepsilon^{-1}$ when $\eta^\varepsilon \in D_{2,\varepsilon}$. Assume first $\delta \geq 2$ (which implies necessarily that $d > 2$). We have, using (4.19):

$$\begin{aligned} I &\leq \int_{\mathbb{R}^{d-1}}\int_1^\infty \int_{|\eta_\perp|\leq 1}\int_1^{\varepsilon^{-1}} d\xi_\perp d\xi_\parallel d\eta_\perp d\eta_\parallel \langle \xi_\perp\rangle^{-n}|\xi_\perp - \varepsilon^{\alpha/2}\eta_\perp|^{2-\delta}\varepsilon^{-\gamma}|\eta_\parallel|^{-1}|\xi_\parallel|^{-1-\gamma}\\ &\quad + \int_{\mathbb{R}^{d-1}}\int_1^\infty \int_{\varepsilon^{-\frac{(4-\alpha)}{6}}\geq|\eta_\perp|\geq 1}\int_1^{\varepsilon^{-1}} d\xi_\perp d\xi_\parallel d\eta_\perp d\eta_\parallel\\ &\qquad\qquad \times\langle \xi_\perp\rangle^{-n}|\xi_\perp - \varepsilon^{\alpha/2}\eta_\perp|^{2-\delta}|\eta_\perp|^{-\delta}\varepsilon^{-\gamma}|\eta_\parallel|^{-1}|\xi_\parallel|^{-1-\gamma},\\ &\leq |\log\varepsilon|\varepsilon^{-\gamma}\int_{\mathbb{R}^{d-1}}\int_{|\eta_\perp|\leq 1} d\xi_\perp d\eta_\perp \langle \xi_\perp\rangle^{-n}|\xi_\perp - \varepsilon^{\alpha/2}\eta_\perp|^{2-\delta}\\ &\quad + |\log\varepsilon|\varepsilon^{-\gamma}\int_{\mathbb{R}^{d-1}}\int_{\varepsilon^{-\frac{(4-\alpha)}{6}}\geq|\eta_\perp|\geq 1} d\xi_\perp d\eta_\perp \langle \xi_\perp\rangle^{-n}|\xi_\perp - \varepsilon^{\alpha/2}\eta_\perp|^{2-\delta}|\eta_\perp|^{-\delta}\end{aligned}$$

This implies, since $\delta < d$ and

$$\int_{\mathbb{R}^{d-1}} d\xi_\perp \langle \xi_\perp\rangle^{-n}|\xi_\perp - \varepsilon^{\alpha/2}\eta_\perp|^{2-\delta} \leq C, \tag{4.20}$$

where the constant C does not depend on $\eta_\perp$, that

$$I \quad \leq \quad C|\log\varepsilon|\varepsilon^{-\gamma}\left(1+\int_1^{-\frac{(4-\alpha)}{6}} dr r^{d-2-\delta}\right) = \mathcal{O}\left(|\log\varepsilon|\varepsilon^{-\gamma-\frac{(4-\alpha)}{6}(d-1-\delta)}\right). \quad (4.21)$$

When $0 \leq \delta < 2$, we have since $|\eta^\varepsilon| \leq \varepsilon^{-(4-\alpha)/6}$,

$$\begin{aligned}\langle\xi^\varepsilon\rangle^{-n}|\xi^\varepsilon-\varepsilon^{\alpha/2}\eta^\varepsilon|^{2-\delta} &\leq C\langle\xi^\varepsilon\rangle^{-(n-1)}(1+|\varepsilon^{\alpha/2}\eta^\varepsilon|^{2-\delta}),\\ &\leq C\langle\xi^\varepsilon\rangle^{-(n-1)}(1+\varepsilon^{(\alpha/2-(4-\alpha)/6)(2-\delta)})\end{aligned} \quad (4.22)$$

and obtain

$$\begin{aligned} I &= \mathcal{O}\left(|\log\varepsilon|\varepsilon^{-\gamma+(\alpha/2-(4-\alpha)/6)(2-\delta)}\int_1^{-\frac{(4-\alpha)}{6}} dr r^{d-\delta-2}\right),\\ &= \mathcal{O}\left(|\log\varepsilon|\varepsilon^{-\gamma+(\alpha/2(2-\delta)-\frac{(4-\alpha)}{6}(d+1-2\delta)}\right). \end{aligned} \quad (4.23)$$

The cases $|\eta_\||\leq 1$ and $|\xi_\|| \leq 1$ are simpler and yield higher order terms than (4.21) and (4.23). Therefore, going back to w_2^L, we find when $\delta \geq 2$:

$$\begin{aligned} w_2^L(\tau) &= \mathcal{O}(|\log\varepsilon|(1+\varepsilon^{(1-4\alpha)/3})\varepsilon^{(d-\delta)(1-\alpha)+(d-\delta)(1-\alpha/2)-\alpha/2-\frac{(4-\alpha)}{6}(d-1-\delta)-\gamma}),\\ &= \mathcal{O}(|\log\varepsilon|(1+\varepsilon^{(1-4\alpha)/3})\varepsilon^{\frac{4}{3}(d-\delta)(1-\alpha)+\frac{2}{3}(1-\alpha)-\gamma}),\\ &= \mathcal{O}(|\log\varepsilon|\varepsilon^{\frac{4}{3}(d-\delta)(1-\alpha)+\frac{2}{3}(1-\alpha)-\gamma}), \quad \text{when } \alpha \leq \frac{1}{4},\\ &= \mathcal{O}(|\log\varepsilon|\varepsilon^{\frac{4}{3}(d-\delta)(1-\alpha)+1-2\alpha-\gamma}), \quad \text{when } \frac{1}{4} \leq \alpha \leq \frac{1}{2}. \end{aligned}$$

Setting for instance $\gamma = \frac{1}{3}(d-\delta)(1-\alpha)$, which is strictly positive since $\delta < d$ and $\alpha \leq \frac{1}{2}$, we have $\forall \delta \in [2, d)$, $\forall \alpha \in [0, \frac{1}{2}]$, pointwise in τ:

$$w_2^L(\tau) \to 0 \qquad \text{as} \qquad \varepsilon \to 0. \quad (4.24)$$

When $0 \leq \delta < 2$, we find

$$\begin{aligned} w_2^L(\tau) &= \mathcal{O}(|\log\varepsilon|(1+\varepsilon^{(1-4\alpha)/3})\varepsilon^{(d-\delta)(1-3\alpha/2)+\alpha/2-\alpha/2\delta-\frac{(4-\alpha)}{6}(d+1-2\delta)-\gamma}),\\ &= \mathcal{O}(|\log\varepsilon|\varepsilon^{\frac{2}{3}(2d-\delta-1)(1-\alpha)-\gamma}), \quad \text{when } \alpha \leq \frac{1}{4},\\ &= \mathcal{O}(|\log\varepsilon|\varepsilon^{\frac{2}{3}(2d-\delta-1)(1-\alpha)+\frac{1}{3}(1-4\alpha)-\gamma}), \quad \text{when } \frac{1}{4} \leq \alpha \leq \frac{1}{2}. \end{aligned}$$

Let $\gamma_0 = 2-\delta > 0$. Then, since $d \geq 2$:

$$\frac{2}{3}(2d-\delta-1)(1-\alpha)+\frac{1}{3}(1-4\alpha)-\gamma \geq 1-2\alpha+\frac{2}{3}\gamma_0(1-\alpha)-\gamma.$$

Setting for instance $\gamma = \frac{1}{3}\gamma_0(1-\alpha)$ then yields (4.24) for $\delta \in [0, 2)$ and $\alpha \geq \frac{1}{4}$. The same holds for the simpler case $\alpha \leq \frac{1}{4}$. Regarding the other scintillation w_2^N and the related term R^ε, we proceed exactly as for w_1^N in section 4.1 and notice that the phase factor Ψ_1 satisfies the following bound from below:

$$\begin{aligned}\Psi_1 &= 2\eta^\star(\eta_\perp)+\varepsilon^{\alpha/2}\eta_\parallel\left(1+\varepsilon^{1-\alpha/2}\eta^\star(\eta_\perp)+\frac{1}{2}\varepsilon\eta_\parallel\right),\\ &\geq \eta^\star(\eta_\perp)\sqrt{1-\varepsilon^{2(1-\alpha/2)}|\eta_\perp|^2}\geq C\varepsilon^{-(1-\alpha/2)},\end{aligned}$$

for ε small enough. Following step by step the method used for w_1^N shows that w_2^N is negligible compared to w_2^L. We do not go into further details.

The conclusion of this section is that, $\forall\delta\in[0,d)$, $\forall\alpha\in[0,\frac{1}{2}]$, pointwise in τ:

$$w_2(\tau)\to 0 \qquad \text{as} \qquad \varepsilon\to 0. \tag{4.25}$$

As for section 4.1, this last result can be generalized to the domain $\mathcal{D}^0_-$ and to w_ε^2 without difficulty.

4.3. **Contribution of $D_3 := B_{\varepsilon^{\gamma_2}}$.** When $\eta\in D_3$, we have

$$0\leq|\eta|\leq\varepsilon^{(1-\frac{\alpha}{2})}. \tag{4.26}$$

Starting from (3.5), we perform the change of variables $\eta\to\eta\varepsilon$, $s\to s^{-\alpha}s$. Then:

$$\begin{aligned}F^\varepsilon_{\sigma_2}(\tau,\xi^\varepsilon,\eta) &= \int_0^\tau\int_0^t dtds\exp\left\{-i\sigma_2\varepsilon^{1-\alpha}s\xi^\varepsilon\cdot\eta/2\right\}\exp\left\{-it\xi_\parallel\right\}\\ &\qquad\times\exp\left\{is\Psi_{\sigma_2}(\eta)\right\}a^\varepsilon\left(\xi^\varepsilon,\varepsilon^{1-2\alpha}t\xi^\varepsilon-\varepsilon^{1-\alpha}s\eta,[z^\varepsilon]\right),\\ \Psi_{\sigma_2}(\eta) &= \eta_\parallel+\frac{\sigma_2\varepsilon}{2}\left(|\eta_\perp|^2+\eta_\parallel^2\right),\qquad [z^\varepsilon]=(\tau,t,s,\varepsilon^{1-\alpha}\xi^\varepsilon,\varepsilon\eta,\sigma_2).\end{aligned}$$

In order to obtain an optimal estimate, we expand a^ε in powers of ε. According to (3.2), we need to expand

$$\begin{aligned}\psi^\varepsilon(x,p,[z^\varepsilon]) = \sum_{\sigma_1=\pm1}\sigma_1\varphi\Big(&\varepsilon^\alpha x+\frac{1}{2}(\tau-t)\sigma_1(\varepsilon^{1-\alpha}\xi-\varepsilon\eta)+\tau(q_0+\varepsilon^{1-\alpha}p)\\ &+\frac{1}{2}\sigma_2(\tau-t+s)\varepsilon\eta,\ q_0+\varepsilon^{1-\alpha}p+\frac{1}{2}\sigma_1(\varepsilon^{1-\alpha}\xi-\varepsilon\eta)+\frac{1}{2}\sigma_2\varepsilon\eta\Big).\end{aligned}$$

Easy calculations yield:

$$\begin{aligned}&\psi^\varepsilon=\psi_0^\varepsilon+\varepsilon\psi_1^\varepsilon,\\ &\psi_0^\varepsilon=\left[(\varepsilon^{1-\alpha}\xi-\varepsilon\eta)\cdot((\tau-t)\nabla_x+\nabla_p)\right]\varphi(\varepsilon^\alpha x+\tau(q_0+\varepsilon^{1-\alpha}p),q_0+\varepsilon^{1-\alpha}p)\end{aligned}$$

where ψ_1^ε satisfies, for all multi-indices i and j, pointwise in η,ξ^ε, for τ,t,s bounded, $k=0,1$:

$$|\partial_t^k\partial_x^i\partial_p^j\psi_1^\varepsilon(x,p,[z^\varepsilon])|\leq C|\eta||\varepsilon^{1-\alpha}\xi^\varepsilon-\varepsilon\eta|. \tag{4.27}$$

In the same way, we have

$$\exp\left\{-i\sigma_2\varepsilon^{1-\alpha}s\xi^\varepsilon\cdot\eta+i\varepsilon\frac{\sigma_2}{2}s|\eta|^2\right\}=1-i\sigma_2\varepsilon^{1-\alpha}s\xi^\varepsilon\cdot\eta+\varepsilon\psi_3,$$

where the function ψ_3 verifies, $\forall n\geq 2$ and for s bounded:

$$\langle\xi^\varepsilon\rangle^{-n}|\psi_3|\leq\langle\xi^\varepsilon\rangle^{-(n-2)}|\eta|^2. \tag{4.28}$$

This implies that

$$\sum_{\sigma_2=\pm 1} \sigma_2 \exp\left\{-i\sigma_2\varepsilon^{1-\alpha}s\xi^\varepsilon\cdot\eta + i\varepsilon\frac{\sigma_2}{2}s|\eta|^2\right\}\psi^\varepsilon = -2i\varepsilon^{1-\alpha}s\xi^\varepsilon\cdot\eta\psi_0^\varepsilon + \varepsilon\psi_4^\varepsilon,$$

where ψ_4^ε satisfies, using (4.27),

$$|\partial_t^k\psi_4^\varepsilon| \leq C\langle\xi^\varepsilon\rangle^{-n}(1+|\eta|^2)|\varepsilon^{1-\alpha}\xi^\varepsilon - \varepsilon\eta|. \tag{4.29}$$

Consequently, we can write

$$\begin{aligned}
F^\varepsilon &= \sum_{\sigma_2=\pm 1} F_{\sigma_2}^\varepsilon = -2i\varepsilon^{1-\alpha}(\xi^\varepsilon\cdot\eta)\int_0^\tau\int_0^t dtdss\exp\left\{-it\xi_\|\right\}\exp\left\{is\eta_\|\right\}(a_0^\varepsilon + \varepsilon a_1^\varepsilon),\\
a_0^\varepsilon &= \mathcal{F}(W_0\psi_0^\varepsilon)(\xi^\varepsilon, \varepsilon^{1-2\alpha}t\xi^\varepsilon - \varepsilon^{1-\alpha}s\eta, [z^\varepsilon]),\\
a_1^\varepsilon &= \mathcal{F}(W_0\psi_4^\varepsilon)(\xi^\varepsilon, \varepsilon^{1-2\alpha}t\xi^\varepsilon - \varepsilon^{1-\alpha}s\eta, [z^\varepsilon]).
\end{aligned}$$

With obvious notation, we recast the latter system as

$$\begin{aligned}
&|F^\varepsilon|^2 = L^\varepsilon + R^\varepsilon, \qquad\qquad R^\varepsilon = \varepsilon^{2-\alpha}r^\varepsilon,\\
&L^\varepsilon = 4\left|\varepsilon^{1-\alpha}(\xi^\varepsilon\cdot\eta)\int_0^\tau\int_0^t dtdss\exp\left\{-it\xi_\|\right\}\exp\left\{is\eta_\|\right\}a_0^\varepsilon\right|^2,
\end{aligned}$$

and the scintillation w_3 reads

$$\begin{aligned}
w_3(\tau) &= \varepsilon^{d(1-\alpha)+d+\alpha-2}\int_{\mathbb{R}^d\times B_{\varepsilon^{-\alpha/2}}}\frac{d\xi d\eta}{(2\pi)^{2d}}\hat{R}_\varepsilon(\xi^\varepsilon, \varepsilon\eta)\hat{R}(\varepsilon\eta)(L^\varepsilon + R^\varepsilon),\\
&:= w_3^L(\tau) + w_3^R(\tau).
\end{aligned}$$

Consider first the leading term w_3^L. The function a_0^ε satisfying the estimate of Lemma 8.1 with $\gamma' = 1-2\alpha$, $r' = r = 1$, $h = 0$, we apply Lemma 8.2 with $\gamma' = 1-2\alpha$, $A = 0$, $B = \xi_\|$, $\Psi = \eta_\|$, $r' = r = 1$, $h = 0$ and find

$$L^\varepsilon \leq C\varepsilon^{2(1-\alpha)}\langle\xi^\varepsilon\rangle^{-n}g(\xi_\|)|\eta|^2|\varepsilon^{1-\alpha}\xi^\varepsilon - \varepsilon\eta|^2 \tag{4.30}$$

where g is an in (4.6). Besides, we have

$$\begin{aligned}
\frac{1}{|\xi^\varepsilon - \varepsilon^\alpha\eta|^{\delta-2}} &= \frac{1}{|\xi^\varepsilon|^{\delta-2}} + \varepsilon^\alpha(2-\delta)\int_0^1\frac{(\xi^\varepsilon - u\varepsilon^\alpha\eta)\cdot\eta}{|\xi^\varepsilon - u\varepsilon^\alpha\eta|^{\delta-4}}du,\\
&= \frac{1}{|\xi^\varepsilon|^{\delta-2}} + \varepsilon^\alpha h(\xi^\varepsilon, \eta),
\end{aligned} \tag{4.31}$$

where h verifies, η and $\xi_\perp$ a.e, since $\delta < d$:

$$\begin{aligned}
\int_{\mathbb{R}^{d-1}}\langle\xi^\varepsilon\rangle^{-n}|h(\xi^\varepsilon,\eta)|d\xi_\perp &\leq C|\eta|\int_{\mathbb{R}^{d-1}}\int_0^1\langle\xi^\varepsilon\rangle^{-n}|\xi^\varepsilon - u\varepsilon^\alpha\eta|^{3-\delta}dud\xi_\perp,\\
&\leq \begin{cases} C|\eta|\displaystyle\int_{\mathbb{R}^{d-1}}\int_0^1\langle\xi_\perp\rangle^{-n}|\xi_\perp - u\varepsilon^\alpha\eta_\perp|^{3-\delta}dud\xi_\perp \leq C|\eta|, \\ \qquad\qquad\qquad\qquad\text{if}\quad 3\leq\delta<d, \\ C|\eta|(1+|\varepsilon^\alpha\eta_\perp|^{3-\delta}), \qquad \text{if} \qquad 0\leq\delta<3. \end{cases}
\end{aligned} \tag{4.32}$$

Using (4.31), we then write for the contribution of L^ε to the scintillation:

$$\begin{aligned}
w_3^L(\tau) &= \varepsilon^{(d-\delta)(1-\alpha)+d-\delta+\alpha-2} \\
&\quad\times \int_{\mathbb{R}^d}\int_{B_{\varepsilon^{-\alpha/2}}} \frac{d\xi d\eta}{(2\pi)^{2d}} S(\varepsilon^{1-\alpha}\xi^\varepsilon-\varepsilon\eta)|\xi^\varepsilon-\varepsilon^\alpha\eta|^{2-\delta} S(\varepsilon\eta)|\eta|^{-\delta}|\xi^\varepsilon-\varepsilon^\alpha\eta|^{-2}L^\varepsilon, \\
&= \varepsilon^{(d-\delta)(1-\alpha)+d-\delta+\alpha-2}\int_{\mathbb{R}^d}\int_{B_{\varepsilon^{-\alpha/2}}} \frac{d\xi d\eta}{(2\pi)^{2d}} S(\varepsilon^{1-\alpha}\xi^\varepsilon-\varepsilon\eta) \\
&\qquad\qquad \times\left(\frac{1}{|\xi^\varepsilon|^{\delta-2}}+\varepsilon^\alpha h(\xi^\varepsilon,\eta)\right) S(\varepsilon\eta)|\eta|^{-\delta}|\xi^\varepsilon-\varepsilon^\alpha\eta|^{-2}L^\varepsilon, \\
&:= \varepsilon^{(d-\delta)(1-\alpha)+d-\delta+\alpha-2}(T_1+T_2),
\end{aligned}$$

with obvious notation. The change of variable $\eta\to\varepsilon^{-\alpha/2}\eta$ in T_1 yields

$$\begin{aligned}
T_1 = \varepsilon^{(\delta-d)\alpha/2}\int_{\mathbb{R}^d}\int_{B_1} &\frac{d\xi d\eta}{(2\pi)^{2d}} S(\varepsilon^{1-\alpha}\xi^\varepsilon-\varepsilon^{1-\alpha/2}\eta) \\
&|\xi^\varepsilon|^{2-\delta} S(\varepsilon^{1-\alpha/2}\eta)|\eta|^{-\delta}|\xi^\varepsilon-\varepsilon^{\alpha/2}\eta|^{-2}L^\varepsilon(\varepsilon^{-\alpha/2}\eta).
\end{aligned}$$

According to (4.30), the integrand is controlled by, $\forall n\geq 0$:

$$\begin{aligned}
&\varepsilon^{4(1-\alpha)-\alpha}\|S\|^2_{L^\infty}|\eta|^{-\delta}|\xi^\varepsilon|^{2-\delta}\langle\xi^\varepsilon\rangle^{-n}g(\xi_\|)|\eta|^2 \\
&\leq C\varepsilon^{4(1-\alpha)-\alpha}\|S\|^2_{L^\infty}|\eta|^{-\delta}\langle\xi_\perp\rangle^{-n}g(\xi_\|)|\eta|^2 \begin{cases} |\xi_\perp|^{2-\delta} & \text{if} \quad 2\leq\delta<d, \\ 1 \quad \text{if} & 0\leq\delta<2. \end{cases} \\
&:= \varepsilon^{4(1-\alpha)-\alpha}H(\xi,\eta).
\end{aligned}$$

This tells us that

$$T_1 = \mathcal{O}(\varepsilon^{(\delta-d)\alpha/2+4(1-\alpha)-\alpha}). \tag{4.33}$$

Assume now $\alpha=\frac{1}{2}$. We will obtain an optimal estimate in this case that shows that w_3^L is of order $\varepsilon^{d-\delta}$. Since the function H belongs to $L^1(\mathbb{R}^d\times B_1)$, we can apply the Lebesgue dominated convergence theorem and obtain that, pointwise in τ:

$$\begin{aligned}
\varepsilon^{(d-\delta)\alpha/2-4(1-\alpha)+\alpha}T_1 \to S(0)^2 &\int_{\mathbb{R}^d}\int_{B_1}\frac{d\xi d\eta}{(2\pi)^{2d}}|\xi_\perp|^{2-\delta}|\eta|^{-\delta} \\
&\times\lim_{\varepsilon\to 0}\varepsilon^{-4(1-\alpha)+\alpha}|\xi^\varepsilon-\varepsilon^{\alpha/2}\eta|^{-2}L^\varepsilon(\varepsilon^{-\alpha/2}\eta)
\end{aligned}$$

with, according to the definition of L^ε and a_0^ε, when $\alpha=\frac{1}{2}$:

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\varepsilon^{-4(1-\alpha)+\alpha}|\xi^\varepsilon-\varepsilon^{\alpha/2}\eta|^{-2}L^\varepsilon(\varepsilon^{-\alpha/2}\eta) \\
&\qquad = 4|\xi_\perp|^{-2}\left|(\xi_\perp\cdot\eta_\perp)\int_0^\tau\int_0^t dtds s\exp\{-it\xi_\|\}\exp\{is\eta_\|\}\,a_0(\xi_\perp,\tau,t)\right|^2, \\
&a_0(\xi_\perp,\tau,t) = \psi_0(\tau,t)\,\mathcal{F}W_0(\xi^0,t\xi^0), \\
&\psi_0 = \left[\xi^0\cdot((\tau-t)\nabla_x+\nabla_p)\right]\varphi(\tau q_0,q_0).
\end{aligned}$$

Above, we used the notation $\xi^0 = (0, \xi_\perp)$. Regarding, T_2, using (4.30)-(4.32) and following step by step the calculations for T_1, we find:

$$T_2 = \mathcal{O}(\varepsilon^{(\delta-d)\alpha/2+\alpha/2+4(1-\alpha)-\alpha}),$$

which is an order $\varepsilon^{\alpha/2}$ smaller than T_1. Together with (4.33), this means first that $\forall \delta \in [0, d)$, $\forall \alpha \in [0, \frac{1}{2}]$, pointwise in τ:

$$w_3^L(\tau) \to 0 \qquad \text{as} \qquad \varepsilon \to 0, \tag{4.34}$$

and that there exists a non-identically vanishing function f such that, when $\alpha = \frac{1}{2}$,

$$\varepsilon^{-(d-\delta)} w_3^L(\tau) \to f(\tau). \tag{4.35}$$

Regarding the remainder R^ε, it is proved to be negligible (when $\alpha > 0$ and same order when $\alpha = 0$) by mimicking the steps for L^ε and using estimate (4.29) together with Lemma 8.2. We leave the details to the reader.

We treat now (2.7). The definition of w_1^ε gives straightforwardly

$$w_3(\tau) \leq w_\varepsilon^1(\tau),$$

so that, together with (4.35) and the fact that w_3^R is of higher order than w_3^L,

$$f(\tau) \leq \liminf_{\varepsilon \to 0} \varepsilon^{-(d-\delta)} w_\varepsilon^1(\tau),$$

which yields (2.7). The main results of this section are therefore (2.7), and the fact that, $\forall \delta \in [0, d)$, $\forall \alpha \in [0, \frac{1}{2}]$, pointwise in τ:

$$w_3(\tau) \to 0 \qquad \text{as} \qquad \varepsilon \to 0.$$

5. Optimal estimates for $\delta = 0$ and $\alpha \in [0, \frac{1}{2}]$.

Since $\hat{R}$ is bounded in L^∞ when $\delta = 0$, we can consider the whole domain D_+^0 without having to decompose it into subdomains to be able to treat the singularity of $\hat{R}$. Assume first $\alpha > 0$. We follow step by the step the lines of section 4.1 and decompose $|F^\varepsilon|^2$ into leading and negligible parts. We already know from the results of section 4.1 that the leading term is given by w_1^L and thus focus on this term. Estimate (4.6) provides the majorizing function

$$\langle \xi_\perp \rangle^{-n} g(\xi_\parallel) f(\eta_\parallel, \eta_\perp)$$

that allows us to use the Lebesgue dominated convergence theorem and pass to the limit in the expression of w_1^L. Recall that

$$\begin{aligned} D_{1,\varepsilon} &= \left\{ (\eta_\parallel, \eta_\perp) \in \mathbb{R}^d, (\eta^\star(\eta_\perp) + \varepsilon^{1-\alpha}\eta_\parallel, \eta_\perp) \in D_+^0 \right\}, \\ D_+^0 &= \left\{ (\eta_\parallel, \eta_\perp) \in \mathbb{R}^d, \quad |\eta_\perp| \leq 1, \quad 0 \leq \eta_\parallel \leq 1 \right\}. \end{aligned}$$

This implies that, pointwise in $\eta^\varepsilon = (\eta^\star(\eta_\perp) + \varepsilon^{1-\alpha}\eta_\parallel, \eta_\perp)$:

$$\mathbb{1}_{D_{1,\varepsilon}}(\eta^\varepsilon) \to \mathbb{1}_{\mathbb{R}}(\eta_\parallel) \mathbb{1}_{B_1}(\eta_\perp), \tag{5.1}$$

where B_1 is $d-1$ dimensional unit ball. Hence, pointwise in τ:

$$\begin{aligned}\lim_{\varepsilon\to 0}\varepsilon^{-d(1-\alpha)-2\alpha+1}w_1^L(\tau) &= \int_{\mathbb{R}^{2d}}\frac{d\xi d\eta}{(2\pi)^{2d}}\lim_{\varepsilon\to 0}\mathbb{1}_{D_{1,\varepsilon}}\hat{R}_\varepsilon(\xi^\varepsilon,\eta^\varepsilon)\hat{R}(\eta^\varepsilon)|F^\varepsilon_{-1}(\tau,\xi^\varepsilon,\eta^\varepsilon)|^2,\\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}\times B_1}\frac{d\xi d\eta}{(2\pi)^{2d}}\hat{R}^2(\eta^-)|F^0(\tau,\xi_\parallel,\xi_\perp,\eta_\parallel,\eta^-)|^2,\end{aligned}$$

where F^ε_{-1} is defined in (3.4), $\eta^-=(\eta^\star(\eta_\perp),\eta_\perp)$ and

$$F^0(\tau,\xi_\parallel,\xi_\perp,\eta_\parallel,\eta^-)=\lim_{\varepsilon\to 0}F^\varepsilon_{-1}(\tau,\xi^\varepsilon,\eta^\varepsilon)\qquad a.e..$$

In order to identify F^0, we need another majorizing function. Assume for the moment that $\alpha<\frac{1}{2}$. We write, for ξ^ε and η^ε fixed:

$$\begin{aligned}a^\varepsilon\left(\xi^\varepsilon,\varepsilon^{1-2\alpha}t\xi^\varepsilon-s\eta\right) &= a^\varepsilon\left(\xi^\varepsilon,-s\eta\right)+\varepsilon^{1-2\alpha}t\xi^\varepsilon\cdot\int_0^1\nabla_2 a^\varepsilon\left(\xi^\varepsilon,u\varepsilon^{1-2\alpha}t\xi^\varepsilon-s\eta\right)du,\\ &:= a^\varepsilon\left(\xi^\varepsilon,-s\eta\right)+\varepsilon^{1-2\alpha}b^\varepsilon.\end{aligned}\tag{5.2}$$

Above, ∇_2 denotes the gradient of $a^\varepsilon(x,y)$ with respect to $y\in\mathbb{R}^d$. Using the definition of a^ε given in (3.2) and the fact that $W_0\in\mathcal{S}(\mathbb{R}^{2d})$, we find, $\forall n\geq 0$, for $k=0$ or $k=1$:

$$|(\nabla_2)^k a^\varepsilon\left(\xi^\varepsilon,y\right)|\leq C\frac{1}{(1+|\xi^\varepsilon|^2+|y|^2)^n}\,|\varepsilon^{1-\alpha}\xi^\varepsilon-\eta^\varepsilon|,\tag{5.3}$$

so that

$$b^\varepsilon\leq C\int_0^1 du\frac{\varepsilon^{1-\alpha}+|\eta^\varepsilon|}{(1+|\xi^\varepsilon|^2+|u\varepsilon^{1-2\alpha}t\xi^\varepsilon-s\eta^\varepsilon|^2)^n}.$$

It is then clear that

$$\int_0^{t\varepsilon^{-\alpha}}b^\varepsilon ds\leq t\varepsilon^{1-2\alpha}+\sum_{i=1}^d\int_0^\infty\int_0^1 dsdu\frac{|\eta_i^\varepsilon|}{(1+|\xi^\varepsilon|^2+|u\varepsilon^{1-2\alpha}t\xi^\varepsilon-s\eta^\varepsilon|^2)^n}\leq C,\tag{5.4}$$

where η_i^ε denotes the i-th component of the vector η^ε and C does not depend on ε. Hence, owing (3.4), (5.2) and (5.4), F^ε_{-1} admits the expression

$$\begin{aligned}&F^\varepsilon_{-1}(\tau,\xi^\varepsilon,\eta^\varepsilon)=\mathcal{O}(\varepsilon^{1-2\alpha})+\\ &\int_0^\tau\int_{\mathbb{R}}dtds\mathbb{1}_{0\leq s\leq\varepsilon^{-\alpha}t}\exp\left\{is\xi^\varepsilon\cdot\eta^\varepsilon/2\right\}\exp\left\{-it\xi_\parallel\right\}\exp\left\{\frac{i}{\varepsilon^{1-\alpha}}s\Psi_{-1}\right\}a^\varepsilon\left(\xi^\varepsilon,-s\eta^\varepsilon\right).\end{aligned}$$

We then perform the change of variables $s\to s|\eta^\varepsilon|^{-1}$ in the expression above and introduce the notation $\tilde{\eta}^\varepsilon=\eta^\varepsilon|\eta^\varepsilon|^{-1}$. According to (5.3), we have, $\forall n\geq 0$

$$|a^\varepsilon\left(\xi^\varepsilon,-s\tilde{\eta}^\varepsilon\right)|\leq C\langle\xi_\perp\rangle^{-n}\langle s\rangle^{-n}.$$

Applying the Lebesgue dominated convergence with the latter majorizing function, together with

$$\frac{i}{\varepsilon^{1-\alpha}}s|\eta^\varepsilon|^{-1}\Psi_{-1}(\eta^\varepsilon)\to is\eta_\parallel|\eta^-|^{-1}(1-|\eta_\perp|^2)^{1/2},\qquad a.e.,$$

we obtain the following limit for F^ε_{-1}:

$$F^0(\tau,\xi_\|,\xi_\perp,\eta_\|,\eta^-) = |\eta^-|^{-1}\int_0^\tau\int_0^\infty dtds\exp\left\{is|\eta^-|^{-1}\xi_\perp\cdot\eta_\perp/2\right\}\exp\left\{-it\xi_\|\right\}$$
$$\exp\left\{is(1-|\eta_\perp|^2)^{1/2}\eta_\||\eta^-|^{-1}\right\}a_0\left(\xi^0,-s\eta^-|\eta^-|^{-1}\right),$$
$$=(1-|\eta_\perp|^2)^{-1/2}\int_0^\tau\int_0^\infty dtds\exp\left\{is(1-|\eta_\perp|^2)^{-1/2}\xi_\perp\cdot\eta_\perp/2\right\}$$
$$\exp\left\{-it\xi_\|\right\}\exp\left\{is\eta_\|\right\}a_0\left(\xi^0,-s(1-|\eta_\perp|^2)^{-1/2}\eta^-\right)$$

where

$$a_0=\psi_0\mathcal{F}(W_0),\quad \psi_0=\sum_{\sigma_1=\pm1}\sigma_1\varphi\left(\tau q_0-\frac12(1+\sigma_1)(\tau-t)\eta^-,q_0-\frac12(1+\sigma_1)\eta^-\right).$$

Then, the Fourier-Plancherel theorem yields the final expression

$$\lim_{\varepsilon\to0}\varepsilon^{-d(1-\alpha)-2\alpha+1}w_1^L(\tau)=$$
$$=\int_0^\tau\int_0^\infty\int_{\mathbb{R}^{d-1}}\int_{B_1}\frac{dtdsd\xi_\perp d\eta_\perp}{(2\pi)^{2(d-1)}}\frac{\hat R^2(\eta^-)}{1-|\eta_\perp|^2}\left|a_0\left(\xi^0,s(1-|\eta_\perp|^2)^{-1/2}\eta^-\right)\right|^2,$$
$$=\int_0^\tau\int_0^\infty\int_{\mathbb{R}^{d-1}}\int_{B_1}\frac{dtdsd\xi_\perp d\eta_\perp}{(2\pi)^{2(d-1)}}\frac{\hat R^2(\eta^-)}{(1-|\eta_\perp|^2)^{1/2}}|a_0\left(\xi^0,s\eta^-\right)|^2. \tag{5.5}$$

When $\alpha=\frac12$, we simply control a^ε by

$$|a^\varepsilon\left(\xi^\varepsilon,t\xi^\varepsilon-s\eta\right)|\le\frac{C}{(1+|\xi^\varepsilon|^2+|\varepsilon^{1-2\alpha}t\xi^\varepsilon-s\eta^\varepsilon|^2)^n}\le\frac{C}{(1+|\xi_\perp|^2+|t\xi_\perp-s\eta_\perp|^2)^n}$$

which provides a majorizing function and allows us to pass to the limit. All calculations done, it comes for $\alpha=\frac12$:

$$\lim_{\varepsilon\to0}\varepsilon^{-d(1-\alpha)-2\alpha+1}w_1^L(\tau)=$$
$$=\int_0^\tau\int_0^\infty\int_{\mathbb{R}^{d-1}}\int_{B_1}\frac{dtdsd\xi_\perp d\eta_\perp}{(2\pi)^{2(d-1)}}\frac{\hat R^2(\eta^-)}{(1-|\eta_\perp|^2)^{1/2}}|a_0\left(\xi^0,t\xi^0-s\eta^-\right)|^2. \tag{5.6}$$

The case $\alpha=0$ is the most direct to treat and yields the result

$$\lim_{\varepsilon\to0}\varepsilon^{-d+1}w_1^L(\tau)=$$
$$\frac{1}{(2\pi)^{2d}}\int_{\mathbb{R}^d}\int_{\mathbb{R}\times B_1}d\xi d\eta\frac{\hat R^2(\eta^-)}{(1-|\eta_\perp|^2)^{1/2}}\left|\int_0^\tau\int_0^t dtdse^{is\xi\cdot\eta^-}e^{-it\xi_\|}e^{is\eta_\|}\mathcal{F}(W_0\psi)\left(\xi,-s\eta^-\right)\right|^2,$$

$$\psi=\sum_{\sigma_1=\pm1}\sigma_1\varphi\Big(x+\frac12(\tau-t)\sigma_1(\xi-\eta^-)+\tau q_0-\frac12(\tau-t+s)\eta^-,$$
$$q_0+\frac12\sigma_1(\xi-\eta^-)-\frac12\eta^-\Big).$$

Using again the Fourier-Plancherel theorem, we find

$$\lim_{\varepsilon\to 0} \varepsilon^{-d+1} w_1^L(\tau) =$$
$$\frac{1}{(2\pi)^{2d-1}} \int_{\mathbb{R}^d} \int_{B_1} \int_0^\tau d\xi d\eta_\perp ds \frac{\hat{R}^2(\eta^-)}{(1-|\eta_\perp|^2)^{1/2}} \left| \int_s^\tau dt e^{-it\xi_\parallel} \mathcal{F}(W_0\psi)\left(\xi, -s\eta^-\right) \right|^2. \quad (5.7)$$

Hence, the limit of the scintillation w_ε^1 corresponding to the domain D_+^0 is given by (5.5)-(5.6)-(5.7). The contribution of D_+^1 admits the same expression with η^- replaced by $\eta^+ = (1+\sqrt{1-|\eta_\perp|^2}, \eta_\perp)$. Moreover, simple symmetry considerations then show that the contribution of $D_-^0 \cup D_-^1$ is the same as that of $D_+^0 \cup D_+^1$. As claimed in the outline, the contributions of the other domains are of higher order while w_ε^2 shares the same limit as w_ε^1 since the two terms have similar expressions up to neglibible quantities. A first look at G_{σ_2} when $\alpha = 0$ seems to indicate that the limit is different since the extra $\varepsilon^\alpha s$ term is now of order one. This term actually disappears in the final expression after using the Fourier-Plancherel equality, leading therefore to the same expression as w_ε^1. This ends the section.

6. **The case** $\alpha \in (\frac{1}{2}, 1)$

The contributions of the domains D_1, D_2 and D_3 can be treated with some modifications of sections 4.1, 4.2 as well as Lemmas 8.1 and 8.2 of the appendix. The case $d \geq 4$ is relatively straightforward while the cases $d = 2$ and $d = 3$ require a little more attention. In all cases, the corresponding scintillation is proved to converge to zero for any $\delta \in (0, d)$. We do not go into further details. We remain below at an informal level for the derivation of optimal estimates when $\delta = 0$.

As already mentioned in section 5, it is not necessary to divide the domain D_+^0 into various subdomains when $\delta = 0$ since the power spectrum is bounded. We then only consider the contribution of the scintillation in D_+^0, that we denote by w_1 and generalize the result to the other domains of interest. We start from expression (3.6). Assume first $\alpha < 1$. It is clear from the discussions of sections 4.1 and 5 that the leading term in $|F^\varepsilon|^2$ is $|F_{-1}^\varepsilon|^2$ and we therefore concentrate our attention on this sole term formally neglecting the remainder. The corresponding scintillation is denoted by w_1^L. We perform the classical change of variables $\eta_\parallel \to \eta_\parallel = \eta^\star(\eta_\perp) + \varepsilon^{1-\alpha}\eta_\parallel$ and define $\eta^\varepsilon = (\eta^\star(\eta_\perp) + \varepsilon^{1-\alpha}\eta_\parallel, \eta_\perp)$ as well as $\eta^- = (\eta^\star(\eta_\perp), \eta_\perp)$, $\xi^0 = (0, \xi_\perp)$. According to (3.2), we have informally

$$a^\varepsilon\left(\xi^\varepsilon, t\xi^\varepsilon - s\eta^\varepsilon, [z^\varepsilon]\right) = a_0(\xi^0, t\xi^0 - s\eta^-, [z]) + o(\varepsilon),$$

where

$$a_0(u, v, [z]) = \psi_0(z)\mathcal{F}(W_0)(u, v), \qquad [z] = (\eta^-, \tau),$$
$$\psi_0(z) = \sum_{\sigma_1=\pm 1} \sigma_1 \varphi\left(-\frac{1}{2}(1+\sigma_1)\tau\eta^- + \tau q_0, q_0 - \frac{1}{2}(1+\sigma_1)\eta^-\right).$$

Besides,

$$\Psi_{-1}(\eta^\varepsilon) = \varepsilon^{1-\alpha}\eta_\parallel(1-|\eta_\perp|^2)^{1/2} + o(\varepsilon^{1-\alpha}).$$

This implies that

$$\begin{aligned}
F^{\varepsilon}_{-1}(\tau,\xi^{\varepsilon},\eta) &= \int_0^{\tau\varepsilon^{1-2\alpha}}\int_0^{t\varepsilon^{\alpha-1}} dtds \exp\{is\xi_\perp\cdot\eta_\perp/2\}\exp\{-it\xi_\parallel\} \\
&\qquad \times\exp\left\{is\eta_\parallel(1-|\eta_\perp|^2)^{1/2}\right\} a_0(\xi^0,t\xi^0-s\eta^-,[z])+o(\varepsilon), \\
&:= F^{\varepsilon}_0(\tau,\xi_\parallel,\xi_\perp,\eta_\parallel,\eta_\perp)+o(\varepsilon).
\end{aligned}$$

Consequently, w_1^L reads

$$\begin{aligned}
w_1^L(\tau) &= \varepsilon^{d(1-\alpha)+2(2\alpha-1)}\int_{\mathbb{R}^{2d}}\frac{d\xi d\eta}{(2\pi)^{2d}}\mathbb{1}_{D_{1,\varepsilon}}\hat{R}^2(\eta^-)|F^{\varepsilon}_{-1}(\tau,\xi^{\varepsilon},\eta^{\varepsilon})|^2, \\
&= \varepsilon^{d(1-\alpha)+2(2\alpha-1)}\int_{\mathbb{R}^{2d}}\frac{d\xi d\eta}{(2\pi)^{2d}}\mathbb{1}_{\mathbb{R}}(\eta_\parallel)\mathbb{1}_{B_1}(\eta_\perp)\hat{R}^2(\eta^-)|F^{\varepsilon}_0(\tau,\xi_\parallel,\xi_\perp,\eta_\parallel,\eta_\perp)|^2 \\
&\quad +o(\varepsilon^{d(1-\alpha)+2(2\alpha-1)}),
\end{aligned}$$

where $D_{1,\varepsilon}=\{\eta\in\mathbb{R}^d,\eta^{\varepsilon}\in D_1\}$. The Fourier-Plancherel theorem then yields

$$\begin{aligned}
&w_1^L(\tau)=\varepsilon^{d(1-\alpha)+2(2\alpha-1)}w_{1,\varepsilon}(\tau)+o(\varepsilon^{d(1-\alpha)+2(2\alpha-1)}), \\
&w_{1,\varepsilon}(\tau)=\int_{\mathbb{R}^{d-1}}\int_{B_1}\int_0^{\tau\varepsilon^{1-2\alpha}}\int_0^{t\varepsilon^{\alpha-1}}\frac{d\xi_\perp d\eta_\perp dtds}{(2\pi)^{2(d-1)}}\frac{\hat{R}^2(\eta^-)}{(1-|\eta_\perp|^2)^{1/2}}|a_0(\xi^0,t\xi^0-s\eta^-,[z])|^2.
\end{aligned}$$

Suppose first that $d\geq 3$. Passing formally to the limit in the latter equation gives

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\varepsilon^{-d(1-\alpha)-2(2\alpha-1)}w_1^L(\tau)= \\
&\int_{\mathbb{R}^{d-1}}\int_{B_1}\int_0^{\infty}\int_0^{\infty}\frac{d\xi_\perp d\eta_\perp dtds}{(2\pi)^{2(d-1)}}\frac{\hat{R}^2(\eta^-)}{(1-|\eta_\perp|^2)^{1/2}}|a_0(\xi^0,t\xi^0-s\eta^-,[z])|^2. \qquad (6.1)
\end{aligned}$$

We claim the term on the right is finite. We only verify it for $t\geq 1$, the remaining part of the integral following directly. Notice first that, since $\varphi\in\mathcal{S}(\mathbb{R}^{2d})$,

$$|\psi_0|\leq C|\eta^-|.$$

This fact, together with the change of variables in order $s\to s(\eta^\star)^{-1}$, $\xi_\perp\to t^{-1}\xi_\perp$, $\xi_\perp\to\xi_\perp+s(\eta^\star)^{-1}\eta_\perp$ implies, $\forall n\geq 0$:

$$\begin{aligned}
&\int_{\mathbb{R}^{d-1}}\int_{B_1}\int_1^{\infty}\int_0^{\infty}\frac{d\xi_\perp d\eta_\perp dtds}{(2\pi)^{2(d-1)}}\frac{1}{t^{d-1}\eta^\star}\frac{\hat{R}^2(\eta^-)}{(1-|\eta_\perp|^2)^{1/2}}|a_0(t^{-1}\xi^0,\xi^0-s\eta^-(\eta^\star)^{-1},[z])|^2 \\
&\leq C\int_{\mathbb{R}^{d-1}}\int_{B_1}\int_1^{\infty}\int_0^{\infty}\frac{dtdsd\xi_\perp d\eta_\perp}{(2\pi)^{2(d-1)}}\frac{1}{t^{d-1}\eta^\star}\frac{\hat{R}^2(\eta^-)}{(1-|\eta_\perp|^2)^{1/2}}\frac{|\eta^-|^2}{(1+|\xi_\perp|^2+s^2)^n}, \\
&\leq C\int_{B_1}d\eta_\perp\frac{|\eta^\star(\eta_\perp)|^2+|\eta_\perp|^2}{(1-|\eta_\perp|^2)^{1/2}\eta^\star(\eta_\perp)}.
\end{aligned}$$

The last integral is finite since by definition $|\eta^\star(\eta_\perp)|^2+|\eta_\perp|^2=2\eta^\star(\eta_\perp)$.

The case $d=2$ requires a little more work. We set $\xi_\perp \to t^{-1}\xi_\perp$ and perform an integration by parts in t when $t \geq 1$. It comes, with obvious notation:

$$\begin{aligned}
w_{1,\varepsilon}(\tau) &= \int_0^1 dt + \int_1^{\tau\varepsilon^{1-2\alpha}} dt := w^1_{1,\varepsilon}(\tau) + w^2_{1,\varepsilon}(\tau),\\
w^2_{1,\varepsilon}(\tau) &= \int_1^{\tau\varepsilon^{1-2\alpha}} \frac{H^\varepsilon(t)}{t} dt = \left(\log \tau\varepsilon^{1-2\alpha}\right) H^\varepsilon(\tau\varepsilon^{1-2\alpha}) - \int_1^{\tau\varepsilon^{1-2\alpha}} (H^\varepsilon)'(t) dt, \quad (6.2)\\
H^\varepsilon(t) &= \int_{\mathbb{R}^{d-1}} \int_{B_1} \int_0^{t\varepsilon^{\alpha-1}} \frac{d\xi_\perp d\eta_\perp ds}{(2\pi)^2} \frac{\hat{R}^2(\eta^-)}{(1-|\eta_\perp|^2)^{1/2}} |a_0(t^{-1}\xi^0, \xi^0 - s\eta^-, [z])|^2.
\end{aligned}$$

It is not difficult to see that $w^1_{1,\varepsilon}$ and the second term on the right of (6.2) are of order one compared to ε. Besides, as $\varepsilon \to 0$, we have

$$H^\varepsilon(\tau\varepsilon^{1-2\alpha}) \to \int_{\mathbb{R}^{d-1}} \int_{B_1} \int_0^\infty \frac{d\xi_\perp d\eta_\perp ds}{(2\pi)^{2(d-1)}} \hat{R}^2(\eta^-) |a_0(0, \xi^0 - s\eta^-, [z])|^2,$$

and is therefore also of order one compared to ε. Therefore, when $d=2$, the leading term is given by the one proportional to $\log \tau\varepsilon^{1-2\alpha}$:

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \varepsilon^{-2(1-\alpha)-2(2\alpha-1)} (\log \tau\varepsilon^{1-2\alpha})^{-1} w^L_1(\tau) =\\
&\int_{\mathbb{R}} \int_{B_1} \int_0^\infty \frac{d\xi_\perp d\eta_\perp ds}{(2\pi)^2} \frac{\hat{R}^2(\eta^-)}{(1-|\eta_\perp|^2)^{1/2}} |a_0(0, \xi^0 - s\eta^-, [z])|^2. \qquad (6.3)
\end{aligned}$$

Hence, the limit of the scintillation w^1_ε corresponding to the domain D^0_+ is given by (6.1)-(6.3). The contribution of D^1_+ admits the same expression with η^- replaced by $\eta^+ = (1 + \sqrt{1-|\eta_\perp|^2}, \eta_\perp)$ and simple symmetry considerations then show that the contribution of $D^0_- \cup D^1_-$ is the same as that of $D^0_+ \cup D^1_+$. To conclude the case $1 > \alpha > \frac{1}{2}$, we finally claim as before that w^2_ε shares the same limit as w^1_ε. This ends the case $\alpha \in (\frac{1}{2}, 1)$.

7. The case $d = 1$.

The case $d=1$ is particular in the sense that ξ and η are always aligned with q_0. Starting from expression (3.5), this implies that F_{σ_2} reads:

$$\begin{aligned}
F^\varepsilon_{\sigma_2}(\tau,\xi,\eta) = \int_0^\tau \int_0^{\varepsilon^{-\alpha}t} & dtds \exp\{-i\sigma_2\varepsilon^\alpha s\xi\eta/2\} \exp\{-it\xi\} \exp\left\{\frac{i}{\varepsilon^{1-\alpha}} s\Psi_{\sigma_2}(\eta)\right\}\\
&\times a^\varepsilon\left(\varepsilon^\alpha\xi, \varepsilon^{1-\alpha}t\xi - s\eta, [z^\varepsilon]\right), \qquad (7.1)
\end{aligned}$$

$$\Psi_{\sigma_2}(\eta) = \eta + \frac{\sigma_2}{2}\eta^2, \qquad [z^\varepsilon] = (\tau, t, \varepsilon^\alpha s, \varepsilon\xi, \eta, \sigma_2).$$

Recall that $|q_0| = 1$, so that the phase Ψ_{σ_2} vanishes a the points $\eta = 0$ and $\eta = -2\sigma_2 = \pm 2q_0$. The origin is therefore a singularity as in the multidimensional case and requires a careful treatment. All the methods of the case $\alpha \leq \frac{1}{2}$ of section (4.1)-(4.2)-(4.3) carry on to $d=1$ with some simplifications, for instance only three subdomains in η are necessary. It is then not difficult to show that, when $\alpha > 0$, $\forall\delta \in (0, d)$, pointwise in τ:

$$w_\varepsilon(\tau) \to 0.$$

When $\delta = 0$, the origin is no longer a singularity and the corresponding contribution is negligible. In this case, when $\alpha \in (0, 1)$, we find for the contribution of the subdomain including $2q_0$ (denoted by w_1):

$$\lim_{\varepsilon\to 0} \varepsilon^{-\alpha} w_1(\tau) = \int_{\mathbb{R}^2} \frac{d\xi d\eta}{(2\pi)^2} \hat{R}^2(2q_0) \left| \int_0^\tau \int_0^\infty \exp\{-it\xi\} \exp\{2is\eta q_0\} a_0(0, -2sq_0) \right|^2,$$

where $a_0 = \mathcal{F}(W_0\psi)$ with

$$\psi(x, p) = \sum_{\sigma_1=\pm 1} \sigma_1 \varphi(-(\tau - t)(1 + \sigma_1)q_0 + \tau q_0, q_0 - (1 + \sigma_1)q_0).$$

Using the Fourier Plancherel equality, it comes

$$\lim_{\varepsilon\to 0} \varepsilon^{-\alpha} w_1(\tau) = \frac{1}{4|q_0|^2} \int_0^\tau \int_0^\infty dt ds \hat{R}^2(2q_0) |a_0(0, s)|^2 := \frac{1}{2} w^\alpha(\tau). \tag{7.2}$$

Since $\hat{R}$ is an even function, it follows that the contribution of the subdomain related to $-2q_0$ satisfies the same limit. Moreover, the second scintillation w_ε^2, whose expression is given at the end of section 3.2 and is equal to that of w_ε^1 up to vanishing terms at the limit, also converges to the same limit. Summing up the contributions, this implies that $\varepsilon^{-\alpha} w_\varepsilon$ converges pointwise to the w_α defined in (7.2).

When $\alpha = 0$, the limiting expression becomes

$$\lim_{\varepsilon\to 0} w_1(\tau) = \frac{\hat{R}^2(2q_0)}{8\pi|q_0|^2} \int_0^\tau \int_{\mathbb{R}} ds d\xi \left| \int_s^\tau \exp\{-it\xi\} a_0(\xi, s) \right|^2 := \frac{1}{2} w^0(\tau), \tag{7.3}$$

where now

$$\psi(x, p) = \sum_{\sigma_1=\pm 1} \sigma_1 \varphi(x - (\tau - t)(1 + \sigma_1)q_0 + (\tau - s)q_0, q_0 - (1 + \sigma_1)q_0).$$

Again the contribution related to $-2q_0$ has the same expression. Regarding w_ε^2, a first look at G_{σ_2} seems to indicate that the limit is different since the extra ε^α terms are now of order one. They actually disappear in the final expression using the Fourier-Plancherel equality, leading therefore to the same expression as w_ε^1.

When $\alpha = 1$, there is no longer localization at the zeros of $\Psi_{\sigma_2}(\eta)$, which renders the analysis simpler. We obtain

$$\lim_{\varepsilon\to 0} \varepsilon^{-1} w_1(\tau) := \int_{\mathbb{R}^2} \frac{d\xi d\eta}{(2\pi)^2} \hat{R}^2(\eta)$$
$$\left| \sum_{\sigma_2=\pm 1} \sigma_2 \int_0^\tau \int_0^\infty dt ds \exp\{-it\xi\} \exp\{is\Psi_{\sigma_2}(\eta)\} a_0(0, t\xi - s\eta) \right|^2 := w^1(\tau).$$

where now

$$\psi(x,p) = \sum_{\sigma_1=\pm 1} \sigma_1 \varphi\left(\frac{1}{2}(\tau - t)(\sigma_2-\sigma_1)\eta + \tau(p+q_0), p+q_0+\frac{1}{2}(\sigma_2-\sigma_1)\eta\right).$$

After the change of variables $s\eta \to s\eta + t\xi$ and the Fourier-Plancherel equality, we find

$$w^1(\tau) = 4\int_{\mathbb{R}}\int_0^\tau \frac{d\eta}{2\pi|\eta|}\hat{R}^2(\eta)\left|\sum_{\sigma_2=\pm 1}\sigma_2\int_0^\infty ds \exp\{is\Psi_{\sigma_2}(\eta)\}\, a_0\,(0,s\eta)\right|^2. \quad (7.4)$$

Again, w_ε^2 converges to the same limit. The final result of the section is therefore that, pointwise in τ:

$$\lim_{\varepsilon\to 0}\varepsilon^{-\alpha} w_\varepsilon(\tau) = w^\alpha(\tau),$$

where w^α is defined in (7.2)-(7.3)-(7.4).

8. **Appendix**

Let $\varphi \in \mathcal{S}(\mathbb{R}^{2d})$, $(x,p,\xi,\eta,\tau,t,s) \in \mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ and define

$$\psi^\varepsilon(x,p,[z]) =$$

$$\sum_{\sigma_1=\pm 1}\sigma_1\varphi\Big(\varepsilon^\alpha x + \frac{1}{2}(\tau-t)\sigma_1(\varepsilon^{1-\alpha}\xi - \varepsilon^r\eta) + \tau(q_0+\varepsilon^{1-\alpha}p) + \frac{1}{2}\sigma_2(\tau - t + \varepsilon^h s)\varepsilon^r\eta,$$

$$q_0 + \varepsilon^{1-\alpha}p + \frac{1}{2}\sigma_1(\varepsilon^{1-\alpha}\xi - \varepsilon^r\eta) + \frac{1}{2}\sigma_2\varepsilon^r\eta\Big), \quad (8.1)$$

for some parameters $(\alpha, r, h) \in [0,1]^3$. Above, we used as usual the shorthand $[z] = (\tau, t, s, \xi, \eta, \sigma_2)$. Let $W_0 \in \mathcal{S}(\mathbb{R}^{2d})$ and

$$a^\varepsilon(u,v,[z]) = \mathcal{F}(W_0(\cdot,\cdot)\psi^\varepsilon(\cdot,\cdot,[z]))(u,v),$$

where $\mathcal{F}$ denotes the Fourier transform with respect to x and p. We have the following lemma:

Lemma 8.1. *$\forall n \geq 0$, there exists $C_n > 0$, such that, for $k,l = 0,1$:*

$$|\partial_t^k\partial_s^l a^\varepsilon\left(\xi, \varepsilon^{\gamma'}t\xi - \varepsilon^{r'}s\eta, [z]\right)| \quad (8.2)$$

$$\leq C_n \frac{\varepsilon^{k\gamma'+lr'}|\xi|^k + \varepsilon^{l(h+r)}(|\varepsilon^{1-\alpha}\xi - \varepsilon^r\eta|^k + |\varepsilon^r\eta|^k)}{(1+|\xi|^2+|\varepsilon^{\gamma'}t\xi - \varepsilon^{r'}s\eta|^2)^n}\,|\eta|^l\,|\varepsilon^{1-\alpha}\xi - \varepsilon^r\eta|,$$

pointwise in $(\xi,\eta,\tau,t,s) \in \mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^3$.

Proof. We drop the dependence on $[z]$ for simplicity. Since $\varphi \in \mathcal{S}(\mathbb{R}^{2d})$, it is clear that there exists a constant $C > 0$, such that, for all multi-indices i and j, for $k,l = 0,1$, uniformly in (x,p,t,s):

$$|\partial_x^i\partial_p^j\partial_t^k\partial_s^l\psi^\varepsilon(x,p)| \leq C\varepsilon^{l(h+r)}(|\varepsilon^{1-\alpha}\xi - \varepsilon^r\eta|^k + |\varepsilon^r\eta|^k)|\eta|^l|\varepsilon^{1-\alpha}\xi - \varepsilon^r\eta|. \quad (8.3)$$

Besides,

$$\begin{aligned}\partial_t^k\partial_s^l a^\varepsilon\left(\xi,\varepsilon^{\gamma'}t\xi-\varepsilon^{r'}s\eta\right) &= \left[(\varepsilon^{\gamma'}\xi\cdot\nabla_v)^k(-\varepsilon^{r'}\eta\cdot\nabla_v)^l a^\varepsilon\right]\left(\xi,\varepsilon^{\gamma'}t\xi-\varepsilon^{r'}s\eta\right)\\ &\quad+\mathcal{F}(W_0\partial_t^k\partial_s^l\psi^\varepsilon)(\varepsilon^{\gamma'}t\xi-\varepsilon^{r'}s\eta),\\ &:= a_1+a_2.\end{aligned}$$

Using (8.3) with $k=l=0$, together with $W_0\in\mathcal{S}(\mathbb{R}^{2d})$, there exists $C_n>0$, such that, $\forall n\geq 0$:

$$|a_1(\xi,\varepsilon^{\gamma'}t\xi-s\varepsilon^{r'}\eta)|\leq C_n\frac{|\varepsilon^{\gamma'}\xi|^k|\varepsilon^{r'}\eta|^l}{(1+|\xi|^2+|\varepsilon^{\gamma'}t\xi-\varepsilon^{r'}s\eta|^2)^n}\,|\varepsilon^{1-\alpha}\xi-\varepsilon^r\eta|.$$

In the same way, we find for a_2:

$$|a_2(\xi,\varepsilon^{\gamma'}t\xi-s\varepsilon^{r'}\eta)|\leq C_n\frac{\varepsilon^{l(h+r)}(|\varepsilon^{1-\alpha}\xi-\varepsilon^r\eta|^k+|\varepsilon^r\eta|^k)|\eta|^l}{(1+|\xi|^2+|\varepsilon^{\gamma'}t\xi-\varepsilon^{r'}s\eta|^2)^n}\,|\varepsilon^{1-\alpha}\xi-\varepsilon^r\eta|.$$

Gathering the last two estimates ends the proof.

□

Given $(A,B,\Psi)\in\mathbb{R}^3_*$, $(\xi,\eta)\in\mathbb{R}^d\times\mathbb{R}^d$, we study parametrized oscillatory integrals for the form

$$\mathfrak{I}(\xi,\eta) = \int_0^\tau\int_0^{\varepsilon^{-a}t} dtds\exp\{-isA\}\exp\{-itB\}\exp\{is\Psi\}f(t,s,\xi,\eta),$$

where $a\geq 0$. Since in our analysis $\mathfrak{I}$ needs to be integrated with respect to (A,B,Ψ,ξ,η), we will obtain explicit bounds according to these parameters. In particular, we are interested in controls for large values of B and Ψ as for stationary phase techniques. We assume that $f\in\mathcal{C}^\infty(\mathbb{R}^{2d+2})$ and satisfies the estimate of lemma 8.1. The case corresponding to $\alpha\in[\frac{1}{2},1]$ involves integrals of the form

$$\int_0^{\tau\varepsilon^{1-2\alpha}}\int_0^{\varepsilon^{\alpha-1}t} dtds\exp\{-isA\}\exp\{-itB\}\exp\{is\Psi\}f(t,s,\xi,\eta)$$

that require some modifications in the analysis that we will not pursue. We have the following result:

Lemma 8.2. *$\forall n\geq 0$, there exists $C_n>0$, such that $\mathfrak{I}$ satisfies the estimate, pointwise in $(A,B,\Psi)\in\mathbb{R}^3_*$, $(\xi,\eta)\in\mathbb{R}^d\times\mathbb{R}^d$:*

$$|\mathfrak{I}|\leq C_n\langle\xi\rangle^{-n}I_1\wedge I_2\wedge I_3\wedge I_4,$$

where:

- *Case 1: when $r=r'=0$, $a=h=\alpha\leq\frac{1}{2}$, $\gamma'=1-2\alpha$*

$$\begin{aligned} I_1 &= 1,\quad I_2 = |\Psi|^{-1}(\varepsilon^\alpha+|A|+|\eta|),\quad I_3 = |B|^{-1}(1+|\xi|+|\eta|),\\ I_4 &= |B\Psi|^{-1}\big(\varepsilon^\alpha|B|+(1+|A|+|\eta|)(1+|\xi|+|\eta|)\big).\end{aligned}$$

- *Case 2:* $a = h = 0$, $\gamma' = 1 - 2\alpha$

$$\begin{aligned} I_1 &= |\varepsilon^{1-\alpha}\xi - \varepsilon^r\eta|, \qquad I_4 = I_1, \\ I_2 &= |\Psi|^{-1}(1 + |A| + \varepsilon^{r\wedge r'}|\eta|)|\varepsilon^{1-\alpha}\xi - \varepsilon^r\eta|, \\ I_3 &= |B|^{-1}(1 + |\xi| + \varepsilon^r|\eta|)|\varepsilon^{1-\alpha}\xi - \varepsilon^r\eta|. \end{aligned}$$

Proof. For convience, we omit the dependence of f on (ξ, η) and recast $\mathfrak{I}$ as

$$\begin{aligned} \mathfrak{I} &= \int_0^\tau dt\, \exp\{-itB\}\, \mathfrak{F}(t), \qquad \mathfrak{F}(t) = \int_0^{\varepsilon^{-\alpha}t} ds\, \mathfrak{G}(t,s), \\ \mathfrak{G}(t,s) &= \exp\{-isA\} \exp\{is\Psi\}\, f(t,s). \end{aligned}$$

Case 1: $r = r' = 0$, $a = h = \alpha \le \frac{1}{2}$, $\gamma' = 1 - 2\alpha$. We estimate first $\mathfrak{I}$ for bounded values of B and Ψ. We have using Lemma 8.1 with $k = l = 0$, $\forall n \ge 0$:

$$\begin{aligned} |\mathfrak{F}| &\le C \int_0^{t\varepsilon^{-\alpha}} ds \frac{|\varepsilon^{1-\alpha}\xi - \eta|}{(1 + |\xi|^2 + |\varepsilon^{\gamma'}t\xi - s\eta|^2)^n}, \\ &\le C \int_0^{t\varepsilon^{-\alpha}} ds \frac{|\varepsilon^{1-\alpha}\xi|}{(1 + |\xi|^2 + |\varepsilon^{\gamma'}t\xi - s\eta|^2)^n} + C \int_0^{t\varepsilon^{-\alpha}} ds \frac{|\eta|}{(1 + |\xi|^2 + |\varepsilon^{\gamma'}t\xi - s\eta|^2)^n}, \\ &:= F_1 + F_2. \end{aligned} \tag{8.4}$$

Since $\alpha \le \frac{1}{2}$ and $t \le \tau$, it comes that

$$F_1 \le Ct\varepsilon^{1-2\alpha}\langle\xi\rangle^{-n} \le C\langle\xi\rangle^{-n}. \tag{8.5}$$

For a vector $v \in \mathbb{R}^d$, and $j = 1, \cdots, d$, we denote by $\tilde{v}^j \in \mathbb{R}^{d-1}$ the vector with components $\tilde{v}^j = (v_1, \cdots, v_{j-1}, v_{j+1}, \cdots, v^d)^T$. Using this notation, we have for F_2, $\forall n \ge 1$:

$$F_2 \le C \sum_{j=1}^d \int_0^{t\varepsilon^{-\alpha}} ds \frac{|\eta_j|}{(1 + |\xi|^2 + |\varepsilon^{\gamma'}t\tilde{\xi}^j - s\tilde{\eta}^j|^2 + |\varepsilon^{\gamma'}t\xi_j - s\eta_j|^2)^n}, \tag{8.6}$$

$$\le C \sum_{j=1}^d \int_{\mathbb{R}} \frac{ds}{(1 + |\xi|^2 + |\varepsilon^{\gamma'}t(\tilde{\xi}^j - \xi_j\eta_j^{-1}\tilde{\eta}^j) - s\eta_j^{-1}\tilde{\eta}^j|^2 + |s|^2)^n} \le C\langle\xi\rangle^{-(n-1)}.$$

Consequently, we obtain the first following bound for $\mathfrak{I}$: for all $n \ge 0$, there exists $C_n > 0$ such that

$$|\mathfrak{I}| \le C_n\langle\xi\rangle^{-n}. \tag{8.7}$$

To control $\mathfrak{I}$ for large values of Ψ, we need another estimate. It is obtained by performing an integration by part w.r.t. s in $\mathfrak{F}$. It comes:

$$\mathfrak{F} := \mathfrak{F}_1 + \mathfrak{F}_2, \tag{8.8}$$

where

$$\mathfrak{F}_1(t) = \frac{1}{i\Psi}\mathfrak{G}(t, \varepsilon^{-\alpha}t), \qquad \mathfrak{F}_2(t) = -\frac{1}{i\Psi}\int_0^{\varepsilon^{-\alpha}t} ds \exp\{is\Psi\}\, \partial_s\left[\exp\{-isA\}\, f(t,s)\right].$$

Using (8.2) with $k=l=0$ and the change of variables $t \to t\varepsilon^{-\alpha}$, $\mathfrak{F}_1$ is estimated as

$$\begin{aligned}\int_0^\tau dt|\mathfrak{F}_1(t)| &\leq C\varepsilon^a|\Psi|^{-1}\int_0^{\tau\varepsilon^{-\alpha}} dt\frac{|\varepsilon^{1-\alpha}\xi-\eta|}{(1+|\xi|^2+t^2|\varepsilon^{\gamma'+\alpha}\xi-\eta|^2)^n},\\ &\leq C\varepsilon^\alpha|\Psi|^{-1}\int_0^\infty \frac{dt}{(1+|\xi|^2+t^2)^n} \leq \varepsilon^\alpha|\Psi|^{-1}\langle\xi\rangle^{-(n-1)}. \quad (8.9)\end{aligned}$$

Above, we used the fact that $\gamma'+a=1-\alpha$. $\mathfrak{F}_2$ is estimated using the same method as (8.6) along with (8.2) with $k=0$, $l=1$, the only difference is that $\exp\{-isA\} f(t,s)$ is replaced by $\partial_s\left[\exp\{-isA\} f(t,s)\right]$. We find

$$\int_0^{\tau\varepsilon^{-b}} dt|\mathfrak{F}_2(t)| \leq C|\Psi|^{-1}\langle\xi\rangle^{-n}(|A|+|\eta|).$$

Together with (8.9), this gives a second estimate for $\mathfrak{I}$, $\forall n\geq 0$:

$$C_n^{-1}|\Psi|\langle\xi\rangle^n|\mathfrak{I}| \leq \varepsilon^\alpha+|A|+|\eta|. \quad (8.10)$$

In order to control $\mathfrak{I}$ for large B, we perform an integration by part w.r.t. t in $\mathfrak{I}$. It comes:

$$\mathfrak{I} := \mathfrak{I}_1+\mathfrak{I}_2+\mathfrak{I}_3,$$

where

$$\begin{aligned}\mathfrak{I}_1 &= \frac{1}{-iB}\exp\{-iB\tau\}\,\mathfrak{F}(\tau), \qquad \mathfrak{I}_2 = \frac{1}{iB\varepsilon^\alpha}\int_0^\tau dt\exp\{-itB\}\,\mathfrak{G}(t,\varepsilon^{-\alpha}t),\\ \mathfrak{I}_3 &= \frac{1}{iB}\int_0^\tau\int_0^{t\varepsilon^{-\alpha}} dtds\exp\{-itB\}\,\partial_t\mathfrak{G}(t,s).\end{aligned}$$

To control $\mathfrak{I}_1$, we use the decomposition (8.4) with t replaced by τ and follow the same lines. We find:

$$|\mathfrak{I}_1| \leq C|B|^{-1}\langle\xi\rangle^{-n}. \quad (8.11)$$

For $\mathfrak{I}_2$, we set $t\to\varepsilon^\alpha t$ and obtain, using (8.2) with $k=l=0$, for any $n\geq 0$:

$$\begin{aligned}|\mathfrak{I}_2| &\leq C|B|^{-1}\langle\xi\rangle^{-n}\int_0^{\tau\varepsilon^{-\alpha}} dt\frac{|\varepsilon^{1-\alpha}\xi-\eta|}{(1+t^2(|\varepsilon^{\gamma'+\alpha}\xi-\eta|)^2)^n},\\ &\leq C|B|^{-1}\langle\xi\rangle^{-n}\int_0^\infty\frac{dt}{(1+t^2)^n} \leq C|B|^{-1}\langle\xi\rangle^{-n}. \quad (8.12)\end{aligned}$$

Above, we used again the fact that $\gamma'+a=1-\alpha$. $\mathfrak{I}_3$ is treated in a similar fashion as the first estimate (8.6). The only difference is that $\mathfrak{G}$ has to be replaced by $\partial_t\mathfrak{G}$. We find, using (8.2) with $k=1$ and $l=0$:

$$|\mathfrak{I}_3| \leq C|B|^{-1}\langle\xi\rangle^{-n}(|\xi|+|\eta|).$$

Together with (8.11) and (8.12), this yields the third estimate:

$$|\mathfrak{I}| \leq C_n(1+|\eta|+|\xi|)|B|\langle\xi\rangle^{-n}. \quad (8.13)$$

It remains now to obtain a bound that allows us to control the large values of both Ψ and B. For this, we perform an integration by parts w.r.t. s in $\mathfrak{I}_1$ and $\mathfrak{I}_3$, and w.r.t. t in $\mathfrak{I}_2$. The term $\mathfrak{I}_1$ involves

$$\mathfrak{F}(\tau) = \frac{1}{i\Psi}\mathfrak{G}(\tau, \varepsilon^{-\alpha}\tau) - \frac{1}{i\Psi}\int_0^{\varepsilon^{-\alpha}\tau} ds \exp\{is\Psi\}\, \partial_s \left[\exp\{-isA\}\, f(\tau, s)\right].$$

The first term is directly estimated by

$$|\mathfrak{G}(\tau, \varepsilon^{-\alpha}\tau)| \leq C\langle\xi\rangle^{-n}|\varepsilon^{1-\alpha}\xi - \eta| \leq C\langle\xi\rangle^{-n}(|\xi| + |\eta|).$$

For the second, we proceed as for (8.5)-(8.6) except that f is replaced by the expression $\partial_s \left[\exp\{-isA\}\, f(\tau, s)\right]$. We then find for $\mathfrak{I}_1$:

$$C^{-1}|B||\Psi|\langle\xi\rangle^n|\mathfrak{I}_1| \quad \leq \quad 1 + |A| + |\eta| + |\xi|. \tag{8.14}$$

Regarding $\mathfrak{I}_2$, after an integration by part in t, we have with $\mathfrak{I}_2 := \mathfrak{I}_2^1 + \mathfrak{I}_2^2$,

$$\begin{aligned}
\mathfrak{I}_2^1 &= -\frac{1}{B\Psi}\exp\{-i\tau\Psi\}\,\mathfrak{G}(\tau, \varepsilon^{-\alpha}\tau),\\
\mathfrak{I}_2^2 &= \frac{1}{B\Psi}\int_0^{\varepsilon^{-\alpha}\tau} dt \exp\{it\Psi\}\, \partial_t \left[\exp\{-i\varepsilon^\alpha tB\}\, f(\varepsilon^\alpha t, t)\right].
\end{aligned}$$

We find directly

$$|\mathfrak{I}_2^1| \leq C|B\Psi|^{-1}\langle\xi\rangle^{-n}(|\xi| + |\eta|). \tag{8.15}$$

For $\mathfrak{I}_2^2$, according to (8.2), we have

$$|\partial_t \left[\exp\{-i\varepsilon^\alpha tB\}\, f(\varepsilon^\alpha t, t)\right]| \quad \leq \quad C\bigg(\varepsilon^\alpha|B| + |\xi| + |\eta|\bigg)\frac{|\varepsilon^{1-\alpha}\xi - \varepsilon^r\eta|}{(1 + t^2(|\varepsilon^{\gamma'+\alpha}\xi - \eta|)^2)^n},$$

so that, following the same technique as (8.9), and the fact that $\alpha + \gamma' = 1 - \alpha$:

$$|\mathfrak{I}_2^2| \leq C|B\Psi|^{-1}\langle\xi\rangle^{-n}\left(\varepsilon^\alpha|B| + |\xi| + |\eta|\right). \tag{8.16}$$

$\mathfrak{I}_3$ is treated in a similar manner as $\mathfrak{F}$ decomposed as $\mathfrak{F}_1 + \mathfrak{F}_2$ in (8.8) in order to obtain (8.10), only $\mathfrak{G}$ needs to be replaced by $\partial_t\mathfrak{G}$. We find, all computations done:

$$C|B\Psi|\langle\xi\rangle^n|\mathfrak{I}_3| \leq (\varepsilon^\alpha + |A| + |\eta|)(|\xi| + |\eta|). \tag{8.17}$$

Gathering (8.14)-(8.16)-(8.16)-(8.17), this provides the last estimate,

$$C_n^{-1}|B\Psi||\mathfrak{I}|\langle\xi\rangle^n \quad \leq \quad \varepsilon^\alpha|B| + (1 + |A| + |\eta|)(1 + |\xi| + |\eta|). \tag{8.18}$$

The lemma is proved by taking the best estimate among (8.7)-(8.10)-(8.13)-(8.18).

Case 2: $a = h = 0$**,** $\gamma' = 1 - 2\alpha$**.** . The proof is simpler and very similar to the first case, so we only underline the differences. Since $a = 0$, the integration in s over $\mathbb{R}$ is not required any longer in order to obtain an estimate uniform in ε, and (8.7) can be replaced with

$$|\mathfrak{I}| \leq C\langle\xi\rangle^{-n}|\varepsilon^{1-\alpha}\xi - \varepsilon^r\eta|. \tag{8.19}$$

For the second estimate, we remark with the help of Lemma 8.1 that

$$\partial_s \left[\exp\left\{-isA\right\} f(t,s)\right] \leq C\frac{|A| + (\varepsilon^r + \varepsilon^{r'})|\eta|}{(1+|\xi|^2 + |\varepsilon^{\gamma'}t\xi - s^{r'}\eta|^2)^n}|\varepsilon^{1-\alpha}\xi - \varepsilon^r\eta|,$$

which transforms (8.10) into

$$|\mathfrak{I}| \quad \leq \quad C_n|\Psi|^{-1}\langle\xi\rangle^n(1+|A| + \varepsilon^{r\wedge r'}|\eta|)|\varepsilon^{1-\alpha}\xi - \varepsilon^r\eta|. \tag{8.20}$$

In the same way, it is not difficult to show that (8.13) becomes

$$|\mathfrak{I}| \quad \leq \quad C_n|B|^{-1}\langle\xi\rangle^n(1+|\xi| + \varepsilon^r|\eta|)|\varepsilon^{1-\alpha}\xi - \varepsilon^r\eta|, \tag{8.21}$$

which concludes the proof of the Lemma. □

Acknowledgments. This work was funded in part by the NSF. OP acknowledges support from the French ANR project QUATRAIN and the ERC Starting Grant NUSIKIMO.

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