Approximate stability estimates in inverse transport theory

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Abstract

The theory of inverse transport consists of reconstructing optical properties of a domain of interest from measurements performed at the boundary of the domain. Using the decomposition of the measurement operator into singular components (ballistic part, single scattering part, multiple scattering part), several stability estimates have been obtained that show what may stably be reconstructed from available measurements. Such stability estimates typically assume that the measurements are in the range of the functional mapping the optical parameters to the measurement operator. In practice, available measurements are rarely in the latter range, which renders the stability estimates of lesser interest. In this paper, we generalize the derivation of the stability estimates to account for general physical noise models. The resulting approximate stability estimates provide a quantitative description of the type of information that may be obtained on the optical parameters.

1 Introduction

The theory of inverse transport has many applications in medical and geophysical imaging. The forward transport equation models the propagation

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of particles or of the energy density of waves in scattering media. The interaction of the particles with the underlying medium is characterized by the optical parameters $\sigma(x)$, the extinction coefficient or total absorption coefficient, and $k(x, v', v)$ the scattering coefficient. The inverse transport problem consists of reconstructing the latter two parameters from redundant measurements of the particle density at the boundary of a domain of interest.

Typical applications of inverse transport in medical imaging are optical tomography [1, 16] and optical molecular imaging [11]. Applications in remote sensing in the atmosphere are considered in [15]. Inverse transport can also be used efficiently for imaging using high frequency waves propagating in highly heterogeneous media; see e.g. [8, 9].

The amount of information that may theoretically be reconstructed depends on the type of available boundary measurements. One may typically consider four categories of measurements, depending on the availability of time dependent versus time independent measurements and the availability of angularly resolved versus angularly averaged measurements. Time independent angularly averaged measurements are what is routinely considered in practice in optical tomography, although the inversion is known mathematically to be severely ill-posed, which implies that the (spatial) resolution one expects from such measurements is relatively low. The availability of time dependent and/or angularly resolved measurements significantly improves the resolution. Section 2 briefly recalls known stability results in inverse transport theory.

The main drawback for the stability results as they are known in the literature is that they account for noise terms that are of a very specific nature and do not account for noise terms as they appear in many practical situations. The objective of this paper is to extend some known stability results to a large variety of practical noise models, such as noise coming from blurred measurements and resolution-limited source terms. This comes at the price that the stability results then become approximate stability results. Section 3 presents the approximate stability results considered in this paper and comments on their relevance. A proof of their derivation is postponed to section 4.

2 Stability in inverse transport

We consider the following time independent linear transport equation

$$\begin{aligned} v \cdot \nabla u + \sigma(x) u &= \int_V k(x, v', v) u(x, v') dv', \qquad && (x, v) \in X \times V \\ u &= g && (x, v) \in \Gamma_-. \end{aligned} \tag{2.1}$$

The solution $u(x,v)$ models the density of particles, such as photons, as a function of space x and velocity v. The spatial domain X is a convex, bounded, open subset of $\mathbb{R}^d$ for dimension $d \geq 2$, with a C^1 boundary ∂X. The space of velocities $V = \mathbb{S}^{d-1}$ is here the unit sphere to simplify the presentation. The sets of incoming conditions Γ_- and outgoing conditions Γ_+ are defined by

$$\Gamma_\pm = \{(x,v) \in X \times V, \text{ s.t. } \pm v \cdot \nu(x) > 0\}, \tag{2.2}$$

where $\nu(x)$ is the outgoing normal vector to X at $x \in \partial X$.

The optical parameters $\sigma(x)$ and $k(x,v',v)$ model the interaction of the particles (e.g. photons) with the underlying medium (e.g. biological tissues). The parameter $\sigma(x)$ models the total absorption of particles caused by intrinsic absorption and by scattering of particles into other directions. The scattering coefficient $k(x,v',v)$ indicates the amount of particles scattering from a direction v into a direction v' at position x. The inverse transport problem consists of reconstructing the optical parameters from boundary measurements of $u(x,v)$ on Γ_+.

We also consider the corresponding transport equation in the time dependent setting

$$\begin{aligned} &\frac{\partial u}{\partial t} + v \cdot \nabla u + \sigma(x) u = \int_V k(x,v',v) u(x,v') dv', \qquad && (t,x,v) \in \mathrm{T} \times X \times V \\ &u = g && (t,x,v) \in \mathrm{T} \times \Gamma_-, \end{aligned} \tag{2.3}$$

where T is an interval of time. Measurements are then performed for $(t,x,v) \in \mathrm{T} \times \Gamma_+$.

Both in the time dependent and time independent settings, we denote by $\mathcal{A}$ the albedo operator, which maps $u_{|\Gamma_-} = g$ on Γ_- to the transport solution u restricted to Γ_+:

$$\begin{array}{rcrcl} \mathcal{A} & : & u_{|\Gamma_-}(t,x,v) & \mapsto & \mathcal{A}u_{|\Gamma_-}(t,x,v) = u_{|\Gamma_+}(t,x,v) \\ \mathcal{A} & : & u_{|\Gamma_-}(x,v) & \mapsto & \mathcal{A}u_{|\Gamma_-}(x,v) = u_{|\Gamma_+}(x,v), \end{array} \tag{2.4}$$

in the time dependent and time independent settings, respectively.

In many applications in medical and geophysical imaging, only partial information about the above albedo operator is accessible. In many practical settings in optical tomography, the source term $g = g(x)$ is independent of the angular variable and the available measurements are of the form $J(x) = \int_V |v \cdot \nu(x)| u(x,v) dv$, i.e., are angularly averaged measurements. The reconstruction of the optical parameters is seriously compromised when only averaged measurements are available. We refer the reader to [7] for inverse transport results in the time independent setting and to [3, 6] for results on the time dependent setting. The main conclusion of these studies is that the reconstruction of the optical parameters in the time independent setting is a severely ill-posed problem. Mathematically, this implies that the spatial resolution is necessarily extremely limited. This is consistent with practical resolutions obtained in optical tomography [2, 10, 17]. In the time dependent setting, accurate measurements in time allow us to stably reconstruction the attenuation coefficient and the spatial distribution (but not the angular phase function) of the scattering coefficient; we refer the reader to [3, 6] for more details.

In this paper, we assume that we are capable of measuring the angularly resolved density of outgoing particles and that the incoming particles are also angularly resolved (which then significantly increases the time of acquisition of the data). In such a setting, the absorption and scattering coefficients are uniquely determined by knowledge of the albedo operator, except in the time independent setting and in spatial dimension $d = 2$, where the scattering coefficient is uniquely determined only when it is sufficiently small. We refer the reader to [4, 5, 12, 13, 18, 20] for uniqueness and stability results in this setting.

The main ingredient in the derivation of the stability analysis is the decomposition of the albedo operator into three terms $\mathcal{A} = \sum_{j=1}^{3} \mathcal{A}_j$, where $\mathcal{A}_1$ denotes the part of the measurements that does not depend on scattering (the ballistic part) and is thus obtained by setting $k = 0$ in the above transport equation; $\mathcal{A}_2$ is the single scattering part of the measurement, which is linear in the scattering coefficient k; and $\mathcal{A}_3$ is the multiple scattering component of $\mathcal{A}$, corresponding to measured particles that have scattered at least twice before they exit the domain X.

The ballistic part $\mathcal{A}_1$ is always more singular than the other contributions and may thus be obtained from knowledge of $\mathcal{A}$. Knowledge of $\mathcal{A}_1$ implies knowledge of the Radon transform of σ, hence knowledge of σ by inverse Radon transform. In dimension $d \geq 3$ in the time independent setting and in dimension $d \geq 2$ in the time dependent setting, $\mathcal{A}_2$ is also more singular than $\mathcal{A}_3$ so that knowledge of $\mathcal{A}$ again uniquely determines $\mathcal{A}_2$. Since $\mathcal{A}_2$ is linear in the scattering coefficient k, a simple inversion formula shows that k is again uniquely determined from $\mathcal{A}_2$. These uniqueness results do not tell us how errors in the measurements propagate in the inversion. Stability estimates quantify this propagation in errors.

Let (σ, k) and $(\tilde{\sigma}, \tilde{k})$ be two sets of continuous optical parameters and such that the transport equation (2.1) is well posed in the time dependent setting; see [4]. Let $\mathcal{A}$ and $\tilde{\mathcal{A}}$ be the corresponding albedo (measurement) operators. We define

$$E(x,y) := \exp\Big(-\int_0^{|x-y|}\sigma\big(x - s\frac{x-y}{|x-y|}\big)ds\Big), \tag{2.5}$$

the optical depth between points $x, y \in \partial X$ and define $\tilde{E}(x,y)$ similarly by replacing σ by $\tilde{\sigma}$. We also define

$$E_+(x,v,w) = \exp\Big(-\int_0^{\tau_-(x,v)}\sigma(x-sv)ds - \int_0^{\tau_+(x,w)}\sigma(x+sw)ds\Big), \tag{2.6}$$

the optical path along the broken line with directions $-v$ and w and changing direction at point x, and define $\tilde{E}_+$ similarly by replacing σ by $\tilde{\sigma}$. Here, $\tau_\pm(x,v)$ is defined as the distance between $x \in X$ and the boundary ∂X in the direction $\pm v$.

Let $(x_0, v_0) \in \Gamma_-$ and define $y_0 = x_0 + \tau_+(x_0,v_0)v_0 \in \partial X$. Then we have the following stability results

$$\begin{aligned} \big|E(x_0,y_0) - \tilde{E}(x_0,y_0)\big| &\leq \|\mathcal{A} - \tilde{\mathcal{A}}\|_{\mathcal{L}(L^1)} \\ \int_V\int_0^{\tau_+(x_0,v_0)}\big|E_+k - \tilde{E}_+\tilde{k}\big|(x_0+sv_0,v_0,v)dsdv &\leq \|\mathcal{A} - \tilde{\mathcal{A}}\|_{\mathcal{L}(L^1)}. \end{aligned} \tag{2.7}$$

In the time dependent setting, we have assumed that the interval of time T was sufficiently large; see [5]. The first estimate holds for $d \geq 2$. The second estimate holds when $d \geq 2$ in the time dependent setting and when $d \geq 3$ in the time independent setting.

The operator norms are defined by $\|\cdot\|_{\mathcal{L}(L^1)} = \|\cdot\|_{\mathcal{L}(L^1(\mathrm{T},L^1(\Gamma_-,d\xi)),L^1(\mathrm{T},L^1(\Gamma_+,d\xi)))}$ in the time dependent setting and $\|\cdot\|_{\mathcal{L}(L^1)} = \|\cdot\|_{\mathcal{L}(L^1(\Gamma_-,d\xi),L^1(\Gamma_+,d\xi))}$ in the time independent setting, where $d\xi = |v\cdot\nu(x)|dvd\mu(x)$ is a measure on $\Gamma_\pm$ and $d\mu(x)$ is the surface measure on ∂X.

Such estimates provide a precise relationship between errors in the measurements and errors on simple functionals of the optical parameters. Provided that additional regularity assumptions are met by the optical parameters, direct stability estimates for σ and k may be obtained as in [4, 5, 20].

The major drawback of such estimates is that they assume that the available measurements are in the *range* of the operator mapping the optical parameters (σ, k) to the albedo operator $\mathcal{A} = \mathcal{A}(\sigma, k)$. In practice, the available measurements are likely to be obtained as blurred versions of the "exact" measurements. Let us denote by $\mathcal{A}_\varepsilon$ a blurred version of $\mathcal{A}$ at a scale ε. It turns out that for most types of blurring one may consider, $\|\mathcal{A} - \mathcal{A}_\varepsilon\|_{\mathcal{L}(L^1)}$ exists but is of order $O(1)$ independent of ε. To understand this, we may look at this simplified example. Let $A = I$ be the identity operator and A_ε

be the convolution by $\varepsilon^{-d}\phi(\frac{x}{\varepsilon})$ for a smooth, compactly supported, function $\phi(x) \geq 0$ such that $\int_{\mathbb{R}^d} \phi(x)dx = 1$. Then A_ε converges to A strongly but not uniformly and it is straightforward to obtain that $\|A - A_\varepsilon\|_{\mathcal{L}(L^1(\mathbb{R}^d))} = 2$ independent of ε. This renders the practical use of estimates such as (2.7) relatively uninteresting as soon as the noisy measurements are not in the range of the operator mapping the optical parameters to the measurement operator.

The analysis of the albedo operator leading to (2.7) may be refined to provide approximate (inexact) stability estimates, which account for noise in the available measurements. This is addressed in the next two sections.

3 Approximate stability results

Since the $\mathcal{L}(L^1)$ norm on $\mathcal{A}-\tilde{\mathcal{A}}$ is too stringent, we need to slightly modify the metric in which we gauge the quality of measurements. What we propose to do here is to assume that the source and the detectors have limited resolution.

Time independent setting. We start with the time independent setting. The source resolution is quantified by the scale $\varepsilon = (\varepsilon_1, \varepsilon_2)$, where ε_1 measures the minimal spatial extension of the source and ε_2 the minimal angular extension. In other words, the source term may be written as a function of the form $\phi(\frac{x-x_0}{\varepsilon_1})\psi(\frac{v-v_0}{\varepsilon_2})$. The detector resolution is quantified by the scale $\eta = (\eta_1, \eta_2)$, where again η_1 is related to spatial resolution and η_2 to angular resolution.

The smoothing of the detectors is quantified by a kernel ϕ_η. Let $\eta_i > 0$, $i = 1, 2$, and $\phi_\eta \in C^1(\Gamma_+ \times \Gamma_+, \mathbb{R})$ be such that

$$\phi_\eta \geq 0, \tag{3.1}$$

$$\mathrm{supp}\phi_\eta \subset \{(x, v, y, w) \in \Gamma_+ \times \Gamma_+ \mid |x - y| < \eta_1 \text{ and } |v - w| < \eta_2\}, \tag{3.2}$$

$$\int_{\Gamma_+} \phi_\eta(x, v, y, w)d\xi(y, w) = 1 \text{ for all } (x, v) \in \Gamma_+, \tag{3.3}$$

$$\int_{\Gamma_+} \phi_\eta(x, v, y, w)d\xi(x, v) \leq C \text{ for all } (y, w) \in \Gamma_+, \tag{3.4}$$

where C is a constant. We denote by R_η the bounded operator from $L^1(\Gamma_+, d\xi)$ to $L^1(\Gamma_+, d\xi)$ defined by

$$R_\eta g(x, v) = \int_{\Gamma_+} \phi_\eta(x, v, y, w)g(y, w)d\xi(y, w), \tag{3.5}$$

for a.e. $(x,v) \in \Gamma_+$ and for $g \in L^1(\Gamma_+, d\xi)$. Note that R_η is a smoothing operator at the spatial scale η_1 and the angular scale η_2. The details of the optical coefficients at scales smaller than η are thus not recoverable in a stable manner.

We now model some reasonable limitations of the source term. Let $(x_0', v_0') \in \Gamma_-$. The point $(x_0', v_0') \in \Gamma_-$ models the incoming condition and is fixed in the analysis that follows. For $\varepsilon := (\varepsilon_1, \varepsilon_2) \in (0,+\infty)^2$, let $f_\varepsilon \in C_0^1(\Gamma_-)$ such that

$$\|f_\varepsilon\|_{L^1(\Gamma_-, d\xi)} = 1, \tag{3.6}$$

$$f_\varepsilon \geq 0, \tag{3.7}$$

$$\mathrm{supp} f_\varepsilon \subset \{(x',v') \in \Gamma_- \mid |x' + \tau_+(x',v')v' - x_0' - \tau_+(x_0', v_0')v_0'| < \varepsilon_1 \text{ and } |v' - v_0'| < \varepsilon_2\}. \tag{3.8}$$

The condition for ε_1 is written as a constraint on Γ_+ rather than a constraint on Γ_-. Yet f_ε above is easily seen as a smooth approximation of the delta function on Γ_- at (x_0', v_0') as $\varepsilon_1 \to 0^+$ and $\varepsilon_2 \to 0^+$ and is thus an admissible incoming condition in $L^1(\Gamma_-, d\xi)$.

Now that the source term has resolution limited by ε and the measurements are convolved measurements at the scale η, we need to select measurements that capture the singularities of the albedo operator while eliminating multiple scattering as efficiently as possible. Since the source term and detector resolution is limited, the separation between different orders of scattering based on the singularities of the albedo operator is no longer feasible exactly. The role of approximate stability estimates is to show what may still be reconstructed stably and with which error. The selection of measurements is different for the ballistic an the single scattering parts. The selection is performed by means of a function ψ whose support indicates which measurements are selected or not. Such a function is different for the selection of the ballistic and the single scattering components as we shall see.

Assume that $(k, \tilde{k}) \in L^\infty(X \times V \times V)^2$. Let $\psi \in L^\infty(\Gamma_+)$ such that $\|\psi\|_{L^\infty(\Gamma_+)} \leq 1$. Then using (4.8), (3.5) and (4.6) we obtain for $\varepsilon > 0$ that

$$\int_{\Gamma_+} \psi(x,v) R_\eta(\mathcal{A} - \tilde{\mathcal{A}}) f_\varepsilon(x,v) d\xi(x,v) = I_1(\psi,\eta,\varepsilon) + I_2(\psi,\eta,\varepsilon) + I_3(\psi,\eta,\varepsilon), \tag{3.9}$$

where

$$I_1(\psi,\eta,\varepsilon) = \int_{\Gamma_+}\psi(x,v)\int_{\Gamma_+}\phi_\eta(x,v,y,w)\Big(e^{-\int_0^{\tau_-(y,w)}\sigma(y-sw,w)ds} \tag{3.10}$$
$$-e^{-\int_0^{\tau_-(y,w)}\tilde\sigma(y-sw,w)ds}\Big) f_\varepsilon(y-\tau_-(y,w)w,w)d\xi(y,w)d\xi(x,v),$$

$$I_2(\psi,\eta,\varepsilon) = \int_{\Gamma_+}\psi(x,v)\int_{V\times\Gamma_+}\phi_\eta(x,v,y,w) \tag{3.11}$$
$$\int_0^{\tau_-(y,w)}\Big((kE_+)(y-tw,w',w)-(\tilde k\tilde E_+)(y-tw,w',w)\Big)$$
$$f_\varepsilon(y-tw-\tau_-(y-tw,w')w',w')dtd\xi(y,w)dw'd\xi(x,v),$$

$$I_3(\psi,\eta,\varepsilon) \le C\int_V\Bigg(\int_{\substack{y\in\partial X\\ \nu(y)\cdot w>0}}\left|\int_{\Gamma_+}\phi_\eta(x,v,y,w)\psi(x,v)d\xi(x,v)\right|$$
$$\times(\nu(y)\cdot w)dy\Bigg)^{\frac{1}{p'}}dw, \tag{3.12}$$

and $C=C(p,X,V\sigma,k,\tilde\sigma,\tilde k)$ for $1<p<\frac{d}{d-1}$ and $p^{-1}+p'^{-1}=1$, and where $E_+(z,w',w)=\exp\Big(-\int_0^{\tau_-(z,w')}\sigma(z-sw',w')ds-\int_0^{\tau_+(z,w)}\sigma(z+sw,w)ds\Big)$ for $(z,w',w)\in X\times V\times V$ (in section 3 and 4 the absorption coefficient $\sigma(x,v)$ is allowed to depend on the velocity variable). Here, I_1 corresponds to the ballistic part we aim at selecting while I_2 and I_3 correspond to the scattering components. The objective is to find some stability for I_1.

Most of the theorems above are presented for $V=\mathbb{S}^{d-1}$ the unit sphere. The results adapt with few modifications to the case where V is a bounded open subset in $\mathbb{R}^d$ with $\inf_{v\in V}|v|>0$. It turns out that that approximate stability estimates depend on the dimension on V. To present estimates that work in both scenarios, we define $\dim V$ as

$$\mathrm{dim}V := \begin{cases} d-1, \text{ if } V:=\mathbb{S}^{d-1},\\ d, \text{ if } V \text{ is an open subset of } \mathbb{R}^d.\end{cases} \tag{3.13}$$

The main approximate stability result is as follows.

Theorem 3.1. *Assume that (σ,k) and $(\tilde\sigma,\tilde k)$ satisfy (4.1) and (4.4). Assume also $(k,\tilde k)\in L^\infty(X\times V\times V)^2$. Let $1<p<\frac{d}{d-1}$ and let $p'=\frac{p}{p-1}>d$. Then the following statements are valid:*

i. there exists a constant $C_1=C_1(X,V,p,\sigma,k,\tilde\sigma,\tilde k)$ such that

$$|I_1(\psi,\eta,\varepsilon)|\le\|R_\eta(\mathcal{A}-\tilde{\mathcal{A}})f_\varepsilon\|_{L^1(\Gamma_+,d\xi)}+C_1(\eta_2+\rho)^{\dim(V)}, \tag{3.14}$$

for $(\rho,\varepsilon_1,\varepsilon_2,\eta_1,\eta_2)\in(0,+\infty)^5$ and for $\psi\in L^\infty(\Gamma_+)$, $\|\psi\|_{L^\infty(\Gamma_+)}\le1$,

$$\mathit{supp}\psi\subset\{(x,v)\in\Gamma_+\mid |v-v_0'|<\rho\}. \tag{3.15}$$

ii. There exists a constant $C_2 = C_2(X, V, p, \sigma, k, \tilde{\sigma}, \tilde{k})$ *such that*

$$
\begin{aligned}
|I_2(\psi,\eta,\varepsilon)| &\le \|R_\eta(\mathcal{A}-\tilde{\mathcal{A}})f_\varepsilon\|_{L^1(\Gamma_+,d\xi)} + C_2\left(\rho_1+\eta_1+\frac{2\mathrm{diam}(X)\check{\eta}_2}{\sqrt{1-\check{\eta}_2}}\right)^{\frac{d-2}{p'}}, \\
\check{\eta}_2 &= \frac{2\eta_2}{\mathrm{v}_0(1-\rho_2^2)^{\frac{1}{2}}}
\end{aligned}
\qquad (3.16)
$$

for $(\rho_1, \rho_2, \varepsilon_1, \varepsilon_2, \eta_1, \eta_2) \in (0, +\infty)^6$, $\check{\eta}_2 < 1$, *and for* $\psi \in L^\infty(\Gamma_+)$, $\|\psi\|_{L^\infty(\Gamma_+)} \le 1$,

$$
supp\psi \subset \{(x,v)\in\Gamma_+ \mid |x-x_0'-\tau_+(x_0',v_0')v_0'| > \eta_1+\varepsilon_1 \ or \ |v-v_0'| > \eta_2+\varepsilon_2\}, \qquad (3.17)
$$

$$
supp\psi \subset \{(x,v)\in\Gamma_+ \mid \inf_{(s,s')\in\mathbb{R}^2} |x-sv-x_0'+s'v_0'| < \rho_1\}, \qquad (3.18)
$$

$$
supp\psi \subset \{(x,v)\in\Gamma_+ \mid |\hat{v}\cdot\hat{v}_0'| < \rho_2\}. \qquad (3.19)
$$

Theorem 3.1 is proved in Section 4. We now comment on its significance. The first result (3.14) applies to all functions ψ supported in the velocity variable in the ρ-vicinity of v_0 as indicated in (3.15). Not all such test functions are of interest. When ρ is much smaller than ε_2 or η_2, then $I_1(\psi,\eta,\varepsilon)$ does not capture the whole ballistic part. This renders the estimate (3.14) useless. The support of ψ thus needs to be sufficiently large so that it captures the ballistic part. With our assumptions on f_ε and ϕ_η, this means that ψ should have a support of size $\varepsilon_1+\eta_1$ in the vicinity of x_0 and of size $\varepsilon_2+\eta_2$ in the vicinity of v_0.

Once the support of ψ is sufficiently large as indicated above, then $I_1(\psi,\eta,\varepsilon)$ captures the ballistic part of the signal up to an error caused by single and multiple scattering as indicated in Theorem 3.1. This is the error made on the Radon transform of σ averaged over the support of ψ. We then have to invert the Radon transform from these smoothed out measurements at the scale of the support of ψ. This is a task that needs to be performed carefully and whose analysis will be carried out elsewhere. At a qualitative level, we expect to reconstruct $\sigma = \sigma(x)$ at the scale limited by the support of ψ. The latter should therefore be sufficiently large (of size $\varepsilon_1+\eta_1$ in the vicinity of x_0 and of size $\varepsilon_2+\eta_2$ in the vicinity of v_0) in order to capture the ballistic front and yet sufficiently small so as to guarantee the best available resolution for the reconstruction of σ. All spatial scales in σ smaller than ε and η cannot be reconstructed stably. What our results says is that all scales larger than these numbers can indeed be reconstructed stably from transport measurements.

The second result (3.16) in Theorem 3.1 addresses the reconstruction of the scattering coefficient. The test function ψ should be supported away from the ballistic part, have a support that is sufficiently large so that it captures all of the single scattering contribution, and yet not too large so that the multiple scattering contribution is small over the support and so that resolution is not lost in the reconstruction of $k(x, v', v)$ from available transport measurements. Note the role of $\check{\eta}_2$ as a combination of η_2 and ρ_2. The term involving $\check{\eta}_2$ shows that the reconstruction of $k(x, v_0, v)$ involves an error of order $\check{\eta}_2 \sim \eta_2$ when v_0 and v are not close to being parallel (i.e., when $v_0 \cdot v$ bounded away from 1). When v_0 and v become parallel, it becomes harder to separate the ballistic part from the single scattering part and $\check{\eta}_2 \gg \eta_2$ when ρ_2 approaches 1.

Time dependent setting. We now consider the time dependent setting. The results are very similar to those in the time independent setting. The main difference is that resolution in the time variable needs to be accounted for. We now have $\eta = (\eta_1, \eta_2, \eta_3) \in (0, +\infty)^3$, where η_1 measures smoothing in time, η_2 smoothing in space, and η_3 smoothing in velocity. Similarly, $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3) \in (0, +\infty)^3$ measures the support of the source term in time, space, and velocity, respectively.

Let $\phi_\eta \in C^1(\mathbb{R} \times \Gamma_+ \times \Gamma_+, \mathbb{R})$ be such that

$$\begin{aligned}
&\phi_\eta \geq 0, && (3.20)\\
&\mathrm{supp}\phi_\eta \subset \{(\tau, x, v, y, w) \in T \times \Gamma_+ \times \Gamma_+ \mid 0 < \tau < \eta_1,\ |x - y| < \eta_2 \\
&\ \text{and } |v - w| < \eta_3\}, && (3.21)\\
&\int_{T\times\Gamma_+} \phi_\eta(\tau, x, v, y, w) d\tau d\xi(y, w) = 1 \text{ for all } (\tau, x, v) \in T \times \Gamma_+, && (3.22)\\
&\int_{T\times\Gamma_+} \phi_\eta(\tau, x, v, y, w) d\tau d\xi(x, v) \leq C \text{ for all } (\tau, y, w) \in T \times \Gamma_+. && (3.23)
\end{aligned}$$

where C is a constant. We denote by R_η the bounded operator in $L^1(T \times \Gamma_+, dtd\xi)$ defined by

$$R_\eta g(t, x, v) = \int_{T\times\Gamma_+} \phi_\eta(t - t', x, v, y, w) g(t', y, w) dt' d\xi(y, w), \qquad (3.24)$$

for a.e. $(t, x, v) \in T \times \Gamma_+$ and for $g \in L^1(T \times \Gamma_+, dtd\xi)$.

Let $(x_0', v_0') \in \Gamma_-$. Let $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3) \in (0, +\infty)^3$ and let $f_\varepsilon \in C^1(T \times \Gamma_-)$ be such that $f_\varepsilon \geq 0$, $\|f_\varepsilon\|_{L^1(T\times\Gamma_-)} = 1$, $\mathrm{supp} f_\varepsilon \subseteq \{(t', x', v') \in T \times \Gamma_- \mid t' < \varepsilon_1, |x' + \tau_+(x', v')v' - x_0' - \tau_+(x_0', v_0')v_0'| < \varepsilon_2,\ |v' - v_0'| < \varepsilon_3\}$ (we assume that $(0, t_0) \subset T$ for some $t_0 > 0$).

Assume that $(k,\tilde{k}) \in L^\infty(X\times V\times V)^2$. From the decomposition of the time-dependent albedo operator we have

$$\int_{T\times\Gamma_+} \psi(t,x,v) R_\eta\left(\mathcal{A}-\tilde{\mathcal{A}}\right) f_\varepsilon(t,x,v)dtd\xi(x,v) = \sum_{1=1}^{3} I_i(\psi,\eta,\varepsilon), \qquad (3.25)$$

where

$$I_1(\psi,\eta,\varepsilon) = \int_{T\times\Gamma_+} \left(e^{-\int_0^{\tau_-(x,v)}\sigma(x-sv,v)ds} - e^{-\int_0^{\tau_-(x,v)}\tilde{\sigma}(x-sv,v)ds}\right)\psi(t,x,v) \qquad (3.26)$$

$$\times \int_{T\times\Gamma_+} \phi_\eta(t-t',x,v,y,w) f_\varepsilon(t'-\tau_-(y,w), y-\tau_-(y,w)w, w)dt'd\xi(y,w)dtd\xi(x,v),$$

$$I_2(\psi,\eta,\varepsilon) = \int_{T\times\Gamma_+}\psi(t,x,v)\int_{T\times\Gamma_+}\phi_\eta(t-t',x,v,y,w) \qquad (3.27)$$

$$\times \int_V\int_0^{\tau_-(y,w)} \Big((kE_+)(y-sw,w',w) - (\tilde{k}\tilde{E}_+)(y-sw,w',w)\Big)$$

$$f_\varepsilon(t'-s-\tau_-(y-sw,w'), y-sw-\tau_-(y-sw,w')w', w')dsdw'd\xi(y,w)dt'd\xi(x,v)dt,$$

$$|I_3(\psi,\eta,\varepsilon)| \le C\int_{T\times V}\left(\int_{\substack{y\in\partial X\\ \nu(y)\cdot w>0}}\left|\int_{(0,T)\times\Gamma_+}\phi_\eta(t-t',x,v,y,w)\psi(t,x,v)dtd\xi(x,v)\right|^{p'}\right.$$

$$\left. \times\ (\nu(y)\cdot w)d\mu(y)\right)^{\frac{1}{p'}} dwdt', \qquad (3.28)$$

where C is a constant that does not depend on ε and η.

Theorem 3.2. *Assume that (σ,k) and $(\tilde{\sigma},\tilde{k})$ satisfy (4.1). Assume also $(k,\tilde{k})\in L^\infty(X\times V\times V)^2$. Let $1<p<\frac{\dim V+1}{\dim V}$ and let $p'=\frac{p}{p-1}$. Then the following statements are valid:*

i. there exists a constant $C_1 = C_1(X,V,p,\sigma,k,\tilde{\sigma},\tilde{k})$ such that

$$|I_1(\psi,\eta,\varepsilon)| \le \|R_\eta(\mathcal{A}-\tilde{\mathcal{A}})f_\varepsilon\|_{L^1(T\times\Gamma_+,dtd\xi)} + C_1(\eta_3+\rho)^{\dim(V)}, \quad (3.29)$$

for $(\rho,\varepsilon_1,\varepsilon_2,\varepsilon_3,\eta_1,\eta_2,\eta_3)\in(0,+\infty)^7$, and for $\psi\in L^\infty(T\times\Gamma_+)$, $\|\psi\|_{L^\infty(T\times\Gamma_+)}\le 1$, $\mathrm{supp}\psi\subset\{(t,x,v)\in T\times\Gamma_+ \mid |v-v_0'|<\rho\}$.

ii. There exists a constant $C_2 = C_2(X,V,p,\sigma,k,\tilde{\sigma},\tilde{k})$ such that

$$\begin{aligned}|I_2(\psi,\eta,\varepsilon)| &\le \|R_\eta(\mathcal{A}-\tilde{\mathcal{A}})f_\varepsilon\|_{L^1(T\times\Gamma_+,dtd\xi)}\\ &\quad + C_2(\eta_1+\rho_1)^{\frac{1}{p'}}\left(\rho_2+\eta_2+\frac{2\mathrm{diam}(X)\check{\eta}_3}{\sqrt{1-\check{\eta}_3}}\right)^{\frac{d-2}{p'}},\\ \check{\eta}_3 &= \frac{2\eta_3}{\mathrm{v}_0(1-\rho_3^2)^{\frac{1}{2}}}\end{aligned} \qquad (3.30)$$

for $(\rho_1,\rho_2,\rho_3,\varepsilon_1,\varepsilon_2,\varepsilon_3,\eta_1,\eta_2,\eta_3)\in(0,+\infty)^9$, $\rho_3<1$, $\check{\eta}_3<1$, *and for* $\psi\in L^\infty(T\times\Gamma_+)$ *which satisfies the conditions* $\|\psi\|_{L^\infty(T\times\Gamma_+)}\le 1$,

$$supp\psi\subset\{(t,x,v)\in T\times\Gamma_+\mid |x-x_0'-\tau_+(x_0',v_0')v_0'|>\eta_2+\varepsilon_2 \text{ or } |v-v_0'|>\eta_3+\varepsilon_3\}, \tag{3.31}$$

$$\begin{aligned} supp\psi \;\subset\; \{(t,x,v)\in T\times\Gamma_+ \mid & |\hat{v}\cdot\hat{v}_0'|<\rho_3, \\ & \inf_{(s,s')\in\mathbb{R}^2}|x-s\hat{v}-x_0'+s'\hat{v}_0'|<\rho_2\}, \end{aligned} \tag{3.32}$$

$$supp\psi\subset\{(t,x,v)\in T\times\Gamma_+\mid |t-s-s'|_{\substack{s=\frac{(x-x_0')\cdot(\hat{v}_0'-(\hat{v}\cdot\hat{v}_0')\hat{v})}{1-(\hat{v}_0'\cdot\hat{v})^2}\\ s'=\frac{(x-x_0')\cdot(\hat{v}-(\hat{v}\cdot\hat{v}_0')\hat{v}_0')}{1-(\hat{v}_0'\cdot\hat{v})^2}}}<\rho_1\}. \tag{3.33}$$

The derivation of the above theorem is also presented in section 4. The conclusions drawn after Theorem 3.1 still hold in the time dependent setting. The main difference between the two settings is the stability of the scattering coefficient in I_2, which is better in the time dependent setting than it is in the time independent setting because of the presence of the term $(\eta_1+\rho_1)^{\frac{1}{p'}}$, where $p'>\dim V+1$. The availability of time dependent measurements allows one to better separate single scattering from multiple scattering than when only in the presence of time dependent measurements. No such effect is observable on the separation between the ballistic and single scattering contributions.

4 Sketch of derivation

In this section we develop the mathematical framework for the well-posedness of the stationary Boltzmann transport equation (2.1) and for the existence of the albedo operator $\mathcal{A}: L^1(\Gamma_-,d\xi)\to L^1(\Gamma_+,d\xi)$. We give results on the decomposition of the albedo operator used in (3.9) and we prove the estimate on the multiple scattering part (3.12). Finally we prove Theorem 3.1.

For the non-stationary case we refer the reader to [5, 12]. Modifying the results of [5], one obtains the decomposition (3.25) with the estimate (3.28). The proof of Theorem 3.2 then follows the same lines as that of Theorem 3.1.

Existence theory for the time independent albedo operator. Recall that $X\subset\mathbb{R}^d$, $d\ge 2$, is an open bounded subset with C^1 boundary ∂X, and that V is $\mathbb{S}^{d-1}$ or an open subset of $\mathbb{R}^d$ which satisfies $\mathrm{v}_0:=\inf_{v\in V}|v|>0$, $V_0:=\sup_{v\in V}|v|<\infty$. We do not assume that X is convex in this section so that one obtains that Theorems 3.1 and 3.2 still hold when X is not assumed to be convex. We assume that (σ, k) is admissible if

$$\begin{array}{l} 0 \le \sigma \in L^\infty(X \times V), \\ k(x, v', v) \text{ is a measurable function on } X \times V \times V, \text{ and} \\ 0 \le k(x, v', .) \in L^1(V) \text{ for a.e. } (x, v') \in X \times V \\ \sigma_p(x, v') = \int_V k(x, v', v) dv \text{ belongs to } L^\infty(X \times V). \end{array} \tag{4.1}$$

We introduce the Banach space

$$\begin{array}{l} W := \{u \in L^1(X \times V); v \cdot \nabla_x u \in L^1(X \times V),\ \tau^{-1} u \in L^1(X \times V)\}, \\ \|u\|_W = \|v \cdot \nabla u\|_{L^1(X \times V)} + \|\tau^{-1} u\|_{L^1(X \times V)}, \end{array}$$

where $v \cdot \nabla_x$ is understood in the distributional sense and $\tau(x, v) = \tau_-(x, v) + \tau_+(x, v)$ for $(x, v) \in \bar{X} \times V$, and we recall that the map $\gamma_\pm : u \to u_{|\Gamma_\pm}$ is continuous from W to $L^1(\Gamma_\pm, d\xi)$ (see Theorem 2.1 of [14]) so that the equation (2.1) makes sense for $u \in W$. The stationary linear Boltzmann transport equation (2.1) is transformed into the following integral equation

$$(I + K)u = Ju_-, \tag{4.2}$$

where K is the bounded operator in $L^1(X \times V, \tau^{-1} dx dv)$ defined by

$$Ku(x, v) = -\int_0^{\tau_-(x,v)} e^{-\int_0^t \sigma(x - sv, v) ds} \int_V k(x - tv, v', v) u(x - tv, v') dv' dt,\ (x, v) \in X \times V,$$

for all $u \in L^1(X \times V, \tau^{-1} dx dv)$, and where J is the continuous lifting of γ_- given by

$$Ju_-(x, v) = e^{-\int_0^{\tau_-(x,v)} \sigma(x - sv, v) ds} u_-(x - \tau_-(x, v) v, v),\ (x, v) \in X \times V, \tag{4.3}$$

for $u_- \in L^1(\Gamma_-, d\xi)$ (see Proposition 2.1 [14] for the continuity of J). Then under the additional assumption

the bounded operator $I + K$ in $L^1(X \times V, \tau^{-1} dx dv)$ admits a bounded inverse in $L^1(X \times V, \tau^{-1} dx dv)$, (4.4)

the integral equation (4.2) is uniquely solvable for all $u_- \in L^1(\Gamma_-, d\xi)$; $u \in$ W; and the operator $\mathcal{A} : L^1(\Gamma_-, d\xi) \to L^1(\Gamma_+, d\xi)$, $u_- \to u_{|\Gamma_+}$ is bounded (see [4, 14]).

Note that condition (4.4) is satisfied when $\|\tau \sigma_p\|_\infty < 1$ or $\sigma - \sigma_p \ge 0$ (see [4, 19]).

Decomposition of the albedo operator and proof of the estimate (3.12). First we note that $-v\cdot\nabla_x(Ku)(v,x)=\sigma(x,v)Ku(x,v)+\int_V k(x,v',v)u(x,v')dv'$ for a.e. $(x,v)\in X\times V$ and for $u\in L^1(X\times V,\tau^{-1}dxdv)$. Therefore the operator $\gamma_+(K^2)$ ($\gamma_+(K^2u)=(K^2u)_{|\Gamma_+}$) is a well-defined and bounded operator from $L^1(X\times V,\tau^{-1}dxdv)\to L^1(\Gamma_+,d\xi)$, and the following Lemma 4.1 gives properties of the distributional kernel of this bounded operator. This Lemma 4.1 is a variant of lemma 2.7 of [4]. The proof of Lemma 4.1 is postponed to the end of this paragraph.

Lemma 4.1. *Assume that $k\in L^\infty(X\times V\times V)$, then there exists a nonnegative measurable function $\beta : X\times V\times X\times V\to[0,+\infty)$ such that*

$$\left(K^2u\right)_{|\Gamma_+}(x,v)=\int_{X\times V}\beta(x,v,x',v')u(x',v')dx'dv' \tag{4.5}$$

for a.e. $(x,v)\in\Gamma_+$ and for any $u\in L^1(X\times V,\tau^{-1}dxdv)$; and for any $1<p<1+\frac{1}{d-1}$, $p^{-1}+p'^{-1}=1$, there exists some nonnegative constant $C(p,X,V,\sigma,k)$ such that

$$\begin{aligned}&\left\|\int_{\Gamma_+}\psi(x,v)\beta(x,v,x',v')\tau(x',v')d\xi(x,v)\right\|_{L^\infty(X_{x'}\times V_{v'})}\\ &\le C\int_V\left(\int_{\substack{x\in\partial X\\ \nu(x)\cdot v>0}}|\psi(x,v)|^{p'}(\nu(x)\cdot v)d\mu(x)\right)^{\frac{1}{p'}}dv,\end{aligned} \tag{4.6}$$

for any $\psi\in L^\infty(\Gamma_+)$.

Note that from condition (4.4) and the integral equation (4.2) it follows that

$$u:=Ju_- - KJu_- + K^2(I+K)^{-1}Ju_-. \tag{4.7}$$

Therefore taking the trace over Γ_+ of both the left hand side of equality (4.7) and each of the three terms of the sum on the right-hand side of (4.7) and taking account of Lemma 4.1 we obtain the following Proposition 4.2.

Proposition 4.2. *Under condition* (4.4)*, the following equality in the distributional sense is valid*

$$\begin{aligned}\mathcal{A}u_-(x,v) &= \sum_{i=1}^{2}\int_{\Gamma_-}\alpha_i(x,v,x',v')u_-(x',v')d\mu(x')dv' \\ &\quad+\int_{X\times V}\beta(x,v,x',v')((I+K)^{-1}Ju_-)(x',v')dx'dv',\end{aligned} \tag{4.8}$$

for a.e. $(x,v)\in\Gamma_+$ *and for any* C^1 *compactly supported function* u_- *on* Γ_-*, where*

$$\alpha_1(x,v,x',v') = e^{-\int_0^{\tau_-(x,v)}\sigma(x-sv,v)ds}\delta_v(v')\delta_{x-\tau_-(x,v)v}(x'), \quad (4.9)$$

$$\begin{aligned}\alpha_2(x,v,x',v') &= \int_0^{\tau_-(x,v)} e^{-\int_0^t\sigma(x-sv,v)ds-\int_0^{\tau_-(x-tv,v')}\sigma(x-tv-sv',v')ds}\\ &\quad\times k(x-tv,v',v)\delta_{x-tv-\tau_-(x-tv,v')v'}(x')dt, \quad (4.10)\end{aligned}$$

for a.e. $(x,v)\in\Gamma_+$ *and* $(x',v')\in\Gamma_-$*, and where* β *is given by* (4.5).

We refer the reader to [4] for more details on the derivation of Proposition 4.2.

Finally the equality (3.9) follows from (4.8) (applied on (σ,k) and $(\tilde\sigma,\tilde k)$), and the estimate (3.12) follows from (4.8), (4.6) (applied on (σ,k) and $(\tilde\sigma,\tilde k)$), the boundedness of $(I+K)^{-1}J$ and $(I+\tilde K)^{-1}\tilde J$ from $L^1(\Gamma_-,d\xi)$ to $L^1(X\times V,\tau^{-1}(x,v)dxdv)$ and equality (3.6).

Proof of Lemma 4.1. Let $k\in L^\infty(X\times V\times V)$ and let $1<p<1+\frac{1}{d-1}$, $p'^{-1}+p^{-1}=1$. Using the estimate $\tau(x',v')\le\frac{\mathrm{diam}(X)}{\mathrm{v}_0}$ for $(x',v')\in X\times V$ and using Hölder inequality, we obtain that the proof of estimate (4.6) is reduced to the proof of (4.15). We introduce the function $\chi:\mathbb{R}^d\times\mathbb{R}^d\to[0,+\infty)$ defined by

$$\chi(x,y)=\begin{cases}0, \text{ if } x+p(y-x)\notin X \text{ for some } p\in(0,1);\\ 1, \text{ if } x+p(y-x)\in X \text{for all } p\in(0,1).\end{cases} \quad (4.11)$$

for $(x,y)\in(\mathbb{R}^d)^2$. From the definition of K, one obtains that $(K^2u_-)_{|\Gamma_+}(x,v)=\int_{X\times V}\beta(x,v,x',v')u_-(x',v')dx'dv'$, where β is a nonnegative measurable function from $X\times V\times X\times V$ to $[0,+\infty)$. More precisely if $V=\mathbb{S}^{d-1}$ then

$$\begin{aligned}\beta(x,v,x',v') &= \int_0^{\tau_-(x,v)}\chi(x-tv,x')\Bigg[\frac{k(x-tv,v_1,v)k(x',v',v_1)}{|x-tv-x'|^{d-1}}\\ &\times e^{-\int_0^t\sigma(x-sv,v)ds-\int_0^{t_1}\sigma(x-tv-sv_1,v_1)ds}\Bigg]_{\substack{t_1=|x-tv-x'|\\ v_1=\frac{x-tv-x'}{t_1}}}dt, \quad (4.12)\end{aligned}$$

for a.e. $(x,v)\in\Gamma_+$, $(x',v')\in X\times V$, where χ is defined by (4.11); and if V is an open bounded subset of $\mathbb{R}^d$ which satisfies $\mathrm{v}_0:=\inf_{v\in V}|v|>0$ and $\mathrm{V}_0:=\sup_{v\in V}|v|<\infty$, then

$$\begin{aligned}\beta(x,v,x',v') &= \int_{\mathrm{v}_0}^{\mathrm{V}_0}r^{d-2}\int_0^{\tau_-(x,v)}\Bigg[\frac{k(x-tv,r\omega,v)k(x',v',r\omega)}{|x-x'-tv|^{d-1}} \quad (4.13)\\ &\times e^{-\int_0^t\sigma(x-sv,v)ds-\int_0^{t_1}\sigma(x-tv-sr\omega,r\omega)ds}\chi_V(r\omega)\Bigg]_{rt_1\omega=x-x'-tv}\chi(x-tv,x')dtdr,\end{aligned}$$

for a.e. $(x,v)\in\Gamma_+$, $(x',v')\in X\times V$, where χ is defined by (4.11) and $\chi_V(w)=1$ for $w\in V$ and $\chi_V(w)=0$ for $w\notin V$. Let $C(V):=1$ when $V=\mathbb{S}^{d-1}$. Let $C(V):=\frac{V_0^d}{d}$ when V is an open bounded subset of $\mathbb{R}^d$. Then using (4.12), (4.13), the estimate $\sigma\ge 0$, Hölder inequality and the estimate $\tau_-(x,v)\le\frac{\mathrm{diam}(X)}{\mathrm{v}_0}$ for $(x,v)\in\Gamma_+$ we obtain

$$|\beta(x,v,x',v')|^p\le\left(\frac{C(V)\|k\|_\infty^2\mathrm{diam}(X)^{\frac{1}{p'}}}{\mathrm{v}_0^{\frac{1}{p'}}}\right)^p\int_0^{\tau_-(x,v)}\frac{dt}{|x-tv-x'|^{p(d-1)}},\tag{4.14}$$

for a.e. $(x,v)\in\Gamma_+$, $(x',v')\in X\times V$. Therefore

$$\begin{aligned}&\int_{\substack{x\in\partial X\\ \nu(x)\cdot v>0}}\beta(x,v,x',v')^p(\nu(x)\cdot v)d\mu(x)\\ &\le C(V)^p\left(\frac{\mathrm{diam}(X)}{\mathrm{v}_0}\right)^{\frac{p}{p'}}\|k\|_\infty^{2p}\int_{y\in X}\frac{1}{|y-x'|^{p(d-1)}}dtdy\\ &\le C(V)^p\left(\frac{\mathrm{diam}(X)}{\mathrm{v}_0}\right)^{\frac{p}{p'}}\|k\|_\infty^{2p}\mathrm{Vol}(\mathbb{S}^{d-1})\frac{\mathrm{diam}(X)^{1+(1-p)(d-1)}}{1+(1-p)(d-1)},\end{aligned}\tag{4.15}$$

for a.e. $(x',v')\in X\times V$ (we performed the change of variables $y=x-tv$, $y\in X$, $(x,v)\in\Gamma_+$, $t\in(0,\tau_-(x,v))$, $dy=(\nu(x)\cdot v)dtd\mu(x)$). $\square$

Proof of Theorem 3.1. Let $\psi\in L^\infty(\Gamma_+)$, $\|\psi\|_{L^\infty(\Gamma_+)}\le 1$. Let $(x,v)\in\Gamma_+$. Using (3.2) we obtain that if $(x,v)\in\mathrm{supp}\psi$ then

$$\phi_\eta(x,v,z,w)=0 \text{ if } \chi_{\mathrm{supp}\psi+B_\eta}(z,w)=0,\tag{4.16}$$

for $(z,w)\in\Gamma_+$, where

$$B_\eta := \{(a,b)\in\mathbb{R}^d\times\mathbb{R}^d \mid |a|<\eta_1 \text{ and } |b|<\eta_2\},\tag{4.17}$$

$$\chi_{\mathrm{supp}\psi+B_\eta}(z,w) = \begin{Bmatrix}1 \text{ if } (z,w)\in\mathrm{supp}\psi+B_\eta,\\ 0 \text{ if } (z,w)\notin\mathrm{supp}\psi+B_\eta.\end{Bmatrix}\tag{4.18}$$

From (3.12), (4.16) and the estimate $\|\psi\|_{L^\infty(\Gamma_+)}\le 1$ it follows that

$$|I_3(\psi,\eta,\varepsilon)|\le C\int_V\left(\int_{\substack{y\in\partial X\\ \nu(y)\cdot w>0}}\chi_{\mathrm{supp}\psi+B_\eta}(y,w)(\nu(y)\cdot w)dy\right)^{\frac{1}{p'}}dw.\tag{4.19}$$

where C is a constant that depends only on p, X, V, (σ,k) and $(\tilde\sigma,\tilde k)$.

In addition using the estimate $\|\psi\|_{L^\infty(\Gamma_+)} \le 1$ we obtain $|\int_{\Gamma_+}\psi(x,v) R_\eta(\mathcal{A}-\tilde{\mathcal{A}})f_\varepsilon(x,v)d\xi(x,v)| \le \|R_\eta(\mathcal{A}-\tilde{\mathcal{A}})f_\varepsilon\|_{L^1(\Gamma_+,d\xi)}$. Therefore using (3.9) we obtain

$$\text{either } |I_1(\psi,\eta,\varepsilon)| \le \|R_\eta(\mathcal{A}-\tilde{\mathcal{A}})f_\varepsilon\|_{L^1(\Gamma_+,d\xi)} + |I_2(\psi,\eta,\varepsilon)| + |I_3(\psi,\eta,\varepsilon)|. \tag{4.20}$$

$$\text{or } |I_2(\psi,\eta,\varepsilon)| \le \|R_\eta(\mathcal{A}-\tilde{\mathcal{A}})f_\varepsilon\|_{L^1(\Gamma_+,d\xi)} + |I_1(\psi,\eta,\varepsilon)| + |I_3(\psi,\eta,\varepsilon)|. \tag{4.21}$$

Assume that ψ satisfies (3.15). We prove (3.14) which reduces to estimating $I_i(\psi,\eta,\varepsilon)$, $i=2,3$ (see (4.20)). Using the changes of variables "$(z,w)=(y-tw,w)$" and "$(z,w')=(y'+t'\hat{w}',w')$"), (3.11) gives

$$\begin{aligned}&I_2(\psi,\eta,\varepsilon)=\\ &\int_{\Gamma_+}\psi(x,v)\int_{V\times\Gamma_-}\int_0^{\tau_+(y',w')}\phi_\eta\left(x,v,y'+t'w'+\tau_+(y'+t'w',w)w,w\right)\\ &(kE_+-\tilde{k}\tilde{E}_+)(y'+t'w',w',w)f_\varepsilon(y',w')dt'd\xi(y',w')dwd\xi(x,v).\end{aligned} \tag{4.22}$$

Combining (3.15) and (3.2) we obtain that if $(x,v)\in\mathrm{supp}\psi$ then

$$\phi_\eta\left(x,v,z,w\right)=0 \text{ for } (z,w)\in\Gamma_+,\ |w-v_0'|\ge\eta_2+\rho. \tag{4.23}$$

From (4.22)–(4.23) and the assumptions $\sigma\ge0$, $\tilde{\sigma}\ge0$, $(k,\tilde{k})\in L^\infty(X\times V\times V)^2$, and (3.4), (3.6) and the estimate $\|\psi\|_{L^\infty(\Gamma_+)}\le1$ it follows that $|I_2(\psi,\eta,\varepsilon)|\le\frac{C\|k+\tilde{k}\|_\infty\mathrm{diam}(X)}{\mathrm{v}_0}\int_{\substack{w\in V\\ |w-v_0'|<\eta_2+\rho}}dw$. Combining (4.19), (3.15), (4.17) and (4.18), we obtain $|I_3(\psi,\eta,\varepsilon)|\le C\int_{\substack{w\in V\\ |w-v_0'|<\eta_2+\rho}}dw$. Combining these latter estimates with (4.20) we obtain (3.14).

Now assume that ψ satisfy (3.17)–(3.19). Let $(\varepsilon_1,\varepsilon_2,\eta_1,\eta_2)\in(0,+\infty)^4$ be such that $\eta_2<\frac{\mathrm{v}_0\sqrt{1-\rho_2^2}}{2}$. We prove (3.16), which reduces to estimating $I_i(\psi,\eta,\varepsilon)$, $i=1,3$ (see (4.21)). From (3.2) and (3.8), it follows that

$$\mathrm{supp}\Phi_{\varepsilon,\eta}\subseteq\{(x,v)\in\Gamma_+\mid |x-x_0'-\tau_+(x_0',v_0')v_0'|<\varepsilon_1+\eta_1 \text{ and } |v-v_0'|<\varepsilon_2+\eta_2\}, \tag{4.24}$$

where $\Phi_{\varepsilon,\eta}$ is the function defined by

$$\begin{aligned}\Phi_{\varepsilon,\eta}(x,v) &= \int_{\Gamma_+}\phi_\eta(x,v,y,w)f_\varepsilon(y-\tau_-(y,w)w,w)\\ &\times\left(e^{-\int_0^{\tau_-(y,w)}\sigma(y-sw,w)ds}-e^{-\int_0^{\tau_-(y,w)}\tilde{\sigma}(y-sw,w)ds}\right)d\xi(y,w),\end{aligned} \tag{4.25}$$

for $(x,v)\in\Gamma_+$. Therefore using (3.10) and (3.17), we obtain $I_1(\psi,\varepsilon,\eta)=0$.

Now we estimate $I_3(\psi,\varepsilon,\eta)$. For $w \in V$ let $\hat{w} \in \mathbb{S}^{d-1}$ be defined by $\hat{w} = \frac{w}{|w|}$. Let $(x,v) \in \Gamma_+$, $\hat{v} \neq \pm\hat{v}_0'$. Let $\mathrm{dist}(x,v)$ denote $\inf_{(s,s')\in\mathbb{R}^2} |x - s\hat{v} - x_0' + s'\hat{v}_0'|$. Then

$$\mathrm{dist}(x,v) = |x - x_0' - ((x-x_0')\cdot\widehat{v_\perp})\,\widehat{v_\perp} - ((x-x_0')\cdot\hat{v}_0')\,\hat{v}_0'|\,, \tag{4.26}$$

where

$$v_\perp := v - (v\cdot\hat{v}_0')\,\hat{v}_0', \tag{4.27}$$

i.e. $\mathrm{dist}(x,v)$ is the orthogonal distance of the vector $x - x_0'$ to the vector plane spanned by v and v_0'. We use the following Lemma 4.3 whose proof is postponed to the end of this section.

Lemma 4.3. *Let $(y,w) \in \Gamma_+$ and $(z,\zeta) \in B_\eta$ such that $(y-z, w-\zeta) \in \mathrm{supp}\psi$. Then*

$$\mathrm{dist}(y,w) \leq \rho_1 + \eta_1 + \frac{4\mathrm{diam}(X)\eta_2}{\mathrm{v}_0(1-\rho_2^2)^{\frac{1}{2}}\left(1 - \frac{2\eta_2}{\mathrm{v}_0\sqrt{1-\rho_2^2}}\right)^{\frac{1}{2}}}. \tag{4.28}$$

Let $\lambda = \rho_1 + \eta_1 + \frac{4\mathrm{diam}(X)\eta_2}{\mathrm{v}_0(1-\rho_2^2)^{\frac{1}{2}}\left(1 - \frac{2\eta_2}{\mathrm{v}_0\sqrt{1-\rho_2^2}}\right)^{\frac{1}{2}}}$. Combining (4.28) and (4.19) we obtain

$$|I_3(\psi,\eta,\varepsilon)| \leq C\int_{w\in V}\left(\int_{\substack{y\in\partial X\\ \mathrm{dist}(y,w)\leq\lambda}} |\nu(y)\cdot w|dy\right)^{\frac{1}{p'}} dw \leq C'\lambda^{\frac{d-2}{p'}}, \tag{4.29}$$

where C' is a constant that depends on p, X, V, (σ,k), $(\tilde{\sigma},\tilde{k})$, and that does not depend on ψ, ε and η.

Proof of Lemma 4.3. From (4.26) and the estimates $|z| \leq \eta_1$ and $\mathrm{dist}(y-z, w-\zeta) < \rho_1$ (see (3.18) for $(y-z,w-\zeta) \in \mathrm{supp}\psi$), it follows that

$$\begin{aligned}
\mathrm{dist}(y,w) &\leq \mathrm{dist}(y,w-\zeta) \\
&\quad + \left|\left((y-x_0')\cdot\widehat{(w-\zeta)_\perp}\right)\widehat{(w-\zeta)_\perp} - ((y-x_0')\cdot\widehat{w_\perp})\,\widehat{w_\perp}\right|, \\
&\leq \mathrm{dist}(y,w-\zeta) + \left|\left((y-x_0')\cdot\left(\widehat{(w-\zeta)_\perp} - \widehat{w_\perp}\right)\right)\widehat{(w-\zeta)_\perp}\right| \\
&\quad + \left|(y-x_0')\cdot\widehat{w_\perp}\left(\widehat{w_\perp} - \widehat{(w-\zeta)_\perp}\right)|\right| \\
&\leq \mathrm{dist}(y,w-\zeta) + 2|y-x_0'|\left|\widehat{(w-\zeta)_\perp} - \widehat{w_\perp}\right|, &(4.30)\\
\mathrm{dist}(y,w-\zeta) &\leq \mathrm{dist}(y-z,w-\zeta) + |z| \leq \rho_1 + \eta_1, &(4.31)
\end{aligned}$$

(we also used the Cauchy-Bunyakovski-Schwarz inequality). Taking account of the estimates $|y - x_0'| \le \mathrm{diam}(X)$, (4.30)–(4.31), estimate (4.28) reduces to proving the following estimate

$$\left|\widehat{(w-\zeta)_\perp} - \widehat{w_\perp}\right| \le \frac{2\eta_2}{\mathrm{v}_0(1-\rho_2^2)^{\frac{1}{2}}\left(1 - \frac{2\eta_2}{\mathrm{v}_0\sqrt{1-\rho_2^2}}\right)^{\frac{1}{2}}}. \tag{4.32}$$

First note that from (4.27), and from the estimates $|(w-\zeta)\cdot\hat{v}_0'| \le \rho_2|w-\zeta|$ (see (3.19) and $(y-z, w-\zeta) \in \mathrm{supp}\psi$) and $|\zeta| \le \eta_2$ it follows that

$$|(w-\zeta)_\perp|^2 = |w-\zeta|^2 - ((w-\zeta)\cdot\hat{v}_0')^2 \ge (1-\rho_2^2)|w-\zeta|^2, \tag{4.33}$$

Note also that using the estimates $|\zeta_\perp| \le |\zeta| \le \eta_2$, $|w_\perp| \ge |(w-\zeta)_\perp| - |\zeta_\perp| \ge |(w-\zeta)_\perp| - \eta_2$, we obtain

$$\left|\widehat{(w-\zeta)_\perp} - \widehat{w_\perp}\right|^2 = \frac{|\zeta_\perp|^2 - (|(w-\zeta)_\perp| - |w_\perp|)^2}{|(w-\zeta)_\perp||w_\perp|} \le \frac{\eta_2^2}{|(w-\zeta)_\perp|^2\left(1 - \frac{\eta_2}{|(w-\zeta)_\perp|}\right)}. \tag{4.34}$$

Note also that $|w-\zeta| \ge \frac{\mathrm{v}_0}{2}$ since $|\zeta| \le \eta_2 \le \frac{\mathrm{v}_0}{2} \le \frac{|w|}{2}$. Therefore combining (4.33)–(4.34) we obtain (4.32). □

Remarks. For the geometry based on $F_\pm$ introduced in [4] we can obtain approximate stability estimates similar to the estimates of Theorem 3.1 for the albedo operator $\mathcal{A} : L^1(F_-) \to L^1(F_+)$. We emphasize that the geometry based on $F_\pm$ may be more practical for application than the geometry based on $\Gamma_\pm$.

One may obtain approximate stability estimates similar to the estimates of Theorem 3.1 for the albedo operator $\mathcal{A} : L^1(\Gamma_-, d\tilde{\xi}) \to L^1(\Gamma_+, d\tilde{\xi})$, where $d\tilde{\xi}(x,v) = \tau(x,v)d\xi(x,v)$ (for the existence of the albedo operator in the spaces $L^1(\Gamma_\pm, d\tilde{\xi})$, we refer the reader to [14, 4]).

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