Homogenization in random media and effective medium theory for high frequency waves

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Abstract

We consider the homogenization of the wave equation with high frequency initial conditions propagating in a medium with highly oscillatory random coefficients. By appropriate mixing assumptions on the random medium, we obtain an error estimate between the exact wave solution and the homogenized wave solution in the energy norm. This allows us to consider the limiting behavior of the energy density of high frequency waves propagating in highly heterogeneous media when the wavelength is much larger than the correlation length in the medium.

1 Introduction

Homogenization in random environment. Homogenization of second-order linear elliptic operators in divergence form with highly oscillatory coefficients has a long history, both when the coefficients are periodic and when they are modeled as random fields; see e.g. [4, 10]. Such results can then be used to approximate solutions to elliptic, hyperbolic, or parabolic equations with oscillatory coefficients by solutions to the same equations with effective constant coefficients.

In the case of random coefficients with proper ergodic properties, the first rigorous results in homogenization theory were obtained in [11, 15]. They are based on the analysis of a local problem that may be written in the form: Find a tensor $\psi = (\psi_{ij})_{ij}$ such that

$$\begin{aligned} &-\nabla\cdot(a\psi)(\mathbf{x},\omega) = 0, \qquad && \mathbf{x}\in\mathbb{R}^d, \omega\in\Omega, \\ &\mathbb{E}\{\psi\} = I, \quad \nabla\times\psi = 0. && \end{aligned} \tag{1}$$

Here, $a(\mathbf{x},\omega)$ is the random diffusion tensor constructed on a probability space $(\Omega,\mathcal{F},P)$, I is the identity tensor, and $\mathbb{E}$ denotes mathematical expectation associated to P. The tensor ψ may be approximated by $\psi^\beta = I + \nabla\otimes\boldsymbol{\theta}^\beta$, where $\boldsymbol{\theta}^\beta$ is the solution of the regularized problem

$$-\nabla\cdot a\nabla\boldsymbol{\theta}^\beta(\mathbf{x},\omega) + \beta\boldsymbol{\theta}^\beta(\mathbf{x},\omega) = \nabla\cdot a(\mathbf{x},\omega), \qquad \mathbf{x}\in\mathbb{R}^d. \tag{2}$$

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Here $0 < \beta \ll 1$ is a regularizing parameter. The properties of $\boldsymbol{\theta}^\beta$ are used in [11, 15] to approximate operators with random ergodic coefficients by homogenized operators involving the constant coefficient $a^* = \mathbb{E}\{a\psi\}$.

Provided that $a(\mathbf{x}, \omega)$ has additional mixing properties, it is shown in [17] when the space dimension $d \geq 3$ that $\boldsymbol{\theta}^\beta(\omega)$ and $a\psi^\beta(\omega)$ satisfy appropriate mixing conditions. Such results, which will be reviewed later in the text, are used to derive error estimates for correctors, which measure the difference between the exact solution of the heterogeneous equation and the solution of the effective medium equation. These error estimates then allow us to address the homogenization of the energy density of high frequency waves propagating in random media. In the one-dimensional case, where explicit expressions for the solutions to the heterogeneous problems are available, optimal error estimates can be found in [5].

This paper reconsiders the derivation of such error estimates for the homogenization corrector in a simplified setting. Let us introduce the harmonic coordinates $\mathbf{z}^\varepsilon$ defined in [11] as

$$\mathbf{z}^\varepsilon(\mathbf{x}) = \varepsilon \mathbf{z}\Big(\frac{\mathbf{x}}{\varepsilon}\Big) = \int_0^1 \psi\Big(\frac{t\mathbf{x}}{\varepsilon}\Big)\mathbf{x}dt. \tag{3}$$

Harmonic coordinates, which verify that $\nabla \cdot a\nabla \mathbf{z}(\mathbf{x}) = 0$ thanks to (1) and the fact that $\nabla \mathbf{z}^\varepsilon(\mathbf{x}) = \psi(\frac{\mathbf{x}}{\varepsilon})$, have also been used successfully to derive efficient algorithms in numerical homogenization [1, 14].

The main assumptions of the simplified setting are that the random fields $\psi(\omega)$ and $a\psi(\omega)$ are mixing. Such assumptions are much stronger than $a(\omega)$ being mixing, except in the one-dimensional case where ψ is proportional to a^{-1}. We will present restrictive cases in which such assumptions are valid and heuristic arguments indicating that they should be satisfied in other practical situations. The advantage of such assumptions is that they considerably simplify the derivation of estimates for the homogenization corrector and that they are independent of space dimension.

High frequency wave equation. Error estimates for the difference between the random and the homogenized solutions allow us to address the homogenization of high frequency waves in highly oscillatory heterogeneous media. We consider here the homogenization of the following wave equation:

$$\begin{aligned}
&\kappa\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big)\frac{\partial^2 p_\varepsilon(t,\mathbf{x},\omega)}{\partial t^2} - \nabla\cdot a\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big)\nabla p_\varepsilon(t,\mathbf{x},\omega) = 0, \qquad t>0, \quad \mathbf{x}\in\mathbb{R}^d,\\
&p_\varepsilon(0,\mathbf{x}) = g_\varepsilon(\mathbf{x}), \qquad \frac{\partial p_\varepsilon}{\partial t}(0,\mathbf{x}) = j_\varepsilon(\mathbf{x}), \qquad \mathbf{x}\in\mathbb{R}^d.
\end{aligned} \tag{4}$$

The compressibility $\kappa(\mathbf{x}, \omega)$ and the inverse density tensor $a(\mathbf{x}, \omega)$ are random fields defined for $\mathbf{x} \in \mathbb{R}^d$, where spatial dimension $d \geq 1$, and $\omega \in \Omega$, a set such that $(\Omega, \mathcal{F}, P)$ is an abstract probability space. We assume that $(\mathbf{x}, \omega) \mapsto \kappa(\mathbf{x}, \omega)$ and the symmetric matrix $(\mathbf{x}, \omega) \mapsto a(\mathbf{x}, \omega) = \{a_{ij}(\mathbf{x}, \omega)\}_{1\leq i,j\leq d}$, are jointly measurable in $(\mathbb{R}^d, \mathcal{B}, d\mathbf{x}) \times (\Omega, \mathcal{F}, P)$, where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}^d$ and $d\mathbf{x}$ is Lebesgue measure, and that they satisfy the uniform ellipticity constraints

$$\begin{aligned}
&0 < \kappa_0 \leq \kappa(\mathbf{x},\omega) \leq \kappa_0^{-1}, \qquad d\mathbf{x}\times P- \text{ a.s.}\\
&0 < a_0 \leq \boldsymbol{\xi}\cdot a(\mathbf{x},\omega)\boldsymbol{\xi} \leq a_0^{-1} \quad d\mathbf{x}\times P- \text{ a.s.} \qquad \text{for all } \boldsymbol{\xi}\in\mathbb{S}^{d-1}.
\end{aligned} \tag{5}$$

The pressure potential $p_\varepsilon(t,\mathbf{x},\omega)$ has for initial conditions the potential field $g_\varepsilon(\mathbf{x})$ and the pressure field $j_\varepsilon(\mathbf{x})$, which we assume are compactly supported. For concreteness (see Theorem 5.1 below for a general statement), let us consider highly oscillatory initial conditions of the form:

$$g_\varepsilon(\mathbf{x}) = \varepsilon^\alpha \varphi_g(\mathbf{x}) g_0\Big(\frac{\mathbf{x}}{\varepsilon^\alpha}\Big), \qquad j_\varepsilon(\mathbf{x}) = \varphi_j(\mathbf{x}) j_0\Big(\frac{\mathbf{x}}{\varepsilon^\alpha}\Big), \tag{6}$$

where $\alpha > 0$, where φ_g and φ_j are compactly supported and where all functions above are of class $\mathcal{C}^n(\mathbb{R}^d)$ for $n \geq 5 + \frac{d}{2}$. The choice of the scaling is meant to ensure that the wave energy density is independent of ε; see section 4.

When the initial conditions are independent of ε, i.e., when $\alpha = 0$, the above problem is a classical effective medium theory problem. Using the techniques developed in [11, 15], we can show that p_ε converges strongly in $L^2(\mathbb{R}^d)$ to the solution p_0 of an effective medium equation with effective compressibility $\kappa^* = \mathbb{E}\{\kappa\}$ and effective diffusion tensor $a^* = \mathbb{E}\{a\psi\}$.

When $\alpha > 0$, it is well-known that the relationship between the typical wavelength of the wave fields and the typical correlation length of the underlying medium characterizes the macroscopic regime of wave propagation. When α is large, high frequency waves strongly interact with the underlying structure and we do not expect the effective medium theory to hold. For instance, when both the correlation length (here ε) and the wavelength (here ε^α) are of the same order, we expect in the so-called weak coupling regime that wave propagation be characterized by a radiative transfer equation; see e.g. [3, 7, 13, 16]. When the correlation length is much larger than the wavelength, then wave propagation is best modeled by a Fokker-Planck equation [2].

In this paper, we address the reverse case, where the correlation length is much smaller than the wavelength. In such a configuration, we expect the following double-limit process to hold. We first replace the heterogeneous wave equation by an effective medium wave equation with constant constitutive coefficients κ^* and a^*, and then address high frequency wave propagation in the medium with constant coefficients. That we are allowed to do so, i.e., that the double-limit process may be justified as a single parameter tends to 0, is one of the main objectives of this paper. Provided that the wavelength is of order ε^α for $\alpha > 0$ sufficiently small, we show that in the limit of $\varepsilon \to 0$, the energy density of the wave equation with random coefficients is indeed approximated by the energy density of waves propagating in the appropriate effective medium. We use the theory of Wigner transforms [9, 12] to do so.

We consider two different scenarios. When $\kappa(\omega)$, $\psi(\omega)$ and $a\psi(\omega)$ are mixing random fields, we show that p_ε converges strongly in the energy norm to the solution of the homogeneous problem provided that:

$$\alpha < \frac{d}{(3+\frac{d}{2})(d+2)} \wedge \frac{1}{2(2+\frac{d}{2})}, \qquad d \geq 1. \tag{7}$$

Here $a \wedge b = \inf\{a, b\}$. We then obtain error estimates between p_ε and the homogenized solution of order $\varepsilon^{\lambda(\alpha,d)}$, where

$$\lambda(\alpha, d) = \Big(\frac{d}{d+2} - (3+\frac{d}{2})\alpha\Big) \wedge \Big(\frac{1}{2} - (2+\frac{d}{2})\alpha\Big) + \eta, \tag{8}$$

for an arbitrary $\eta > 0$.

In the second scenario, which has a more general applicability, we assume that $\kappa(\omega)$ and $a(\omega)$ are mixing random fields. Based on results obtained in [17], which are restricted to spatial dimensions $d \geq 3$, and assuming to simplify that $a(\omega)$ is mixing exponentially rapidly (the mixing coefficient $\rho(r)$ defined in (24) below decays exponentially fast), we can show that the error estimates between p_ε and the homogenized solution of order $\varepsilon^{\lambda_\beta(\alpha,d)}$, where $\lambda_\beta(\alpha, d)$ is defined as

$$\lambda_\beta(\alpha, d) = \max_{0<\xi<1} \Big[\Big(\frac{\xi d}{d+2} - (3 + \frac{d}{2})\alpha \Big) \wedge \big(\xi \wedge \gamma_1(\xi) - (2 + \frac{d}{2})\alpha \big) \Big]. \tag{9}$$

Here $\gamma_1(\xi)$ is defined for all fixed $0 < \gamma < \frac{1}{4}$ as

$$\gamma_1(\xi) = \frac{\gamma(d-2) - \xi(1 + \gamma(d-1))}{1 + \gamma d}. \tag{10}$$

We then obtain convergence of p_ε to the homogenized solution in the energy norm provided that α is chosen small enough so that $\lambda_\beta(\alpha, d) > 0$.

Outline for the rest of the paper. Section 2 describes the main assumptions on the random medium and analyzes the solutions $\boldsymbol{\theta}^\beta$ and ψ of the local problems. Section 3 describes the main mixing assumptions imposed on the random medium and derives decorrelation estimates for the solutions to the local problems. Error estimates for the homogenization of the wave equation are obtained in section 4. These error estimates are then used in section 5 to establish the explicit limiting behavior of the energy density of high frequency waves propagating in random media provided that the wavelength of the waves is sufficiently larger than the correlation length of the medium.

2 Random medium and local problems

Following [15], we recall the construction of the random processes and some of their properties. We first define $\mathcal{H} = L^2((\Omega, \mathcal{F}, P))$, the Hilbert space of square integrable random variables on Ω with inner product $\mathbb{E}\{fg\} = \int_\Omega f(\omega)g(\omega)dP(\omega)$. We define $H = L^2(\mathbb{R}^d; \mathcal{H}) = L^2(\mathbb{R}^d \times \Omega)$ equipped with its natural inner product $(u, v) = \int_{\mathbb{R}^d} \mathbb{E}\{uv\}d\mathbf{x}$, and $X = \mathcal{C}(0, T; H)$. We also define $H^1(\mathbb{R}^d; \mathcal{H}) \equiv L^2(\Omega; H^1(\mathbb{R}^d))$ by Fubini as the space of $\mathcal{H}$ valued functions whose spatial derivatives are also square integrable, and equip this Hilbert space with the usual inner product $(u, v)_1 = (u, v) + \sum_{j=1}^d (\frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_j})$.

We assume the existence of a group of one-to-one transformations $\tau_\mathbf{x} : \Omega \to \Omega$ for $\mathbf{x} \in \mathbb{R}^d$, which leave the probability measure P invariant. The group property is that $\tau_{\mathbf{x}+\mathbf{y}} = \tau_\mathbf{x}\tau_\mathbf{y}$. The invariance property is that for all $F \in \mathcal{F}$, $P(\tau_\mathbf{x}^{-1}F) = P(F)$, where $\tau_\mathbf{x}^{-1}F$ is the set of ω' such that $\tau_\mathbf{x}\omega' \in F$. The group of transformations $\tau_\mathbf{x}$ generates a unitary group of operators $T_\mathbf{x} : \mathcal{H} \to \mathcal{H}$ defined by

$$(T_\mathbf{x} f)(\omega) = f(\tau_{-\mathbf{x}}\omega), \quad \mathbf{x} \in \mathbb{R}^d. \tag{11}$$

By varying one coordinate of $\mathbf{x}$ at a time in the group $T_\mathbf{x}$, we obtain d one-parameter strongly continuous unitary groups in $\mathcal{H}$. They thus admit infinitesimal generators D_j,

$1 \leq j \leq d$, which are closely and densely defined in their domain $\mathcal{D}(D_i) \subset \mathcal{H}$. For $f \in \mathcal{D}(D_j)$, we have

$$(D_j f)(\omega) = \frac{\partial}{\partial x_j}(T_{\mathbf{x}} f)(\omega)\Big|_{\mathbf{x}=0}. \tag{12}$$

We define $\mathbf{D} = (D_1, \ldots, D_d)^t$ the vector-valued infinitesimal generator. Because $T_{\mathbf{x}}$ is unitary we verify the skew-adjointness property $\mathbb{E}\{gD_j f\} = -\mathbb{E}\{fD_j g\}$, for all $f, g \in \mathcal{D}(D_j)$. We now define the linear subspace $\mathcal{H}^1 = \bigcap_{j=1}^d \mathcal{D}(D_j)$, which is dense in $\mathcal{H}$ and is a Hilbert space, since the D_j are closed, when equipped with the inner product $\mathbb{E}\{fg\} + \mathbb{E}\{\mathbf{D}f \cdot \mathbf{D}g\} = \mathbb{E}\{fg\} + \sum_{j=1}^d \mathbb{E}\{D_j f D_j g\}$, and associated norm $\|f\|_{\mathcal{H}^1} = (\|f\|_{\mathcal{H}}^2 + \|\mathbf{D}f\|_{\mathcal{H}}^2)^{1/2}$.

For any $f(\omega) \in \mathcal{H}$, we can associate the *stationary* process

$$f(\mathbf{x}, \omega) = (T_{\mathbf{x}} f)(\omega) = f(\tau_{-\mathbf{x}}\omega), \tag{13}$$

where the same notation f is used for the function in $\mathcal{H}$ and the function on $\Omega \times \mathbb{R}^d$. Let $H_{\text{loc}}(\mathbb{R}^d; \mathcal{H})$ be the space of functions from $\mathbb{R}^d$ into $\mathcal{H}$ with inner product $\int_{\mathcal{O}} \mathbb{E}\{f(\mathbf{x})g(\mathbf{x})\}d\mathbf{x}$ for every bounded open set $\mathcal{O}$. Then $H_S(\mathbb{R}^d; \mathcal{H})$, the space of stationary random processes on $\mathbb{R}^d$, i.e., processes such that $f(\mathbf{x}, \omega) = f(\tau_{-\mathbf{x}}\omega)$, is a closed subset of $H_{\text{loc}}(\mathbb{R}^d; \mathcal{H})$ in one-to-one correspondence with $\mathcal{H}$. The processes in H_S are thus strictly stationary, in the sense that their joint distribution at n arbitrary points $\mathbf{x}_i$ is the same as the distribution at the points $\mathbf{x}_i + \mathbf{h}$ for an arbitrary $\mathbf{h} \in \mathbb{R}^d$. Similarly, one may identify $H_S^1(\mathbb{R}^d; \mathcal{H})$, the set of mean square differentiable stationary processes, with $\mathcal{H}^1$. We then verify that for $f \in H_S^1$,

$$\frac{\partial f(\mathbf{x}, \omega)}{\partial x_j} = D_j f(\mathbf{x}, \omega) \qquad d\mathbf{x} \times P \ a.e. \tag{14}$$

For stationary processes, $\nabla_{\mathbf{x}}$ and $\mathbf{D}$ are thus identified.

One of the main assumption on $\tau_{\mathbf{x}}$ is that it is P-*ergodic* on Ω. This means that for any invariant set $A \in \mathcal{F}$, i.e., a set such that $\tau_{\mathbf{x}} A \subset A$, we have either $P(A) = 0$ or $P(A) = 1$. The unitary group $T_{\mathbf{x}}$ is then ergodic on $\mathcal{H}$. This implies that the only functions in $\mathcal{H}$ that are invariant under $T_{\mathbf{x}}$ are the constant functions.

We now assume that the random variables $a_{ij}(\mathbf{x}, \omega)$ and $\kappa(\mathbf{x}, \omega)$ are stationary processes. Since they are uniformly bounded, both processes belong to $\mathcal{H}$ at $\mathbf{x}$ fixed and we assume that

$$a(\mathbf{x}, \omega) = a(\tau_{-\mathbf{x}}\omega), \qquad \kappa(\mathbf{x}, \omega) = \kappa(\tau_{-\mathbf{x}}\omega). \tag{15}$$

The above assumptions are sufficient to obtain existence and uniqueness of a solution to the wave equation (4) under classical assumptions on the initial conditions; see Proposition 4.1 below. When $\alpha = 0$, minor modifications of the theory developed in [11, 15] allow us to obtain the convergence of the wave solution to its homogenized limit (see the proof of Thm. 4.4 below). To obtain error estimates for the homogenization corrector and address high frequency initial conditions, additional mixing and smoothness conditions are required.

Local Problem. The construction of the homogenized coefficient a^* requires one to solve the following *local problem*: Find $\psi = (\psi_{ij})_{ij} \in \mathcal{H}$ such that

$$\begin{aligned} &\mathbf{D} \cdot (a\psi)(\omega) = 0 \\ &\mathbb{E}\psi = I, \qquad \mathbf{D} \times \psi = 0. \end{aligned} \tag{16}$$

Here, the curl operator $\mathbf{D}\times$ is defined component-wise by $(\mathbf{D}\times\psi)_i = \sum_{j,k} D_k\psi_{ij} - D_j\psi_{ik}$. The matrix-valued solution $\psi(\omega)$ is uniquely defined. The proof of such a result may be found in [10, 11, 15]. The stationary field ψ may be extended in $H_S(\mathbb{R}^d; \mathcal{H})$ as $\psi(\mathbf{x}, \omega) = \psi(\tau_{-\mathbf{x}}\omega)$. Note that $\nabla \cdot (a\psi)(\mathbf{x}, \omega) = 0$ and $\nabla \times \psi = 0$ as in (1).

The existence of a solution ψ is obtained by a limiting absorption principle and requires one to solve the following local problems:

$$-\nabla \cdot a\nabla\boldsymbol{\theta}^\beta(\mathbf{x}, \omega) + \beta\boldsymbol{\theta}^\beta(\mathbf{x}, \omega) = \nabla \cdot a(\mathbf{x}, \omega), \qquad \mathbf{x} \in \mathbb{R}^d, \tag{17}$$

for almost all $\omega \in \Omega$, where $\beta > 0$. We use the convention that $\nabla\boldsymbol{\theta}^\beta$ and $\mathbf{D}\boldsymbol{\theta}^\beta$ are $d \times d$ matrices. Solutions are sought among stationary vector fields $\boldsymbol{\theta}^\beta(\mathbf{x}, \omega) = \boldsymbol{\theta}^\beta(\tau_{-\mathbf{x}}\omega)$, so that the equation may be recast, using (14), as

$$-\mathbf{D} \cdot a\mathbf{D}\boldsymbol{\theta}^\beta(\omega) + \beta\boldsymbol{\theta}^\beta(\omega) = \mathbf{D} \cdot a(\omega), \tag{18}$$

and more precisely, as the following variational problem: Find $\boldsymbol{\theta}^\beta \in \mathcal{H}^1$ such that

$$\mathbb{E}\{\mathrm{Tr}[a\mathbf{D}\boldsymbol{\theta}^\beta\mathbf{D}\boldsymbol{\phi}]\} + \beta\mathbb{E}\{\boldsymbol{\theta}^\beta \cdot \boldsymbol{\phi}\} = -\mathbb{E}\{\mathrm{Tr}[a\mathbf{D}\boldsymbol{\phi}]\}, \qquad \text{for all } \boldsymbol{\phi} \in (\mathcal{H}^1)^d. \tag{19}$$

Here Tr stands for matrix trace. The above variational problem admits a unique solution [15] by application of the Lax-Milgram theorem in the Hilbert space $\mathcal{H}$.

The properties of $\boldsymbol{\theta}^\beta$ are central in the results obtained in [15, 17]. We easily derive from (19) that $\mathbf{D}\boldsymbol{\theta}^\beta$ and $\beta^{\frac{1}{2}}\boldsymbol{\theta}^\beta$ are bounded in $\mathcal{H}$ independent of β. We call

$$\psi = \lim_{\beta\to 0} \psi^\beta, \qquad \text{where} \quad \psi^\beta = (I + \mathbf{D}\boldsymbol{\theta}^\beta), \tag{20}$$

and verify that ψ indeed solves (16). Note that $\mathbf{D}\boldsymbol{\theta}^\beta$ is in gradient form. In the limit however, it is not guaranteed that $\psi - I$ can indeed be written as the gradient of a stationary process. However, we verify that its solenoidal component $\mathbf{D} \times \psi = 0$.

It is shown in [15] that $\sqrt{\beta}\boldsymbol{\theta}^\beta$ converges to 0 strongly in $\mathcal{H}$. In homogenization in periodic media, the local problem (18) is replaced by a problem on a cell of periodicity Y, and then $\boldsymbol{\theta}^\beta$, whose average over the cell vanishes, is bounded in $L^2_\#(Y)$ independent of β; see e.g. [4, 10]. Such a uniform bound generally does not hold in random media. The asymptotic behavior as $\beta \to 0$ dictates the speed of convergence of the heterogeneous solution to its homogenized limit. The best available results on error estimates for $\boldsymbol{\theta}^\beta$ can be found in [17]. Following [11], we also introduce the corrector

$$\mathbf{z}^\varepsilon(\mathbf{x}, \omega) = \int_0^1 \psi(\frac{t\mathbf{x}}{\varepsilon})\mathbf{x}dt. \tag{21}$$

We verify that

$$\nabla_\mathbf{x}\mathbf{z}^\varepsilon(\mathbf{x}, \omega) = \psi(\frac{\mathbf{x}}{\varepsilon}, \omega) = \psi(\tau_{-\frac{\mathbf{x}}{\varepsilon}}\omega).$$

The gradient of $\mathbf{z}^\varepsilon$ may thus be written as a stationary field although $\mathbf{z}^\varepsilon$ itself is not stationary. The gradient of $(\mathbf{z}^\varepsilon - \mathbf{x})$ is equal to $\psi - I$, whose statistical average vanishes. We thus find that $\mathbf{z}^\varepsilon - \mathbf{x}$ plays a similar role to that of $\varepsilon\boldsymbol{\theta}^\beta(\frac{\mathbf{x}}{\varepsilon})$. It is shown in [11] that $\mathbf{z}^\varepsilon$ is a Hölder function P-a.s. and that $\mathbf{z}^\varepsilon(\mathbf{x}) - \mathbf{x}$ converges to $\mathbf{0}$ as $\varepsilon \to 0$ P-a.s. uniformly on compact sets $K \subset \mathbb{R}^d$.

Homogenized coefficients. We have defined the tensor ψ in (16) and the corrector $\boldsymbol{\theta}^\beta$ in (18). The effective medium coefficients that appear in the limit of solutions to (4) are then defined by

$$a^* = \mathbb{E}\{a\psi\}, \qquad a^{\beta*} = \mathbb{E}\{a\psi^\beta\}, \qquad \kappa^* = \mathbb{E}\{\kappa\}. \tag{22}$$

It is a classical result [11, 15] that

Lemma 2.1 *The homogenized matrices $a^{\beta*}$ and a^* are positive definite and satisfy the relations:*

$$a^* = \mathbb{E}\{\psi^t a\psi\}, \quad a^*\boldsymbol{\xi}\cdot\boldsymbol{\xi} \geq a_0|\boldsymbol{\xi}|^2, \qquad a^{\beta*} = \mathbb{E}\{(\psi^\beta)^t a\psi^\beta\}, \quad a^{\beta*}\boldsymbol{\xi}\cdot\boldsymbol{\xi} \geq a_0|\boldsymbol{\xi}|^2. \tag{23}$$

κ^* is also a positive constant by assumption on $\kappa(\omega)$.

3 Mixing properties and decorrelations

The notation and hypotheses introduced so far allow us to show that when $\alpha = 0$, i.e., when the initial conditions do not oscillate rapidly, the heterogeneous solution to (4) converges to the solution of the homogeneous problem (34) below, where $a(\mathbf{x})$ and $\kappa(\mathbf{x})$ are replaced by a^* and κ^*. Such a convergence result is obtained using the techniques developed in [11, 15] and the stability result stated in Proposition 4.1 below.

To obtain error estimates, the random medium needs to verify additional assumptions. Very few results exist in this direction. In [17], error estimates between the heterogeneous solution and the homogenized solution of diffusion equations are obtained provided that the coefficient $a(\omega)$ is (strongly) *mixing* (see definition below), in addition to being ergodic. The approach hinges on analyzing the mixing properties of $\boldsymbol{\theta}^\beta$, the solution of (2). Such a mixing is obtained from the exponential decay of the Green's function associated to (2). The decay is however very slow, as $\exp(-\sqrt{\beta}|\mathbf{x}|)$, since β is a small regularizing parameter. The mixing properties on $\boldsymbol{\theta}^\beta$ thus provide very slow and presumably sub-optimal convergence estimates. The results in [17] are however the best results available in the literature at present.

In addition to the estimates obtained in [17], we also consider a simpler setting, where we *assume* that other local random fields are also mixing. More precisely, we assume that both $\psi(\omega)$ defined in (16) and $(a\psi)(\omega)$, whose average provides the homogenized coefficient a^*, are mixing. We also assume that the compressibility coefficient $\kappa(\omega)$ is mixing. We shall come back to the rationale for these mixing assumptions at the end of the section. By mixing, we mean here the following strong mixing condition.

For two Borel sets $A, B \subset \mathbb{R}^d$, we denote by $\mathcal{F}_A$ and $\mathcal{F}_B$ the sub-σ algebras of $\mathcal{F}$ generated by the fields $a_{ij}(\mathbf{x},\omega)$, $\psi_{ij}(\mathbf{x},\omega)$, $(a\psi)_{ij}(\mathbf{x},\omega)$, and $\kappa(\mathbf{x},\omega)$ for $\mathbf{x} \in A$ and $\mathbf{x} \in B$, respectively. We denote by $L^2_P(\Omega, \mathcal{F}_A)$ the space of square integrable random variables on $(\Omega, \mathcal{F}_A, P)$. We then define the $\rho-$ mixing coefficient as:

$$\rho(r) = \sup_{\substack{A,B\in\mathbb{R}^d \\ \mathrm{dist(A,B)}\geq \mathrm{r}}} \; \sup_{\substack{\eta\in L^2_P(\Omega,\mathcal{F}_A) \\ \xi\in L^2_P(\Omega,\mathcal{F}_B)}} \frac{\mathbb{E}\big\{(\eta - \mathbb{E}\{\eta\})(\xi - \mathbb{E}\{\xi\})\big\}}{\big(\mathbb{E}\{\eta^2\}\mathbb{E}\{\xi^2\}\big)^{\frac{1}{2}}}. \tag{24}$$

We assume that $\rho(r)$ decays sufficiently fast as $r \to \infty$ so that $\rho(r) \lesssim r^{-\upsilon}$ for some $\upsilon > 0$. The notation $a \lesssim b$ means that there exists a positive constant C such that

$a \leq Cb$. In most of the paper, we will assume short range correlations of the form $\rho(r) \lesssim e^{-r}$ to simplify some expressions. In other words, the correlation of two variables ξ and η, which depend on events restricted on spatial domains separated by a certain distance, decays rapidly with that distance.

The mixing assumptions are used to show results of the following type, which have already appeared in the literature under various forms, see e.g. [17].

Lemma 3.1 *Let $f(\omega)$ be a random field in $\mathcal{H}$ with mean zero, $\mathbb{E}\{f\} = 0$, and such that the correlation function $\mathbb{E}\{f(\omega)f(\tau_{-\mathbf{y}}\omega)\}$ is bounded by the function $\rho(|\mathbf{y}|)$ defined above. Let $\mathcal{C}$ be a cube of length $M \geq 1$ and thus of volume $|\mathcal{C}| = M^d$. Then we find that*

$$\begin{aligned} \mathbb{E}\Big|\frac{1}{|\mathcal{C}|}\int_{\mathcal{C}} f(\mathbf{x},\omega)d\mathbf{x}\Big|^2 &\lesssim \frac{1}{|\mathcal{C}|}M^{(d-\upsilon)\vee 0}, \qquad \upsilon \neq d \\ &\lesssim \frac{1}{|\mathcal{C}|}|\ln M|, \qquad \upsilon = d. \end{aligned} \tag{25}$$

Here, $a \vee b = \sup(a,b)$. In other words, for sufficiently rapidly decaying mixing coefficients with $\upsilon > d$, which we assume for now on for simplicity, the above variance is inversely proportional to the volume $|\mathcal{C}|$.

Proof. We write

$$\mathbb{E}\Big(\int_C f(\mathbf{x},\omega)d\mathbf{x}\Big)^2 = \mathbb{E}\int_C\int_C f(\mathbf{x},\omega)f(\mathbf{y},\omega)d\mathbf{x}d\mathbf{y} = \mathbb{E}\int_C\int_C f(\omega)T_{\mathbf{y}-\mathbf{x}}f(\omega)d\mathbf{x}d\mathbf{y}.$$

We deduce from the mixing assumptions on f that the above term is bounded by

$$\|f\|_{\mathcal{H}}^2\int_C\int_C \rho(|\mathbf{x}-\mathbf{y}|)d\mathbf{x}d\mathbf{y}.$$

Classical estimates for the above integral written in spherical coordinates allow us to conclude the proof of the lemma. □

We now state the following result, which will be important in the analysis of the error estimates in the next section.

Theorem 3.2 *Let $f(\mathbf{x},\omega)$ be a stationary random field as in Lemma 3.1 and assume that $\upsilon > d$ to simplify. Let K be a compact cube in $\mathbb{R}^d$ and $\phi(\mathbf{x},\omega)$ a random field in $L^2(\Omega; H^1(K))$ equipped with its usual norm $\|\cdot\|_{1,K}$. Then we have that:*

$$\mathbb{E}\Big|\int_K f\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big)\phi(\mathbf{x},\omega)d\mathbf{x}\Big| \lesssim \varepsilon^{\frac{d}{d+2}}\|\phi\|_{1,K}\|f\|_{\mathcal{H}}. \tag{26}$$

Note that by Fubini, $L^2(\Omega; H^1(K)) \equiv H^1(K;\mathcal{H})$. We define more generally by $\|\phi\|_{s,K}$ (in this paper for $s = -1, 0, 1$) the norm of ϕ in $L^2(\Omega; H^s(K)) \equiv H^s(K;\mathcal{H})$. The previous theorem provides an error estimate for $\|f\|_{-1,K}$.

Proof. The proof of the theorem is similar to what is obtained in e.g. [17]. We break up K into a finite number of non-overlapping identical cubes K_i of length $l \ll 1$ for $1 \leq i \leq I \approx l^{-d}$. We denote by $\bar{\phi}_i$ the average of ϕ on K_i and by $\overline{\phi_i^2}$ the average of ϕ^2 on K_i, and calculate

$$\begin{aligned} \Big|\int_{K_i} f\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big)\phi(\mathbf{x},\omega)d\mathbf{x}\Big| &\leq \Big|\int_{K_i} f\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big)\bar{\phi}_i(\omega)d\mathbf{x}\Big| + \Big|\int_{K_i} f\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big)(\phi-\bar{\phi}_i)(\mathbf{x},\omega)d\mathbf{x}\Big| \\ &\lesssim \Big|\frac{1}{|K_i|^{\frac12}}\int_{K_i} f\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big)d\mathbf{x}\Big||K_i|^{\frac12}(\overline{\phi_i^2})^{\frac12} + l\Big|\int_{K_i} f^2\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big)d\mathbf{x}\Big|^{\frac12}\Big|\int_{K_i}|\nabla\phi|^2(\mathbf{x},\omega)d\mathbf{x}\Big|^{\frac12}. \end{aligned}$$

The last estimate results from using the Poincaré-Friedrichs inequality. Upon summing all contributions, on the order of $|K_i|^{-1} = l^{-d}$ of them, we deduce from the Cauchy-Schwarz inequality that

$$\Big|\int_K f\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big)\phi(\mathbf{x},\omega)d\mathbf{x}\Big| \lesssim \Big[\Big(\sum_i \frac{1}{|K_i|}\Big(\int_{K_i} f\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big)d\mathbf{x}\Big)^2\Big)^{\frac{1}{2}} + l\|f\|_{L^2(K)}\Big]\|\phi\|_{H^1(K)}$$
$$\lesssim \Big[\sum_i |K_i|\Big(\frac{1}{|K_i|}\int_{K_i} f\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big)d\mathbf{x}\Big)^2 + l^2\|f\|^2_{L^2(K)}\Big]^{\frac{1}{2}}\|\phi\|_{H^1(K)}.$$

Upon taking expectation and using the Cauchy-Schwarz inequality, we obtain

$$\mathbb{E}\Big|\int_K f\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big)\phi(\mathbf{x},\omega)d\mathbf{x}\Big|$$
$$\lesssim \Big(\Big(\sum_i |K_i|\mathbb{E}\Big\{\Big[\frac{1}{|K_i|}\int_{K_i} f\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big)d\mathbf{x}\Big]^2\Big\}\Big)^{\frac{1}{2}} + l\big(\mathbb{E}\|f\|^2_{L^2(K)}\big)^{\frac{1}{2}}\Big)\big(\mathbb{E}\|\phi\|^2_{H^1(K)}\big)^{\frac{1}{2}}.$$

Note that the above term on the right hand side does not depend on i since f is a stationary process. We now use (25) to obtain that

$$\mathbb{E}\Big\{\Big[\frac{1}{|K_i|}\int_{K_i} f\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big)d\mathbf{x}\Big]^2\Big\} = \mathbb{E}\Big\{\Big[\frac{1}{|\frac{K_i}{\varepsilon}|}\int_{\frac{K_i}{\varepsilon}} f(\mathbf{x},\omega)d\mathbf{x}\Big]^2\Big\} \lesssim \frac{\varepsilon^d}{|K_i|}\|f\|^2_{\mathcal{H}} \lesssim \frac{\varepsilon^d}{l^d}\|f\|^2_{\mathcal{H}},$$

independent of the index $1 \le i \le I$. This allows us to deduce that

$$\mathbb{E}\Big|\int_K f\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big)\phi(\mathbf{x},\omega)d\mathbf{x}\Big| \lesssim \Big(\Big(\frac{\varepsilon}{l}\Big)^{\frac{d}{2}} + l\Big)\|\phi\|_{1,K}\|f\|_{\mathcal{H}}.$$

It remains to choose l so that both terms on the right hand side are of the same order, namely $l = \varepsilon^{\frac{d}{d+2}} \ll 1$, to conclude the proof of (26). □

The above theorem applies to the mean zero random fields $\hat{a}(\omega) = (a\psi)(\omega) - a^*$ and $\hat{\kappa}(\omega) = \kappa(\omega) - \kappa^*$ when *by assumption*, they satisfy the required mixing conditions. Note that the error estimates given above are not optimal. For instance, in dimension $d = 1$, we obtain here an error estimate of order $\varepsilon^{\frac{1}{3}}$ whereas we can actually obtain an estimate of order $\varepsilon^{\frac{1}{2}}$ using more sophisticated techniques.

We also want to obtain similar estimates in the H^{-1}-norm for random processes that do not satisfy the mixing hypotheses stated in Lemma 3.1. Estimates of the form (26) may still be established when the average of the process decays with the size of the domain on which averaging takes place. We have the following result.

Theorem 3.3 *Let $f(\mathbf{x},\omega)$ be a stationary random field such that*

$$\mathbb{E}\Big|\frac{1}{|\mathcal{C}|}\int_{\mathcal{C}} f(\mathbf{x},\omega)d\mathbf{x}\Big|^2 \lesssim \frac{1}{\delta^2}\frac{1}{|\mathcal{C}|^{2\zeta}}\|f\|^2_{\mathcal{H}}, \tag{27}$$

for some positive constants δ and ζ. Let $\phi(\mathbf{x},\omega)$ be a random field in $L^2(\Omega; H^1(K))$, where K is a compact cube in $\mathbb{R}^d$. Then we have that

$$\mathbb{E}\Big|\int_K f\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big)\phi(\mathbf{x},\omega)d\mathbf{x}\Big| \lesssim \Big(\frac{\varepsilon^\zeta}{\delta}\Big)^{\frac{1}{1+d\zeta}}\|\phi\|_{1,K}\|f\|_{\mathcal{H}}. \tag{28}$$

In other words, we get an error estimate for $\|f\|_{-1,K}$.

Proof. The proof of the preceding theorem applies until we arrive at the estimate

$$\mathbb{E}\Big\{\Big[\frac{1}{|\frac{K_i}{\varepsilon}|}\int_{\frac{K_i}{\varepsilon}} f(\mathbf{x},\omega)d\mathbf{x}\Big]^2\Big\} \lesssim \frac{1}{\delta^2}\frac{\varepsilon^{2\zeta}}{l^{2d\zeta}}\|f\|^2_{\mathcal{H}}.$$

Therefore,

$$\mathbb{E}\Big|\int_K f\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big)\phi(\mathbf{x},\omega)d\mathbf{x}\Big| \lesssim \Big(\frac{1}{\delta}\frac{\varepsilon^{\zeta}}{l^{d\zeta}}+l\Big)\|\phi\|_{1,K}\|f\|_{\mathcal{H}}.$$

It remains to optimize the choice of l to conclude the proof of the theorem. □

We will now apply the above theorem to the vector field $\frac{\beta}{\varepsilon}\boldsymbol{\theta}^\beta$ and to the tensor field $\hat{a}^\beta = a\psi^\beta - \mathbb{E}\{a\psi^\beta\}$. The following results are proved in Lemmas 2.4 and 2.5 in [17]; see also the appendix in [6].

Lemma 3.4 *Let $\boldsymbol{\theta}^\beta$ the mean-zero random field defined in* (18) *and $\hat{a}^\beta = a\mathbf{D}\boldsymbol{\theta}^\beta - \mathbb{E}\{a\mathbf{D}\boldsymbol{\theta}^\beta\}$ a tensor-valued mean-zero random field. We assume that the mixing coefficient $\rho(r)$, defined in* (24) *for the sub-σ algebras of $\mathcal{F}$ generated by the fields $a_{ij}(\mathbf{x},\omega)$, decays exponentially fast, $\rho(r)\lesssim e^{-r}$. Then we have the estimates:*

$$\mathbb{E}\Big|\frac{1}{|\mathcal{C}|}\int_{\mathcal{C}}\boldsymbol{\theta}^\beta(\mathbf{x},\omega)d\mathbf{x}\Big|^2 \lesssim \frac{1}{(\beta|\mathcal{C}|)^{2\gamma}}\|\boldsymbol{\theta}^\beta\|^2_{\mathcal{H}}, \tag{29}$$

for all $0<\gamma<\frac{1}{4}$, and

$$\mathbb{E}\Big|\frac{1}{|\mathcal{C}|}\int_{\mathcal{C}}\hat{a}^\beta(\mathbf{x},\omega)d\mathbf{x}\Big|^2 \lesssim \frac{1}{\beta^d}\frac{1}{|\mathcal{C}|^2}\|\hat{a}^\beta\|^2_{\mathcal{H}}. \tag{30}$$

Less accurate estimates are also available when $\rho(r)$ has longer range correlations; we refer to [17] for the details. A direct application of the previous lemma and Theorem (3.3) allows us to deduce the following estimates:

Corollary 3.5 *Let us define*

$$\beta=\varepsilon^{2(1-\xi)}, \qquad 0<\xi<1. \tag{31}$$

Under the assumptions of the previous lemma and for each random field $\phi(\mathbf{x},\omega)\in L^2(\Omega;H^1(K))$, where K is a compact cube in $\mathbb{R}^d$, we obtain that

$$\begin{aligned}\mathbb{E}\Big|\int_K \frac{\beta}{\varepsilon}\boldsymbol{\theta}^\beta\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big)\phi(\mathbf{x},\omega)d\mathbf{x}\Big| &\lesssim \varepsilon^{\gamma_1}\|\beta^{\frac{1}{2}}\boldsymbol{\theta}^\beta\|_{\mathcal{H}}\|\phi\|_{1,K},\\ \mathbb{E}\Big|\int_K \hat{a}^\beta_{ij}\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big)\phi(\mathbf{x},\omega)d\mathbf{x}\Big| &\lesssim \varepsilon^{\frac{\xi d}{d+2}}\|\hat{a}^\beta\|_{\mathcal{H}}\|\phi\|_{1,K},\end{aligned} \tag{32}$$

independent of $1\le i,j\le d$, where $\gamma_1=\gamma_1(\xi)$ is defined in (10).

Note that the term γ_1 may be chosen positive provided that ξ is sufficiently small.

It remains to perform a statistical analysis of the corrector $\mathbf{z}^\varepsilon(\mathbf{x})-\mathbf{x}$.

Lemma 3.6 *Assuming that* (24) *and that $\upsilon>d$, we have*

$$\Big|\mathbb{E}\{(\mathbf{z}^\varepsilon(\mathbf{x})-\mathbf{x})\cdot(\mathbf{z}^\varepsilon(\mathbf{y})-\mathbf{y})\}\Big| \lesssim \varepsilon(|\mathbf{x}|+|\mathbf{y}|). \tag{33}$$

Proof. By assumption, $\rho(r)$ is bounded by some $\tilde\rho(r)$, a bounded and decreasing function bounded by $Cr^{-d+\eta}$ for $C>0$ and $\eta>0$ as $r\to\infty$. We verify that

$$\begin{aligned}&\Big|\mathbb{E}\{(\mathbf{z}^\varepsilon(\mathbf{x})-\mathbf{x})\cdot(\mathbf{z}^\varepsilon(\mathbf{y})-\mathbf{y})\}\Big| = \Big|\mathbf{x}^t\Big(\int_0^1\int_0^1 \mathbb{E}\{(\psi-I)^t(\frac{t\mathbf{x}}{\varepsilon})(\psi-I)(\frac{s\mathbf{y}}{\varepsilon})\}dtds\Big)\mathbf{y}\Big| \\ &\lesssim |\mathbf{x}||\mathbf{y}|\int_0^1\int_0^1\rho\Big(\Big|\frac{t\mathbf{x}-s\mathbf{y}}{\varepsilon}\Big|\Big)ds\,dt \lesssim \int_0^{|\mathbf{x}|}\int_0^{|\mathbf{y}|}\tilde\rho\Big(\frac{\sqrt{(t-s)^2+2st\cos\theta}}{\varepsilon}\Big)ds\,dt \\ &\lesssim \int_0^{|\mathbf{x}|}\int_0^{|\mathbf{y}|}\tilde\rho\Big(\frac{|t-s|}{\varepsilon}\Big)ds\,dt \lesssim \varepsilon(|\mathbf{x}|+|\mathbf{y}|),\end{aligned}$$

since $\tilde\rho$ is integrable, where we have defined $\cos\theta=\frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}||\mathbf{y}|}$. This concludes the proof. ∎

This shows that $\mathbf{z}^\varepsilon(\mathbf{x})-\mathbf{x}$ is of order $O(\sqrt{\varepsilon})$. Note that the above estimate is not optimal when $\mathbf{x}\neq\mathbf{y}$. It will however be sufficient in the sequel.

Remark on the mixing assumptions. The assumptions that $\psi(\omega)$ and $a\psi(\omega)$ are mixing processes are probably very difficult to verify in practical settings. We want to present here some simple cases where they are verified and some heuristic arguments as to why they look reasonable.

These assumptions are verified in the one-dimensional setting, where we find that $\psi(\omega)=(\mathbb{E}\{a^{-1}\})^{-1}a^{-1}(\omega)$ is inversely proportional to $a(\omega)$ and $a\psi$ is deterministic. When $a(\omega)$ is mixing, then so are $\psi(\omega)$ and $a\psi(\omega)$. The theory presented in the paper thus applies in that setting.

More generally, we can construct $a(\mathbf{x},\omega)=a(\tau_{-\mathbf{x}}\omega)$ as $a(\mathbf{x},\omega)=\mathrm{Diag}(a_k(x_k,\omega))$. In such a setting, we verify that $\psi(\mathbf{x},\omega)=\mathrm{Diag}(\frac{a_k^{-1}(x_k)}{\mathbb{E}\{a_k^{-1}(x_k)\}})$. Again, $a\psi$ is a deterministic diagonal tensor, so that both ψ and $a\psi$ are mixing when a is mixing. A slightly more interesting example is the case where $a(\mathbf{x},\omega)=\prod_k a_k(x_k,\omega)I$. We can show that $\psi(\mathbf{x},\omega)=\mathrm{Diag}(\frac{\alpha_k}{a_k(x_k)})$, where $\alpha_k=\mathbb{E}\{a_k^{-1}\}^{-1}$. As a consequence, $a\psi(\mathbf{x},\omega)=\mathrm{Diag}(\frac{\alpha_k a(\mathbf{x})}{a_k(x_k)})$. When all the coefficients $a_k(x_k,\omega)$ are mixing, we deduce that both ψ and $a\psi$, which is no longer deterministic, are mixing as well. These examples are of limited practical interest because of their "Cartesian grid" effects.

Other multi-dimensional processes that satisfy the hypotheses may also be constructed as follows. Let us assume that $a(\omega)$ takes the form

$$a(\omega)=\frac{I}{I+\mathbf{D}\otimes\mathbf{D}\gamma(\omega)},$$

for some scalar-valued process γ such that $|\mathbf{D}\otimes\mathbf{D}\gamma(\omega)|\leq\gamma_0<1$ $P-$a.s., so that $a(\omega)$ is a symmetric positive-definite matrix $P-$a.s. Then $\psi(\omega)=I+\mathbf{D}\otimes\mathbf{D}\gamma(\omega)$ satisfies (16). It remains to assume that γ has appropriate mixing conditions to deduce that both $\psi(\omega)$ and $a\psi(\omega)$, which is deterministic, satisfy the required assumptions. In this case however, ψ may be written as $I+\mathbf{D}\boldsymbol{\theta}$ with $\boldsymbol{\theta}=\mathbf{D}\gamma$. The corrector $\boldsymbol{\theta}^\beta$ is therefore bounded independently of β in $L^\infty(\Omega)$ as in the periodic case [4, 10].

More generally, let us assume that

$$a(\omega)=\frac{I}{I+\lambda(\mathbf{D}\boldsymbol{\gamma}+\mathbf{D}\times\delta)(\omega)},$$

which is the Weyl decomposition [10] of a^{-1} for a tensor whose average is the identity matrix. We assume that the vector $\boldsymbol{\gamma}$, the matrix δ, and λ are chosen so that $a(\omega)$ is a symmetric, positive definite matrix. Let us assume that $\psi(\omega) = I + \lambda\phi(\omega)$ and let us expand $\phi = \phi_0 + \lambda\phi_1 + O(\lambda^2)$. Upon performing an asymptotic expansion of $\mathbf{D}\cdot(a\psi) = 0$, we obtain that $\phi_0 = \mathbf{D}\boldsymbol{\gamma}$ and that

$$\mathbf{D}\cdot\Big(\phi_1 + (\mathbf{D}\boldsymbol{\gamma} + \mathbf{D}\times\delta)(\mathbf{D}\times\delta)\Big) = 0,$$

whose solution, since $\mathbb{E}\phi = 0$ and $\mathbf{D}\times\phi = 0$, is given by

$$\phi_1 = \mathbb{E}\{(\mathbf{D}\boldsymbol{\gamma} + \mathbf{D}\times\delta)(\mathbf{D}\times\delta)\} + \mathbf{P}_{\rm pot}\{(\mathbf{D}\boldsymbol{\gamma} + \mathbf{D}\times\delta)(\mathbf{D}\times\delta)\},$$

where $\mathbf{P}_{\rm pot}$ is the L^2-orthogonal projection onto potential stationary vector fields [10].

Up to higher-order terms in powers of λ, we thus obtain the requested mixing assumptions on $\psi \approx I + \lambda\mathbf{D}\boldsymbol{\gamma} + \lambda^2\phi_1$ and $a\psi \approx I + \lambda^2(\mathbf{D}\boldsymbol{\gamma} + \mathbf{D}\times\delta)(\mathbf{D}\times\delta)$ are mixing stationary processes. Note that $a\psi$ is no longer deterministic. It remains to see whether higher-order expansions remain mixing under appropriate assumptions on $\boldsymbol{\gamma}$ and δ. These formal calculations tend to indicate that imposing mixing conditions on ψ and $a\psi$ is not unreasonable, even though rigorous proofs are not available at the moment.

Let us conclude this section by mentioning that the only properties on the local solutions we use in subsequent sections are the results stated in Theorem 3.3 for the field $\hat{a}$ and in Lemma 3.6 for the corrector $\mathbf{z}^\varepsilon(\mathbf{x}) - \mathbf{x}$. In other words, we need the correlation function of $\psi(\omega)$ to decay rapidly and the average of $\hat{a}$ over a domain D to converge to 0 sufficiently rapidly when the size of D increases. These properties are simpler to verify than the strong mixing assumption stated in (24).

4 Homogenization of high frequency waves

Let us now come back to the homogenization of the wave equation (4). We have defined two types of homogenized problems: a first type based on the corrector $\mathbf{z}^\varepsilon$ and the homogenized coefficient a^*, and a second type based on the corrector $\boldsymbol{\theta}^\beta$ and the corresponding homogenized tensor $a^{*\beta}$. The first homogenized problem is useful when we assume (or can demonstrate) that ψ and $a\psi$ are mixing random processes. The second homogenized problem needs to be considered when mixing assumptions are imposed only on the tensor a.

Scenario with ψ and $a\psi$ mixing. The homogenized solution is given here by p_0^ε, which solves the following constant-coefficient wave equation:

$$\begin{aligned}&\kappa^*\frac{\partial^2 p_0^\varepsilon(t,\mathbf{x},\omega)}{\partial t^2} - \nabla\cdot a^*\nabla p_0^\varepsilon(t,\mathbf{x},\omega) = 0, \qquad t>0, \quad \mathbf{x}\in\mathbb{R}^d,\\ &p_0^\varepsilon(0,\mathbf{x}) = g_\varepsilon(\mathbf{x}), \qquad \frac{\partial p_0^\varepsilon}{\partial t}(0,\mathbf{x}) = j_\varepsilon(\mathbf{x}), \qquad \mathbf{x}\in\mathbb{R}^d.\end{aligned} \tag{34}$$

We introduce the following ansatz

$$p_\varepsilon(t,\mathbf{x},\omega) = p_0^\varepsilon(t,\mathbf{x}) + (\mathbf{z}^\varepsilon(\mathbf{x},\omega) - \mathbf{x})\cdot\nabla p_0^\varepsilon(t,\mathbf{x}) + \zeta_\varepsilon(t,\mathbf{x},\omega). \tag{35}$$

Note that

$$\nabla p_\varepsilon(t,\mathbf{x},\omega) = \psi(\frac{\mathbf{x}}{\varepsilon})\nabla p_0^\varepsilon(t,\mathbf{x}) + (\mathbf{z}^\varepsilon(\mathbf{x})-\mathbf{x})\cdot\nabla\otimes\nabla p_0^\varepsilon + \nabla\zeta_\varepsilon(t,\mathbf{x},\omega).$$

Using the equation for $p_0^\varepsilon(t,\mathbf{x})$, we find as in e.g. [11] that ζ_ε solves the following equation:

$$\begin{aligned}
&\kappa\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big)\frac{\partial^2\zeta_\varepsilon(t,\mathbf{x},\omega)}{\partial t^2} - \nabla\cdot a\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big)\nabla\zeta_\varepsilon(t,\mathbf{x},\omega) = S_1^\varepsilon + \nabla\cdot a\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big)\mathbf{S}_2^\varepsilon,\\
&\zeta_\varepsilon(0,\mathbf{x},\omega) = -(\mathbf{z}^\varepsilon(\mathbf{x})-\mathbf{x})\cdot\nabla g_\varepsilon(\mathbf{x}), \qquad \frac{\partial\zeta_\varepsilon}{\partial t}(0,\mathbf{x},\omega) = -(\mathbf{z}^\varepsilon(\mathbf{x})-\mathbf{x})\cdot\nabla j_\varepsilon(\mathbf{x}),\\
&S_1^\varepsilon = \Big(\kappa^* - \kappa\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big)\Big)\frac{\partial^2 p_0^\varepsilon}{\partial t^2} - \kappa\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big)(\mathbf{z}^\varepsilon(\mathbf{x})-\mathbf{x})\cdot\nabla\frac{\partial^2 p_0^\varepsilon}{\partial t^2}\\
&\qquad -\Big(a^* - (a\psi)\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big)\Big) : \nabla\otimes\nabla p_0^\varepsilon(t,\mathbf{x}),\\
&\mathbf{S}_2^\varepsilon = (\mathbf{z}^\varepsilon(\mathbf{x})-\mathbf{x})\cdot\nabla\otimes\nabla p_0^\varepsilon.
\end{aligned} \tag{36}$$

We will show, using the results of Theorem 3.2 and Lemma 3.6 that each term above is small as $\varepsilon\to0$. This will be sufficient to obtain an error estimate for ζ_ε by using the stability result to be established in Proposition 4.1 below.

Scenario with a mixing. We now consider the approach to homogenization based on the corrector $\boldsymbol{\theta}^\beta$ and the homogenized tensor $a^{*\beta}$. The appropriate homogenized equation is

$$\begin{aligned}
&\kappa^*\frac{\partial^2 p_0^\beta(t,\mathbf{x},\omega)}{\partial t^2} - \nabla\cdot a^*\nabla p_0^\beta(t,\mathbf{x},\omega) = 0, \qquad t>0, \quad \mathbf{x}\in\mathbb{R}^d,\\
&p_0^\beta(0,\mathbf{x}) = g_\varepsilon(\mathbf{x}), \qquad \frac{\partial p_0^\beta}{\partial t}(0,\mathbf{x}) = j_\varepsilon(\mathbf{x}), \qquad \mathbf{x}\in\mathbb{R}^d.
\end{aligned} \tag{37}$$

In what follows, we assume that ε and β as related as in (31) for some ξ to be determined so that λ_β in (9) is maximized. The appropriate ansatz now becomes

$$p_\varepsilon(t,\mathbf{x},\omega) = p_0^\beta(t,\mathbf{x}) + \varepsilon\boldsymbol{\theta}^\beta\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big)\cdot\nabla p_0^\beta(t,\mathbf{x}) + \zeta_\beta(t,\mathbf{x},\omega). \tag{38}$$

As in e.g. [15, 17], the equation for ζ_β is:

$$\begin{aligned}
&\kappa\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big)\frac{\partial^2\zeta_\beta(t,\mathbf{x},\omega)}{\partial t^2} - \nabla\cdot a\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big)\nabla\zeta_\beta(t,\mathbf{x},\omega) = S_1^\beta + \nabla\cdot a\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big)\mathbf{S}_2^\beta,\\
&\zeta_\beta(0,\mathbf{x},\omega) = -\varepsilon\boldsymbol{\theta}^\beta\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big)\cdot\nabla g_\varepsilon(\mathbf{x}), \qquad \frac{\partial\zeta_\beta}{\partial t}(0,\mathbf{x},\omega) = -\varepsilon\boldsymbol{\theta}^\beta\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big)\cdot\nabla j_\varepsilon(\mathbf{x}),\\
&S_1^\beta = \Big(\kappa^* - \kappa\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big)\Big)\frac{\partial^2 p_0^\beta}{\partial t^2} - \kappa\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big)\varepsilon\boldsymbol{\theta}^\beta\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big)\cdot\nabla\frac{\partial^2 p_0^\beta}{\partial t^2}\\
&\qquad -\Big(a^{*\beta} - (a\psi^\beta)\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big)\Big) : \nabla\otimes\nabla p_0^\beta(t,\mathbf{x}) - \frac{\beta}{\varepsilon}\boldsymbol{\theta}^\beta\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big)\cdot\nabla p_0^\beta\\
&\mathbf{S}_2^\beta = \varepsilon\boldsymbol{\theta}^\beta\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big)\cdot\nabla\otimes\nabla p_0^\beta.
\end{aligned} \tag{39}$$

We will show, using Theorem 3.3 and Corollary 3.5, that all of the source terms above are small as $\varepsilon\to0$. The analysis of both error terms ζ_ε and ζ_β will then be based on the following stability result:

Proposition 4.1 *Let us consider the wave equation*

$$\kappa\frac{\partial^2 z}{\partial t^2} - \nabla\cdot a\nabla z = S_1 + \nabla\cdot a\mathbf{S}_2,$$
$$z(0,\mathbf{x}) = g(\mathbf{x},\omega), \qquad \frac{\partial z}{\partial t}(0,\mathbf{x},\omega) = j(\mathbf{x},\omega), \tag{40}$$

with compactly supported source terms. We assume that $\kappa(\mathbf{x},\omega)$ and $a(\mathbf{x},\omega)$ are smooth processes on $(\Omega,\mathcal{F},P)$, which satisfy the constraints in (5). Let $X = \mathcal{C}(0,T;L^2(\mathbb{R}^d\times\mathcal{H}))$ and $X^{-1} = \mathcal{C}(0,T;H^{-1}(\mathbb{R}^d;\mathcal{H}))$ equipped with their natural norms. Then we have the following estimate:

$$\begin{aligned}
\|z\|_X + \|\frac{\partial z}{\partial t}\|_X + \|\nabla z\|_X &\leq C(T,a_0,\kappa_0)\Big((\mathbb{S}\wedge\mathbb{S}') + \mathbb{S}''\Big),\\
\mathbb{S} &= \|S_1\|_{X^{-1}} + \|\frac{\partial S_1}{\partial t}\|_{X^{-1}} + \|\mathbf{S}_2\|_X + \|\frac{\partial \mathbf{S}_2}{\partial t}\|_X\\
\mathbb{S}' &= \|S_1\|_X + \|\mathbf{S}_2\|_X + \|\frac{\partial \mathbf{S}_2}{\partial t}\|_X\\
\mathbb{S}'' &= \|g\|_{H^1(\mathbb{R}^d;\mathcal{H})} + \|j\|_{L^2(\mathbb{R}^d;\mathcal{H})}.
\end{aligned} \tag{41}$$

Proof. Classical theories show that the equation is well posed for almost every realization $\omega\in\Omega$. We consider the bound involving $\mathbb{S}$, the bound for $\mathbb{S}'$ being similar and somewhat simpler. Let us define the energy

$$\mathcal{E}(t,\omega) = \int_{\mathbb{R}^d}\Big(\kappa(\frac{\partial z}{\partial t})^2 + a\nabla z\cdot\nabla z\Big)d\mathbf{x}. \tag{42}$$

We find, using equation (40), that

$$\dot{\mathcal{E}}(t,\omega) = \int_{\mathbb{R}^d}\frac{\partial z}{\partial t}\Big(S_1 + \nabla\cdot a\mathbf{S}_2\Big)d\mathbf{x}.$$

Let us first assume that $g=j=0$. Since $\mathcal{E}(0,\omega)=0$, we deduce by integration that

$$\mathcal{E}(t,\omega) = \int_0^t\int_{\mathbb{R}^d}\Big(-z\frac{\partial S_1}{\partial t} + \nabla z\cdot a\frac{\partial \mathbf{S}_2}{\partial t}\Big)d\mathbf{x}dt + \int_{\mathbb{R}^d}(S_1 z(t) - \nabla z\cdot a\mathbf{S}_2(t))d\mathbf{x}.$$

Using the properties of κ and a and the Cauchy-Schwarz inequality, we obtain that

$$\begin{aligned}
\kappa_0\|\frac{\partial z}{\partial t}(t)\|_H^2 + a_0\|\nabla z(t)\|_H^2 \leq\ &\int_0^t\Big(\|\frac{\partial S_1}{\partial t}(s)\|_{H^{-1}}\|z(s)\|_{H^1} + a_0^{-1}\|\nabla z(s)\|_H\|\frac{\partial \mathbf{S}_2}{\partial t}(s)\|_H\Big)ds\\
&+a_0^{-1}\|\nabla z(t)\|_H\|\mathbf{S}_2(t)\|_H + \|z(t)\|_{H^1}\|S_1(t)\|_{H^{-1}}.
\end{aligned}$$

Since $\|z(t)\|_H \leq \int_0^t\|\frac{\partial z}{\partial t}(s)\|_H ds$ as $\|z(0)\|_H = 0$, we verify, using bounds of the form

$$\mathbb{E}\int_0^t\|\frac{\partial S_1}{\partial t}(s)\|_{H^{-1}}\|z(s)\|_{H^1}ds \lesssim \mathbb{S}^2 + \mathbb{E}\int_0^t\Big(\|\frac{\partial z}{\partial t}\|_H + \|\nabla z\|_H\Big)^2(s)ds,$$

and $\|\nabla z(t)\|_H\|\mathbf{S}_2(t)\|_H \leq \varepsilon\|\nabla z(t)\|_H^2 + \varepsilon^{-1}\|\mathbf{S}_2(t)\|_H^2$ with ε sufficiently small, that

$$\mathbb{E}\Big(\|\frac{\partial z}{\partial t}\|_H + \|\nabla z\|_H\Big)^2(t) \lesssim \mathbb{S}^2 + \mathbb{E}\int_0^t\Big(\|\frac{\partial z}{\partial t}\|_H + \|\nabla z\|_H\Big)^2(s)ds.$$

Application of the Gronwall lemma shows that (41) holds for the time and spatial derivatives of z. It remains to integrate in time to obtain (41) for $\|z\|_X$. The same energy method allows us to obtain the estimate in the absence of volume source term with non-vanishing initial conditions. □

Note that the constants appearing in the preceding result depend only on the constants of uniform ellipticity of a and κ so that the result holds with $a(\frac{\mathbf{x}}{\varepsilon})$ and $\kappa(\frac{\mathbf{x}}{\varepsilon})$. Because the initial conditions for p_0^ε and p_0^β are compactly supported, then so are p_0^ε and p_0^β by finite speed of propagation. Let K be a sufficiently large cube so that $p_0^\varepsilon(t)$ and $p_0^\beta(t)$ are supported on K for all $0 \le t \le T$. We define $K_T = (0,T) \times K$. We are now ready to obtain our main error estimates on ζ_ε and ζ_β.

Lemma 4.2 *Let ζ_ε be the solution to* (36). *Then we have that:*

$$\|\zeta_\varepsilon\|_X + \|\frac{\partial \zeta_\varepsilon}{\partial t}\|_X + \|\nabla\zeta_\varepsilon\|_X \lesssim \varepsilon^{\frac{d}{d+2}}\|p_0^\varepsilon\|_{C^4(K_T)} + \sqrt{\varepsilon}\|p_0^\varepsilon\|_{C^3(K_T)}. \tag{43}$$

Lemma 4.3 *Let ζ_β be the solution to* (39). *Then we have that:*

$$\|\zeta_\beta\|_X + \|\frac{\partial \zeta_\beta}{\partial t}\|_X + \|\nabla\zeta_\beta\|_X \lesssim \varepsilon^{\frac{\xi d}{d+2}}\|p_0^\beta\|_{C^4(K_T)} + \varepsilon^{\xi\wedge\gamma_1}\|p_0^\beta\|_{C^3(K_T)}, \tag{44}$$

where ξ is defined in (31) *and γ_1 in* (10).

Proof [Lemma 4.2]. Let us consider the first source term in S_1^ε in (36). Using Theorem 3.2, we verify that

$$\mathbb{E}\Big|\int_K \hat{\kappa}\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big)\frac{\partial^3 p_0^\varepsilon}{\partial t^3}\phi d\mathbf{x}\Big| \lesssim \varepsilon^{\frac{d}{d+2}}\|p_0^\varepsilon\|_{C^4(K_T)}\|\phi\|_{1,K}.$$

Similarly, the third contribution in S_1^ε yields

$$\mathbb{E}\Big|\int_K \hat{a}\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big)\frac{\partial^3 p_0^\varepsilon}{\partial t^3}\phi d\mathbf{x}\Big| \lesssim \varepsilon^{\frac{d}{d+2}}\|p_0^\varepsilon\|_{C^4(K_T)}\|\phi\|_{1,K},$$

thanks to (26). The other contributions involve terms proportional to $\mathbf{z}^\varepsilon(\mathbf{x}) - \mathbf{x}$, which is of order $\sqrt{\varepsilon}$ in $L^2(K;\mathcal{H})$. This local estimate is sufficient since by the finite speed of propagation, both the initial conditions and the solution p_0^ε are compactly supported in K. We thus get additional error terms of order at most $\sqrt{\varepsilon}\|p_0^\varepsilon\|_{C^3(K_T)}$. The initial conditions are dealt with in a similar fashion. □

Proof [Lemma 4.3]. We also find that p_0^β is compactly supported in K for all finite time $0 \le t \le T$. Using the stability result in Proposition 4.1, the source term $\mathbf{S}_2^\beta$ provides a contribution bounded by

$$\|\varepsilon\boldsymbol{\theta}_\varepsilon^\beta\|_{0,K}\|p_0^\beta\|_{C^3(K_T)} \lesssim \varepsilon^\xi\|p_0^\beta\|_{C^3(K_T)},$$

with the choice of β in (31). Here and below, we use the notation $f_\varepsilon(\mathbf{x},\omega) = f(\frac{\mathbf{x}}{\varepsilon},\omega)$ for arbitrary stationary fields $f(\mathbf{x},\omega) = f(\tau_{-\mathbf{x}}\omega)$. As in the proof of the preceding lemma, the terms in S_1^β involving $\hat{a}^\beta$ and $\hat{\kappa}$ produce a contribution bounded by

$$\|\hat{a}_\varepsilon^\beta\|_{-1,K}\|p_0^\beta\|_{C^4(K_T)} \lesssim \varepsilon^{\frac{\xi d}{d+2}}\|p_0^\beta\|_{C^4(K_T)}.$$

Here, we have used the second estimate in Corollary 3.5. Using the $\mathbb{S}'$-estimate in Proposition 4.1, we obtain that

$$\|\kappa_\varepsilon \boldsymbol{\theta}_\varepsilon^\beta \cdot \nabla \frac{\partial^2 p_0^\beta}{\partial t^2}\|_{0,K} \lesssim \varepsilon^\xi \|p_0^\beta\|_{C^3(K_T)}.$$

It remains to address the term involving the most delicate estimate in Corollary 3.5 and obtain a contribution of the form

$$\|\frac{\beta}{\varepsilon}\boldsymbol{\theta}_\varepsilon^\beta\|_{-1,K}\|p_0^\beta\|_{C^3(K_T)} \lesssim \varepsilon^{\gamma_1}\|p_0^\beta\|_{C^3(K_T)}.$$

We verify that the initial conditions do not generate higher-order contributions than those already considered above. This concludes the proof of the lemma. □
The above lemmas allow us to obtain the following error estimates:

Theorem 4.4 *Let $p_\varepsilon(t,\mathbf{x},\omega)$ be the solution of* (4) *with initial conditions in* (6)*, i.e., oscillating at frequencies of order $\varepsilon^{-\alpha}$ with $\alpha > 0$. Let $p_0^\varepsilon(t,\mathbf{x})$ be the solution of the homogenized problem* (34)*. We assume that $\kappa(\omega)$, $\psi(\omega)$, and $a\psi(\omega)$ are strongly mixing with mixing coefficient $\rho(r) \lesssim r^{-d-\eta}$ for some $\eta > 0$. Then we have the error estimate:*

$$\|p_\varepsilon - p_0^\varepsilon\|_X + \Big\|\frac{\partial p_\varepsilon}{\partial t} - \frac{\partial p_0^\varepsilon}{\partial t}\Big\|_X + \Big\|\nabla_{\mathbf{x}} p_\varepsilon - \psi\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big)\nabla_{\mathbf{x}} p_0^\varepsilon\Big\|_X \lesssim \varepsilon^{\lambda(\alpha,d)}, \tag{45}$$

where $\lambda(\alpha,d)$ is defined in (8)*.*

Theorem 4.5 *Let $p_\varepsilon(t,\mathbf{x},\omega)$ be the solution of* (4) *with initial conditions in* (6)*, i.e., oscillating at frequencies of order $\varepsilon^{-\alpha}$ with $\alpha > 0$. Let $p_0^\beta(t,\mathbf{x})$ be the solution of the homogenized problem* (37) *with $\beta = \beta(\varepsilon)$ chosen as in* (31)*. We assume that $a(\omega)$ and $\kappa(\omega)$ are strongly mixing with mixing coefficient $\rho(r) \lesssim e^{-r}$. Then we have the error estimate:*

$$\|p_\varepsilon - p_0^\beta\|_X + \Big\|\frac{\partial p_\varepsilon}{\partial t} - \frac{\partial p_0^\beta}{\partial t}\Big\|_X + \Big\|\nabla_{\mathbf{x}} p_\varepsilon - \psi^\beta\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big)\nabla_{\mathbf{x}} p_0^\beta\Big\|_X \lesssim \varepsilon^{\lambda_\beta(\alpha,d)}, \tag{46}$$

where $\lambda_\beta(\alpha,d)$ is defined in (9)*.*

Proof [Theorem 4.4]. The solution p_0^ε of the wave equation with constant coefficients and sufficiently smooth initial conditions given by (6) satisfies that $\varepsilon^{\alpha(|m|-1)}\partial^m p_0^\varepsilon$ is bounded in X independent of ε and is supported on $K_T = (0,T)\times K$ for all multi-index $m = (m_0,\dots,m_d)$ of length $0 \le |m| \le 5 + d/2$. To obtain this result, we use the energy estimate

$$\int_{\mathbb{R}^d} n_{0\varepsilon}(t,\mathbf{x})d\mathbf{x} = \int_{\mathbb{R}^d} n_{0\varepsilon}(0,\mathbf{x})d\mathbf{x}; \qquad n_{0\varepsilon}(t,\mathbf{x}) = \kappa^*\Big(\frac{\partial p_0^\varepsilon}{\partial t}\Big)^2 + a^*\nabla p_0^\varepsilon \cdot \nabla p_0^\varepsilon,$$

and the fact that $\partial^m p_0^\varepsilon$ solves the same partial differential equation with appropriately modified initial conditions. Because of concentration effects, uniform bounds require additional regularity. Using Sobolev inequalities (in space) [8], we obtain that

$$\|p_0^\varepsilon\|_{\mathcal{C}^n(K)} \lesssim \|p_0^\varepsilon\|_{H^{n+\frac{d}{2}+\eta}(K)} \lesssim \varepsilon^{-\alpha(n+\frac{d}{2}+\eta-1)}, \quad \text{for all } \eta > 0,$$

and deduce that $\|p_0^\varepsilon\|_{\mathcal{C}^n(K_T)} \lesssim \varepsilon^{-\alpha(n+\frac{d}{2}+\eta-1)}$. Similar results may be obtained by using the explicit expression of the Green's function for the constant-coefficient wave equation [8]. The advantage of the proposed method is that it generalizes to homogenized wave equation with smooth spatially varying coefficients $\kappa^*(\mathbf{x})$ and $a^*(\mathbf{x})$. It remains to apply Lemma 4.2 and Proposition 4.1 to conclude the proof of the theorem. □

Proof [Theorem 4.5]. The proof is identical to that of the preceding theorem based on the estimates obtained in Lemma 4.3. □

The above error estimates are of interest only when α is small enough so that $\lambda(\alpha, d) > 0$ in the case of $\psi(\omega)$ and $a\psi(\omega)$ mixing and so that $\lambda_\beta(\alpha, d) > 0$ in the case of only $a(\omega)$ mixing. Under either assumption, we obtain below that the infinite frequency limit of high frequency waves propagating in randomly heterogeneous media corresponds to the infinite frequency limit of waves propagating in the proper homogeneous medium provided that the wavelength and the correlation length of the medium are sufficiently well-separated.

5 Convergence of the energy densities

We have seen in the preceding section that the pressure field and its temporal and spatial gradients converged in an appropriate sense as $\varepsilon \to 0$. We now consider the convergence of the energy density associated to the wave fields. The random energy density is defined as:

$$n_\varepsilon(t, \mathbf{x}, \omega) = \kappa\Big(\frac{\mathbf{x}}{\varepsilon}, \omega\Big)\Big(\frac{\partial p_\varepsilon}{\partial t}\Big)^2(t, \mathbf{x}, \omega) + a\Big(\frac{\mathbf{x}}{\varepsilon}, \omega\Big)\nabla p_\varepsilon \cdot \nabla p_\varepsilon(t, \mathbf{x}, \omega). \tag{47}$$

It is bounded in $\mathcal{C}(0, T; L^1(\mathbb{R}^d \times \Omega))$ and energy conservation takes the form

$$\int_{\mathbb{R}^d} n_\varepsilon(t, \mathbf{x}, \omega) d\mathbf{x} = \int_{\mathbb{R}^d} n_\varepsilon(0, \mathbf{x}, \omega) d\mathbf{x}, \qquad t > 0, \quad P - a.s. \tag{48}$$

Using Theorem 4.4, we deduce that

$$n_\varepsilon = \kappa\Big(\frac{\mathbf{x}}{\varepsilon}, \omega\Big)\Big(\frac{\partial p_0^\varepsilon}{\partial t}\Big)^2 + a\Big(\frac{\mathbf{x}}{\varepsilon}, \omega\Big)\psi\Big(\frac{\mathbf{x}}{\varepsilon}, \omega\Big)\nabla p_0^\varepsilon \cdot \psi\Big(\frac{\mathbf{x}}{\varepsilon}, \omega\Big)\nabla p_0^\varepsilon + \varepsilon^{\lambda(\alpha,d)} r_\varepsilon, \tag{49}$$

where $r_\varepsilon(t, \mathbf{x}, \omega)$ is bounded in $C(0, T; L^1(\mathbb{R}^d))$ independent of ε. This implies the following error estimate:

$$\|\mathbb{E} n_\varepsilon(t, \mathbf{x}, \omega) - n_{0\varepsilon}(t, \mathbf{x})\|_{C(0,T;L^1(\mathbb{R}^d))} \lesssim \varepsilon^{\lambda(\alpha,d)}, \tag{50}$$

where we have defined

$$\begin{aligned} n_{0\varepsilon}(t, \mathbf{x}) &= \mathbb{E}\Big\{\kappa\Big(\frac{\mathbf{x}}{\varepsilon}, \omega\Big)\Big(\frac{\partial p_0^\varepsilon}{\partial t}\Big)^2 + a\Big(\frac{\mathbf{x}}{\varepsilon}, \omega\Big)\psi\Big(\frac{\mathbf{x}}{\varepsilon}, \omega\Big)\nabla p_0^\varepsilon \cdot \psi\Big(\frac{\mathbf{x}}{\varepsilon}, \omega\Big)\nabla p_0^\varepsilon\Big\} \\ &= \kappa^*\Big(\frac{\partial p_0^\varepsilon}{\partial t}\Big)^2 + a^*\nabla p_0^\varepsilon \cdot \nabla p_0^\varepsilon, \end{aligned} \tag{51}$$

thanks to Lemma 2.1. Similarly, we can define the energy density:

$$n_{0\beta}(t, \mathbf{x}) = \kappa^*\Big(\frac{\partial p_0^\beta}{\partial t}\Big)^2 + a^{*\beta}\nabla p_0^\beta \cdot \nabla p_0^\beta, \tag{52}$$

and deduce from Theorem 4.5 that

$$\|\mathbb{E}n_\varepsilon(t,\mathbf{x},\omega) - n_{0\beta}(t,\mathbf{x})\|_{C(0,T;L^1(\mathbb{R}^d))} \lesssim \varepsilon^{\lambda_\beta(\alpha,d)}. \tag{53}$$

Since for the choice of initial conditions (6), $n_{0\varepsilon}(t,\cdot)$ is uniformly bounded in $L^1(\mathbb{R}^d)$, we have thus, up to the extraction of a subsequence, that as $\varepsilon \to 0$,

$$n_{0\varepsilon}(t,\mathbf{x}) \to \nu(t,\mathbf{x}), \tag{54}$$

weakly as bounded measures on $\mathbb{R}^d$. Since $|a^{*\beta} - a^*|$ converges to 0 as $\varepsilon \to 0$ [6, 17], the sequences $n_{0\varepsilon}(t,\mathbf{x})$ and $n_{0\beta}(t,\mathbf{x})$ (with $\beta = \beta(\varepsilon)$ as in (31)) have the same accumulation points. For concreteness, we thus analyze the limit of $n_{0\varepsilon}(t,\mathbf{x})$ as $\varepsilon \to 0$.

Wigner measures [9, 12] may then be used to obtain the limit of $n_{0\varepsilon}(t,\mathbf{x})$ as follows. We assume to simplify that a^* is scalar and define $\rho^* = (a^*)^{-1}$. We define

$$\pi_\varepsilon(t,\mathbf{x}) = \partial_t p_0^\varepsilon(t,\mathbf{x}), \qquad \mathbf{v}_\varepsilon(t,\mathbf{x}) = (\rho^*)^{-1}\nabla p_0^\varepsilon(t,\mathbf{x}),$$

with initial conditions $\pi_{I\varepsilon}(\mathbf{x}) = j_\varepsilon(\mathbf{x})$ and $\mathbf{v}_{I\varepsilon}(\mathbf{x}) = (\rho^*)^{-1}\nabla g_\varepsilon(\mathbf{x})$.

We introduce the Wigner transform of two fields on $\mathbb{R}^d$ as

$$W_{\varepsilon^\alpha}[\phi,\psi](\mathbf{x},\mathbf{k}) = \int_{\mathbb{R}^d} e^{i\mathbf{k}\cdot\mathbf{y}}\psi(\mathbf{x} - \varepsilon^\alpha\frac{\mathbf{y}}{2})\phi(\mathbf{x} + \varepsilon^\alpha\frac{\mathbf{y}}{2})\frac{d\mathbf{y}}{(2\pi)^d}, \tag{55}$$

and define $W_{\varepsilon^\alpha}[\phi] = W_{\varepsilon^\alpha}[\phi,\phi]$. Then we have the following result [9]:

Theorem 5.1 *Let us assume that $\pi_{I\varepsilon}$ and $\mathbf{v}_{I\varepsilon}$ (in gradient form) are $\varepsilon^\alpha-$oscillatory and compact at infinity [9]. A sufficient condition for this is that $\varepsilon^\alpha\nabla\pi_{I\varepsilon}$ and $\varepsilon^\alpha\nabla\mathbf{v}_{I\varepsilon}$ are compactly supported and bounded in $L^2(\mathbb{R}^d)$ with bound independent of α. Let us define*

$$a_0(\mathbf{x},\mathbf{k}) = \lim_{\varepsilon\to 0}\frac{1}{2}W_{\varepsilon^\alpha}[\sqrt{\rho^*}\hat{\mathbf{k}}\cdot\mathbf{v}_{I\varepsilon} + \sqrt{\kappa^*}\pi_{I\varepsilon}](\mathbf{x},\mathbf{k}), \qquad \hat{\mathbf{k}} = \frac{\mathbf{k}}{|\mathbf{k}|}, \tag{56}$$

where the above limit, after possible extraction of a subsequence still denoted by $(\pi_{I\varepsilon},\mathbf{v}_{I\varepsilon})$, is supposed to exist.

Then the solution $a(t,\mathbf{x},\mathbf{k}) = a_0(\mathbf{x} - c^*t\mathbf{k},\mathbf{k})$ of the Liouville equation

$$\frac{\partial a}{\partial t} + c^*\hat{\mathbf{k}}\cdot\nabla_{\mathbf{x}}a = 0, \qquad a(0,\mathbf{x},\mathbf{k}) = a_0(\mathbf{x},\mathbf{k}), \tag{57}$$

may be interpreted as a phase-space energy density. More precisely, it satisfies that

$$\int_{\mathbb{R}^d} a(t,d\mathbf{x},d\mathbf{k}) = \nu(d\mathbf{x}), \quad \int_{\mathbb{R}^{2d}} a(t,d\mathbf{x},d\mathbf{k}) = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^d} n_{0\varepsilon}(t,\mathbf{x})d\mathbf{x} = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^d}\mathbb{E}\{n_\varepsilon\}(t,\mathbf{x})d\mathbf{x}. \tag{58}$$

In other words, the spatial energy density $\nu(d\mathbf{x})$ may be recovered from the average of $a(t,d\mathbf{x},d\mathbf{k})$ over wavenumbers $\mathbf{k}$. Moreover, the second set of equalities in (58) shows that no energy is lost when ε^α converges to 0. The acoustic energy density present in the system at time $t = 0$ propagates along straight lines in the direction $\hat{\mathbf{k}}$ with homogenized speed $c^* = (\kappa^*\rho^*)^{-1/2}$, at least in an ensemble averaged sense. In the case where the initial conditions are of the specific form (6), and denoting by $W_0[\phi,\psi]$ the possible limits of $W_{\varepsilon^\alpha}[\phi,\psi]$ as $\varepsilon \to 0$ and $W_0[\phi] = W_0[\phi,\phi]$, it can be verified that

$$\begin{aligned} a_0(\mathbf{x},\mathbf{k}) &= \frac{1}{2\rho^*}|\phi_p(\mathbf{x})|^2 W_0[\hat{\mathbf{k}}\cdot\nabla p_0](\mathbf{x},\mathbf{k}) + \frac{1}{2}\kappa^*|\phi_j(\mathbf{x})|^2 W_0[j_0](\mathbf{x},\mathbf{k}) \\ &\quad + \sqrt{\frac{\kappa^*}{\rho^*}}|\phi_p(\mathbf{x})\phi_j(\mathbf{x})|\Re\{W_0[\hat{\mathbf{k}}\cdot\nabla p_0, j_0](\mathbf{x},\mathbf{k})\}. \end{aligned} \tag{59}$$

The average energy density $\mathbb{E}\{n_\varepsilon\}$ of high frequency waves propagating in random media with much smaller correlation length than the wavelength may thus be approximated by the energy density of waves propagating with the average sound speed $c^* = (\kappa^*\rho^*)^{-1/2}$. However, $n_\varepsilon(t,\mathbf{x},\omega)$ remains pointwise a non-deterministic random variable in the limit $\varepsilon \to 0$. Indeed, we deduce from (49) that $\mathbb{E}\{n_\varepsilon^2\}(t,\mathbf{x})$, which can always be defined provided that the initial conditions for the wave equation are sufficiently smooth, involves the variance of the coefficients $\kappa(\omega)$ and $\psi^t a\psi(\omega)$ and is different from $\big(\mathbb{E}\{n_\varepsilon\}\big)^2(t,\mathbf{x})$. Higher-order moments of $n_\varepsilon(t,\mathbf{x},\omega)$ may be calculated similarly.

Thus Theorem 5.1 characterizes the ensemble average of $n_\varepsilon(t,\mathbf{x},\omega)$, which as a random variable does not converge pointwise (in $\mathbf{x}$) to a deterministic limit. In many regimes of wave propagation, statistical stability of the wave energy density can be obtained in a weak sense, i.e., after appropriate spatial averaging; see e.g. [2, 3]. Such a result can also be obtained in the homogenization setting is we make the following additional assumptions.

We assume that $\psi^t a\psi(\omega)$ has integrable correlation function. More precisely, we assume that

$$R_{ijkl}(\mathbf{x}) = \mathbb{E}\{(\psi^t a\psi - a^*)_{ij}(\omega)(\psi^t a\psi - a^*)_{kl}(\tau_{-\mathbf{x}}\omega)\}, \tag{60}$$

is an integrable function for all indices $1 \le i,j,k,l \le d$. This can be seen as a consequence of imposing that $\psi^t a\psi$ is mixing with integrable strong mixing coefficient. Such a result holds in the one-dimensional setting for instance where $\psi^t a\psi(\omega)$ is inversely proportional to $a(\omega)$. We also assume that the correlation of κ:

$$R(\mathbf{x}) = \mathbb{E}\{(\kappa(\omega) - \kappa^*)(\kappa(\tau_{-\mathbf{x}}\omega) - \kappa^*)\} \tag{61}$$

is integrable, which is a consequence of the mixing property of κ when the strong mixing coefficient is itself integrable. Following (49) and (51), we deduce that

$$n_\varepsilon(t,\mathbf{x},\omega) = n_{0\varepsilon}(t,\mathbf{x}) + \varepsilon^{\lambda(\alpha,d)} r_\varepsilon(t,\mathbf{x},\omega) + s_\varepsilon(t,\mathbf{x},\omega),$$

and we want to show that the error term

$$s_\varepsilon = \Big[\kappa\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big) - \kappa^*\Big]\Big(\frac{\partial p_0^\varepsilon}{\partial t}\Big)^2 + \Big[(\psi^t a\psi - a^*)\Big(\frac{\mathbf{x}}{\varepsilon},\omega\Big) : \nabla p_0^\varepsilon \otimes \nabla p_0^\varepsilon\Big], \tag{62}$$

converges to 0 in a weak sense, since it does not converge to 0 in a pointwise sense. More precisely, we have

Theorem 5.2 *Let $\phi(\mathbf{x})$ be a smooth non-negative compactly supported test function of total mass $\int_{\mathbb{R}^d}\phi(\mathbf{x})d\mathbf{x} = 1$. We assume that the correlation functions $R_{ijkl}(\mathbf{x})$ and $R(\mathbf{x})$ defined in* (60) *and* (61) *are integrable.*

Let $\theta > 0$ be such that $1 - \min(2\alpha + \theta, 2\theta + \alpha) > 0$. We define $\phi_\varepsilon(\mathbf{x}) = \varepsilon^{-\theta d}\phi(\frac{\mathbf{x}-\mathbf{x}_0}{\varepsilon^\theta})$ a localized test function of mass also equal to 1 and centered around an arbitrary point $\mathbf{x}_0 \in \mathbb{R}^d$. Then we find that

$$\int_{\mathbb{R}^d} s_\varepsilon(t, \mathbf{x}, \omega)\phi_\varepsilon(\mathbf{x})d\mathbf{x} \to 0 \qquad \varepsilon \to 0, \tag{63}$$

uniformly in time and P-a.s. As a consequence, we find that

$$\int_{\mathbb{R}^d} n_\varepsilon(t, \mathbf{x}, \omega)\phi_\varepsilon(\mathbf{x})d\mathbf{x} - \int_{\mathbb{R}^{2d}} \phi_\varepsilon(\mathbf{x})a(t, d\mathbf{x}, d\mathbf{k}) \to 0 \qquad \varepsilon \to 0, \tag{64}$$

uniformly in time and P-a.s.

Proof. Since the two terms defining s_ε in (62) can be treated similarly, we only consider the convergence of the term

$$I_\varepsilon(t, \omega) = \int_{\mathbb{R}^d} \Big[\kappa\Big(\frac{\mathbf{x}}{\varepsilon}, \omega\Big) - \kappa^*\Big]\Big(\frac{\partial p_0^\varepsilon}{\partial t}\Big)^2(\mathbf{x})\phi_\varepsilon(\mathbf{x})d\mathbf{x}.$$

We observe that $\mathbb{E}\{I_\varepsilon\} = 0$ and want to show that $\mathbb{E}\{I_\varepsilon^2\} \to 0$ as $\varepsilon \to 0$. This is sufficient to conclude the proof of the theorem. We thus calculate that

$$\mathbb{E}\{I_\varepsilon^2\} = \int_{\mathbb{R}^{2d}} R\Big(\frac{\mathbf{x}-\mathbf{y}}{\varepsilon}\Big)\psi_\varepsilon(\mathbf{x})\psi_\varepsilon(\mathbf{y})d\mathbf{x}d\mathbf{y}, \qquad \psi_\varepsilon(\mathbf{x}) = \Big(\frac{\partial p_0^\varepsilon}{\partial t}\Big)^2(\mathbf{x})\phi_\varepsilon(\mathbf{x}).$$

Passing to the Fourier domain $\mathcal{F}_{\mathbf{x}\to\boldsymbol{\xi}}$, this is

$$\mathbb{E}\{I_\varepsilon^2\} \lesssim \int_{\mathbb{R}^d} \varepsilon^d \hat{R}(\varepsilon\boldsymbol{\xi})|\widehat{\psi_\varepsilon}(\boldsymbol{\xi})|^2 d\boldsymbol{\xi} \lesssim R_0\varepsilon^d \int_{\mathbb{R}^d} |\widehat{\psi_\varepsilon}(\boldsymbol{\xi})|^2 d\boldsymbol{\xi} \lesssim R_0\varepsilon^d \int_{\mathbb{R}^d} \psi_\varepsilon^2(\mathbf{x})d\mathbf{x}.$$

Here, R_0 is the supremum of $\hat{R}(\boldsymbol{\xi}) \geq 0$, which is finite since $R(\mathbf{x})$ is integrable. From the Hölder inequality, we thus obtain that

$$\mathbb{E}\{I_\varepsilon^2\} \lesssim \|\Big(\frac{\partial p_0^\varepsilon}{\partial t}\Big)^4\|_{L^p(\mathbb{R}^d)}\|\phi_\varepsilon^2(\mathbf{x})\|_{L^{p'}(\mathbb{R}^d)},$$

for all $1 \leq p, p' \leq \infty$ and $p^{-1} + (p')^{-1} = 1$. Now since p_0^ε is compactly supported, we obtain from the Sobolev inequality that

$$\|\frac{\partial p_0^\varepsilon}{\partial t}\|_{L^p(K)} \lesssim \|\frac{\partial p_0^\varepsilon}{\partial t}\|_{H^{d(\frac{1}{2}-\frac{1}{p})}(K)} \lesssim \varepsilon^{-\alpha d(\frac{1}{2}-\frac{1}{p})},$$

uniformly in time on compact sets so that

$$\|\Big(\frac{\partial p_0^\varepsilon}{\partial t}\Big)^4\|_{L^p(\mathbb{R}^d)} \lesssim \varepsilon^{-\alpha d(2-\frac{1}{p})}.$$

Using the definition of ϕ_ε, we find that $\|\phi_\varepsilon^2(\mathbf{x})\|_{L^{p'}(\mathbb{R}^d)} \lesssim \varepsilon^{-\theta d(2-\frac{1}{p'})}$. We obtain that the bound on $\mathbb{E}\{I_\varepsilon^2\}$ is minimized by choosing $p = 1$ when $\theta < \alpha$ and $p = \infty$ when $\theta > \alpha$, and that

$$\mathbb{E}\{I_\varepsilon^2\} \lesssim \begin{cases} \varepsilon^{d(1-2\alpha-\theta)} & \theta > \alpha, \\ \varepsilon^{d(1-\alpha-2\theta)} & \theta < \alpha. \end{cases} \tag{65}$$

In either case, $\mathbb{E}\{I_\varepsilon^2\}$ converges to 0 with ε uniformly in time on compact intervals and so by the Chebyshev inequality, $I_\varepsilon(t,\omega)$ converges to 0 $P-$a.s. uniformly in time on compact intervals. The estimate is also uniform with respect to the central point $\mathbf{x}_0$. □

Choosing $\theta = 0$, we thus obtain that the random variable $\int_{\mathbb{R}^d} n_\varepsilon(t,\mathbf{x},\omega)\phi(\mathbf{x}-\mathbf{x}_0)d\mathbf{x}$ converges to the deterministic variable $\int_{\mathbb{R}^{2d}} \phi(\mathbf{x}-\mathbf{x}_0)a(t,d\mathbf{x},d\mathbf{k})$ as $\varepsilon \to 0$. The above result shows that the stability of the energy density is obtained as soon as it is integrated over a much smaller domain, of size ε^θ with e.g. $\theta < \frac{1-\alpha}{2}$ when $\alpha < \frac{1}{3}$.

We finally mention that all the results presented above generalize to the case of a compressibility κ of the form $\kappa(\mathbf{x},\frac{\mathbf{x}}{\varepsilon},\omega) = \kappa(\mathbf{x},\tau_{-\frac{\mathbf{x}}{\varepsilon}}\omega)$ in (4) provided that the function is sufficiently smooth in its first variable. The equation (34) would then involve $\kappa^*(\mathbf{x}) = \mathbb{E}\{\kappa(\mathbf{x},\omega)\}$ and the Liouville equation for $a(t,\mathbf{x},\mathbf{k})$ would be

$$\frac{\partial a}{\partial t} + \{c^*(\mathbf{x})|\mathbf{k}|, a\} = 0, \quad \{a,b\} = \nabla_{\mathbf{k}} a \cdot \nabla_{\mathbf{x}} b - \nabla_{\mathbf{k}} b \cdot \nabla_{\mathbf{x}} a, \quad c^*(\mathbf{x}) = \frac{1}{\sqrt{\kappa^*(\mathbf{x})\rho^*}}, \quad (66)$$

with the same initial conditions $a_0(\mathbf{x},\mathbf{k})$ in (57) and (59) with κ^* replaced by $\kappa^*(\mathbf{x})$. We can then assume that $c^*(\mathbf{x})$ is a random field (with statistics independent of that of $a(\omega)$) of the form $c^*(\mathbf{x}/\delta(\varepsilon))$, where $\delta(\varepsilon)$ is sufficiently large with respect to ε^α, and then obtain a limiting Fokker-Planck equation for the above solution as was done in e.g. [2].

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