Homogenization of the Schrödinger equation with large, random potential

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Abstract

We study the behavior of solutions to a Schrödinger equation with large, rapidly oscillating, mean zero, random potential with Gaussian distribution. We show that in high dimension $d > \mathfrak{m}$, where $\mathfrak{m}$ is the order of the spatial pseudo-differential operator in the Schrödinger equation (with $\mathfrak{m} = 2$ for the standard Laplace operator), the solution converges in the L^2 sense uniformly in time over finite intervals to the solution of a deterministic Schrödinger equation as the correlation length ε tends to 0. This generalizes to long times the convergence results obtained for short times and for the heat equation in [2]. The result is based on the decomposition of multiple scattering contributions introduced in [6]. In dimension $d < \mathfrak{m}$, the random solution converges to the solution of a stochastic partial differential equation; see [1, 13].

1 Introduction

There is a long list of derivations of macroscopic models for solutions to equations involving small scale heterogeneities $\varepsilon \ll 1$. One very successful framework is that of homogenization theory. In the limit $\varepsilon \to 0$, it is shown that the heterogeneous solution converges to the deterministic solution of an effective medium equation [5, 9]. There are cases, however, where the solutions in the limit $\varepsilon \to 0$ remain stochastic, typically in low spatial dimensions; see e.g. [1, 7, 8, 11].

The results of [1, 11] apply to parabolic equations of the form of a heat equation with large, mean-zero, highly oscillatory, random potentials (zeroth-order terms). In [2], it was shown that for large dimensions, the random solution converged to a deterministic solution, which is consistent with the homogenization framework. However, such results could only be obtained for short times, and it is unclear whether they hold for larger times.

In this paper, we revisit the homogenization limit for the (time-dependent) Schrödinger equation. Because the solution operator is unitary in this case, we expect to be able to

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control the long-time asymptotic behavior of the solution. The method of proof, as in [1, 2], is based on a Duhamel expansion of the random solution in terms involving increasing numbers of scattering events. As the number of terms grows exponentially with the number of scattering events, we need to assume that the potential is Gaussian in order to control such a growth. The summation technique used in [2] cannot extend to long time controls. A similar difficulty occurs in the derivation of radiative transfer equations for the energy density of high frequency waves propagating in highly oscillatory media [4, 6, 12]. A precise summation of the scattering terms was introduced in [6] to allow for long-time expansions. We adapt this technique to the asymptotic analysis of Schrödinger equations with large potentials.

We now present in more detail the model considered in this paper and the main convergence result. Let $\mathfrak{m} \geq 2$. We consider the following Schrödinger equation in dimension $d > \mathfrak{m}$:

$$\begin{cases} \Big(i\dfrac{\partial}{\partial t} + \big(P(D) - \dfrac{1}{\varepsilon^{\mathfrak{m}/2}} q(\dfrac{x}{\varepsilon})\big)\Big) u_\varepsilon(t,x) = 0, & t > 0,\ x \in \mathbb{R}^d \\ u_\varepsilon(0,x) = u_0(x), & x \in \mathbb{R}^d. \end{cases} \tag{1.1}$$

Here, $P(D)$ is the pseudo-differential operator with symbol $\hat{p}(\xi) = |\xi|^{\mathfrak{m}}$. We assume that $q(x)$ is a real valued mean zero stationary Gaussian process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with correlation function $R(x) = \mathbb{E}\{q(y)q(x+y)\}$, and the non-negative power spectrum $\hat{R}(\xi)$ is radially symmetric, smooth, and decays fast. For simplicity, we assume $\hat{R} \in \mathcal{S}(\mathbb{R}^d)$. In fact, $\|R\|_{12d,12d} < +\infty$, where

$$\|f\|_{d_1,d_2} := \|\langle x\rangle^{d_1} \langle \nabla_x \rangle^{d_2} f(x)\|_2, \qquad \langle x\rangle := (1+x^2)^{1/2}, \tag{1.2}$$

is enough. We choose the initial condition $u_0(x)$ to be smooth such that $\hat{u}_0(\xi)\langle\xi\rangle^{6d} \in L^2(\mathbb{R}^d)$. For any finite time $T > 0$, the existence of a weak solution $u_\varepsilon(t,x) \in L^2(\Omega \times \mathbb{R}^d)$ uniformly in time $t \in (0,T)$ and $0 < \varepsilon < \varepsilon_0$ can be proved by using a method based on Duhamel expansion.
As $\varepsilon \to 0$, we show that the solution $u_\varepsilon(t)$ to (1.1) converges strongly in $L^2(\Omega \times \mathbb{R}^d)$ uniformly in $t \in (0,T)$ to its limit $u(t)$ solution of the following homogenized equation

$$\begin{cases} (i\dfrac{\partial}{\partial t} + P(D) - \rho) u(t,x) = 0, & t > 0,\ x \in \mathbb{R}^d \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.3}$$

where the potential is given by

$$\rho = \int_{\mathbb{R}^d} \frac{\hat{R}(\xi)}{|\xi|^{\mathfrak{m}}} d\xi. \tag{1.4}$$

The main result of this paper is the following convergence result:

Theorem 1.1. *There exists a solution to (1.1) $u_\varepsilon(t)$ uniformly in $0 < \varepsilon < \varepsilon_0$ for $t > 0$. Moreover, we have the convergence results for all $t \in (0,T)$*

$$\lim_{\varepsilon \to 0} \mathbb{E}\|(u_\varepsilon - u)(t)\|_2^2 = 0. \tag{1.5}$$

The rest of the paper is organized as follows. Section 2 recasts the solution to (1.1) as a Duhamel series expansion in the frequency domain. We estimate the L^2 norm of the first n_0 terms by calculating the contributions of graphs in three categories similar to those defined in [2]. Section 3 estimates the L^2 norm of the error term by first subdividing the time integration into time intervals of smaller sizes, and then using Duhamel formula in each time interval. This method, introduced in [6], significantly improves the error estimates compared to the direct estimates of infinite Duhamel terms and enables the elimination of the restriction to short times. The estimates given in these sections are used in section 4 to characterize the limit of the solution $u_\varepsilon(t,x)$. Section 5 provides the proofs for the inequalities used for justifying the estimates in the previous sections.

The analysis of a parabolic equation of the type of the heat equation (with $i\partial_t$ replaced by ∂_t) is performed in [2] for $d \geq \mathfrak{m}$. Up to a logarithmic correction, we expect the limit of the solution to (1.1) to be deterministic also for the critical dimension $\mathfrak{m} = d$. In [2], the random fluctuations about the deterministic limit are also analyzed for the heat equation. We expect a similar behavior to occur for the Schrödinger equation (1.1) for short times. We do not know the behavior of the random fluctuations for arbitrary times $t \in (0,T)$.

In lower spatial dimension $d < \mathfrak{m}$, the limit of the solutions to (1.1) as $\varepsilon \to 0$ remains stochastic. This behavior was analyzed for the heat equation in [1, 10]. The limit of u_ε is then shown to be the solution of a stochastic partial differential equation with multiplicative noise (written as a Stratonovich product). The analysis of (1.1) for $d < \mathfrak{m}$ is performed in [13]. Note that several results of convergence may be extended to the case of random potential with long range correlations or random potentials that display both temporal and spatial fluctuations [3].

2 Duhamel expansion

We denote by e^{itH} the propagator for the equation (1.1). The Duhamel expansion then states that for any $n_0 \geq 1$

$$u_\varepsilon(t) = e^{itH}u_0 = \sum_{n=0}^{n_0-1} u_{n,\varepsilon}(t) + \Psi_{n_0,\varepsilon}(t), \tag{2.1}$$

where for $H_0 := (-\Delta)^{\frac{\mathfrak{m}}{2}}$, we have defined

$$u_{n,\varepsilon}(t) := (-i)^n(\frac{1}{\varepsilon^{\mathfrak{m}/2}})^n \int_0^t \cdots \int_0^t (\prod_{k=0}^n ds_k)\delta(t - \sum_{k=0}^n s_k)e^{is_0H_0}q(\frac{x}{\varepsilon})\cdots q(\frac{x}{\varepsilon})e^{is_nH_0}u_0, \tag{2.2}$$

and the error term is given by

$$\Psi_{n_0,\varepsilon}(t) = (-i)\frac{1}{\varepsilon^{\mathfrak{m}/2}}\int_0^t ds e^{i(t-s)H}q(\frac{x}{\varepsilon})u_{n_0-1,\varepsilon}(s). \tag{2.3}$$

We shall choose

$$n_0 = n_0(\varepsilon) := \frac{\gamma|\log\varepsilon|}{\log|\log\varepsilon|}, \tag{2.4}$$

for some fixed $0 < \gamma \ll \lambda$ sufficiently small, where λ is defined as

$$\lambda = \begin{cases} d - \mathfrak{m} & \mathfrak{m} < d \leq 2\mathfrak{m} \\ \mathfrak{m} & d > 2\mathfrak{m}. \end{cases} \tag{2.5}$$

For any subset $I \subset \mathbb{N}$, we define the kernel for the evolution in the Fourier space as

$$K(t; \boldsymbol{\xi}, I) := (-i)^{|I|-1} \int_0^t (\prod_{k\in I} ds_k)\delta(t - \sum_{k\in I} s_k) \prod_{k\in I} e^{is_k \xi_k^{\mathfrak{m}}}. \tag{2.6}$$

Hereafter, we use the notation $\xi^{\mathfrak{m}} = |\xi|^{\mathfrak{m}}$. In the special case where $I = \{0, \cdots, n\}$, we denote by $K(t; \boldsymbol{\xi}, n) := K(t; \boldsymbol{\xi}, \{0, \cdots, n\})$ and $\boldsymbol{\xi}_n := \boldsymbol{\xi}_{I_n}$. Denote $\boldsymbol{\xi}_{n,\hat{0}} = \boldsymbol{\xi}_{I_n \setminus \{0\}}$.
Let us introduce $\hat{q}_\varepsilon(\xi) = \varepsilon^{d-\frac{\mathfrak{m}}{2}} \hat{q}(\varepsilon\xi)$, the Fourier transform of $\varepsilon^{-\frac{\mathfrak{m}}{2}} q(\frac{x}{\varepsilon})$. Denote the contribution from the potential term by

$$L(\boldsymbol{\xi}, n) := \prod_{k=1}^{n} \hat{q}_\varepsilon(\xi_k - \xi_{k-1}). \tag{2.7}$$

We may rewrite the n^{th} order wave function as

$$\hat{u}_{\varepsilon,n}(t, \xi_0) = \int K(t; \boldsymbol{\xi}, n) L(\boldsymbol{\xi}, n) \hat{u}_0(\xi_n) d\boldsymbol{\xi}_{n,\hat{0}}. \tag{2.8}$$

We need to introduce the following moments

$$U_\varepsilon^n(t, \xi_0) = \mathbb{E}\{\hat{u}_{\varepsilon,n}\}, \tag{2.9}$$

which are given by

$$U_\varepsilon^n(t, \xi_0) = (-i)^n \int K(t, \boldsymbol{\xi}, n) \mathbb{E}\{L(\boldsymbol{\xi}, n)\} \hat{u}_0(\xi_n) d\boldsymbol{\xi}_{n,\hat{0}}, \tag{2.10}$$

and

$$U_\varepsilon^{n,m}(t, \xi_0, \zeta_0) = \mathbb{E}\{\hat{u}_{\varepsilon,n}(t, \xi)\overline{\hat{u}_{\varepsilon,m}}(t, \zeta)\}, \tag{2.11}$$

which are given by

$$U_\varepsilon^{n,m}(t, \xi_0, \zeta_0) = (-i)^{n+m} \int K(t, \boldsymbol{\xi}, n)\overline{K(t, \boldsymbol{\zeta}, m)} \mathbb{E}\{L(\boldsymbol{\xi}, n)\overline{L(\boldsymbol{\zeta}, m)}\} \hat{u}_0(\xi_n)\overline{\hat{u}_0(\zeta_m)} d\boldsymbol{\xi}_{n,\hat{0}} d\boldsymbol{\zeta}_{m,\hat{0}}. \tag{2.12}$$

We need to estimate moments of the Gaussian process $\hat{q}_\varepsilon$. The expectation in $U_\varepsilon^{n,m}$ vanishes unless there is $\bar{n} \in \mathbb{N}$ such that $n + m = 2\bar{n}$ is even. The moments are thus given as a sum of products of the expectation of pairs of terms $\hat{q}_\varepsilon(\xi_k - \xi_{k+1})$, where the sum runs over all possible pairings. We define the pair (ξ_k, ξ_l), $1 \leq k < l$, as the contribution in the product given by

$$\begin{aligned} \mathbb{E}\{\hat{q}_\varepsilon(\xi_{k-1} - \xi_k)\hat{q}_\varepsilon(\xi_{l-1} - \xi_l)\} &= \varepsilon^{d-\mathfrak{m}} \hat{R}(\varepsilon(\xi_k - \xi_{k-1}))\delta(\xi_k - \xi_{k-1} + \xi_l - \xi_{l-1}) \\ &= \varepsilon^{d-\mathfrak{m}} r(\varepsilon(\xi_k - \xi_{k-1})) r(\varepsilon(\xi_l - \xi_{l-1}))\delta(\xi_k - \xi_{k-1} + \xi_l - \xi_{l-1}) \end{aligned} \tag{2.13}$$

with $r(\xi) := \hat{R}(\xi)^{1/2}$.

Define

$$F(\boldsymbol{\xi}, n) := \prod_{k=1}^{n} r(\varepsilon(\xi_k - \xi_{k-1}))\hat{u}_0(\xi_n). \tag{2.14}$$

Denote by Δ_π the product of delta functions associated with the pairing π. Our analysis is based on the estimate of

$$U_\varepsilon^n(t, \xi) = \sum_{\pi \in \Pi(n)} I_\pi \tag{2.15}$$

with

$$I_\pi := (-i)^n \int K(t, \boldsymbol{\xi}, n)\Delta_\pi(\boldsymbol{\xi})F(\boldsymbol{\xi}, n)d\boldsymbol{\xi}, \tag{2.16}$$

and

$$\int U_\varepsilon^{n,m}(t, \xi, \xi)d\xi = \sum_{\pi \in \Pi(n,m)} C_\pi \tag{2.17}$$

with

$$C_\pi := (-i)^{n+m} \int K(t, \boldsymbol{\xi}, n)\overline{K(t, \boldsymbol{\zeta}, m)}\Delta_\pi(\boldsymbol{\xi}, \boldsymbol{\zeta})\delta(\xi_0 - \zeta_0)F(\boldsymbol{\xi}, n)\overline{F(\boldsymbol{\zeta}, m)}d\boldsymbol{\xi}_n d\boldsymbol{\zeta}_n. \tag{2.18}$$

By Lemma 5.1, $K(t; \boldsymbol{\xi}, I)$ can also be written as

$$K(t; \boldsymbol{\xi}, I) = ie^{t\eta} \int d\alpha e^{-i\alpha t} \prod_{k \in I} \frac{1}{\alpha + \xi_k^{\mathfrak{m}} + i\eta}. \tag{2.19}$$

We let $\eta = t^{-1}$ in this section. Therefore, I_π in (2.16) and C_π in (2.18) can be written explicitly as

$$I_\pi = -i^{n+1}e^{t\eta} \int d\alpha e^{-i\alpha t} \prod_{k=1}^{n} \frac{r(\varepsilon(\xi_k - \xi_{k-1}))}{\alpha + \xi_k^{\mathfrak{m}} + i\eta}\Delta_\pi(\xi)\hat{u}_0(\xi_n)d\boldsymbol{\xi}_{n,\hat{0}}, \tag{2.20}$$

and

$$\begin{aligned} C_\pi = -i^{n+m}e^{2t\eta} \int d\alpha d\beta e^{-i(\alpha-\beta)t} \prod_{k=0}^{n} \frac{1}{\alpha + \xi_k^{\mathfrak{m}} + i\eta} \prod_{l=0}^{m} \frac{1}{\beta + \zeta_l^{\mathfrak{m}} + i\eta} \prod_{k=1}^{n} r(\varepsilon(\xi_k - \xi_{k-1})) \\ \times \prod_{l=1}^{m} r(\varepsilon(\zeta_l - \zeta_{l-1})\Delta_\pi(\boldsymbol{\xi}, \boldsymbol{\zeta})\hat{u}_0(\xi_n)\overline{\hat{u}(\zeta_m)}\delta(\xi_0 - \zeta_0)d\boldsymbol{\xi}_n d\boldsymbol{\zeta}_m. \end{aligned} \tag{2.21}$$

In order to consider the two sets of momenta in a unified way, we introduce the notation

$$\alpha_k = \begin{cases} \alpha & 0 \le k \le n \\ \beta & n+1 \le k \le n+m+1, \end{cases} \tag{2.22}$$

and define $\xi_{n+k+1} = \zeta_{m-k}$ for $0 \le k \le m$. (2.21) can then be rewritten as

$$\begin{aligned} C_\pi = -i^{n+m}e^{2t\eta} \int d\alpha d\beta e^{-i(\alpha-\beta)t} \prod_{k=0}^{n+m+1} \frac{1}{\alpha_k + \xi_k^{\mathfrak{m}} + i\eta} \prod_{k=1, k \neq n+1}^{n+m+1} r(\varepsilon(\xi_k - \xi_{k-1})) \\ \hat{u}_0(\xi_n)\overline{\hat{u}_0(\xi_m)}\delta(\xi_0 - \xi_{n+m+1})d\boldsymbol{\xi}_{n+m+1}. \end{aligned} \tag{2.23}$$

In each instance of the pairings, we have $\bar{n}$ terms k and $\bar{n}$ terms $l \equiv l(k)$. Note that $l(k) \geq k+1$. The collection of pairs $(\xi_k, \xi_{l(k)})$ for $\bar{n}$ values of k and $\bar{n}$ values of $l(k)$ constitutes a graph $\pi \in \Pi(n,m)$, where $\Pi(n,m)$ denotes the set of all graphs π with n copies of $\hat{q}$ and m copies of $\bar{\hat{q}}$. The graphs are defined similarly in the calculation of $U_\varepsilon^n(t,\xi_0)$ in (2.10) for $n = 2\bar{n}$, and we denote by $\Pi(n)$ the set of graphs with n copies of $\hat{q}$. We denote by $A_0 = A_0(\pi)$ the collection of the $\bar{n}$ values of k and by $B_0 = B_0(\pi)$ the collection of the $\bar{n}$ values of $l(k)$.

Now we introduce several classes of graphs for C_π. We say that the graph has a crossing if there is a $k \leq n$ such that $l(k) \geq n+2$. We denote by $\Pi_c(n,m) \subset \Pi(n,m)$ the set of graphs with at least one crossing and by $\Pi_{nc}(n,m) = \Pi(n,m) \backslash \Pi_c(n,m)$ the non-crossing graphs. We denote by simple pairs the pairs such that $l(k) = k+1$, which thus involve a delta function of the form $\delta(\xi_{k+1} - \xi_{k-1})$. The unique graph with only simple pairs is called the simple graph, which is denoted by $\Pi_s(n,m)$. $\Pi_{ncs}(n,m) = \Pi_{nc}(n,m) \backslash \Pi_s(n,m)$ denotes the set of non-crossing, non-simple graphs.

We also use the notation $\Pi_s(n)$ and $\Pi_{ncs}(n)$ defined for I_π, which denote the simple graph and the set of non-crossing, non-simple graphs, respectively.

We shall estimate $F(\boldsymbol{\xi}, n)$ before we proceed to analyze the graphs. For the initial condition, we define

$$\Phi(\xi) = \langle \xi \rangle^{6d} |\hat{u}_0(\xi)|. \tag{2.24}$$

From our assumption on u_0 given in Section 1, we have $\Phi(\xi) \in L^2(\mathbb{R}^d)$. Since $\|\hat{R}\|_{12d,12d} < +\infty$, we have

$$r(\xi) \leq \frac{C}{\langle \xi \rangle^{6d}}. \tag{2.25}$$

Hence, we obtain the estimate for $F(\boldsymbol{\xi}, n)$

$$\begin{aligned} |F(\boldsymbol{\xi}, n)| &\leq \prod_{k=1}^{n} \frac{1}{\langle \varepsilon(\xi_k - \xi_{k-1}) \rangle^{6d}} \frac{\Phi(\xi_n)}{\langle \xi_n \rangle^{6d}} \\ &\leq \prod_{k=1}^{n} \frac{1}{\langle \varepsilon(\xi_k - \xi_{k-1}) \rangle^{6d}} \frac{\Phi(\xi_n)}{\langle \xi_n \rangle^{2d} \langle \varepsilon \xi_n \rangle^{4d}} \\ &\leq C^n \Phi(\xi_n) \frac{1}{\langle \xi_n \rangle^{2d}} \prod_{i=1}^{2} \frac{1}{\langle \varepsilon \xi_{l_i} \rangle^{2d}} \prod_{k=1}^{n} \frac{1}{\langle \varepsilon(\xi_k - \xi_{k-1}) \rangle^{2d}}. \end{aligned} \tag{2.26}$$

We have the freedom to choose l_1 and l_2 between 0 and n.

Analysis of the crossing graphs

Lemma 2.1. *If $\pi \in \Pi_c(n,m)$, we have*

$$|C_\pi| \leq (C \log \varepsilon)^{\bar{n}} \varepsilon^{\lambda} \|\Phi(\xi)\|_2^2. \tag{2.27}$$

Proof. Denote by $(\xi_{q_m}, \xi_{l(q_m)})$, $1 \leq m \leq M$, the crossing pairs and define $Q = \max_m \{q_m\}$. Let us define $A' = A_0 \backslash \{Q\}$. From (2.26), we have

$$|F(\boldsymbol{\xi}, n) \overline{F(\boldsymbol{\zeta}, m)}| \leq |\Phi(\xi_n)| |\Phi(\xi_{n+1})| \frac{1}{\langle \varepsilon \xi_0 \rangle^{2d}} \frac{1}{\langle \xi_n \rangle^{2d}} \prod_{k \in A'} \frac{1}{\langle \varepsilon(\xi_k - \xi_{k-1}) \rangle^{2d}}. \tag{2.28}$$

The terms $\frac{1}{|\alpha_k+\xi_k^{\mathfrak{m}}+it^{-1}|}$ for $k \notin A' \cup \{0, n, n+1, n+m+1\}$ are bounded by C. This allows us to obtain

$$\begin{aligned}|C_\pi| \leq & C^{\bar{n}} \int d\alpha d\beta \frac{1}{|\alpha+\xi_0^{\mathfrak{m}}+it^{-1}|}\frac{1}{|\beta+\xi_0^{\mathfrak{m}}+it^{-1}|}\frac{1}{\langle\varepsilon\xi_0\rangle^{2d}} \\ & \prod_{k\in A'}\frac{1}{|\alpha_k+\xi_k^{\mathfrak{m}}+it^{-1}|}\frac{1}{\langle\varepsilon(\xi_k-\xi_{k-1})\rangle^{2d}}\delta(\xi_k-\xi_{k-1}+\xi_{l(k)}-\xi_{l(k)-1}) \\ & \delta(\xi_Q-\xi_{Q-1}+\xi_{l(Q)}-\xi_{l(Q)-1}) \\ & \frac{1}{|\alpha+\xi_n^{\mathfrak{m}}+it^{-1}|}\frac{1}{|\beta+\xi_n^{\mathfrak{m}}+it^{-1}|}|\Phi(\xi_n)|^2 d\boldsymbol{\xi}_{n+m}.\end{aligned} \tag{2.29}$$

For each $k \in A' \cup \{0\}$, we perform the change of variables $\xi_k \to \frac{\xi_k}{\varepsilon}$, and define

$$\xi_k^\varepsilon = \begin{cases} \xi_k & k \neq A' \cup \{0\} \\ \frac{\xi_k}{\varepsilon} & k \in A' \cup \{0\}. \end{cases} \tag{2.30}$$

We then find the estimate

$$\begin{aligned}|C_\pi| \leq & C^{\bar{n}} \int d\alpha d\beta \frac{1}{|\alpha+\xi_0^{\mathfrak{m}}/\varepsilon^{\mathfrak{m}}+it^{-1}|}\frac{1}{|\beta+\xi_0^{\mathfrak{m}}/\varepsilon^{\mathfrak{m}}+it^{-1}|}\varepsilon^{-\mathfrak{m}}\frac{1}{\langle\xi_0\rangle^{2d}} \\ & \prod_{k\in A'}\frac{1}{|\alpha_k+\xi_k^{\mathfrak{m}}/\varepsilon^{\mathfrak{m}}+it^{-1}|}\varepsilon^{-\mathfrak{m}}\frac{1}{\langle\xi_k-\varepsilon\xi_{k-1}^\varepsilon\rangle^{2d}} \\ & \delta(\frac{\xi_k}{\varepsilon}-\xi_{k-1}^\varepsilon+\xi_{l(k)}-\xi_{l(k)-1}^\varepsilon)\delta(\xi_Q-\xi_{Q-1}+\xi_{l(Q)}-\xi_{l(Q)-1}) \\ & \frac{1}{|\alpha+\xi_n^{\mathfrak{m}}+it^{-1}|}\frac{1}{|\beta+\xi_n^{\mathfrak{m}}+it^{-1}|}\frac{1}{\langle\xi_n\rangle^{2d}}|\Phi(\xi_n)|^2 d\boldsymbol{\xi}_{n+m}.\end{aligned} \tag{2.31}$$

We now estimate the above product. Assume $Q < n$ and $n = l(k_0)$. When $Q = n$ or $l(Q) = n+m+1$, the derivation of the same estimates is simpler and left to the reader. Define k_1 such that $l(k_1) = n+m+1$. For each $k \in A'(\pi)\backslash(k_0 \cup k_1)$, we use (5.25) below to find the estimate

$$\varepsilon^{-\mathfrak{m}} \int \frac{1}{\langle\xi_k-\varepsilon\xi_{k-1}^\varepsilon\rangle^{2d}}\frac{1}{|\alpha_k+\xi_k^{\mathfrak{m}}/\varepsilon^{\mathfrak{m}}+it^{-1}|}\delta(\frac{\xi_k}{\varepsilon}-\varepsilon\xi_{k-1}^\varepsilon+\xi_{l(k)}-\xi_{l(k)-1}^\varepsilon)d\xi_k d\xi_{l(k)} \leq C|\log\varepsilon|. \tag{2.32}$$

The integration in the $\xi_{l(Q)}$ variable is estimated by using the above delta function. The delta function for $k = k_0 \in A'(\pi)$ may be written in the form $\delta(\xi_Q - \xi_{Q-1} + \xi_0 - \xi_n + \sum_{m=1}^{M-1} \xi_{q_m} - \xi_{q_m-1})$, and is thus used to integrate in the variable ξ_Q. The term $\frac{1}{\langle\xi_{k_0}-\varepsilon\xi_{k_0-1}^\varepsilon\rangle^{2d}}$ is used to integrate in the variable ξ_{k_0}. The integral in α and β is estimated using (5.3) by

$$\int \frac{1}{|\alpha+\xi_0^{\mathfrak{m}}/\varepsilon^{\mathfrak{m}}+it^{-1}|}\frac{1}{|\alpha+\xi_n^{\mathfrak{m}}+it^{-1}|}d\alpha \leq C\frac{1}{|\xi_0^{\mathfrak{m}}/\varepsilon^{\mathfrak{m}}-\xi_n^{\mathfrak{m}}+it^{-1}|}\left[1+\log_+\left|\frac{\xi_0^{\mathfrak{m}}/\varepsilon^{\mathfrak{m}}-\xi_n^{\mathfrak{m}}}{t^{-1}}\right|\right]. \tag{2.33}$$

The integral in ξ_0 is estimated using (5.18) by

$$\varepsilon^{-\mathfrak{m}} \int \frac{1}{\langle\xi_0\rangle^{2d}}\frac{\left[1+\log_+\left|\frac{\xi_0^{\mathfrak{m}}/\varepsilon^{\mathfrak{m}}-\xi_n^{\mathfrak{m}}}{t^{-1}}\right|\right]^2}{|\xi_0^{\mathfrak{m}}/\varepsilon^{\mathfrak{m}}-\xi_n^{\mathfrak{m}}+it^{-1}|^2}d\xi_0 \leq C\varepsilon^\lambda(\xi_n^\lambda \vee 1). \tag{2.34}$$

Following the usual convention, we use $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$. The delta function $\delta(\xi_{k_1} - \xi_{k_1-1} - \xi_{n+m+1} - \xi_{n+m})$ is seen to be equivalent to $\delta(\xi_n - \xi_{n+1})$, which handles the integration in the variable ξ_{n+1}. Finally we integrate in ξ_n

$$\int \frac{1}{\langle \xi_n \rangle^{2d}} |(\xi_n^\lambda \vee 1)\Phi(\xi_n)|^2 d\xi_n \leq \|\Phi(\xi)\|_2^2, \tag{2.35}$$

and obtain

$$|C_\pi| \leq (C|\log \varepsilon|)^{\bar{n}} \varepsilon^\lambda \|\Phi(\xi)\|_2^2. \tag{2.36}$$

□

Using Stirling's formula, we find that $|\Pi_c(n, m)| \leq \frac{(2\bar{n}-1)!}{2^{\bar{n}-1}(\bar{n}-1)!}$ is bounded by $\left(\frac{2\bar{n}}{e}\right)^{\bar{n}}$. After summation in $n, m \leq n_0$, we obtain

$$\begin{aligned} \mathbb{E}\| \sum_{n=0}^{n_0} (\hat{u}_{n,\varepsilon} - U_\varepsilon^n)(t)\|_2^2 \leq & \sum_{n=0}^{n_0-1} \sum_{m=0}^{n_0-1} (\frac{2\bar{n}}{e})^{\bar{n}} (C|\log \varepsilon|)^{\bar{n}} \varepsilon^\lambda \|\Phi(\xi)\|_2^2 \\ \leq & n_0^{n_0} (C|\log \varepsilon|)^{n_0} \varepsilon^\lambda \|\Phi(\xi)\|_2^2 \lesssim \varepsilon^{\lambda - 3\gamma}, \end{aligned} \tag{2.37}$$

where $a \lesssim b$ means $a \leq Cb$ for some $C > 0$.

Analysis of the non-crossing graphs

Lemma 2.2. *If $\pi \in \Pi_{ncs}(n)$, we have*

$$|I_\pi| \leq (C \log \varepsilon)^{\frac{n}{2}} \varepsilon^{\lambda(1-\delta)} |\Phi(\xi)|, \tag{2.38}$$

where $0 < \delta \ll 1$. Moreover, if $\pi \in \Pi_{ncs}(n, m)$,

$$|C_\pi| \leq (C \log \varepsilon)^{\bar{n}} \varepsilon^{2\lambda(1-\delta)} \|\Phi(\xi)\|_2^2. \tag{2.39}$$

Proof. In a graph $\pi \in \Pi_{ncs}(n)$, the delta function

$$\delta(\xi_0 - \xi_n) \tag{2.40}$$

is obtained by adding up all the delta functions in Δ_π. We perform the change of variables for all $k \in A_0$, $\xi_k \to \frac{\xi_k}{\varepsilon}$ and define as before

$$\xi_k^\varepsilon = \begin{cases} \xi_k & k \notin A_0 \\ \frac{\xi_k}{\varepsilon} & k \in A_0. \end{cases} \tag{2.41}$$

We shall solve the following two cases in different ways.

(i) If there exists $k_2 \in A_0$, such that for all k satisfying $k_2 + 1 \leq k \leq l(k_2) - 2, k \in A_0$, $(\xi_k, \xi_{l(k)})$ are simple pairs, then the delta function $\delta(\frac{\xi_{k_2}}{\varepsilon} - \xi_{l(k_2)-1})$ is present. From (2.26), we have

$$F(\boldsymbol{\xi}, n) \leq C^n |\Phi(\xi_n)| \frac{1}{\langle \xi_{k_2} \rangle^{2d}} \frac{1}{\langle \xi_n \rangle^{2d}} \prod_{k \in A_0} \frac{1}{\langle \xi_k - \varepsilon \xi_{k-1}^\varepsilon \rangle^{2d}}. \tag{2.42}$$

The estimate of integration in ξ_{k_2} is then obtained by using (5.18) below:

$$\int \varepsilon^{-\mathfrak{m}} \frac{1}{\langle \xi_{k_2} \rangle^{2d}} \frac{1}{|\alpha + \xi_{k_2}^{\mathfrak{m}}/\varepsilon^{\mathfrak{m}} + it^{-1}|^2} d\xi_{k_2} \leq C\varepsilon^{\lambda(1-\delta)}(\alpha^{\frac{\lambda(1-\delta)}{\mathfrak{m}}} \vee 1). \tag{2.43}$$

The delta function in which ξ_n is involved is equivalent to $\delta(\xi_n - \xi_0)$, which we use to integrate in ξ_n:

$$\int \frac{1}{\langle \xi_n \rangle^{2d}} \frac{1}{|\alpha + \xi_n^{\mathfrak{m}} + it^{-1}|} \delta(\xi_n - \xi_0) d\xi_n \leq \frac{1}{\langle \xi_0 \rangle^{2d}} \frac{1}{|\alpha + \xi_0^{\mathfrak{m}} + it^{-1}|}. \tag{2.44}$$

For $k \in A_0, k \neq k_2$, we have the estimate

$$\int \varepsilon^{-\mathfrak{m}} \frac{1}{\langle \xi_k - \varepsilon \xi_{k-1}^{\varepsilon} \rangle^{2d}} \frac{1}{|\alpha + \xi_k^{\mathfrak{m}}/\varepsilon^{\mathfrak{m}} + it^{-1}|} \delta(\frac{\xi_k}{\varepsilon} - \xi_{k-1}^{\varepsilon} + \xi_{l(k)} - \xi_{l(k)-1}^{\varepsilon}) d\xi_k d\xi_{l(k)} \leq C|\log \varepsilon|. \tag{2.45}$$

The estimate of integration in the variable α is then given by

$$\int \frac{\alpha^{\frac{\lambda(1-\delta)}{\mathfrak{m}}} \vee 1}{|\alpha + \xi_0^{\mathfrak{m}} + it^{-1}|^2} \leq C\xi_0^{\lambda(1-\delta)}. \tag{2.46}$$

The extra term $\xi_0^{\lambda(1-\delta)}$ that arises in the last estimate can be canceled by the term $1/\langle \xi_0 \rangle^{2d}$ in (2.44), which concludes (2.38).

(ii) If there exists no such $k_2 \in A_0$ satisfying the condition in case (i), then we first delete all simple pairs that exist in the graph π. In fact, the simple pairs can be handled first by using the bound as in (2.58). Therefore without loss of generality, we need only to consider a graph π with no simple pair. Let us define $k_4 = \min\{k | k \in A_0, l(k) - 1 \in A_0\}$, and $k_5 = l(k_4) - 1$. Note that $k_5 \geq k_4 + 1$.

We have from (2.26) that

$$F(\boldsymbol{\xi}, n) \leq C^n |\Phi(\xi_n)| \frac{1}{\langle \xi_{l(k_4)} \rangle^{2d}} \frac{1}{\langle \xi_{k_5} \rangle^{2d}} \frac{1}{\langle \xi_n \rangle^{2d}} \prod_{k \in A_0} \frac{1}{\langle \xi_k - \varepsilon \xi_{k-1}^{\varepsilon} \rangle^{2d}}. \tag{2.47}$$

The integration in $\xi_{l(k_4)}$ provides the terms which we will need for integration in ξ_{k_5}

$$\begin{aligned}&\int \frac{1}{\langle \varepsilon \xi_{l(k_4)} \rangle^{2d}} \frac{1}{|\alpha + \xi_{l(k_4)}^{\mathfrak{m}} + it^{-1}|} \delta(\frac{\xi_{k_4}}{\varepsilon} - \xi_{k_4-1}^{\varepsilon} + \xi_{l(k_4)} - \frac{\xi_{k_5}}{\varepsilon}) d\xi_{l(k_4)} \\ =&\frac{1}{\langle \xi_{k_5} - \xi_{k_4} + \varepsilon \xi_{k_4-1}^{\varepsilon} \rangle^{2d}} \frac{1}{|\alpha + |\xi_{k_5} - \xi_{k_4} + \varepsilon \xi_{k_4-1}^{\varepsilon}|^{\mathfrak{m}}/\varepsilon^{\mathfrak{m}} + it^{-1}|}.\end{aligned} \tag{2.48}$$

We can estimate the integration in ξ_{k_5} using (5.18)

$$\begin{aligned}&\int \varepsilon^{-\mathfrak{m}} \frac{1}{\langle \xi_{k_5} \rangle^{2d}} \frac{1}{\langle \xi_{k_5} - \xi_{k_4} + \varepsilon \xi_{k_4-1}^{\varepsilon} \rangle^{2d}} \frac{1}{|\alpha + \xi_{k_5}^{\mathfrak{m}}/\varepsilon^{\mathfrak{m}} + it^{-1}|} \\ &\frac{1}{|\alpha + |\xi_{k_5} - \xi_{k_4} + \varepsilon \xi_{k_4-1}^{\varepsilon}|^{\mathfrak{m}}/\varepsilon^{\mathfrak{m}} + it^{-1}|} d\xi_{k_5} \leq C\varepsilon^{\lambda(1-\delta)}(\alpha^{\frac{\lambda(1-\delta)}{\mathfrak{m}}} \vee 1).\end{aligned} \tag{2.49}$$

All the other integrations are handled the same way as in case (i). In order to make sure that no integration above is affected by other integrands we plan to use for integrating in other variables, we just need to first integrate in $\xi_{l(k)}$ with index in decreasing order and then integrate in ξ_k with index in decreasing order. This gives (2.38).
If $\pi \in \Pi_{ncs}(n, m)$, we may denote the pairings for $k \leq n$ and for $n+1 \leq k \leq n+m+1$ by π_1 and π_2 and then $\pi = \pi_1 \cup \pi_2$, since there is no crossing in π. Hence it follows that

$$|C_\pi| \leq \int |I_{\pi_1}(\xi) I_{\pi_2}(\xi)| d\xi \leq (C \log \varepsilon)^{\bar{n}} \varepsilon^{2\lambda(1-\delta)} \|\Phi(\xi)\|_2^2. \tag{2.50}$$

□

Similarly to (2.37), we obtain

$$\begin{aligned} \|\sum_{n=0}^{n_0-1} (U_\varepsilon^n - U_{\varepsilon,s}^n)(t)\|_2^2 \leq & \sum_{n=0}^{n_0-1} \sum_{m=0}^{n_0-1} (\frac{2\bar{n}}{e})^{\bar{n}} (C|\log \varepsilon|)^{\bar{n}} \varepsilon^{2\lambda(1-\delta)} \|\Phi(\xi)\|_2^2 \\ \leq & n_0^{n_0} (C|\log \varepsilon|)^{n_0} \varepsilon^{2\lambda(1-\delta)} \|\Phi(\xi)\|_2^2 \lesssim \varepsilon^{2\lambda(1-\delta)-3\gamma}, \end{aligned} \tag{2.51}$$

where

$$U_{\varepsilon,s}^n := I_\pi \tag{2.52}$$

for $\pi \in \Pi_s(n)$.
Collecting the results obtained in (2.37) and (2.51), we have shown that

$$\mathbb{E}\| \sum_{n=0}^{n_0-1} (\hat{u}_{\varepsilon,n} - U_{\varepsilon,s}^n)(t)\|_2^2 \lesssim \varepsilon^{2\lambda(1-\delta)-3\gamma}. \tag{2.53}$$

Analysis of the simple Graphs

Lemma 2.3. *If $\pi \in \Pi_s(n)$, we have*

$$|I_\pi| \leq (\frac{C^n}{(n/2)!} + O(C^n \varepsilon^{\mathtt{m}/2})) |\hat{u}_0(\xi)|. \tag{2.54}$$

Moreover, if $\pi \in \Pi_s(n,m)$, we have

$$|C_\pi| \leq (\frac{C^{n+m}}{(n/2)!(m/2)!} + O(C^{n+m} \varepsilon^{\mathtt{m}/2})) \|\hat{u}_0(\xi)\|_2^2. \tag{2.55}$$

Proof. In the case of a simple graph π, we can explicitly write out the product of delta functions

$$\Delta_\pi = \prod_{k \in A_0} \delta(\xi_{k-1} - \xi_{k+1}), \tag{2.56}$$

which is independent of ξ_k for all $k \in A_0$, and forces $\xi_k = \xi_0$ for all $k \notin A_0$.
Integrating in ξ_k for all $k \in B_0$ using delta functions, we obtain

$$I_\pi = \int K(t, \xi_0, \cdots, \xi_0, \xi_{k_{a_1}}, \cdots, \xi_{k_{a_{n/2}}}) \prod_{k \in A_0} \varepsilon^{d-\mathtt{m}} \hat{R}(\varepsilon(\xi_k - \xi_0)) \hat{u}_0(\xi_0) d\boldsymbol{\xi}_{A_0}, \tag{2.57}$$

where $\{k_{a_1}, \cdots, k_{a_{n/2}}\} = A_0$.
This implies

$$|I_\pi| \leq \int d\alpha \frac{1}{|\alpha + \xi_0^{\mathfrak{m}} + i\eta|^{n/2+1}} \prod_{k \in A_0} \left| \int \frac{\varepsilon^{d-\mathfrak{m}} \hat{R}(\varepsilon(\xi_k - \xi_0))}{\alpha + \xi_k^{\mathfrak{m}} + i\eta} d\xi_k \right| |\hat{u}_0(\xi_0)|. \tag{2.58}$$

Define

$$\Theta_{\alpha,\eta}(\xi_0) = \int \frac{\varepsilon^{d-\mathfrak{m}} \hat{R}(\varepsilon(\xi_k - \xi_0))}{\alpha + \xi_k^{\mathfrak{m}} + i\eta} d\xi_k. \tag{2.59}$$

Perform the change of variable $\xi_k \to \frac{\xi_k}{\varepsilon}$. This gives

$$\Theta_{\alpha,\eta}(\xi_0) = \int \frac{\hat{R}(\xi_k - \varepsilon\xi_0)}{\varepsilon^{\mathfrak{m}}\alpha + \xi_k^{\mathfrak{m}} + i\varepsilon^{\mathfrak{m}}\eta} d\xi_k. \tag{2.60}$$

It is clear from Lemma 5.8 that

$$|\Theta_{\alpha,\eta}(\xi_0)| \leq C. \tag{2.61}$$

Thus (2.58) already implies

$$|I_\pi| \leq C^n |\hat{u}_0(\xi)|. \tag{2.62}$$

However, this estimate is not sufficient and there is in fact a term $1/(n/2)!$ missing. We now recover this factor.

Introduce the notation

$$\Theta(\xi) = \lim_{\eta \to 0} \Theta_{\xi^{\mathfrak{m}},\eta}(\xi). \tag{2.63}$$

From the estimate in (5.36), we have

$$|\Theta_{\alpha,\eta}(\xi_0) - \Theta(\xi_k)| \leq \varepsilon^{\mathfrak{m}/2}(|\alpha + \xi_0^{\mathfrak{m}}||\eta|^{-1/2} + |\eta|^{1/2}). \tag{2.64}$$

We shall show that the leading term of I_π is

$$K(t, \xi_0, \cdots, \xi_0)\Theta(\xi_0)^{n/2}\hat{u}_0(\xi_0). \tag{2.65}$$

In fact, the error term is bounded by

$$\int d\alpha \frac{1}{|\alpha + \xi_0^{\mathfrak{m}} + i\eta|^{n/2+1}} |\Theta_{\alpha,\eta}^{n/2}(\xi_0) - \Theta^{n/2}(\xi_0)||\hat{u}_0(\xi_0)|. \tag{2.66}$$

Using the uniform bound on Θ in (2.61) and the estimate in (2.64), we can bound (2.66) by $O(\varepsilon^{\mathfrak{m}/2}C^n|\hat{u}_0(\xi_0)|)$.

We can now use Lemma 5.1 to bound (2.65) by

$$\frac{C^n}{(n/2)!}|\hat{u}_0(\xi_0)|. \tag{2.67}$$

Finally, we obtain (2.55) as an immediate consequence of (2.54). This concludes Lemma 2.3. □

3 Partial Time Integration

In this section, we estimate the L^2 norm of $\Psi_{n_0,\varepsilon}$. The central idea is to subdivide the time integration into smaller time intervals of size $t/\kappa(\varepsilon)$ with

$$\kappa(\varepsilon) := |\log \varepsilon|^{1/\gamma^2}. \tag{3.1}$$

We then use the Duhamel formula to estimate the evolution in each time interval.
Recall the error term $\Psi_{n_0,\varepsilon} = \sum_{n=n_0}^{+\infty} u_{n,\varepsilon}$. The Duhamel formula states that

$$\Psi_{n_0,\varepsilon}(t) = (-i)\frac{1}{\varepsilon^{\mathfrak{m}/2}}\int_0^t ds e^{i(t-s)H} q(\frac{x}{\varepsilon}) u_{n_0-1,\varepsilon}(s), \tag{3.2}$$

where e^{itH} denotes the propagator of equation (1.1).
Let $\theta_j = jt/\kappa$ for $j = 0, 1, \cdots, \kappa$. Rewrite

$$\Psi_{n_0,\varepsilon}(t) = (-i)\frac{1}{\varepsilon^{\mathfrak{m}/2}}\sum_{j=0}^{\kappa-1} e^{i(t-\theta_{j+1})H}\int_{\theta_j}^{\theta_{j+1}} e^{i(\theta_{j+1}-s)H} q(\frac{x}{\varepsilon}) u_{n_0-1,\varepsilon}(s) ds. \tag{3.3}$$

Define the n-th term of the Duhamel expansion for the operator $e^{i(\theta_{j+1}-s)H}$ in (3.3) as

$$\begin{aligned} u_{n,n_0,\theta_j} =& (-i)^{n-n_0+1}(\frac{1}{\varepsilon^{\mathfrak{m}/2}})^{n-n_0+1}\int_{\theta_j}^{\theta_{j+1}}\int_0^t\cdots\int_0^t (\prod_{k=0}^{n-n_0} ds_k)\delta(\theta_{j+1}-s-\sum_{k=0}^{n-n_0} s_k) \\ & e^{is_0H_0} q(\frac{x}{\varepsilon})\cdots q(\frac{x}{\varepsilon}) e^{is_{n-n_0+1}H_0} u_{n_0-1,\varepsilon}(s) ds. \end{aligned} \tag{3.4}$$

We may further obtain the form of u_{n,n_0,θ_j} in terms of u_0 by writing $u_{n_0-1,\varepsilon}(s)$ out explicitly using (2.2)

$$\begin{aligned} u_{n,n_0,\theta_j} =& (-i)^n(\frac{1}{\varepsilon^{\mathfrak{m}/2}})^n\int_{\theta_j}^{\theta_{j+1}}\int_0^t\cdots\int_0^t (\prod_{k=0}^{n} ds_k) \\ & \delta(\theta_{j+1}-s-\sum_{k=0}^{n-n_0} s_k)\delta(s-\sum_{k=n-n_0+1}^{n} s_k) e^{is_0H_0} q(\frac{x}{\varepsilon})\cdots q(\frac{x}{\varepsilon}) e^{is_nH_0} u_0 ds. \end{aligned} \tag{3.5}$$

The amputated versions of these functions are defined as

$$\tilde{u}_{4n_0,n_0,\theta_j}(x) = \frac{1}{\varepsilon^{\mathfrak{m}/2}} q(\frac{x}{\varepsilon}) u_{4n_0-1,n_0,\theta_j}(x). \tag{3.6}$$

Then Duhamel formula then gives

$$\Psi_{n_0,\varepsilon} = U_1 + U_2, \tag{3.7}$$

where

$$\begin{aligned} U_1(t) &= \sum_{n_0\le n<4n_0}\sum_{j=0}^{\kappa-1} e^{i(t-\theta_{j+1})H} u_{n,n_0,\theta_j}(\theta_{j+1}), \\ U_2(t) &= (-i)\sum_{j=0}^{\kappa-1} e^{i(t-\theta_{j+1})H}\int_{\theta_j}^{\theta_{j+1}} \tilde{u}_{4n_0,n_0,\theta_j}(s) ds. \end{aligned} \tag{3.8}$$

From the unitarity of $e^{i(t-\theta_{j+1})H}$ and the triangle inequality, we can bound U_1 by

$$\|U_1\|_2^2 \leq Cn_0\kappa \sum_{n_0\leq n<4n_0} \sum_{j=0}^{\kappa-1} \|u_{n,n_0,\theta_j}(\theta_{j+1})\|_2^2 \leq Cn_0^2\kappa^2 \sup_{n_0\leq n<4n_0} \|u_{n,n_0,\theta_j}\|_2^2. \tag{3.9}$$

Applying the Cauchy-Schwarz inequality, we can bound U_2 by

$$\|U_2\|_2^2 \leq t\sum_{j=0}^{\kappa-1} \int_{\theta_j}^{\theta_{j+1}} \|\tilde{u}_{4n_0,n_0,\theta_j}(s)\|_2^2 ds \leq t^2 \sup_j \sup_{\theta_j\leq s\leq\theta j+1} \|\tilde{u}_{4n_0,n_0,\theta_j}(s)\|_2^2. \tag{3.10}$$

Denote

$$I_{n-n_0+1,n} := \{n-n_0+1,\cdots,n\}. \tag{3.11}$$

Define the free evolution operator with constraint given by the parameters n_0 and θ as

$$K^{\#}(\theta_{j+1},\theta_j;\boldsymbol{\xi},n,n_0) := \int_{\theta_j}^{\theta_{j+1}} K(\theta_{j+1}-s;\boldsymbol{\xi},n-n_0)K(s,\boldsymbol{\xi},I_{n-n_0+1,n})ds. \tag{3.12}$$

We can write the wave function in Fourier space $\hat{u}_{n,n_0,\theta_j}$ as

$$\hat{u}_{n,n_0,\theta_j}(\theta_{j+1},\xi_0) = \int K^{\#}(\theta_{j+1},\theta_j;\boldsymbol{\xi},n,n_0)L(\boldsymbol{\xi},n)\hat{u}_0(\xi_n)d\boldsymbol{\xi}_{n,\hat{0}}. \tag{3.13}$$

We then write

$$\mathbb{E}\|u_{n,n_0,\theta_j}(\theta_{j+1})\|^2 = \sum_{\pi\in\Pi(n,n)} C_\pi^{\#}, \tag{3.14}$$

where

$$C_\pi^{\#} := \int d\boldsymbol{\xi}d\boldsymbol{\zeta}K^{\#}(\theta_{j+1},\theta_j;\xi,n,n_0)\overline{K^{\#}(\theta_{j+1},\theta_j;\zeta,n,n_0)}\Delta_\pi F(\xi,n)\overline{F(\zeta,n)}. \tag{3.15}$$

For the amputated function, we have

$$\mathbb{E}\|\tilde{u}_{n,n_0,\theta_j}(\theta_{j+1})\|^2 = \sum_{\pi\in\Pi(n,n)} \tilde{C}_\pi^{\#}, \tag{3.16}$$

$$\tilde{C}_\pi^{\#} := \int d\boldsymbol{\xi}d\boldsymbol{\zeta}\tilde{K}^{\#}(\theta_{j+1},\theta_j;\boldsymbol{\xi},n,n_0)\overline{\tilde{K}^{\#}(\theta_{j+1},\theta_j;\boldsymbol{\zeta},n,n_0)}\Delta_\pi F(\boldsymbol{\xi},n)\overline{F(\boldsymbol{\zeta},n)}, \tag{3.17}$$

where

$$\tilde{K}^{\#}(\theta_{j+1},\theta_j;\boldsymbol{\xi},n,n_0) := \int_{\theta_j}^{\theta_{j+1}} K(\theta_{j+1}-s;\boldsymbol{\xi},I_{1,n-n_0})K(s,\boldsymbol{\xi},I_{n-n_0+1,n})ds. \tag{3.18}$$

Recall Lemma 5.3. We can extend it to the following identity for $K^{\#}(\theta_{j+1},\theta_j;\boldsymbol{\xi},n,n_0)$:

$$\begin{aligned} K^{\#}(\theta_{j+1},\theta_j;\boldsymbol{\xi},n,n_0) = &-\int_{\theta_j}^{\theta_{j+1}} dse^{(\theta_{j+1}-s)\eta}e^{s\tilde{\eta}} \int_{-\infty}^{+\infty} d\alpha d\tilde{\alpha}e^{-i\alpha(\theta_{j+1}-s)}e^{-i\tilde{\alpha}s} \\ &\times \prod_{k=0}^{n-n_0} \frac{1}{\alpha+\xi_k^{\mathfrak{m}}+i\eta} \prod_{k\in I_{n-n_0+1,n}} \frac{1}{\tilde{\alpha}+\xi_k^{\mathfrak{m}}+i\tilde{\eta}}, \end{aligned} \tag{3.19}$$

where we choose $\eta := (t/\kappa)^{-1}$,$\tilde{\eta} := t^{-1}$. We can integrate in s to have

$$K^{\#}(\theta_{j+1}, \theta_j; \boldsymbol{\xi}, n, n_0) = i \int_{-\infty}^{+\infty} d\alpha d\tilde{\alpha} \frac{e^{-i\theta_{j+1}\tilde{\alpha}+\theta_{j+1}\tilde{\eta}} - e^{-i\alpha(\theta_{j+1}-\theta_j)-i\theta_j\tilde{\alpha}+(\theta_{j+1}-\theta_j)\eta+\theta_j\tilde{\eta}}}{\alpha - \tilde{\alpha} + i(\eta - \tilde{\eta})} \times \prod_{k=0}^{n-n_0} \frac{1}{\alpha + \xi_k^{\mathfrak{m}} + i\eta} \prod_{k \in I_{n-n_0+1,n}} \frac{1}{\tilde{\alpha} + \xi_k^{\mathfrak{m}} + i\tilde{\eta}}. \tag{3.20}$$

Hence we can bound $C_\pi^{\#}$ by

$$|C_\pi^{\#}| \le \int d\xi \int_{-\infty}^{+\infty} d\alpha d\tilde{\alpha} d\beta d\tilde{\beta} \frac{1}{|\alpha - \tilde{\alpha} + i(\eta - \tilde{\eta})|} \frac{1}{|\beta - \tilde{\beta} + i(\eta - \tilde{\eta})|} \times \prod_{0 \le k \le 2n+1} \frac{1}{|\alpha_k + \xi_k^{\mathfrak{m}} + i\eta_k|} \prod_{k=1, k \neq n+1}^{2n+1} r(\varepsilon(\xi_k - \xi_{k-1})), \tag{3.21}$$

where α_k and η_k are defined as

$$\alpha_k := \begin{cases} \tilde{\alpha} & if\ k \le n - n_0 \\ \alpha & if\ n - n_0 + 1 \le k \le n \\ \beta & if\ n + 1 \le k \le n + n_0 \\ \tilde{\beta} & if\ k \ge n + n_0 + 2 \end{cases} \tag{3.22}$$

$$\eta_k := \begin{cases} \tilde{\eta} & if\ k \le n - n_0 \\ \eta & if\ n - n_0 + 1 \le k \le n \\ \eta & if\ n + 1 \le k \le n + n_0 \\ \tilde{\eta} & if\ k \ge n + n_0 + 2. \end{cases} \tag{3.23}$$

We present the following lemmas to show $\lim_{\varepsilon \to 0} \mathbb{E}\|\Psi_{n_0,\varepsilon}\|_2^2 \to 0$.

Lemma 3.1. *Let $n = 4n_0$. For any $\pi \in \Pi(n, n)$ we have*

$$|\tilde{C}_\pi^{\#}| \le \frac{(C|\log \varepsilon|)^{4n_0}}{\kappa^{n_0}}. \tag{3.24}$$

Proof. The following bound can be easily obtained by using Lemma

$$|\tilde{C}_\pi^{\#}| \le (C|\log \varepsilon|)^{4n_0}. \tag{3.25}$$

To recover the denominator in (3.24), notice that among the η_k for $k \in B_0$, there are at least $n - 2n_0 - 2 (\ge n_0)$ of them with $\eta_k = \kappa/t$. Hence

$$\prod_{k \in B_0} \frac{1}{|\alpha_k + \xi_k^{\mathfrak{m}} + i\eta_k|} \le \prod_{k \in B_0} |\eta_k|^{-1} \le t^n \left(\frac{t}{\kappa}\right)^{n_0} \le \frac{t^n}{\kappa^{n_0}}. \tag{3.26}$$

□

Lemma 3.2. *If $\pi \in \Pi_c(n,n)$, then we have*

$$|C_\pi^\#| \le (C\log\varepsilon)^n \varepsilon^\lambda \|\Phi(\xi)\|_2^2. \tag{3.27}$$

Proof. The proof of Lemma 3.2 is essentially the same as in Lemma 2.1. The only difference is that the integral in α and $\tilde\alpha$ (β and $\tilde\beta$) is estimated using Proposition 5.3 by

$$\int\int \frac{1}{|\tilde\alpha - \alpha + i(\eta - \tilde\eta)|}\frac{1}{|\tilde\alpha + \xi_n^{\mathtt{m}} + i\eta|}\ \frac{1}{|\alpha + \xi_0^{\mathtt{m}}/\varepsilon^{\mathtt{m}} + i\tilde\eta|}d\alpha d\tilde\alpha \le \frac{C\left[1+\log_+\left|\frac{\xi_n^{\mathtt{m}} - \xi_0^{\mathtt{m}}/\varepsilon^{\mathtt{m}}}{\tilde\eta}\right|\right]^2}{|\xi_n^{\mathtt{m}} - \xi_0^{\mathtt{m}}/\varepsilon^{\mathtt{m}} + i\tilde\eta|}, \tag{3.28}$$

and the integral in ξ_0 is estimated using Proposition 5.6

$$\varepsilon^{-\mathtt{m}}\int \frac{1}{\langle\xi_0\rangle^{2d}}\frac{\left[1+\log_+\left|\frac{\xi_0^{\mathtt{m}}/\varepsilon^{\mathtt{m}} - \xi_n^{\mathtt{m}}}{\tilde\eta}\right|\right]^4}{|\xi_0^{\mathtt{m}}/\varepsilon^{\mathtt{m}} - \xi_n^{\mathtt{m}} + i\tilde\eta|^2}d\xi_0 \le C\varepsilon^\lambda(\xi_n^\lambda \vee 1). \tag{3.29}$$

□

Lemma 3.3. *If $\pi \in \Pi_{ncs}(n,n)$, then we have*

$$|C_\pi^\#| \le (C\log\varepsilon)^n \varepsilon^{2\lambda(1-\delta)} \|\Phi(\xi)\|_2^2. \tag{3.30}$$

Proof. The proof of Lemma 3.3 is similar to that of Lemma 2.2. The only difference is that the integral in α and $\tilde\alpha$ (β and $\tilde\beta$) is estimated by

$$\begin{aligned}&\int\int (\alpha^{\frac{\lambda(1-\delta)}{\mathtt{m}}} \vee 1)\frac{1}{|\tilde\alpha - \alpha + i(\eta-\tilde\eta)|}\frac{1}{|\tilde\alpha + \xi_0^{\mathtt{m}} + i\tilde\eta|}\frac{1}{|\alpha + \xi_0^{\mathtt{m}} + i\eta|}d\alpha d\tilde\alpha \\ \le & C\int (\alpha^{\frac{\lambda(1-\delta)}{\mathtt{m}}} \vee 1)\frac{1}{|\alpha + \xi_0^{\mathtt{m}} + i\tilde\eta|^2}\left[\log_+\left|\frac{\alpha + \xi_0^{\mathtt{m}}}{\tilde\eta}\right| + 1\right]d\alpha \le C\xi_0^{\lambda(1-\delta)}.\end{aligned} \tag{3.31}$$

□

Lemma 3.4. *If $\pi \in \Pi_s(n,n)$, then we have*

$$|C_\pi^\#| \le \frac{C^n}{(n!)^{1/2}}\|\hat u_0(\xi)\|_2^2. \tag{3.32}$$

Proof. The proof is essentially the same as in Lemma 2.3. We shall not repeat the argument here. □

We now apply the above lemmas to estimate $\Psi_{n_0,\varepsilon}$. From Lemma 3.2, 3.3, and 3.4 we have

$$\|U_1\|_2^2 \le n_0^2\kappa^2[(C\log\varepsilon)^{4n_0}\varepsilon^\lambda](\frac{8n_0}{e})^{4n_0} + \frac{C^{n_0}n_0^2\kappa^2}{(n_0!)^{1/2}}. \tag{3.33}$$

From Lemma 3.1 we have

$$\|U_2\|_2^2 \le \frac{(C|\log\varepsilon|)^{4n_0}}{\kappa^{n_0}}(\frac{8n_0}{e})^{4n_0}. \tag{3.34}$$

The L^2 estimate of $\Psi_{n_0,\varepsilon}$ is therefore given by

$$\mathbb{E}\|\sum_{n_0}^{\infty}\hat{u}_{\varepsilon,n}(t)\|_2^2 = \mathbb{E}\|\Psi_{n_0,\varepsilon}\|_2^2 \le 2(\mathbb{E}\|U_1\|_2^2 + \mathbb{E}\|U_2\|_2^2) \to 0 \tag{3.35}$$

as $\varepsilon \to 0$.

4 Homogenization

We come back to the analysis of $U_{\varepsilon,s}(t,\xi)$. We find that $U_{\varepsilon,s}$ is the solution to the following equation

$$\begin{aligned} U_{\varepsilon,s} &= e^{it\xi^{\mathfrak{m}}}\hat{u}_0(\xi) - \int_0^t e^{is\xi^{\mathfrak{m}}}\int_0^{t-s} e^{is_1\xi_1^{\mathfrak{m}}}\int \varepsilon^{d-\mathfrak{m}}\hat{R}(\varepsilon(\xi_1-\xi))U_{\varepsilon,s}(t-s-s_1,\xi)d\xi_1 ds ds_1 \\ &:= e^{it\xi^{\mathfrak{m}}}\hat{u}_0(\xi) + A_\varepsilon U_{\varepsilon,s}(t,\xi). \end{aligned} \tag{4.1}$$

Lemma 4.1. *Let us define $\mathfrak{U}_\epsilon(t,\xi)$ to be the solution to*

$$\begin{aligned} (i\frac{\partial}{\partial t} + \xi^{\mathfrak{m}} - \rho_\epsilon(\xi))\mathfrak{U}_\epsilon(t,\xi) = 0 \\ \mathfrak{U}_\epsilon(0,\xi) = \hat{u}_0(\xi), \end{aligned} \tag{4.2}$$

with $\rho_\epsilon = \int_{\mathbb{R}^d}\frac{\hat{R}(\xi_1-\epsilon\xi)}{\xi_1^{\mathfrak{m}}}d\xi_1$. We have the convergence results

$$|(U_{\varepsilon,s} - \mathfrak{U}_\varepsilon)(t)| \lesssim \max\{\varepsilon^{\mathfrak{m}}, \varepsilon^{d-\mathfrak{m}}|\log\varepsilon|\}. \tag{4.3}$$

Proof. (1) We obtain from Duhamel's principle that

$$\begin{aligned} \mathfrak{U}_\varepsilon(t,\xi) &= e^{it\xi^{\mathfrak{m}}}\hat{u}_0(\xi) - i\int_0^t e^{i\xi^{\mathfrak{m}}v}\int_{\mathbb{R}^d}\frac{\hat{R}(\xi_1-\varepsilon\xi)}{\xi_1^{\mathfrak{m}}}d\xi_1\mathfrak{U}_\varepsilon(t-v,\xi)dv \\ &:= e^{it\xi^{\mathfrak{m}}}\hat{u}_0(\xi) + B_\varepsilon\mathfrak{U}_\varepsilon(t,\xi) \\ &:= e^{it\xi^{\mathfrak{m}}}\hat{u}_0(\xi) + A_\varepsilon\mathfrak{U}_\varepsilon(t,\xi) + E_\varepsilon\mathfrak{U}_\varepsilon(t,\xi), \end{aligned} \tag{4.4}$$

where the operator A_ε is defined in (4.1) and may be recast as

$$\begin{aligned} A_\varepsilon\mathfrak{U}_\varepsilon &= -\int_0^t e^{i\xi^{\mathfrak{m}}v}\int_{\mathbb{R}^d}\int_0^{\frac{v}{\varepsilon^{\mathfrak{m}}}} e^{is_1(\xi_1^{\mathfrak{m}}-\varepsilon^{\mathfrak{m}}\xi^{\mathfrak{m}})}ds_1\hat{R}(\xi_1-\varepsilon\xi)d\xi_1\mathfrak{U}_\varepsilon(t-v,\xi)dv \\ &= -\int_0^t e^{i\xi^{\mathfrak{m}}v}\int_{\mathbb{R}^d}\frac{1}{i(\xi_1^{\mathfrak{m}}-\varepsilon^{\mathfrak{m}}\xi^{\mathfrak{m}})}(e^{i\frac{v}{\varepsilon^{\mathfrak{m}}}(\xi_1^{\mathfrak{m}}-\varepsilon^{\mathfrak{m}}\xi^{\mathfrak{m}})}-1)\hat{R}(\xi_1-\varepsilon\xi)d\xi_1\mathfrak{U}_\varepsilon(t-v,\xi)dv, \end{aligned} \tag{4.5}$$

and the remainder E_ε is then given by

$$\begin{aligned} E_\varepsilon\mathfrak{U}_\varepsilon &= i\int_0^t\int_{\mathbb{R}^d}\frac{1}{\xi_1^{\mathfrak{m}}-\varepsilon^{\mathfrak{m}}\xi^{\mathfrak{m}}}e^{i\frac{v}{\varepsilon^{\mathfrak{m}}}\xi_1^{\mathfrak{m}}}\hat{R}(\xi_1-\varepsilon\xi)d\xi_1\mathfrak{U}_\varepsilon(t-v,\xi)dv \\ &+\ i\int_0^t\int_{\mathbb{R}^d}\frac{\varepsilon^{\mathfrak{m}}\xi^{\mathfrak{m}}}{\xi_1^{\mathfrak{m}}(\xi_1^{\mathfrak{m}}-\varepsilon^{\mathfrak{m}}\xi^{\mathfrak{m}})}(e^{i\xi^{\mathfrak{m}}v}-e^{i\frac{\xi_1^{\mathfrak{m}}}{\varepsilon^{\mathfrak{m}}}v})\hat{R}(\xi_1-\varepsilon\xi)d\xi_1\mathfrak{U}_\varepsilon(t-v,\varepsilon)dv \\ &:= I_1 + I_2. \end{aligned} \tag{4.6}$$

For the calculation of I_1, note that equation (4.2) has the explicit solution:

$$\mathfrak{U}(t,\xi) = e^{it(\xi^{\mathtt{m}}-\rho_\varepsilon(\xi))}\hat{u}_0(\xi). \tag{4.7}$$

On one hand, we may obtain the following expression of integral in v using the method of separation of variables:

$$\begin{aligned}\int_0^t e^{i\frac{v}{\varepsilon^{\mathtt{m}}}(\xi_1^{\mathtt{m}}-\varepsilon^{\mathtt{m}}\xi^{\mathtt{m}})}\mathfrak{U}_\varepsilon(t-v,\xi)dv =& \frac{\varepsilon^{\mathtt{m}}}{\xi_1^{\mathtt{m}}-\varepsilon^{\mathtt{m}}\xi^{\mathtt{m}}}\int_0^t \mathfrak{U}_\varepsilon(t-v,\xi)d(e^{i\frac{v}{\varepsilon^{\mathtt{m}}}(\xi_1^{\mathtt{m}}-\varepsilon^{\mathtt{m}}\xi^{\mathtt{m}})})\\ =& \frac{\varepsilon^{\mathtt{m}}}{\xi_1^{\mathtt{m}}-\varepsilon^{\mathtt{m}}\xi^{\mathtt{m}}}[e^{i\frac{t}{\varepsilon^{\mathtt{m}}}(\xi_1^{\mathtt{m}}-\varepsilon^{\mathtt{m}}\xi^{\mathtt{m}})}\hat{u}_0(\xi) - e^{i(t-v)(\xi^{\mathtt{m}}-\rho_\varepsilon(\xi))}\hat{u}_0(\xi)\\ &+ i\int_0^t e^{i\frac{v}{\varepsilon^{\mathtt{m}}}(\xi_1^{\mathtt{m}}-\varepsilon^{\mathtt{m}}\xi^{\mathtt{m}})}e^{i(t-v)(\xi^{\mathtt{m}}-\rho_\varepsilon(\varepsilon))}\hat{u}_0(\xi)dv].\end{aligned} \tag{4.8}$$

On the other hand, we have the following simple estimate of this integral:

$$\left|\int_0^t e^{i\frac{v}{\varepsilon^{\mathtt{m}}}(\xi_1^{\mathtt{m}}-\varepsilon^{\mathtt{m}}\xi^{\mathtt{m}})}\mathfrak{U}_\varepsilon(t-v,\xi)dv\right| \le \int_0^t |e^{i\frac{v}{\varepsilon^{\mathtt{m}}}(\xi_1^{\mathtt{m}}-\varepsilon^{\mathtt{m}}\xi^{\mathtt{m}})}\mathfrak{U}_\varepsilon(t-v,\xi)|dv \le C|\hat{u}_0(\xi)|. \tag{4.9}$$

This gives that

$$\left|\int_0^t e^{i\frac{v}{\varepsilon^{\mathtt{m}}}(\xi_1^{\mathtt{m}}-\varepsilon^{\mathtt{m}}\xi^{\mathtt{m}})}\mathfrak{U}_\varepsilon(t-v,\xi)dv\right| \le C\varepsilon^{\mathtt{m}}(\frac{1}{\xi_1^{\mathtt{m}}-\varepsilon^{\mathtt{m}}\xi^{\mathtt{m}}}\wedge\frac{1}{\varepsilon^{\mathtt{m}}})\max\{|\hat{u}_0(\xi)|,|\xi^{\mathtt{m}}\hat{u}_0(\xi)|\}. \tag{4.10}$$

We therefore find the estimate of $|I_1|$ using Lemma 5.9:

$$\begin{aligned}|I_1| &\le C\varepsilon^{\mathtt{m}}\int_{\mathbb{R}^d}(\frac{1}{\xi_1^{\mathtt{m}}-\varepsilon^{\mathtt{m}}\xi^{\mathtt{m}}}\wedge\frac{1}{\varepsilon^{\mathtt{m}}})\max\{|\hat{u}_0(\xi)|,|\xi^{\mathtt{m}}\hat{u}_0(\xi)|\}\hat{R}(\xi_1-\varepsilon\xi)d\xi_1\\ &\le C\max\{1,\xi^{\mathtt{m}}\}\max\{\varepsilon^{\mathtt{m}},\varepsilon^{d-\mathtt{m}}|\log\varepsilon|(\xi^{\mathtt{m}}+1)^{\frac{d}{\mathtt{m}}-2},\varepsilon^{d-\mathtt{m}}(\xi^{\mathtt{m}}+1)^{\frac{d}{\mathtt{m}}-2}\}|\hat{u}_0(\xi)|.\end{aligned} \tag{4.11}$$

For the calculation of I_2, we first estimate the integral in v:

$$\varepsilon^{\mathtt{m}}\left|\int_0^t \frac{e^{i\xi^{\mathtt{m}}v}-e^{i\frac{\xi_1^{\mathtt{m}}}{\varepsilon^{\mathtt{m}}}v}}{\xi_1^{\mathtt{m}}-\varepsilon^{\mathtt{m}}\xi^{\mathtt{m}}}\mathfrak{U}_\varepsilon(t-v,\xi)dv\right| \le C(\frac{1}{\xi_1^{\mathtt{m}}-\varepsilon^{\mathtt{m}}\xi^{\mathtt{m}}}\wedge\frac{1}{\varepsilon^{\mathtt{m}}})\max\{|\hat{u}_0(\xi)|,|\xi^{\mathtt{m}}\hat{u}_0(\xi)|\}. \tag{4.12}$$

Hence

$$\begin{aligned}|I_2| &\le C\int_{\mathbb{R}^d}\frac{\xi^{\mathtt{m}}}{\xi_1^{\mathtt{m}}}(\frac{1}{\xi_1^{\mathtt{m}}-\varepsilon^{\mathtt{m}}\xi^{\mathtt{m}}}\wedge\frac{1}{\varepsilon^{\mathtt{m}}})\hat{R}(\xi_1-\varepsilon\xi)\max\{|\hat{u}_0(\xi)|,|\xi^{\mathtt{m}}\hat{u}_0(\xi)|\}d\xi_1\\ &\le C\max\{1,\xi^{\mathtt{m}}\}\max\{\varepsilon^{\mathtt{m}},\varepsilon^{d-\mathtt{m}}|\log\varepsilon|(\xi^{\mathtt{m}}+1)^{\frac{d}{\mathtt{m}}-2},\varepsilon^{d-\mathtt{m}}(\xi^{\mathtt{m}}+1)^{\frac{d}{\mathtt{m}}-2}\}|\xi^{\mathtt{m}}\hat{u}_0(\xi)|.\end{aligned} \tag{4.13}$$

Note that

$$|A_\varepsilon U(t,\xi)| \le C\int_0^t |U(s,\xi)|ds, \tag{4.14}$$

over a bounded interval in time. The equation

$$(I-A_\varepsilon)U(t,\xi) = S(t,\xi) \tag{4.15}$$

therefore admits a unique solution by Gronwall's Lemma, which is bounded by

$$|U(t,\xi)| \leq \|S\|_\infty e^{Ct}. \tag{4.16}$$

We verify that the solution to

$$(I - B_\varepsilon)\mathfrak{U}_\varepsilon = e^{it\xi^{\mathfrak{m}}}\hat{u}_0(\xi), \tag{4.17}$$

is given by

$$\mathfrak{U}_\varepsilon(t,\xi) = e^{it(\xi^{\mathfrak{m}} - \rho_\varepsilon(\xi))}\hat{u}_0(\xi). \tag{4.18}$$

The error $V_\varepsilon(t,\xi) = (U_{\varepsilon,s}(t,\xi) - \mathfrak{U}_\varepsilon(t,\xi))$ is a solution to

$$(I - A_\varepsilon)V_\varepsilon = E_\varepsilon \mathfrak{U}_\varepsilon(t,\xi), \tag{4.19}$$

so that over bounded intervals in time, we find that

$$|U_{\varepsilon,s}(t,\xi) - \mathfrak{U}_\varepsilon(t,\xi)| = |V_\varepsilon(t,\xi)| \lesssim \max\{\varepsilon^{\mathfrak{m}}, \varepsilon^{d-\mathfrak{m}}|\log\varepsilon|\}. \tag{4.20}$$

□

From our assumption that $\hat{R}(\xi) \in \mathcal{C}^2(\mathbb{R}^d)$, we find that

$$|e^{it(\xi^{\mathfrak{m}} - \rho_\varepsilon(\xi))} - e^{it(\xi^{\mathfrak{m}} - \rho)}| \leq t|\rho_\varepsilon(\xi) - \rho)| \leq C\varepsilon^2\xi^2. \tag{4.21}$$

The reason for the second-order accuracy is that $\hat{R}(-\xi) = \hat{R}(\xi)$ and $\nabla\hat{R}(0) = 0$ so that first-order terms in the Taylor expansion vanish.

In terms of the solutions of PDE we defined in (1.3), we may recast the above result as

$$|(U_{\varepsilon,s} - \mathcal{U})(t)| \lesssim \max\{\varepsilon^2, \varepsilon^{\mathfrak{m}}, \varepsilon^{d-\mathfrak{m}}|\log\varepsilon|\} \qquad \mathcal{U}(t,\xi) = e^{i(\xi^{\Updownarrow} - \rho)t}\hat{u}_0(\xi), \tag{4.22}$$

We now prove Theorem 1.1. By the triangle inequality, we have the estimate

$$\begin{aligned} &\mathbb{E}\|\hat{u}_\varepsilon(t,\xi) - \mathcal{U}(t,\xi)\|_2 \\ \leq &\mathbb{E}\| \sum_{n=0}^{n_0(\varepsilon)-1} (u_{\varepsilon,n} - U^n_{\varepsilon,s})(t)\|_2 + \mathbb{E}\| \sum_{n_0(\varepsilon)}^{+\infty} \hat{u}_{\varepsilon,n}(t)\|_2 + \|(U_{\varepsilon,s} - \mathcal{U})(t)\|_2 + \| \sum_{n_0(\varepsilon)}^{+\infty} U^n_{\varepsilon,s}(t)\|_2. \end{aligned} \tag{4.23}$$

The vanishing of the first three terms on right hand side of this inequality when ε goes to zero follows from (2.53), (3.35) and (4.22) respectively. The fourth term also vanishes because of the L^2 convergence of $U_{\varepsilon,s}$.

5 Inequalities and Proofs

In this section, we present and prove several inequalities used in earlier sections. There are similar versions of Lemma 5.2 and 5.7 in [6]. The proofs are given below for the convenience of the reader. The proofs of similar versions of Lemma 5.1 and 5.8 can be found in [6].

Lemma 5.1. *We have the following identity for* $\eta > 0$*:*

$$K(t, \boldsymbol{\xi}, I) = ie^{t\eta} \int d\alpha e^{-i\alpha t} \prod_{k \in I} \frac{1}{\alpha + \xi_k^{\mathfrak{m}} + i\eta}. \tag{5.1}$$

We also claim the following estimate with $n := |I| - 1$*:*

$$|K(t, \boldsymbol{\xi}, I)| \le \frac{t^n}{n!}. \tag{5.2}$$

Lemma 5.2. *Assume* $\eta > 0$*. We have the following inequality:*

$$\int_{-\infty}^{\infty} \frac{d\alpha}{|\alpha + A + i\eta||\alpha + B + i\eta|} \le \frac{C}{|A - B + i\eta|} \left[1 + \log_+ \left|\frac{A - B}{\eta}\right|\right], \tag{5.3}$$

where $\log_+ x := \max\{0, \log x\}$ *for* $x > 0$ *and* $\log_+ 0 := 0$*.*

Proof. Without loss of generality, we assume $B > A$. We split the integration over $(-\infty, -\frac{A+B}{2})$ as follows:

$$\begin{aligned}
&\int_{-\infty}^{-\frac{A+B}{2}} \frac{d\alpha}{|\alpha + A + i\eta||\alpha + B + i\eta|} \\
&= \int_{-B}^{-\frac{A+B}{2}} \frac{d\alpha}{|\alpha + A + i\eta||\alpha + B + i\eta|} + \int_{-B-(B-A)}^{-B} \frac{d\alpha}{|\alpha + A + i\eta||\alpha + B + i\eta|} \\
&+ \int_{-\infty}^{-B-(B-A)} \frac{d\alpha}{|\alpha + A + i\eta||\alpha + B + i\eta|}.
\end{aligned} \tag{5.4}$$

The first term is estimated as

$$\begin{aligned}
\int_{-B}^{-\frac{A+B}{2}} \frac{d\alpha}{|\alpha + A + i\eta||\alpha + B + i\eta|} &\le \frac{1}{|-\frac{A+B}{2} + A + i\eta|} \int_{-B}^{-\frac{A+B}{2}} \frac{d\alpha}{|\alpha + B + i\eta|} \\
&\le \frac{2}{|A - B + i\eta|} \int_0^{\frac{B-A}{2\eta}} \frac{d\alpha}{\sqrt{\alpha^2 + 1}} \\
&\le \frac{C}{|A - B + i\eta|} \left[1 + \log_+ \left|\frac{A - B}{\eta}\right|\right].
\end{aligned} \tag{5.5}$$

Likewise, the second term is estimated as

$$\begin{aligned}
\int_{-B-(B-A)}^{-B} \frac{d\alpha}{|\alpha + A + i\eta||\alpha + B + i\eta|} &\le \frac{1}{|A - B + i\eta|} \int_{-B-(B-A)}^{-B} \frac{d\alpha}{|\alpha + B + i\eta|} \\
&\le \frac{1}{|A - B + i\eta|} \int_0^{\frac{B-A}{\eta}} \frac{d\alpha}{\sqrt{\alpha^2 + 1}} \\
&\le \frac{C}{|A - B + i\eta|} \left[1 + \log_+ \left|\frac{A - B}{\eta}\right|\right].
\end{aligned} \tag{5.6}$$

We obtain the bound for the third term by using the inequality $|\alpha + A + i\eta| \leq |\alpha + B + i\eta|$ on $(-\infty, -B - (B - A))$.

$$\begin{aligned}\int_{-\infty}^{-B-(B-A)} \frac{d\alpha}{|\alpha + A + i\eta||\alpha + B + i\eta|} &\leq \int_{-\infty}^{-B-(B-A)} \frac{d\alpha}{|\alpha + B + i\eta|^2} \\ &= \frac{1}{\eta} \int_{\frac{B-A}{\eta}}^{+\infty} \frac{d\alpha}{\alpha^2 + 1}.\end{aligned} \tag{5.7}$$

If $B - A \geq \eta$, we have

$$\int_{\frac{B-A}{\eta}}^{+\infty} \frac{d\alpha}{\alpha^2 + 1} \leq \int_{\frac{B-A}{\eta}}^{+\infty} \frac{d\alpha}{\alpha^2} \leq \frac{\eta}{B - A}, \tag{5.8}$$

in which case

$$\int_{-\infty}^{-B-(B-A)} \frac{d\alpha}{|\alpha + A + i\eta||\alpha + B + i\eta|} \leq \frac{1}{B - A} \leq \frac{\sqrt{2}}{|A - B + i\eta|}. \tag{5.9}$$

If $B - A < \eta$, we have

$$\int_{\frac{B-A}{\eta}}^{+\infty} \frac{d\alpha}{\alpha^2 + 1} \leq \int_{1}^{+\infty} \frac{d\alpha}{\alpha^2 + 1} + \int_{\frac{B-A}{\eta}}^{1} \frac{d\alpha}{\alpha^2 + 1} \leq \frac{\pi}{4} + (1 - \frac{B - A}{\eta}), \tag{5.10}$$

in which case

$$\int_{-\infty}^{-B-(B-A)} \frac{d\alpha}{|\alpha + A + i\eta||\alpha + B + i\eta|} \leq \frac{1}{\eta}(\frac{\pi}{4} + 1) \leq \frac{\sqrt{2}(\pi/4 + 1)}{|A - B + i\eta|}. \tag{5.11}$$

By symmetry, the integration over $(-\infty, -\frac{A+B}{2})$ admits an identical bound. □

Proposition 5.3. *Assume $\eta, \tilde{\eta} > 0$, $\tilde{\eta} > 2\eta$, and η^{-1} bounded. We have the following inequality:*

$$\int\int \frac{1}{|\tilde{\alpha} - \alpha + i(\eta - \tilde{\eta})|} \frac{1}{|\tilde{\alpha} + A + i\eta|} \frac{1}{|\alpha + B + i\tilde{\eta}|} d\alpha d\tilde{\alpha} \leq \frac{C\left[1 + \log_+ \left|\frac{A-B}{\tilde{\eta}}\right|\right]^2}{|A - B + i\tilde{\eta}|}. \tag{5.12}$$

Proof. Without loss of generality, we assume $B > A$.
We first integrate in $\tilde{\alpha}$ by using Lemma 5.2:

$$\int \frac{1}{|\tilde{\alpha} - \alpha + i(\eta - \tilde{\eta})|} \frac{1}{|\tilde{\alpha} + A + i\eta|} d\tilde{\alpha} \leq \frac{1}{|\alpha + A + i\tilde{\eta}|} \left[1 + \log_+ \left|\frac{\alpha + A}{\tilde{\eta}}\right|\right]. \tag{5.13}$$

The integration over $(-\infty, -\frac{A+B}{2})$ is split as in (5.4) into three pieces: $(-B, -\frac{A+B}{2})$, $(-B - (B - A), -B)$ and $(-\infty, -B - (B - A))$.
The term $1 + \log_+ \left|\frac{\alpha+A}{\tilde{\eta}}\right|$ is bounded by $1 + \log_+ \left|\frac{A-B}{\tilde{\eta}}\right|$ on the interval $(-B, -\frac{A+B}{2})$. The integral over $(-B, -\frac{A+B}{2})$ is then bounded by

$$\int_{-B}^{-\frac{A+B}{2}} \frac{1 + \log_+ \left|\frac{\alpha+A}{\tilde{\eta}}\right|}{|\alpha + A + i\tilde{\eta}||\alpha + B + i\tilde{\eta}|} d\alpha \leq \frac{C}{|A - B + i\tilde{\eta}|} \left[1 + \log_+ \left|\frac{A - B}{\tilde{\eta}}\right|\right]^2. \tag{5.14}$$

The same bound holds for the integral over $(-B-(B-A), -B)$.
It remains to estimate the integral over $(-\infty, -B-(B-A))$. The domain of integration can be divided into two parts. On the set $|\alpha + A| < \tilde{\eta}$, we have $\log_+ \left|\frac{\alpha+A}{\tilde{\eta}}\right| = 0$, so that applying Lemma 5.2 immediately gives the desired result. On the set $|\alpha + A| \geq \tilde{\eta}$, we use the simple bound on this interval

$$1 + \log_+ \left|\frac{\alpha + A}{\tilde{\eta}}\right| \leq \left|\frac{\alpha + A}{\tilde{\eta}} + i\right|^{1/2}. \tag{5.15}$$

It then follows that

$$\begin{aligned}\int_{-\infty}^{-B-(B-A)} \frac{1 + \log_+ \left|\frac{\alpha+A}{\tilde{\eta}}\right|}{|\alpha + A + i\tilde{\eta}||\alpha + B + i\tilde{\eta}|} d\alpha &\leq C \int_{-\infty}^{-B-(B-A)} \frac{1}{|\alpha + B + i\tilde{\eta}|^{3/2}} d\alpha \\ &= C \int_{\frac{B-A}{\tilde{\eta}}}^{+\infty} \frac{d\alpha}{(\alpha^2+1)^{3/2}}.\end{aligned} \tag{5.16}$$

After performing an analysis similar to that from (5.8) to (5.11) we find the estimate:

$$\int_{-\infty}^{-B-(B-A)} \frac{1 + \log_+ \left|\frac{\alpha+A}{\tilde{\eta}}\right|}{|\alpha + A + i\tilde{\eta}||\alpha + B + i\tilde{\eta}|} d\alpha \leq \frac{C}{|A - B + i\tilde{\eta}|} \left[1 + \log_+ \left|\frac{A-B}{\tilde{\eta}}\right|\right], \tag{5.17}$$

which concludes the proof. □

Lemma 5.4. *Assume $\eta > 0$ and η^{-1} is bounded. We have the following inequality:*

$$\int \varepsilon^{\mathfrak{m}} \frac{1}{\langle \xi \rangle^{2d}} \frac{1}{|\varepsilon^{\mathfrak{m}}\alpha + \xi^{\mathfrak{m}} + i\varepsilon^{\mathfrak{m}}\eta|^2} d\xi \leq C\varepsilon^{\lambda}(\alpha^{\frac{\lambda}{\mathfrak{m}}} \vee 1). \tag{5.18}$$

Proof. The domain of integration may be split into two parts. On the set $|\varepsilon^{\mathfrak{m}}\alpha + \xi^{\mathfrak{m}}| \geq 1$, the integral is bounded by $\varepsilon^{\mathfrak{m}} \|1/\langle \xi \rangle^{2d}\|_1$. On the set $|\varepsilon^{\mathfrak{m}}\alpha + \xi^{\mathfrak{m}}| < 1$, we change to polar coordinates

$$\int_{|\varepsilon^{\mathfrak{m}}\alpha + \xi^{\mathfrak{m}}|<1} \varepsilon^{\mathfrak{m}} \frac{1}{\langle \xi \rangle^{2d}} \frac{1}{|\epsilon^{\mathfrak{m}}\alpha + \xi^{\mathfrak{m}} + i\varepsilon^{\mathfrak{m}}\eta|^2} d\xi \leq C \int_{|\varepsilon^{\mathfrak{m}}\alpha + \xi^{\mathfrak{m}}|<1} \varepsilon^{\mathfrak{m}} \frac{1}{\langle \xi \rangle^{2d}} \frac{\xi^{d-1}}{|\epsilon^{\mathfrak{m}}\alpha + \xi^{\mathfrak{m}} + i\varepsilon^{\mathfrak{m}}\eta|^2} d|\xi|. \tag{5.19}$$

When $\mathfrak{m} < d < 2\mathfrak{m}$, we use the inequality $|1/\langle \xi \rangle^{2d}| \leq 1$ and change variables to $Q = \varepsilon^{\mathfrak{m}}\alpha + \xi^{\mathfrak{m}}$

$$\int_{|\varepsilon^{\mathfrak{m}}\alpha + \xi^{\mathfrak{m}}|<1} \varepsilon^{\mathfrak{m}} \frac{\xi^{d-1}}{|\epsilon^{\mathfrak{m}}\alpha + \xi^{\mathfrak{m}} + i\varepsilon^{\mathfrak{m}}\eta|^2} d|\xi| = \int_0^1 \varepsilon^{\mathfrak{m}} \frac{(Q - \varepsilon^{\mathfrak{m}}\alpha)^{\frac{d}{\mathfrak{m}}-1}}{|Q + i\varepsilon^{\mathfrak{m}}\eta|^2} dQ. \tag{5.20}$$

On the set $[0, \varepsilon^{\mathfrak{m}}\eta]$, we bound $|Q + i\varepsilon^{\mathfrak{m}}\eta|$ from below by its imaginary part and estimate the integral

$$\int_0^{\varepsilon^{\mathfrak{m}}\eta} \varepsilon^{\mathfrak{m}} \frac{(Q - \varepsilon^{\mathfrak{m}}\alpha)^{\frac{d}{\mathfrak{m}}-1}}{|\varepsilon^{\mathfrak{m}}\eta|^2} dQ \leq \int_0^{\varepsilon^{\mathfrak{m}}\eta} \varepsilon^{\mathfrak{m}} \frac{Q^{\frac{d}{\mathfrak{m}}-1} + (\varepsilon^{\mathfrak{m}}\alpha)^{\frac{d}{\mathfrak{m}}-1}}{|\varepsilon^{\mathfrak{m}}\eta|^2} dQ \leq C\varepsilon^{d-\mathfrak{m}}(\alpha \vee 1)^{\frac{d}{\mathfrak{m}}-1}. \tag{5.21}$$

On the set $[\varepsilon^{\mathfrak{m}}\eta, 1]$, we bound $|Q+i\varepsilon^{\mathfrak{m}}\eta|$ from below by its real part and estimate the integral

$$\int_0^{\varepsilon^{\mathfrak{m}}\eta} \varepsilon^{\mathfrak{m}} \frac{(Q-\varepsilon^{\mathfrak{m}}\alpha)^{\frac{d}{\mathfrak{m}}-1}}{Q^2} dQ \leq \int_0^{\varepsilon^{\mathfrak{m}}\eta} \varepsilon^{\mathfrak{m}} \frac{Q^{\frac{d}{\mathfrak{m}}-1} + (\varepsilon^{\mathfrak{m}}\alpha)^{\frac{d}{\mathfrak{m}}-1}}{Q^2} dQ \leq C\varepsilon^{d-\mathfrak{m}}(\alpha \vee 1)^{\frac{d}{\mathfrak{m}}-1}. \quad (5.22)$$

This concludes the proof for $\mathfrak{m} < d < 2\mathfrak{m}$. When $d \geq 2\mathfrak{m}$, we use the inequality $|\xi^{d-2\mathfrak{m}}/\langle\xi\rangle^{2d}| \leq 1$ instead of $|1/\langle\xi\rangle^{2d}| \leq 1$ and the rest of the proof is similar. □

Proposition 5.5. *Assume $\eta > 0$ and η^{-1} is bounded. We have the following inequality:*

$$\int \varepsilon^{\mathfrak{m}} \frac{1}{\langle\xi\rangle^{2d}} \frac{1}{|\varepsilon^{\mathfrak{m}}\alpha + \xi^{\mathfrak{m}} + i\varepsilon^{\mathfrak{m}}\eta|^2} d\xi \leq C\varepsilon^{\lambda(1-\delta)}(\alpha^{\frac{\lambda(1-\delta)}{\mathfrak{m}}} \vee 1). \quad (5.23)$$

Proof. To prove (5.23), we just need to use $1/\langle\xi\rangle^{2d}$ to bound an additional $\xi^{(d-\mathfrak{m})\delta}$ and the rest of the proof is the same as in Lemma 5.4. As a result, we lose a contribution $\varepsilon^{(d-\mathfrak{m})\delta}$ for lowering the order of α by $\frac{\beta\delta}{\mathfrak{m}}$. □

Proposition 5.6. *Assume $\eta > 0$, η^{-1} bounded, and $k \in \mathbb{N}$. We have the following inequality:*

$$\int \varepsilon^{\mathfrak{m}} \frac{1}{\langle\xi\rangle^{2d}} \frac{\left[1 + \log_+ \left|\frac{\alpha+\xi^{\mathfrak{m}}/\varepsilon^{\mathfrak{m}}}{\eta}\right|\right]^k}{|\varepsilon^{\mathfrak{m}}\alpha + \xi^{\mathfrak{m}} + i\varepsilon^{\mathfrak{m}}\eta|^2} d\xi \leq C\varepsilon^{\lambda}(\alpha \vee 1)^{\frac{\lambda}{\mathfrak{m}}}. \quad (5.24)$$

Proof. The integral on the set $\left|\frac{\alpha+\xi^{\mathfrak{m}}/\varepsilon^{\mathfrak{m}}}{\eta}\right| < 1$ is immediately bounded by $C\varepsilon^{\lambda}\alpha^{\frac{\lambda}{\mathfrak{m}}}$ using Lemma 5.4. The integral on the set $\left|\frac{\alpha+\xi^{\mathfrak{m}}/\varepsilon^{\mathfrak{m}}}{\eta}\right| \geq 1$ is bounded by $C\varepsilon^{\mathfrak{m}}\|1/\langle\xi\rangle^{2d}\|_1$. □

Lemma 5.7. *Assume $\eta > 0$. We have the following inequalities:*

$$\sup_{\omega} \int \frac{1}{|\alpha + \xi^{\mathfrak{m}} + i\eta|} \frac{1}{\langle\xi-\omega\rangle^{2d}} \frac{1}{\langle\xi\rangle^{2d}} d\xi \leq C\frac{|\log\eta|}{\langle\alpha\rangle}, \quad (5.25)$$

where the $\langle\alpha\rangle$ in the denominator again has to be dropped if we remove the factor $\langle\xi\rangle^{2d}$.

Proof. We will first show that

$$\sup_{\omega} \int \frac{1}{|\alpha + \xi^{\mathfrak{m}} + i\eta|} \frac{1}{\langle\xi-\omega\rangle^{2d}} d\xi \leq C|\log\eta|. \quad (5.26)$$

If $\eta \geq 1$, the integral is bounded by

$$\int \frac{1}{|\alpha + \xi^{\mathfrak{m}} + i\eta|} \frac{1}{\langle\xi-\omega\rangle^{2d}} d\xi \leq \frac{1}{\eta} \int \frac{1}{\langle\xi-\omega\rangle^{2d}} d\xi \leq \frac{C}{\eta} \leq C|\log\eta|. \quad (5.27)$$

Therefore, we only need to look at the case for $\eta < 1$. The integral over $|\alpha + \xi^{\mathfrak{m}}| \geq 1$ is bounded by

$$\int_{|\alpha+\xi^{\mathfrak{m}}|\geq 1} \frac{1}{|\alpha + \xi^{\mathfrak{m}} + i\eta|} \frac{1}{\langle\xi-\omega\rangle^{2d}} d\xi \leq C \int \frac{1}{\langle\xi-\omega\rangle^{2d}} d|\xi| \leq C \leq C|\log\eta|. \quad (5.28)$$

The integral over $|\alpha+\xi^{\mathfrak{m}}|<1$ can be estimated by splitting the integration domain according to the size $k \le |\xi-\omega| \le k+1$ for $k=0,1,\cdots$

$$\begin{aligned}
&\int_{|\alpha+\xi^{\mathfrak{m}}|<1} \frac{1}{|\alpha+\xi^{\mathfrak{m}}+i\eta|} \frac{1}{\langle \xi-\omega\rangle^{2d}} d\xi \\
&= \sum_{k=0}^{+\infty} \int_{\{|\alpha+\xi^{\mathfrak{m}}|<1\}\cap\{k\le|\xi-\omega|\le k+1\}} \frac{1}{|\alpha+\xi^{\mathfrak{m}}+i\eta|} \frac{1}{\langle \xi-\omega\rangle^{2d}} d\xi \\
&= \sum_{k=0}^{+\infty} \int_{|\alpha+\xi^{\mathfrak{m}}|<1} \frac{d|\xi|}{|\alpha+\xi^{\mathfrak{m}}+i\eta|} \int_{k\le|\xi-\omega|\le k+1} \frac{J(|\xi|,\theta_1,\cdots,\theta_{d-1})}{\langle \xi-\omega\rangle^{2d}} d\theta_1\cdots d\theta_{d-1},
\end{aligned} \tag{5.29}$$

where $J(|\xi|,\theta_1,\cdots,\theta_{d-1})$ denotes the Jacobian to change to polar coordinates. For fixed $|\xi|$, we have the inequality

$$\begin{aligned}
\int_{k\le|\xi-\omega|\le k+1} \frac{J(|\xi|,\theta_1,\cdots,\theta_{d-1})}{\langle \xi-\omega\rangle^{2d}} d\theta_1\cdots d\theta_{d-1} &\le \int_{k\le|\xi-\omega|\le k+1} \frac{J(|\xi|,\theta_1,\cdots,\theta_{d-1})}{(1+k^2)^d} d\theta_1\cdots d\theta_{d-1} \\
&\le \begin{cases} C\frac{(k+1)^{d-1}}{(1+k^2)^d}|\xi|^{\mathfrak{m}-1}, & |\xi|\ge 1 \\ C\frac{1}{(1+k^2)^d}|\xi|^{\mathfrak{m}-1}, & |\xi|<1. \end{cases}
\end{aligned} \tag{5.30}$$

Summing up the terms over $k=0,1,\cdots$ gives

$$\int_{|\alpha+\xi^{\mathfrak{m}}|<1} \frac{1}{|\alpha+\xi^{\mathfrak{m}}+i\eta|} \frac{1}{\langle \xi-\omega\rangle^{2d}} d\xi \le C\int_{|\alpha+\xi^{\mathfrak{m}}|<1} \frac{|\xi|^{\mathfrak{m}-1}d|\xi|}{|\alpha+\xi^{\mathfrak{m}}+i\eta|} \le C|\log\eta|. \tag{5.31}$$

We now prove (5.25). Without loss of generality, we may assume $|\alpha|\ge 2$.
The integral over the domain $|\alpha+\xi^{\mathfrak{m}}|\ge \alpha/2$ is easily bounded by

$$\begin{aligned}
\int_{|\alpha+\xi^{\mathfrak{m}}|\le\frac{|\alpha|}{2}} \frac{1}{|\alpha+\xi^{\mathfrak{m}}+i\eta|} \frac{1}{\langle \xi-\omega\rangle^{2d}} \frac{1}{\langle\xi\rangle^{2d}} d\xi &\le \frac{C}{|\alpha|} \int_{|\alpha+\xi^{\mathfrak{m}}|\le\frac{|\alpha|}{2}} \frac{1}{\langle \xi-\omega\rangle^{2d}} \frac{1}{\langle\xi\rangle^{2d}} d\xi \\
&\le \frac{C}{|\alpha|} \le \frac{C|\log\eta|}{\langle\alpha\rangle}.
\end{aligned} \tag{5.32}$$

On the domain $|\alpha+\xi^{\mathfrak{m}}|\le|\alpha|/2$, we have $\xi^{\mathfrak{m}}\ge|\alpha|/2$. Note also that $\langle\xi\rangle^{2d}\ge\langle\xi^{\mathfrak{m}}\rangle$. The integral over this domain is therefore bounded by

$$\begin{aligned}
\int_{|\alpha+\xi^{\mathfrak{m}}|\le\frac{|\alpha|}{2}} \frac{1}{|\alpha+\xi^{\mathfrak{m}}+i\eta|} \frac{1}{\langle \xi-\omega\rangle^{2d}} \frac{1}{\langle\xi\rangle^{2d}} d\xi &\le \frac{C}{\langle\alpha\rangle} \int_{|\alpha+\xi^{\mathfrak{m}}|\le\frac{|\alpha|}{2}} \frac{1}{|\alpha+\xi^{\mathfrak{m}}+i\eta|} \frac{1}{\langle \xi-\omega\rangle^{2d}} d\xi \\
&\le \frac{C|\log\eta|}{\langle\alpha\rangle}.
\end{aligned} \tag{5.33}$$

□

Lemma 5.8. *Let*

$$(Y_z\hat{R})(\xi) := \int \frac{\hat{R}(\xi-y)}{z-\xi^{\mathfrak{m}}} d\xi \tag{5.34}$$

be a family of linear operators parametrized by a complex parameter $z = \alpha + i\eta$ *with* $\eta > 0$.
(i) For $d \geq 3$ *we have*

$$|Y_z \hat{R}| \leq C \|\hat{R}\|_{2d,2d}. \tag{5.35}$$

(ii) If $z' = \alpha' + i\eta'$ *and* $\eta \geq \eta'$, *then for* $d \geq 3$

$$|Y_z \hat{R} - Y_{z'} \hat{R}| \leq C|z - z'|\eta^{-(1-\frac{1}{\mathfrak{m}})}\|\hat{R}\|_{2d,2d}. \tag{5.36}$$

Lemma 5.9. *Let* $\hat{R}$ *satisfy the smoothness condition defined in Section 1. We have the following inequality:*

$$\varepsilon^{\mathfrak{m}} \int (\frac{1}{\xi_1^{\mathfrak{m}} - \varepsilon^{\mathfrak{m}}\xi^{\mathfrak{m}}} \wedge \frac{1}{\varepsilon^{\mathfrak{m}}}) \frac{\hat{R}(\xi_1 - \varepsilon\xi)}{\xi_1^{\mathfrak{m}}} d\xi_1 \leq C \max\{\varepsilon^{\mathfrak{m}}, \varepsilon^{d-\mathfrak{m}}|\log\varepsilon|(\xi^{\mathfrak{m}}+1)^{\frac{d}{\mathfrak{m}}-2}, \varepsilon^{d-\mathfrak{m}}(\xi^{\mathfrak{m}}+1)^{\frac{d}{\mathfrak{m}}-1}\}. \tag{5.37}$$

Proof. We decompose the integral into three integrals on $\{\xi_1^{\mathfrak{m}} \geq \varepsilon^{\mathfrak{m}}\xi^{\mathfrak{m}} + 1\}$, $\{\varepsilon^{\mathfrak{m}}(\xi^{\mathfrak{m}} + 1) \leq \xi_1^{\mathfrak{m}} \leq \varepsilon^{\mathfrak{m}}\xi^{\mathfrak{m}} + 1\}$ and $\{0 \leq \xi_1^{\mathfrak{m}} \leq \varepsilon^{\mathfrak{m}}(\xi^{\mathfrak{m}} + 1)\}$. Clearly, the first integral in bounded by $\varepsilon^{\mathfrak{m}} \int \frac{\hat{R}(\xi_1-\varepsilon\xi)}{\xi^{\mathfrak{m}}} d\xi_1$. For the second integral, note that $\hat{R}$ is bounded, and use $\frac{1}{\xi_1^{\mathfrak{m}}-\varepsilon^{\mathfrak{m}}\xi^{\mathfrak{m}}}$ to bound the term in the bracket on the left hand side of (5.37). We then change variable to polar coordinates and let $Q = \xi_1^{\mathfrak{m}}$ to obtain

$$\begin{aligned} \varepsilon^{\mathfrak{m}} \int_{\{\varepsilon^{\mathfrak{m}}(\xi^{\mathfrak{m}}+1) \leq \xi_1^{\mathfrak{m}} \leq \varepsilon^{\mathfrak{m}}\xi^{\mathfrak{m}}+1\}} (\frac{1}{\xi_1^{\mathfrak{m}} - \varepsilon^{\mathfrak{m}}\xi^{\mathfrak{m}}} \wedge \frac{1}{\varepsilon^{\mathfrak{m}}}) \frac{\hat{R}(\xi_1 - \varepsilon\xi)}{\xi_1^{\mathfrak{m}}} d\xi_1 &\leq C\varepsilon^{\mathfrak{m}} \int_{\varepsilon^{\mathfrak{m}}(\xi^{\mathfrak{m}}+1)}^{\varepsilon^{\mathfrak{m}}\xi^{\mathfrak{m}}+1} \frac{Q^{\frac{d}{\mathfrak{m}}-2}}{Q - \varepsilon^{\mathfrak{m}}\xi^{\mathfrak{m}}} dQ \\ &\leq C\varepsilon^{d-\mathfrak{m}}|\log\varepsilon|(\xi^{\mathfrak{m}} + 1)^{\frac{d}{\mathfrak{m}}-2}. \end{aligned} \tag{5.38}$$

For the third part, we use $1/\varepsilon^{\mathfrak{m}}$ to bound the term in the bracket on the left hand side of (5.37) and apply the same type of change of variable to obtain

$$\begin{aligned} \varepsilon^{\mathfrak{m}} \int_{\{0 \leq \xi_1^{\mathfrak{m}} \leq \varepsilon^{\mathfrak{m}}(\xi^{\mathfrak{m}}+1)\}} (\frac{1}{\xi_1^{\mathfrak{m}} - \varepsilon^{\mathfrak{m}}\xi^{\mathfrak{m}}} \wedge \frac{t}{\varepsilon^{\mathfrak{m}}}) \frac{\hat{R}(\xi_1 - \varepsilon\xi)}{\xi_1^{\mathfrak{m}}} d\xi_1 &\leq t \int_0^{\varepsilon^{\mathfrak{m}}(\xi^{\mathfrak{m}}+1)} Q^{\frac{d}{\mathfrak{m}}-2} dQ \\ &\leq C\varepsilon^{d-\mathfrak{m}}(\xi^{\mathfrak{m}} + 1)^{\frac{d}{\mathfrak{m}}-1}. \end{aligned} \tag{5.39}$$

□

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