# Inverse source problems in transport equations

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**Abstract.** This paper proposes an iterative technique to reconstruct the source term in transport equations, which account for scattering effects, from boundary measurements. In the two-dimensional setting, the full outgoing distribution in the phase space (position and direction) needs to be measured. In three space dimensions, we show that measurements for angles that are orthogonal to a given direction are sufficient. In both cases, the derivation is based on a perturbation of the inversion of the two-dimensional attenuated Radon transform, and requires that (the anisotropic part of) scattering be sufficiently small. We present an explicit iterative procedure, which converges to the source term we want to reconstruct. Applications of the inversion procedure include optical molecular imaging, an increasingly popular medical imaging modality.

## 1. Introduction

Optical molecular imaging (OMI) is being increasingly studied as a powerful detection method in medical imaging. New biochemical markers are currently being engineered to attach to specific molecules and thus be used to detect faulty genes and other molecular processes, which precede the development of certain diseases. This makes possible the detection of such diseases long before phenotypical symptoms appear. In optical molecular imaging, the markers are light-emitting molecules, such as fluorophores or luminophores. Compared to other molecular imaging techniques, such as single photon emission tomography (SPECT) or positron emission tomography (PET), optical markers emit low-energy near-infrared photons that are relatively harmless to the human body. Other advantages are their high sensitivity to oxygen levels, metal ion concentrations, pH, lipid composition, for instance; see [30, 31, 38, 39] for recent references in the biomedical literature.

The inverse problem consists of reconstructing the spatial distribution of the markers from measurements of light intensities at the boundary of the object we wish to image. Two main types of markers are used in OMI, namely bioluminescent and fluorescent markers. In both cases, the propagation in human tissues of the photons emitted by the markers can quite satisfactorily be modeled as inverse source problems of time-harmonic and steady-state radiative transfer equations [11, 20, 40]. To simplify the presentation, we only consider the steady-state problem here, for which relatively few results exist in the mathematical literature.

Our main result consists in providing an explicit (and converging) iterative technique to reconstruct the source term from boundary measurements of the photon intensity in the phase space, i.e., as a function of position and angular direction. We consider the two-dimensional and the three-dimensional settings. In both cases, we have to assume that the *anisotropic part* of scattering is sufficiently regular and small (in the sense that a certain operator linear in the anisotropic part of the scattering term must have norm bounded by one in appropriate spaces). In three dimensions, we show that measurements of the photon intensity for directions orthogonal to an arbitrary given vector is sufficient. Both results are based on perturbations of the Novikov inversion formula to invert the attenuated Radon transform; see [2, 6, 8, 9, 14, 15, 19, 25, 28, 29] for some references on that problem, and we thus show that the Novikov inversion formula is stable under perturbations by a scattering operator. How small scattering has to be in terms of the absorbing and geometric properties of the domain is somewhat characterized in Corollary 3.7.

Several imaging techniques such as SPECT and PET are based on the inversion of the Radon transform or the attenuated Radon transform. Because optical markers emit low-energy light, the photons scatter before they are measured. This renders the inversion more difficult than in the higher energy methods SPECT and PET and necessitates the use of transport equations that account for scattering effects. For earlier works on the inverse source problem of transport equations based on different methods, we refer to [1, 21, 32, 34, 35]. In this paper, we are interested only in the source reconstruction and assume that the absorption and scattering coefficients are known; see [3, 4, 5, 12, 18, 22, 23, 36, 37] for references on the determination of these parameters.

The iterative procedure presented here will not work in the highly scattering regime (unless that scattering is fully isotropic), in which case the diffusion approximation should be used [3]. It should be mostly effective in situations where scattering needs to be accounted for to obtain a desired accuracy in the reconstruction, and yet is not too strong for a method based on a perturbation of a non-scattering inversion technique to converge. Practically, we expect this situation to arise in OMI in small domains (on the order of $5-10$ mean free paths) such as small animals, and in SPECT and PET where moderate scattering is accounted for.

The rest of the paper is organized as follows. Section 2 introduces the inverse source problem in transport equations and presents our main results. The derivation of the results is postponed to section 3 for the two-dimensional case and section 4 for the three-dimensional extension.

## 2. An inverse source problem

The distribution of photons emitted by the markers is denoted by $f(\mathbf{x})$, where position $\mathbf{x} \in \Omega \subset \mathbb{R}^d$. Here $\Omega$ is a bounded open convex domain and $d = 2, 3$ is the space dimension. We normalize the light speed to unity and denote by $\boldsymbol{\theta} \in S^{d-1}$ the direction of the photons. Notice that $d = 3$ is the physical model, whereas $d = 2$ is not physical as photons are only allowed to travel in a two-dimensional plane.

Let $u(\mathbf{x}, \boldsymbol{\theta})$ be the density of photons at position $x$ moving in the direction $\boldsymbol{\theta}$ and let

$$\Gamma_\pm = \{(\mathbf{x}, \boldsymbol{\theta}) \in \partial\Omega \times S^{d-1}, \quad \pm\boldsymbol{\theta} \cdot \mathbf{n}(\mathbf{x}) > 0\}, \tag{1}$$

denote the boundary spaces. Here $\mathbf{n}(\mathbf{x})$ is the outward normal to $\Omega$ at $\mathbf{x} \in \partial\Omega$. The density of particles satisfies the radiation transfer (transport) equation

$$\begin{aligned} &\boldsymbol{\theta} \cdot \nabla_{\mathbf{x}} u(\mathbf{x}, \boldsymbol{\theta}) + a(\mathbf{x}) u(\mathbf{x}, \boldsymbol{\theta}) = Ku(\mathbf{x}, \boldsymbol{\theta}) + f(\mathbf{x}), \quad \text{ in } \Omega \times S^{d-1} \\ &u(\mathbf{x}, \boldsymbol{\theta}) = 0, \quad \text{ on } \Gamma_-, \end{aligned} \tag{2}$$

where the measure $d\boldsymbol{\theta}$ is the usual surface measure on the unit sphere normalized such that $\int_{S^{d-1}} d\boldsymbol{\theta} = 1$. Photon interaction with the underlying medium is modeled by an absorption parameter $a(\mathbf{x})$ and a scattering operator

$$Ku(\mathbf{x}, \boldsymbol{\theta}) = \int_{S^{d-1}} k(\mathbf{x}, \boldsymbol{\theta} \cdot \boldsymbol{\theta}') u(\mathbf{x}, \boldsymbol{\theta}') d\boldsymbol{\theta}', \tag{3}$$

where $k(\mathbf{x}, \mu)$ is the scattering coefficient. Both the absorption and scattering coefficients are assumed to be non-negative, sufficiently smooth functions such that

$$a(\mathbf{x}) - \int_{S^{d-1}} k(\mathbf{x}, \boldsymbol{\theta} \cdot \boldsymbol{\theta}') d\boldsymbol{\theta}' \geq \delta > 0, \tag{4}$$

for some positive constant $\delta$. This sub-criticality condition ensures that the above problem is well-posed in $L^2(\Omega \times S^{d-1})$ provided that the source term $f(\mathbf{x}) \in L^2(\Omega)$; see [13] for instance. Moreover, the outgoing photon distribution, defined as the trace of $u(\mathbf{x}, \boldsymbol{\theta})$ on $\Gamma_+$ is well-defined and belongs to $L^2_{\boldsymbol{\theta}\cdot\mathbf{n}}(\Gamma_+)$, in the sense that $\int_{\Gamma_+} \boldsymbol{\theta}\cdot\mathbf{n}\, u^2(\mathbf{x}, \boldsymbol{\theta}) d\sigma(\mathbf{x}) d\boldsymbol{\theta} < \infty$, where $d\sigma$ is the surface measure on $\partial\Omega$. Further regularity and smallness assumptions on $k$ will be stated within the theorems. For references on the mathematical theory of the transport equation (2), see for instance [10, 13, 24].

It is convenient in the analysis to have unbounded spatial domains. We extended $f(\mathbf{x})$ and $k(\mathbf{x}, \mu)$ by 0 on $\mathbb{R}^d \backslash \Omega$ and extend $a(\mathbf{x})$ on $\mathbb{R}^d$ by preserving its smoothness and compact support. The transport equation is now recast as

$$\begin{aligned} &\boldsymbol{\theta} \cdot \nabla_{\mathbf{x}} u(\mathbf{x}, \boldsymbol{\theta}) + a(\mathbf{x}) u(\mathbf{x}, \boldsymbol{\theta}) = Ku(\mathbf{x}, \boldsymbol{\theta}) + f(\mathbf{x}), \quad \text{ in } \mathbb{R}^d \times S^{d-1} \\ &\lim_{t\to\infty} u(\mathbf{x} - t\boldsymbol{\theta}, \boldsymbol{\theta}) = 0, \quad \text{ on } \mathbb{R}^d \times S^{d-1}. \end{aligned} \tag{5}$$

The restriction of the above solution on $\Omega \times S^{d-1}$ clearly solves (2).

Our main results are that in dimension $d = 2$, knowledge of

$$m(s, \boldsymbol{\theta}) = \lim_{t\to\infty} u(t\boldsymbol{\theta} + s\boldsymbol{\theta}^\perp, \boldsymbol{\theta}), \tag{6}$$

on $\mathbb{R} \times S^1$ uniquely determines $f(\mathbf{x})$ compactly supported on the bounded domain $\Omega$ provided that the scattering kernel $k(\mathbf{x}, \mu)$ is sufficiently small. Moreover the reconstruction is explicit, in the sense that $f(\mathbf{x})$ is obtained as the limit of a converging Neumann series expansion.

In three dimensions, $d = 3$, the above result generalizes as follows. Let an arbitrary vector in $\mathbb{R}^3$ be given, which after possible rotation of $\Omega$ we denote $\mathbf{e}_z$. For $\boldsymbol{\theta} = (\cos\theta, \sin\theta, 0)$, we define $\boldsymbol{\theta}^\perp = (-\sin\theta, \cos\theta, 0)$. Then knowledge of

$$m(z, s, \theta) = \lim_{t\to\infty} u(t\boldsymbol{\theta} + s\boldsymbol{\theta}^\perp + z\mathbf{e}_z, \boldsymbol{\theta}), \tag{7}$$

for $(z, s, \theta) \in \mathbb{R} \times \mathbb{R} \times (0, 2\pi)$ uniquely determines $f(\mathbf{x})$ compactly supported on the bounded domain $\Omega$. This result also requires that $k(\mathbf{x}, \mu)$ be sufficiently small in an appropriate sense and the reconstruction is explicit in the sense mentioned above. This implies that the outgoing measurements are known only for directions orthogonal to $\mathbf{e}_z$. Note that in both cases, the problem is formally determined since both the measurements as well as the unknown source term are $d-$dimensional.

To state the regularity and smallness assumption of the scattering, we introduce the following notation. When $d = 2$, we identify $k(\mathbf{x}, \boldsymbol{\theta}\cdot\boldsymbol{\theta}') = k(\mathbf{x}, \cos(\theta - \theta')) = \tilde{k}(\mathbf{x}, \theta - \theta')$ and define the Fourier coefficients $k_n(\mathbf{x})$ by

$$k_n(\mathbf{x}) = \frac{1}{2\pi}\int_0^{2\pi} \tilde{k}(\mathbf{x}, \theta)e^{-in\theta}d\theta. \tag{8}$$

By $\hat{k}_n(\boldsymbol{\xi})$, we mean its Fourier transform $\hat{k}_n(\boldsymbol{\xi}) = \int_{\mathbb{R}^2} e^{-i\boldsymbol{\xi}\cdot\mathbf{x}}k_n(\mathbf{x})d\mathbf{x}$.

When $d = 3$, we use the Legendre polynomials expansion in $L^2[-1, 1]$:

$$k(\mathbf{x}, t) = \sum_{n=0}^{\infty} k_n(\mathbf{x})P_n(t). \tag{9}$$

By $\hat{k}_n(\boldsymbol{\xi}', z)$, we mean the restricted Fourier transform to the horizontal plane $\hat{k}_n(\boldsymbol{\xi}, z) = \int_{\mathbb{R}^2} e^{-i\mathbf{x}'\cdot\boldsymbol{\xi}'}k_n(\mathbf{x}', z)d\mathbf{x}'$. For $\boldsymbol{\theta} = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi) \in S^2$, $0 \le \theta < 2\pi$, $0 \le \phi < \pi$ let

$$Y_{nm}(\boldsymbol{\theta}) = C_{nm}^{1/2}e^{im\theta}P_n^m(\cos\phi) \tag{10}$$

denote the spherical harmonics on the sphere $S^2$. Here, $P_n^m$ are the associated Legendre polynomials and $C_{nm} = (2n+1)(n-m)!/(n+m)!$; see [17] for details. We only need to consider horizontal directions

$$\boldsymbol{\theta} \in S_H^2 = \{\boldsymbol{\theta} \in S^2 : \boldsymbol{\theta}\cdot e_z = 0\}. \tag{11}$$

We are ready to formulate our main results, whose proof is postponed to the following sections. The decay uses the usual notation $\langle n\rangle = (1 + |n|^2)^{1/2}$.

**Theorem 2.1 (two-dimensional case)** *Let $f(\mathbf{x}) \in L^2(\mathbb{R}^2)$ be a source term of compact support in $\Omega$ and $a(\mathbf{x})$ a sufficiently smooth absorption coefficient of compact* *support. Then there exists $\epsilon > 0$ depending on the size of the support and on the smoothness of $a$ such that, for a scattering coefficient $k$ with*

$$\max_{n\in\mathbb{Z}} \langle n\rangle^{\alpha} \|\hat{k}_n\|^2_{L^1(\mathbb{R}^2)} < \epsilon, \tag{12}$$

*for some $\alpha > 1$, the measurements $m(s,\theta)$ in* (6) *uniquely determine the source term $f(\mathbf{x})$. Moreover, the source term $f(\mathbf{x})$ can be obtained as the limit of the explicit convergent Neumann series in* (44) *below; see also Remark 3.12. A more explicit expression for $\epsilon$ in* (12) *can be found in Corollary 3.7 below.*

**Theorem 2.2 (three-dimensional case)** *Let $f(\mathbf{x}) \in L^2(\mathbb{R}^3)$ be a source term of compact support and $a(\mathbf{x})$ a sufficiently smooth absorption of compact support. Then there exist an $\epsilon > 0$ depending on the size of the support and on the smoothness of $a$ such that, for a scattering kernel $k$ with*

$$\max_{n\in\mathbb{N}} \left( \langle n\rangle^{\alpha-1} \max_{|m|\le n} \max_{\boldsymbol{\theta}\in S^2_H} |Y_{nm}(\boldsymbol{\theta})|^2 \int_{\mathbb{R}} \|\hat{k}_n(\cdot,z)\|^2_{L^1(\mathbb{R}^2)} dz \right) \le \epsilon \tag{13}$$

*for some $\alpha > 1$, the measurements $m(z,s,\theta)$ in* (7) *uniquely determine the source term $f(\mathbf{x})$. Moreover, the source term $f(\mathbf{x})$ can be obtained as the limit of the convergent Neumann series expansion in* (66) *below.*

Note that each theorem requires smallness as well as (weak) smoothness on the scattering kernel $k$. That $k$ is not arbitrary is already apparent in the existence theory for the forward problem, where we have assumed (4). The $L^1$-norm in the Fourier variables implies continuity of the scattering in the horizontal plane. The Sobolev-type decay property implies smoothness in the angular variable. For instance, $\alpha > 2$ in (12) already implies continuity of $k$ in $\theta$. The size of $\epsilon$ depends on the (operator) norm of the operator $N_K$ introduced below and is not explicit. This restriction is the price to pay to obtain a reconstruction as a perturbation of the inversion of the attenuated Radon transform, where there is no scattering.

Let us conclude this section by noting, as was mentioned in the introduction, that only the *anisotropic* part of the scattering need be small:

**Corollary 2.3** *The results stated in Theorems 2.1 and 2.2 are still valid when* (12) *and* (13)*, respectively, hold only for $n \neq 0$.*

*Proof.* Indeed, let us decompose $K = K_0 + K_1$, where $K_0u(\mathbf{x}) = k_0(\mathbf{x}) \int_{S^{d-1}} u(\mathbf{x},\boldsymbol{\theta}')d\boldsymbol{\theta}'$ is the isotropic part of $K$, and $K_1 = K - K_0$ the anisotropic part. We can then define the source term

$$F(\mathbf{x}) = f(\mathbf{x}) + K_0 u(\mathbf{x}), \tag{14}$$

where $u(\mathbf{x},\boldsymbol{\theta})$ is the solution to (5). We then verify that $u$ also solves

$$\begin{aligned} &\boldsymbol{\theta}\cdot\nabla_{\mathbf{x}} u(\mathbf{x},\boldsymbol{\theta}) + a(\mathbf{x})u(\mathbf{x},\boldsymbol{\theta}) = K_1 u(\mathbf{x},\boldsymbol{\theta}) + F(\mathbf{x}), \quad \text{in } \mathbb{R}^d\times S^{d-1} \\ &\lim_{t\to\infty} u(\mathbf{x}-t\boldsymbol{\theta},\boldsymbol{\theta}) = 0, \quad \text{on } \mathbb{R}^d\times S^{d-1}. \end{aligned} \tag{15}$$

We can then apply Theorems 2.1 or 2.2, depending on dimension $d$, based on the smallness assumptions on $K_1$, and conclude that $F(\mathbf{x})$ can be reconstructed from the boundary measurements. Once $F(\mathbf{x})$ is known, we can solve for $u$ in (15) and thus calculate $K_0 u(\mathbf{x})$. It remains to identify $f(\mathbf{x}) = F(\mathbf{x}) - K_0 u(\mathbf{x})$ to conclude the proof of the corollary. □

## 3. Derivation in two space dimensions

This section is devoted to the derivation of the inversion procedure in two space dimensions and on the proof of Theorem 2.1. For any unit vector $\boldsymbol{\theta} \in S^1$, we introduce the representation $\boldsymbol{\theta} = (\cos\theta, \sin\theta)$ for $0 \leq \theta < 2\pi$ and identify any function $f(\boldsymbol{\theta}) \equiv f(\theta)$.

We define the classical beam transform $S$ and the symmetrized beam transform $D$ (independently of the spatial dimension $d$) as

$$Sa(\mathbf{x}, \boldsymbol{\theta}) = \int_{-\infty}^{0} a(\mathbf{x} + s\boldsymbol{\theta}) ds, \tag{16}$$

$$Da(\mathbf{x}, \boldsymbol{\theta}) = \frac{1}{2}\Big( \int_{-\infty}^{0} a(\mathbf{x} + s\boldsymbol{\theta}) ds - \int_{0}^{\infty} a(\mathbf{x} + s\boldsymbol{\theta}) ds \Big). \tag{17}$$

Since $a(\mathbf{x})$ is smooth and compactly supported, $Da(\mathbf{x}, \theta)$ and $(e^{Da})(\mathbf{x}, \theta)$ are well-defined smooth functions. We next introduce

$$w(\mathbf{x}, \theta) = (e^{Da} u)(\mathbf{x}, \theta). \tag{18}$$

We know from the existence theory for the forward problem that $w$ is well-defined. We verify that it solves the equivalent integral equation

$$w(\mathbf{x}, \theta) = Se^{Da} K e^{-Da} w(\mathbf{x}, \theta) + Se^{Da} f(\mathbf{x}, \theta). \tag{19}$$

The transport operator inverting the transport equation can be written as

$$T = [I - Se^{Da} K e^{-Da}]^{-1} Se^{Da}. \tag{20}$$

Under the sub-critical assumption (4) $T$ is bounded from $L^2(\Omega)$ to $L^2(\mathbb{R}^2 \times S^1)$. We have then

$$w(\mathbf{x}, \theta) = Tf(\mathbf{x}, \theta) = Se^{Da} f + Se^{Da} K e^{-Da} Tf. \tag{21}$$

Let us introduce the operator $L$ acting on functions $w(\mathbf{x}, \theta)$ as

$$Lw(s, \theta) = \lim_{t \to \infty} w(t\boldsymbol{\theta} + s\boldsymbol{\theta}^{\perp}, \theta). \tag{22}$$

The product $LS$ is the usual Radon transform

$$Rf(s, \theta) = LSf(s, \theta). \tag{23}$$

It is convenient to work with slightly modified measurements. Let us introduce

$$g(s, \theta) = Lw(s, \theta) = \lim_{t \to \infty} (e^{Da} u)(t\boldsymbol{\theta} + s\boldsymbol{\theta}^{\perp}, \theta) = e^{\frac{1}{2} Ra}(s, \theta) m(s, \theta), \tag{24}$$

where $m(s,\theta)$ was defined in (6). Since $a$ is known, then so are the new "measurements" $g(s,\theta)$.

Let us finally introduce the attenuated $X-$ray transform operator

$$R_a f(s,\theta) = LSe^{Da}f(s,\theta). \tag{25}$$

Applying $L$ to (19), we deduce that the measurements $g(s,\theta)$ are given by

$$g(s,\theta) = R_a f(s,\theta) + Re^{Da}Ke^{-Da}Tf(s,\theta). \tag{26}$$

An inversion for $R_a$ was recently obtained in [28]; see also [2, 6, 8, 9, 25, 29] for recent works on the attenuated $X-$ray transform. We define the inversion operator $N$, acting on functions $g(s,\theta)$ defined on $\mathbb{R}\times S^1$, by

$$Ng(\mathbf{x}) = \frac{1}{4\pi}\int_0^{2\pi} \boldsymbol{\theta}^\perp\cdot\nabla_{\mathbf{x}}(R^*_{-a,\theta}H_a g)(\mathbf{x},\theta)d\theta, \tag{27}$$

where

$$\begin{aligned} R^*_{a,\theta}g(\mathbf{x}) &= e^{D_\theta a(\mathbf{x})}g(\mathbf{x}\cdot\boldsymbol{\theta}^\perp), & & \\ H_a &= C_cHC_c + C_sHC_s, & Hu(t) &= \frac{1}{\pi}\int_{\mathbb{R}}\frac{u(s)}{t-s}ds, \\ C_cg(s,\theta) &= g(s,\theta)\cos(\frac{HRa(s,\theta)}{2}), & C_sg(s,\theta) &= g(s,\theta)\sin(\frac{HRa(s,\theta)}{2}). \end{aligned} \tag{28}$$

The integral in the Hilbert transform $H$, which acts in $C_c$ and $C_s$ on the $s$ variable, has to be understood in the principal value sense. Note that $H_a = H$ in the absence of absorption ($a\equiv 0$) and that the above formula then becomes the usual inversion of the Radon transform [26].

We thus formally apply the operator $N$ to (26) and obtain the equation for $f(\mathbf{x})$ of Fredholm type:

$$Ng(\mathbf{x}) = f(\mathbf{x}) + NRe^{Da}Ke^{-Da}Tf(\mathbf{x}) = (I - N_K)f(\mathbf{x}), \tag{29}$$

Let $\chi(\mathbf{x})$ be a cut-off function supported on $\Omega$ and such that $\chi\equiv 1$ on the support of $f$. The equation above is recast as

$$\chi(\mathbf{x})Ng(\mathbf{x}) = f(\mathbf{x}) + \chi NRe^{Da}Ke^{-Da}Tf(\mathbf{x}) = (I - N_K)f(\mathbf{x}), \tag{30}$$

where we have introduced the operator $N_K = -\chi NRe^{Da}Ke^{-Da}T$.

The proof of Theorem 2.1 is based on the following result

**Theorem 3.1** *The operator $N_K$ defined above is bounded from $L^2(\Omega)$ to $L^2(\Omega)$.*

We study first the mapping properties of the scattering operator $K$. For this we introduce the functional spaces

$$L^{\hat{2}}(\mathbb{R}^2; C^0(S^1)) = \Big\{u(\mathbf{x},\boldsymbol{\theta}) \text{ s.t. } \hat{u}(\boldsymbol{\xi},\boldsymbol{\theta}) \in L^2(\mathbb{R}^2; C^0(S^1))\Big\}, \tag{31}$$

$$L^{\hat{2}}(\Omega; C^0(S^1)) = \Big\{u \in L^{\hat{2}}(\mathbb{R}^2; C^0(S^1)) \text{ s.t. } \mathrm{supp} u(\cdot,\theta)\subseteq\Omega\Big\}, \tag{32}$$

endowed with the norm

$$\|u\|^2_{L^{\hat{2}}(\mathbb{R}^2;C^0(S^1))} = \|\hat{u}\|^2_{L^2(\mathbb{R}^2;C^0(S^1))} = \int_{\mathbb{R}^2} \max_{\boldsymbol{\theta}\in S^1} |\hat{u}(\boldsymbol{\xi},\boldsymbol{\theta})|^2 d\boldsymbol{\xi}.$$

Since $K$ is a convolution in the angular variable, it is decomposed as

$$Ku(\mathbf{x},\boldsymbol{\theta}) = \sum_{n=-\infty}^{\infty} k_n(\mathbf{x}) u_n(\mathbf{x}) e^{in\theta}. \tag{33}$$

**Lemma 3.2** *Assume that* supp $k(\cdot,\theta) \subset \Omega$ *and that* $\max_{n\in\mathbb{Z}} \langle n\rangle^\alpha \|k_n\|^2_{L^1(\mathbb{R}^2)} < C$ *for some* $\alpha > 1$. *Then the operator* $K$ *maps* $L^2(\mathbb{R}^2 \times S^1)$ *to* $L^{\hat{2}}(\Omega; C^0(S^1))$.

*Proof.* Taking the Fourier transform in the space variable in (33) we get

$$|\widehat{Ku}(\xi,\boldsymbol{\theta})|^2 = \left|\sum_{-\infty}^{\infty} (\widehat{k_n} * \widehat{u_n})(\xi) e^{in\theta}\right|^2 \le \left(\sum_{-\infty}^{\infty} \left|\widehat{k_n} * \widehat{u_n}\right|(\boldsymbol{\xi})\right)^2$$
$$\le \left(\sum_{-\infty}^{\infty} \frac{1}{\langle n\rangle^\alpha}\right)\left(\sum_{-\infty}^{\infty} \langle n\rangle^\alpha |\hat{k}_n * \hat{u}_n|^2(\boldsymbol{\xi})\right).$$

Now we take the maximum over $\boldsymbol{\theta} \in S^1$ on both sides and integrate in $\xi \in \mathbb{R}^2$ to get

$$\|Ku\|^2_{L^{\hat{2}}(\mathbb{R}^2;C^0(S^1))} \le \left(\sum_{-\infty}^{\infty} \frac{1}{\langle n\rangle^\alpha}\right) \sum_{-\infty}^{\infty} \langle n\rangle^\alpha \|\hat{k}_n * \hat{u}_n\|^2_{L^2(\mathbb{R}^2)}$$
$$\le \left(\sum_{-\infty}^{\infty} \frac{1}{\langle n\rangle^\alpha}\right) \sum_{-\infty}^{\infty} \langle n\rangle^\alpha \|\hat{k}_n\|^2_{L^1(\mathbb{R}^2)} \|\hat{u}_n\|^2_{L^2(\mathbb{R}^2)}$$
$$\le \left(\sum_{-\infty}^{\infty} \frac{1}{\langle n\rangle^\alpha}\right) \max_{n\in\mathbb{N}} \langle n\rangle^\alpha \|\hat{k}_n\|^2_{L^1(\mathbb{R}^2)} \sum_{-\infty}^{\infty} \|\hat{u}_n\|^2_{L^2(\mathbb{R}^2)} \le C\left(\sum_{-\infty}^{\infty} \frac{1}{\langle n\rangle^\alpha}\right) \|u\|^2_{L^2(\mathbb{R}^2\times S^1)}.$$

□

**Lemma 3.3** *Let* $h : \mathbb{R}^2 \times S^1 \to \mathbb{R}$ *be a smooth map such that*

$$\int_{\mathbb{R}^2} \max_{\boldsymbol{\theta}} |\hat{h}(\xi,\boldsymbol{\theta})| d\xi < \infty.$$

*Then the operator* $M_h$ *of multiplication by* $h$ *is bounded from* $L^{\hat{2}}(\Omega; C^0(S^1))$ *to itself.*

*Proof.*

$$\|M_h f\|^2_{\hat{2},\infty} = \|hf\|^2_{\hat{2},\infty} = \int_{\mathbb{R}^2} \max_{\boldsymbol{\theta}\in S^1} |\widehat{hf}|^2(\xi,\boldsymbol{\theta}) d\xi = \int_{\mathbb{R}^2} \max_{\theta\in S^1} |\hat{h} *_\xi \hat{f}|^2(\xi,\theta) d\xi$$
$$\le \left\| \max_{\boldsymbol{\theta}\in S^1} |\hat{h}| * \max_{\boldsymbol{\theta}\in S^1} |\hat{f}| \right\|^2_{L^2} \le \left\| \max_{\boldsymbol{\theta}\in S^1} |\hat{h}| \right\|^2_{L^1} \|f\|^2_{\hat{2},\infty}. \tag{34}$$

□

Recall that, when acting on maps $f(x,\boldsymbol{\theta})$, $R$ denotes the Radon transform in $x \in \mathbb{R}^2$. The following smoothing property holds.

**Lemma 3.4** *The operator $R : L^{\hat{2},\infty}(\Omega\times S^1) \to H^{1/2}(\mathbb{R}\times S^1)$ is bounded, more precisely*

$$\int_{S^1}\int_{\mathbb{R}}\left|\widehat{Rg}(\rho,\theta)\right|^2(1+|\rho|)d\rho d\theta \le (4\pi|\Omega|^2+3)\|g\|^2_{\hat{2},\infty},$$

*where $|\Omega|$ denotes the volume of $\Omega$.*

*Proof.* Notice first that the Fourier slice theorem $\widehat{Rg}(\rho,\theta) = \hat{g}(\rho\boldsymbol{\theta}^\perp,\theta)$ holds. In the right hand side the Fourier transform is taken with respect of the space-variable only. The following inequalities hold:

$$\begin{aligned}\int_0^{2\pi}\int_0^\infty \left|\widehat{Rg}(\rho,\theta)\right|^2\rho d\rho d\theta &= \int_0^{2\pi}\int_0^\infty \left|\hat{g}(\rho\boldsymbol{\theta}^\perp,\theta)\right|^2\rho d\rho d\theta\\ &\le \int_0^{2\pi}\int_0^\infty \max_{\nu\in S^1}\left|\hat{g}(\rho\boldsymbol{\theta}^\perp,\nu)\right|^2\rho d\rho d\theta = \|g\|^2_{\hat{2},\infty},\\ \int_0^{2\pi}\int_{-\infty}^0 \left|\widehat{Rg}(\rho,\theta)\right|^2|\rho| d\rho d\theta &= \int_0^{2\pi}\int_0^\infty \left|\hat{g}(-\rho\boldsymbol{\theta}^\perp,\theta)\right|^2\rho d\rho d\theta\\ &\le \int_0^{2\pi}\int_0^\infty \max_{\nu\in S^1}\left|\hat{g}(-\rho\boldsymbol{\theta}^\perp,\nu)\right|^2\rho d\rho d\theta = \|g\|^2_{\hat{2},\infty}.\end{aligned}$$

Also we have

$$\int_0^{2\pi}\int_0^\infty|\widehat{Rg}(\rho,\theta)|^2 d\rho d\theta \le \int_0^{2\pi}\int_0^1|\hat{g}(\rho\boldsymbol{\theta}^\perp,\theta)|^2 d\rho d\theta + \int_0^{2\pi}\int_0^\infty|\widehat{Rg}(\rho,\theta)|^2\rho d\rho d\theta$$

and

$$\int_0^{2\pi}\int_0^1|\hat{g}(\rho\boldsymbol{\theta}^\perp,\theta)|^2 d\rho d\theta \le \int_0^{2\pi}\int_0^1 \max_{\nu\in S^1}|\hat{g}(\rho\boldsymbol{\theta}^\perp,\nu)|^2 d\rho d\theta \le 2\pi\max_{|\boldsymbol{\xi}|\le 1}|\hat{g}(\boldsymbol{\xi},\nu_0)|^2,$$

where to simplify notation we have defined

$$|\hat{g}(\boldsymbol{\xi},\nu_0)| = \max_{\nu\in S^1}|\hat{g}(\boldsymbol{\xi},\nu)|.$$

Let $\{\chi_n(\mathbf{x})\}_{n\ge1}$ be a sequence of smooth cut-off functions equal to 1 on $\Omega$, the support of $g$, and equal to 0 at $\mathbf{x}$ such that $d(\mathbf{x},\Omega) > n^{-1}$; and let $\chi(\mathbf{x};\boldsymbol{\xi}) = e^{i\mathbf{x}\cdot\boldsymbol{\xi}}\chi(\mathbf{x})$. Then we verify that

$$\hat{g}(\boldsymbol{\xi},\nu_0) = \int_{\mathbb{R}^2} e^{-i\mathbf{x}\cdot\boldsymbol{\xi}}\chi_n(\mathbf{x})g(\mathbf{x},\nu_0)d\mathbf{x} = \int_{\mathbb{R}^2}\overline{\chi_n(\mathbf{x};\boldsymbol{\xi})}g(\mathbf{x},\nu_0)d\mathbf{x} = \int_{\mathbb{R}^2}\overline{\hat{\chi_n}(\boldsymbol{\eta};\boldsymbol{\xi})}\hat{g}(\boldsymbol{\eta},\nu_0)d\mathbf{x},$$

from which we deduce the following bound:

$$|\hat{g}(\boldsymbol{\xi},\nu_0)| \le \inf_n ||\hat{\chi_n}(\cdot;\xi)||_{L^2}||\hat{g}(\cdot,\nu_0)||_{L^2} = \inf_n ||\chi_n||_{L^2}\|g\|_{\hat{2},\infty} \le |\Omega|\|g\|_{\hat{2},\infty}. \qquad (35)$$

Similarly, we have

$$\int_0^{2\pi}\int_{-\infty}^0|\widehat{Rg}(\rho,\theta)|^2 d\rho d\theta \le 2\pi|\Omega|^2\|g\|^2_{\hat{2},\infty} + \int_0^{2\pi}\int_{-\infty}^0|\widehat{Rg}(\rho,\theta)|^2|\rho| d\rho d\theta.$$

This concludes the proof of the lemma. □

**Lemma 3.5** *Let $f \in H^{1/2}(\mathbb{R} \times S^1)$ and $\phi(\mathbf{x}, \theta)$ be a smooth function such that*

$$\int_{\mathbb{R}^2} \max_{\nu \in S^1} |\hat{\phi}(\boldsymbol{\xi};\nu)|^2 (1+|\boldsymbol{\xi}|^2) d\boldsymbol{\xi} < \infty. \tag{36}$$

*Then the map $(\mathbf{x}, \theta) \to \phi(\mathbf{x}, \theta) f(\mathbf{x} \cdot \boldsymbol{\theta}^\perp, \boldsymbol{\theta})$ is in $H^{1/2}(\mathbb{R}^2 \times S^1)$.*

*Proof.* We have the following sequence of inequalities:

$$\begin{aligned}
&\int_0^{2\pi} \int_{\mathbb{R}^2} (1+|\boldsymbol{\xi}|^2)^{\frac{1}{2}} |\widehat{\phi f}(\boldsymbol{\xi}, \theta)|^2 d\boldsymbol{\xi} d\theta = \int_0^{2\pi} \int_{\mathbb{R}^2} (1+|\boldsymbol{\xi}|^2)^{\frac{1}{2}} \left| \int_{\mathbb{R}} \hat{\phi}(\boldsymbol{\xi} \cdot \boldsymbol{\theta}, t; \theta) \hat{f}(\boldsymbol{\xi} \cdot \boldsymbol{\theta}^\perp - t, \theta) dt \right|^2 d\boldsymbol{\xi} d\theta \\
&\le \int_0^{2\pi} \left\{ \int_{\mathbb{R}} \left[ \int_{\mathbb{R}^2} (1+|\boldsymbol{\xi}|^2)^{\frac{1}{2}} |\hat{\phi}(\xi_1, t; \theta)|^2 |\hat{f}(\xi_2 - t; \theta)|^2 d\xi_1 d\xi_2 \right]^{\frac{1}{2}} dt \right\}^2 d\theta \\
&\le \int_0^{2\pi} \left\{ \int_{\mathbb{R}} \left[ \int_{\mathbb{R}^2} (1+|\xi_1|^2)^{\frac{1}{2}} (1+|\xi_2|^2)^{\frac{1}{2}} |\hat{\phi}(\xi_1, t; \theta)|^2 |\hat{f}(\xi_2 - t; \theta)|^2 d\xi_1 d\xi_2 \right]^{\frac{1}{2}} dt \right\}^2 d\theta \\
&= \int_0^{2\pi} \left\{ \int_{\mathbb{R}} (1+t^2)^{\frac{1}{4}} \left[ \int_{\mathbb{R}^2} (1+|\xi_1|^2)^{\frac{1}{2}} |\hat{\phi}(\xi_1, t; \theta)|^2 \left( \frac{1+|\xi_2+t|^2}{1+|t|^2} \right)^{\frac{1}{2}} |\hat{f}(\xi_2; \theta)|^2 d\xi_1 d\xi_2 \right]^{\frac{1}{2}} dt \right\}^2 d\theta \\
&\le \int_0^{2\pi} \left\{ \int_{\mathbb{R}} (1+t^2)^{\frac{1}{4}} \left[ \int_{\mathbb{R}^2} (1+|\xi_1|^2)^{\frac{1}{2}} |\hat{\phi}(\xi_1, t; \theta)|^2 (1+|\xi_2|^2)^{\frac{1}{2}} |\hat{f}(\xi_2; \theta)|^2 d\xi_1 d\xi_2 \right]^{\frac{1}{2}} dt \right\}^2 d\theta \\
&= \int_0^{2\pi} \int_{\mathbb{R}} (1+|\xi_2|^2)^{\frac{1}{2}} |\hat{f}(\xi_2; \theta)|^2 d\xi_2 \left\{ \int_{\mathbb{R}} (1+t^2)^{\frac{1}{4}} \left[ \int_{\mathbb{R}} (1+|\xi_1|^2)^{\frac{1}{2}} |\hat{\phi}(\xi_1, t; \theta)|^2 d\xi_1 \right]^{\frac{1}{2}} dt \right\}^2 d\theta \\
&\le \int_0^{2\pi} \left( \int_{\mathbb{R}} (1+|\xi_2|^2)^{\frac{1}{2}} |\hat{f}(\xi_2; \theta)|^2 d\xi_2 \int_{\mathbb{R}^2} (1+t^2)^{\frac{1}{2}} (1+|\xi_1|^2)^{\frac{1}{2}} \max_{\nu \in S^1} |\hat{\phi}(\xi_1, t; \nu)|^2 dt \xi_1 \right) d\theta \\
&\le \int_0^{2\pi} \int_{\mathbb{R}} (1+|\xi_2|^2)^{\frac{1}{2}} |\hat{f}(\xi_2; \theta)|^2 d\xi_2 d\theta \int_{\mathbb{R}^2} (1+|\boldsymbol{\xi}|^2) \max_{\nu \in S^1} |\hat{\phi}(\boldsymbol{\xi}; \nu)|^2 d\boldsymbol{\xi},
\end{aligned}$$

where we have used the Minkowsky and Cauchy inequalities. From the second line onwards, we have used the $\theta$ dependent coordinates $\xi_1 = \boldsymbol{\xi} \cdot \boldsymbol{\theta}$ and $\xi_2 = \boldsymbol{\xi} \cdot \boldsymbol{\theta}^\perp$. □
Recall that $\chi(\mathbf{x})$ defined before (29) is a smooth cut-off function supported in $\Omega$. To simplify notation, let

$$\begin{aligned}
f_1(\mathbf{x}, \theta) &= e^{Da}(\mathbf{x}, \theta) \mathrm{trig}(HRa(\mathbf{x} \cdot \boldsymbol{\theta}^\perp, \theta)/2), \\
f_2(s, \theta) &= \mathrm{trig}(HRa(s, \theta)/2) \\
f_3(\mathbf{x}, \theta) &= \chi(\mathbf{x}) (\boldsymbol{\theta}^\perp \cdot \nabla_{\mathbf{x}}) f_1(\mathbf{x}, \theta).
\end{aligned}$$

be smooth functions depending on the attenuation $a$ only, where *trig* stands for either *sin* or *cos*. The composition operator $\chi NR$ becomes

$$\chi NRw(\mathbf{x}) = \frac{\chi(\mathbf{x})}{4\pi} \int_0^{2\pi} \boldsymbol{\theta}^\perp \cdot \nabla \Big( f_1(\mathbf{x}, \theta) H[f_2(s, \theta) R[w](s, \theta)](\mathbf{x} \cdot \boldsymbol{\theta}^\perp, \theta) \Big) d\theta,$$

where the above is understood as a sum over the values that the functions *trig* can take in $f_1$ and $f_2$. We now follow ideas from [5] though we derive some of the estimates in the Fourier domain to characterize the norm of the operator. Let $\phi_n(\mathbf{x})$, $n = 1, 2, ...$ be an orthonormal basis of $L^2(\Omega)$ and let

$$\chi(\mathbf{x})f_1(\mathbf{x},\theta) = \sum_{n=1}^{\infty} \alpha_n(\theta)\phi_n(\mathbf{x}), \qquad \alpha_n(\theta) = \int_\Omega \chi(\mathbf{x})f_1(\mathbf{x},\theta)\phi_n(\mathbf{x})d\mathbf{x}. \tag{37}$$

**Proposition 3.6** *The composition operator $\chi NR$ maps $L^{\hat{2}}(\Omega; C^0(S^1))$ to $L^2(\Omega)$. Moreover, we have the more explicit characterization:*

$$\|\chi NRw\|_{L^2} \le \|w\|_{\hat{2},\infty} \int_{\mathbb{R}^2} \max_{\boldsymbol{\theta}} |\widehat{e^{Da}}(\boldsymbol{\xi},\boldsymbol{\theta})| d\boldsymbol{\xi} \Bigg[ \int_{\mathbb{R}^2} \max_{\nu\in S^1} |\hat{f}_3(\boldsymbol{\xi};\nu)|^2 (1+|\boldsymbol{\xi}|^2) d\boldsymbol{\xi} + \tag{38}$$
$$+2\left(\sum_{n=1}^{\infty} \max_{\nu\in S^1} |\alpha_n(\nu)|^2\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} |\hat{f}_2(s,\nu_0)|(2+4\pi|\Omega|s^2) ds \Bigg].$$

*Proof.* Using lemma 3.4 we obtain that $Rw \in H^{1/2}(\mathbb{R}\times S^1)$. From lemma 3.5, provided that $\chi(\mathbf{x})\left(\boldsymbol{\theta}^\perp \cdot \nabla_x f_1(\mathbf{x},\theta)\right)$ satisfies the smoothness condition (36), we get that the map $(\mathbf{x},\theta) \to \chi(\mathbf{x})\left(\boldsymbol{\theta}^\perp \cdot \nabla_x f_1(\mathbf{x},\theta)\right) H[f_2(s,\theta)R[w](s,\theta)](\mathbf{x}\boldsymbol{\theta}^\perp,\theta)$ is in $H^{1/2}(\mathbb{R}^2\times S^1)$.

Next we show that the operator $M$ defined by

$$Mw(\mathbf{x}) = \int_0^{2\pi} \chi(\mathbf{x})f_1(\mathbf{x},\theta)\left(\boldsymbol{\theta}^\perp \cdot \nabla_x\right) H[f_2(s,\theta)R(w)(s,\theta)](\mathbf{x}\cdot\boldsymbol{\theta}^\perp,\theta)d\theta$$

is bounded from $L^{\hat{2}}(\Omega; C^0(S^1))$ in $L^2(\Omega)$. We have

$$\begin{aligned}
&\left\| \int_0^{2\pi} \sum_{n=1}^{\infty} \alpha_n(\theta)\phi_n(\mathbf{x})(\boldsymbol{\theta}\cdot\nabla)H(f_2Rw(s,\theta))(\mathbf{x}\cdot\boldsymbol{\theta}^\perp,\theta)d\theta \right\|_{L^2_x}^2 \\
&= \sum_{n=1}^{\infty} \langle \phi_n; \int_0^{2\pi} \alpha_n(\theta)(\boldsymbol{\theta}^\perp\cdot\nabla)H(f_2Rw(s,\theta)(\mathbf{x}\cdot\boldsymbol{\theta}^\perp,\theta))d\theta\rangle_{L^2_x}^2 \\
&\le \sum_{n=1}^{\infty} \left\| \int_0^{2\pi} \alpha_n(\theta)(\boldsymbol{\theta}^\perp\cdot\nabla)H(f_2Rw(s,\theta))(\mathbf{x}\cdot\boldsymbol{\theta}^\perp,\theta)d\theta \right\|_{L^2_x}^2 \\
&\le \sum_{n=1}^{\infty} \left\| \mathcal{F}_{\mathbf{x}\to\boldsymbol{\xi}} \left\{ \int_0^{2\pi} \alpha_n(\theta)(\boldsymbol{\theta}^\perp\cdot\nabla)H(f_2Rw(s,\theta))(\mathbf{x}\cdot\boldsymbol{\theta}^\perp,\theta)d\theta \right\}(\boldsymbol{\xi}) \right\|_{L^2_\xi}^2 \\
&= \sum_{n=1}^{\infty} \left\| \alpha_n(\xi_-)\widehat{f_2Rw}(-|\boldsymbol{\xi}|,\xi_-) + \alpha_n(\xi_+)\widehat{f_2Rw}(|\boldsymbol{\xi}|,\xi_+) \right\|_{L^2_\xi}^2 \\
&\le \left(\sum_{n=1}^{\infty} \max_{\nu\in S^1} |\alpha_n(\nu)|^2\right) \left( \|\widehat{f_2Rw}(-|\boldsymbol{\xi}|,\xi_-)\|_{L^2_\xi}^2 + \|\widehat{f_2Rw}(|\boldsymbol{\xi}|,\xi_+)\|_{L^2_\xi}^2 \right).
\end{aligned}$$

In the above expressions, the angles $\xi_\pm$ are defined such that $\boldsymbol{\xi}\cdot\boldsymbol{\theta} = 0$; see notably [6, pp.413&415] for the details of calculations that are not reproduced here. By the Fourier slice theorem we verify that

$$\widehat{f_2Rw}(|\boldsymbol{\xi}|,\xi_+) = \int_{\mathbb{R}} \hat{f}_2(|\boldsymbol{\xi}|-s,\xi_+)\hat{w}\left(s\frac{\xi}{|\xi|},\xi_+\right) ds.$$

Let us now define $\nu_0$ and $\nu_1$ as the values of the angles where $\max_{\nu\in S^1}|\hat{f}_2(\rho,\nu)|$ and $\max_{\nu\in S^1}|\hat{w}(\rho,\nu)|$ are achieved, respectively. We compute:

$$\begin{aligned}
\left\{\int_{\mathbb{R}^2}|\widehat{f_2Rw}(|\boldsymbol{\xi}|,\xi_+)|^2 d\boldsymbol{\xi}\right\}^{\frac{1}{2}} &= \left\{\int_0^{2\pi}\int_0^\infty|\widehat{f_2Rw}(r,-\boldsymbol{\theta}^\perp)|^2 r dr d\theta\right\}^{\frac{1}{2}} \qquad (39)\\
&= \left\{\int_0^{2\pi}\int_0^\infty\left|\int_{\mathbb{R}}\hat{f}_2(s,-\boldsymbol{\theta}^\perp)\hat{w}((r-s)\boldsymbol{\theta},-\boldsymbol{\theta}^\perp)|ds\right|^2 rdrd\theta\right\}^{\frac{1}{2}}\\
&\le \left\{\int_0^{2\pi}\int_0^\infty\left(\int_{\mathbb{R}}\max_{\nu\in S^1}|\hat{f}_2(s,\nu)|\max_{\nu\in S^1}|\hat{w}((r-s)\boldsymbol{\theta},\nu)|ds\right)^2 rdrd\theta\right\}^{\frac{1}{2}}\\
&= \left\{\int_0^{2\pi}\int_0^\infty\left(\int_{\mathbb{R}}|\hat{f}_2(s,\nu_0)||\hat{w}((r-s)\boldsymbol{\theta},\nu_1)|ds\right)^2 rdrd\theta\right\}^{\frac{1}{2}}\\
&\le \int_{\mathbb{R}}\left\{\int_0^{2\pi}\int_0^\infty|\hat{f}_2(s,\nu_0)|^2|\hat{w}((r-s)\boldsymbol{\theta},\nu_1)|^2 rdrd\theta\right\}^{\frac{1}{2}} ds\\
&= \int_{\mathbb{R}}|\hat{f}_2(s,\nu_0)|\left\{\int_0^{2\pi}\int_0^\infty|\hat{w}((r-s)\boldsymbol{\theta},\nu_1|^2 rdrd\theta\right\}^{\frac{1}{2}} ds.
\end{aligned}$$

We evaluate the last term by splitting the integral $\int_0^\infty(...)ds+\int_{-\infty}^0(...)ds$. To estimate

$$\int_0^\infty|\hat{f}_2(s,\nu_0)|\left\{\int_0^{2\pi}\int_0^\infty|\hat{w}((r-s)\boldsymbol{\theta},\nu_1)|^2 rdrd\theta\right\}^{\frac{1}{2}} ds,$$

we further split the inner integral into $\int_0^{2s}(...)drd\theta+\int_{2s}^\infty(...)drd\theta$. We obtain that

$$\begin{aligned}
\int_0^{2\pi}\int_{2s}^\infty|\hat{w}((r-s)\boldsymbol{\theta},\nu_1)|^2 rdrd\theta &\le 2\int_0^{2\pi}\int_{2s}^\infty|\hat{w}((r-s)\boldsymbol{\theta},\nu_1)|^2(r-s)drd\theta\\
&= 2\int_{|\boldsymbol{\xi}|\ge s}|\hat{w}(\boldsymbol{\xi},\nu_1)|^2 d\boldsymbol{\xi},
\end{aligned}$$

and

$$\begin{aligned}
\int_0^{2\pi}\int_0^{2s}|\hat{w}((r-s)\boldsymbol{\theta},\nu_1)|^2 rdrd\theta &= \int_0^{2\pi}\int_{-s}^{s}|\hat{w}(t\boldsymbol{\theta},\nu_1)|^2(t+s)dtd\theta\\
&= 2\int_{|\boldsymbol{\xi}|\le s}|\hat{w}(\boldsymbol{\xi},\nu_1)|^2 d\boldsymbol{\xi}+2s\int_0^{2\pi}\int_0^s|\hat{w}(t\boldsymbol{\theta},\nu_1)|^2 dtd\theta\\
&\le 2\int_{|\boldsymbol{\xi}|\le s}|\hat{w}(\boldsymbol{\xi},\nu_1)|^2 d\boldsymbol{\xi}+4\pi s^2\max_{|\boldsymbol{\xi}|\le s}|\hat{w}(\boldsymbol{\xi},\nu_1)|^2\\
&\le 2\int_{|\boldsymbol{\xi}|\le s}|\hat{w}(\boldsymbol{\xi},\nu_1)|^2 d\boldsymbol{\xi}+4\pi s^2|\Omega|\|\hat{w}(\boldsymbol{\xi},\nu_1)\|^2_{L^2_\xi}.
\end{aligned}$$

The last inequality uses in a crucial way the estimate (35) and the fact that $w$ is compactly supported. We have obtained so far that

$$\int_0^\infty |\hat{f}_2(s,\nu_0)| \left\{ \int_0^{2\pi} \int_0^\infty |\hat{w}((r-s)\boldsymbol{\theta},\nu_1)|^2 r dr d\theta \right\}^{\frac{1}{2}} ds \\ \leq \|w\|_{\hat{2},\infty} \int_0^\infty |\hat{f}_2(s,\nu_0)(2+4\pi|\Omega|s^2) ds.$$

The other contribution is handled similarly:

$$\begin{aligned}
&\int_{-\infty}^0 |\hat{f}_2(s,\nu_0)| \left\{ \int_0^{2\pi} \int_0^\infty |\hat{w}((r-s)\boldsymbol{\theta},\nu_1)|^2 r dr d\theta \right\}^{\frac{1}{2}} ds \\
&\quad = \int_0^\infty |\hat{f}_2(-s,\nu_0)| \left\{ \int_0^{2\pi} \int_0^\infty |\hat{w}((r+s)\boldsymbol{\theta},\nu_1)|^2 r dr d\theta \right\}^{\frac{1}{2}} ds \\
&\quad \leq \int_0^\infty |\hat{f}_2(-s,\nu_0)| \left\{ \int_0^{2\pi} \int_0^\infty |\hat{w}((r+s)\boldsymbol{\theta},\nu_1)|^2 (r+s) dr d\theta \right\}^{\frac{1}{2}} ds \\
&\quad = \|w\|_{\hat{2},\infty} \int_0^\infty |\hat{f}_2(-s,\nu_0)| ds = \|w\|_{\hat{2},\infty} \int_{-\infty}^0 |\hat{f}_2(s,\nu_0)| ds.
\end{aligned}$$

Combined with the estimate in (39), we have obtained that

$$\left\{ \int_{\mathbb{R}^2} |\widehat{f_2 Rw}|(|\boldsymbol{\xi}|,\xi_+)|^2 d\boldsymbol{\xi} \right\}^{\frac{1}{2}} \leq \|w\|_{\hat{2},\infty} \int_{-\infty}^\infty |\hat{f}_2(s,\nu_0)|(2+4\pi|\Omega|s^2) ds. \tag{40}$$

A similar calculation shows that

$$\left\{ \int_{\mathbb{R}^2} |\widehat{f_2 Rw}(-|\boldsymbol{\xi}|,\xi_-)|^2 d\boldsymbol{\xi} \right\}^{\frac{1}{2}} \leq \|w\|_{\hat{2},\infty} \int_{-\infty}^\infty |\hat{f}_2(s,\nu_0)|(2+4\pi|\Omega|s^2) ds. \tag{41}$$

In summary, we have obtained the following estimate for the operator $M$:

$$\|Mw\|_{L^2} \leq 2\|w\|_{\hat{2},\infty} \left( \sum_{n=1}^\infty \max_{\nu\in S^1} |\alpha_n(\nu)|^2 \right)^{\frac{1}{2}} \int_{-\infty}^\infty |\hat{f}_2(s,\nu_0)|(2+4\pi|\Omega|s^2) ds. \tag{42}$$

This concludes the proof of the proposition. □

Note that $N$ has range in $L^2_{\text{loc}}(\mathbb{R}^2)$ and not necessarily in $L^2(\mathbb{R}^2)$. This is where the assumption of the compactness of the support of the source term $f(\mathbf{x})$, which is natural in practice, comes into play. The above estimate shows the role played by the size of the support of the source term.

The proof of the Theorem 3.1 follows from the preceding lemmas and proposition. As we have seen, $T$ maps $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$ and let $|||T|||_{L^2\to L^2}$ denote its operator norm. The preceding calculations allow us to obtain the more explicit version of Theorem 3.1:

**Corollary 3.7** *Assume that $K$ is such that* $\max_n < n >^{\alpha/2} \|\hat{k}_n\|_{L^1(\mathbb{R}^2)} \leq C_\alpha$ *for some* $\alpha > 1$*. Then the operator norm* $|||N_K|||_{L^2\to L^2}$ *is bounded by the following expression*

$$\left[\int_{\mathbb{R}^2} \max_{\nu\in S^1} |\hat{f}_3(\boldsymbol{\xi};\nu)|^2(1+|\boldsymbol{\xi}|^2)d\xi + 2\left(\sum_{n=1}^{\infty}\max_{\nu\in S^1}|\alpha_n(\nu)|^2\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}|\hat{f}_2(s,\nu_0)|(2+4\pi|\Omega|s^2)ds\right] \tag{43}$$

$$\times\, C_\alpha \left(\sum_{-\infty}^{\infty}\frac{1}{< n >^\alpha}\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2}\max_{\boldsymbol{\theta}}|\widehat{e^{Da}}(\xi,\boldsymbol{\theta})|d\xi\right)\|e^{-Da}\|_{L^\infty(\mathbb{R}^2\times S^1)}|||T|||_{L^2\to L^2}.$$

Except for the norm of $T$ (and of course the constant $C_\alpha$), all the other terms involved above are independent of scattering. Upon replacing $k(\mathbf{x},\mu)$ in the definition of the scattering operator by $\lambda k(\mathbf{x},\mu)$ for $\lambda > 0$, we deduce from the proposition that the operator norm of $N_K$ is bounded by a constant less than one in $\mathcal{L}(L^2(\Omega))$ provided that $\lambda$ is sufficiently small. This proves the first part of Theorem 2.1. Note that the constraint on the norm of $N_K$ is only sufficient to solve (30) and by no means necessary. Reconstructions based on (30) thus have a larger domain of validity than what we consider in Theorem 2.1. As for the constructive aspect of the reconstruction, we easily verify that for $\lambda$ sufficiently small, the following Neumann series expansion

$$f(\mathbf{x}) = \sum_{n=0}^{\infty} N_K^n N g(\mathbf{x}), \tag{44}$$

converges in $L^2(\Omega)$ strongly to the solution $f(\mathbf{x})$. This provides us with an explicit reconstruction formula to recover $f(\mathbf{x})$ from the measurements $m(s,\theta) = e^{-\frac{1}{2}Ra}(s,\theta)g(s,\theta)$ and concludes the proof of Theorem 2.1. Let us conclude this section by a few remarks.

**Remark 3.8** The measurements $m(s,\theta)$ for $s\in\mathbb{R}$ and $0\leq\theta<2\pi$ are redundant. Indeed in the case $a\equiv 0$ and $k\equiv 0$, the measurements satisfy $m(s,\theta)=m(-s,\theta+\pi)$ so that the source term can be reconstructed from knowledge of $m(s,\theta)$ on $Z=\mathbb{R}\times(0,\pi)$. When $a\neq 0$, such a redundancy still exists, although it is harder to characterize. Under certain smallness assumptions on $a(\mathbf{x})$, an explicit procedure to reconstruct the source term from $m$ on $Z$ when $k=0$ was proposed in [6] and implemented in [7]. That measurements on $Z$ suffice to determine the source term was recently obtained in [33]; see also [27] in the case of constant absorption. The explicit procedure proposed [6] can be extended to the case of scattering kernels so that provided that $k$ is sufficiently small, the source term is uniquely determined by $m(s,\theta)$ on $Z$.

**Remark 3.9** We could have considered more general scattering kernels of the form $k(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\theta}')$ so long as the smoothing of the scattering kernel $K$ imposed in Lemma 3.2 still holds. The description of this smoothing effect in terms of the scattering coefficients is simplified for kernels of the form $k(\mathbf{x},\boldsymbol{\theta}\cdot\boldsymbol{\theta}')$. However this is the only place where the specific structure of the kernel has been used (except for the subcriticality condition

(4), which should hold with $k(\mathbf{x}, \boldsymbol{\theta}\cdot\boldsymbol{\theta}')$ replaced by both $k(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\theta}')$ and $k(\mathbf{x}, \boldsymbol{\theta}', \boldsymbol{\theta})$; see [13]).

**Remark 3.10** The smoothing effect of the scattering kernel described in 3.2 is rendered necessary (at least some sort of smoothing is) by the behavior of the Radon transform and the inversion operator $N$. Although $NR$ maps functions in $L^2(\Omega)$ to functions in $L^2(\Omega)$ (since the operator $NR$ is then identity), this is no longer the case for functions in $L^2(\Omega \times S^1)$ that depend non-trivially on $\theta$. We need to map functions from the smaller space $L^{\hat{2}}(\mathbb{R}^2; C^0(S^1))$, which is made possible by the regularizing effect of $K$.

**Remark 3.11** Under appropriate assumptions on the scattering kernel $K$, the equation (29) is indeed of Fredholm type as the operator $N_K$ can be shown to be compact. Indeed $NR$ is a bounded operator, whereas the operator $KT$ (as well as $Ke^{-Da}T$ for smooth absorption $a(\mathbf{x})$) can be shown to be compact under general assumptions. We refer to [24] for such results and to [16] for connected result on averaging lemmas.

**Remark 3.12** The reconstruction of the source term can be obtained by the following iterative scheme. We consider the setting of Corollary 2.3. Let $g(s,\theta) = e^{Ra/2}m(s,\theta)$ be the measurements. We initialize the algorithm as

$$F^{(0)}(\mathbf{x}) = Ng(\mathbf{x}). \tag{45}$$

Provided that $F^{(k)}(\mathbf{x})$ is known, we solve for $u^{(k)}$ in

$$\begin{aligned} &\boldsymbol{\theta}\cdot\nabla_{\mathbf{x}} u^{(k)}(\mathbf{x},\theta) + a(\mathbf{x})u^{(k)}(\mathbf{x},\theta) = K_1 u^{(k)}(\mathbf{x},\theta) + F^{(k)}(\mathbf{x}), \qquad \text{in } \mathbb{R}^2\times S^1 \\ &\lim_{t\to\infty} u^{(k)}(\mathbf{x} - t\boldsymbol{\theta}, \theta) = 0, \qquad \text{on } \mathbb{R}^2\times S^1. \end{aligned} \tag{46}$$

We next solve for $v^{(k)}(\mathbf{x},\theta)$ in

$$\begin{aligned} &\boldsymbol{\theta}\cdot\nabla_{\mathbf{x}} v^{(k)}(\mathbf{x},\theta) + a(\mathbf{x})v^{(k)}(\mathbf{x},\theta) = K_1 u^{(k)}(\mathbf{x},\theta), \qquad \text{in } \mathbb{R}^2\times S^1 \\ &\lim_{t\to\infty} v^{(k)}(\mathbf{x} - t\boldsymbol{\theta}, \theta) = 0, \qquad \text{on } \mathbb{R}^2\times S^1. \end{aligned} \tag{47}$$

We then compute the new data

$$g^{(k)}(s,\theta) = e^{Ra/2} R_a v^{(k)}(s,\theta). \tag{48}$$

Finally we set the new source term

$$F^{(k+1)}(\mathbf{x}) = N(g - g^{(k)})(\mathbf{x}). \tag{49}$$

We verify that $F^{(k)}(\mathbf{x})$ converges to $F(\mathbf{x}) = K_0 u(\mathbf{x}) + f(\mathbf{x})$ in $L^2(\Omega)$ as the above algorithm is equivalent to the Neumann series expansion (44). We then solve for $u(\mathbf{x},\theta)$ and reconstruct the source term $f(\mathbf{x}) = F(\mathbf{x}) - K_0 u(\mathbf{x})$.

## 4. Derivation in three space dimensions

The derivation in the three-dimensional case is very similar to that of the preceding section. The main observation is that the inversion of the $X-$ray transform can be performed "slice by slice", i.e., "z by z", using outgoing information for angles perpendicular to $\mathbf{e}_z$ only. The inversion with scattering coefficient is again considered as a perturbation of the inversion of the $X-$ray transform. Mathematically, the main novelty compared to the two-dimensional case is that we need to control the amount of photons scattered into the directions orthogonal to $\mathbf{e}_z$.

Upon defining $w(\mathbf{x},\boldsymbol{\theta}) = (e^{Da}u)(\mathbf{x},\boldsymbol{\theta})$, we still obtain that

$$w(\mathbf{x},\boldsymbol{\theta}) = Se^{Da}Ke^{-Da}w(\mathbf{x},\boldsymbol{\theta}) + Se^{Da}f(\mathbf{x},\boldsymbol{\theta}). \tag{50}$$

We define now the trace operator $P$ onto the horizontal directions $S^2_H$ defined in (11). More precisely $P$ takes functions on $\Omega\times S^2$ onto function on $\Omega\times S^2_H$ as follows. For $\boldsymbol{\theta} = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$,

$$P[w(\mathbf{x},\boldsymbol{\theta})] = w(\mathbf{x},(\cos\theta,\sin\theta,0)). \tag{51}$$

For $\boldsymbol{\theta}\in S^2_H$, we define the orthogonal vector $\boldsymbol{\theta}^\perp = (-\sin\theta,\cos\theta,0)$ and the transversal $X-$ray transform

$$Rf(z,s,\theta) = LSf(z,s,\theta), \tag{52}$$

where the trace operator at infinity is defined by

$$Lw(z,s,\theta) = \lim_{t\to\infty} w(t\boldsymbol{\theta} + s\boldsymbol{\theta}^\perp + z\mathbf{e}_z,\boldsymbol{\theta}). \tag{53}$$

Finally the transversal attenuated $X-$ray transform is defined by

$$R_af(z,s,\theta) = Re^{Da}f(z,s,\theta). \tag{54}$$

Note that

$$R = LS = LPS = LSP = RP, \qquad \text{and} \qquad Pe^{Da} = (Pe^{Da})P,$$

so that the rescaled measurements are given by

$$g(z,s,\theta) = e^{\frac{1}{2}Ra}(z,s,\theta)m(z,s,\theta) = R_af(z,s,\theta) + Re^{Da}PKe^{-Da}Tf(z,s,\theta). \tag{55}$$

Now the operator $R_af(z,s,\theta)$ can be inverted at each fixed $z$ by using the Novikov formula. Namely, for $\mathbf{x} = (\mathbf{x}',z)$, we define

$$N_3g(\mathbf{x}) = N[g(z,\cdot,\cdot)](\mathbf{x}'), \tag{56}$$

by applying the two-dimensional operator $N$ to $(s,\theta)\to g(s,\theta,z)$ for each $z\in\mathbb{R}$. We verify that $N_3R_a = Id$ on functions of $\mathbf{x}\in\mathbb{R}^3$. As in the two-dimensional case however, $N_3R_a$ is no longer identity when applied to functions that depend on the variable $\boldsymbol{\theta}$. Thus formally applying the operator $N_3$ to (55), we obtain that

$$N_3g(\mathbf{x}) = f(\mathbf{x}) + N_3Re^{Da}PKe^{-Da}Tf(\mathbf{x}) = (I - N_K)f(\mathbf{x}), \tag{57}$$

where now $N_K = -N_3Re^{Da}PKe^{-Da}T$. The results of section 3 extend as follows.

**Proposition 4.1** *The operator $N_K$ defined above is bounded from $L^2(\Omega)$ to $L^2(\Omega)$.*

Similarly to the planar case, let $\hat{u}(\boldsymbol{\xi}', z, \theta) = \int_{\mathbb{R}^2} e^{-i\mathbf{x}'\cdot\boldsymbol{\xi}'} u(\mathbf{x}', z, \theta) d\mathbf{x}'$ denote the Fourier transform in the first two components of the spatial variable only. We work with the functional space

$$L^{\hat{2}}(\mathbb{R}^2_{\mathbf{x}'} \times \mathbb{R}_z; C^0(S^1)) = \Big\{ u(\mathbf{x}', z, \theta) \text{ s.t. } \hat{u}(\boldsymbol{\xi}', z, \theta) \in L^2(\mathbb{R}^2_{\boldsymbol{\xi}'} \times \mathbb{R}_z; C^0(S^1)) \Big\}, \qquad (58)$$

where $L^2(\mathbb{R}^2_{\boldsymbol{\xi}'} \times \mathbb{R}_z; C^0(S^1))$ is endowed with the norm

$$\|\hat{u}\|^2_{L^2(\mathbb{R}^2_{\boldsymbol{\xi}'} \times \mathbb{R}_z; C^0(S^1))} = \int_{\mathbb{R}^3} \max_{\boldsymbol{\theta}\in S^1} |\hat{u}(\boldsymbol{\xi}', z, \theta)|^2 d\boldsymbol{\xi}' dz.$$

The proposition is based on the following lemmas.

**Lemma 4.2** *Consider the decomposition of $k(\mathbf{x}, \cdot) \in L^2[-1, 1]$ in Legendre polynomials*

$$k(\mathbf{x}, t) = \sum_{n=0}^{\infty} k_n(\mathbf{x}) P_n(t), \qquad (59)$$

*and assume that, for some $\alpha > 1$,*

$$\max_{n\in\mathbb{N}} \left( \langle n \rangle^{\alpha-1} \max_{|m|\le n} \max_{\boldsymbol{\theta}\in S^2_z} |Y_{nm}(\boldsymbol{\theta})|^2 \int_{\mathbb{R}} \|\hat{k}_n(\cdot, z)\|^2_{L^1(\mathbb{R}^2)} dz \right) \le C. \qquad (60)$$

*Then the operator $PK$ maps $L^2(\mathbb{R}^3 \times S^2)$ to $L^{\hat{2}}(\mathbb{R}^2_{\mathbf{x}'} \times \mathbb{R}_z; C^0(S^1))$.*

*Proof.* Using the summation formula $P_n(\boldsymbol{\theta}\cdot\boldsymbol{\theta}') = \frac{1}{2n+1} \sum_{m=-n}^{n} Y_{nm}(\boldsymbol{\theta}) Y^*_{nm}(\boldsymbol{\theta}')$ (see [17] for instance), we get the following decomposition of the scattering operator

$$Ku(\mathbf{x}, \boldsymbol{\theta}) = \int_{S^2} k(\mathbf{x}, \boldsymbol{\theta}\cdot\boldsymbol{\theta}') u(\mathbf{x}, \boldsymbol{\theta}') d\boldsymbol{\theta}' = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{1}{2n+1} k_n(\mathbf{x}) u_{nm}(\mathbf{x}) Y_{nm}(\boldsymbol{\theta}), \qquad (61)$$

where

$$u_{nm}(\mathbf{x}) = \int_{S^2} u(\mathbf{x}, \boldsymbol{\theta}') \overline{Y_{nm}(\boldsymbol{\theta}')} d\boldsymbol{\theta}'. \qquad (62)$$

The Plancherel identity for the spherical harmonics gives

$$\|u\|^2_{L^2(\mathbb{R}^3\times S^2)} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \|u_{nm}\|^2_{L^2(\mathbb{R}^3)}. \qquad (63)$$

In what follows we consider $\boldsymbol{\theta} \in S^2_H$, i.e., only horizontal directions. To simplify the notation, we denote by

$$\beta_n = \max_{|m|\le n} \max_{\boldsymbol{\theta}\in S^2_H} |Y_{nm}(\boldsymbol{\theta})|. \qquad (64)$$

Taking the Fourier transform with respect to the horizontal variables in (61), we obtain

$$
\begin{aligned}
|\widehat{Ku}(\boldsymbol{\xi}, z, \boldsymbol{\theta})|^2 &= \left|\sum_{n=0}^{\infty}\sum_{m=-n}^{n} \frac{1}{2n+1}(\hat{k}_n *_{\boldsymbol{\xi}} \hat{u}_{nm})(\boldsymbol{\xi}, z) Y_{nm}(\boldsymbol{\theta})\right|^2 \\
&\leq \left(\sum_{n=0}^{\infty}\sum_{m=-n}^{n} \frac{\beta_n}{2n+1}\left|(\hat{k}_n * \hat{u}_{nm})(\boldsymbol{\xi}, z)\right|\right)^2 \\
&\leq \left(\sum_{n=0}^{\infty}\sum_{m=-n}^{n} \frac{1}{\langle n\rangle^\alpha(2n+1)}\right)\left(\sum_{n=0}^{\infty}\sum_{m=-n}^{n} \frac{\langle n\rangle^\alpha \beta_n^2}{2n+1}\left|(\hat{k}_n * \hat{u}_{nm})(\boldsymbol{\xi}, z)\right|^2\right).
\end{aligned}
$$

We now take the maximum in $\boldsymbol{\theta} \in S_z^2$ then integrate in $\boldsymbol{\xi} \in \mathbb{R}^2$. We deduce that

$$
\begin{aligned}
\int_{\mathbb{R}^2} \max_{\boldsymbol{\theta}\in S_z^2} |\widehat{Ku}(\boldsymbol{\xi}, z, \boldsymbol{\theta})|^2 d\boldsymbol{\xi} &\leq \left(\sum_{n=0}^{\infty} \frac{1}{\langle n\rangle^\alpha}\right)\left(\sum_{n=0}^{\infty}\sum_{m=-n}^{n} \frac{\beta_n^2\langle n\rangle^\alpha}{2n+1}\|\hat{k}_n * \hat{u}_{nm}(\cdot, z)\|^2_{L^2(\mathbb{R}^2)}\right) \\
&\leq \left(\sum_{n=0}^{\infty} \frac{1}{\langle n\rangle^\alpha}\right)\left(\sum_{n=0}^{\infty}\sum_{m=-n}^{n} \frac{\beta_n^2\langle n\rangle^\alpha}{2n+1}\|\hat{k}_n(\cdot, z)\|^2_{L^1(\mathbb{R}^2)}\|\hat{u}_{nm}(\cdot, z)\|^2_{L^2(\mathbb{R}^2)}\right) \\
&\leq \left(\max_{n\in\mathbb{N}} \frac{\beta_n^2\langle n\rangle^\alpha}{2n+1}\|\hat{k}_n(\cdot, z)\|^2_{L^1(\mathbb{R}^2)}\right)\left(\sum_{n=0}^{\infty} \frac{1}{\langle n\rangle^\alpha}\right)\left(\sum_{n=0}^{\infty}\sum_{m=-n}^{n} \|\hat{u}_{nm}(\cdot, z)\|^2_{L^2(\mathbb{R}^2)}\right).
\end{aligned}
$$

It remains to integrate in $z \in \mathbb{R}$ to obtain that

$$
||Ku||^2_{L^{\hat{2}}(\mathbb{R}^2_{\mathbf{x}'}\times\mathbb{R}_z; C^0(S^1))} \leq \left(\max_{n\in\mathbb{N}} \frac{\beta_n^2\langle n\rangle^\alpha}{2n+1}\int_{\mathbb{R}} \|\hat{k}_n(\cdot, z)\|^2_{L^1(\mathbb{R}^2)} dz\right)\left(\sum_{n=0}^{\infty} \frac{1}{\langle n\rangle^\alpha}\right)||u||^2_{L^2(\mathbb{R}^3\times S^2)}.
$$

This concludes the proof of the lemma. □

**Lemma 4.3** *The operator $N_3R$ maps $L^{\hat{2}}(\mathbb{R}^2_{\mathbf{x}'} \times \mathbb{R}_z; C^0(S^1))$ to $L^2(\Omega)$.*

*Proof.* This is a direct consequence of Lemma 3.6:

$$
\begin{aligned}
\|N_3Rf\|^2_{L^2(\Omega)} &= \int_{\mathbb{R}}\int_{\mathbb{R}^2} |N_3Rf(\mathbf{x}', z)|^2 \, d\mathbf{x}'dz = \int_{\mathbb{R}}\int_{\mathbb{R}^2} |[NRf(\cdot,\cdot,z)](\mathbf{x}')|^2 \, d\mathbf{x}'dz \\
&\leq C\int_{\mathbb{R}} dz\left\{\int_{\mathbb{R}^2} \max_{\boldsymbol{\theta}\in S_z^2}\left|\hat{f}(\boldsymbol{\xi}, \boldsymbol{\theta}, z)\right|^2 d\boldsymbol{\xi}\right\} = \|f\|^2_{L^{\hat{2}}(\mathbb{R}^2_{\mathbf{x}'}\times\mathbb{R}_z; C^0(S^1))}. \qquad (65)
\end{aligned}
$$

□

The rest of the proof of Theorem 2.2 is similar to that of Theorem 2.1. Provided that scattering is sufficiently small, the following Neumann series expansion

$$
f(\mathbf{x}) = \sum_{n=0}^{\infty} N_K^n N_3 g(\mathbf{x}), \qquad (66)
$$

converges in $L^2(\Omega)$ strongly to the solution $f(\mathbf{x})$.

The remarks at the end of section 3 still hold in the three dimensional setting. The main difference between the two-dimensional and three-dimensional theories is that the scattering operator is required to be more regularizing in three dimensions than in two dimensions. This is so because the three dimensional reconstruction is based on measurements of the outgoing distribution for directions that are orthogonal to the vertical axis $\mathbf{e}_z$. The influence of the geometry on the norm of $N_K$ could be characterized in the three dimensional setting as we have for the two dimensional setting in Corollary 3.7, althgouh we shall not do so here.

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