Introduction to Inverse Problems

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Contents

1 What is an Inverse Problem **1**
- 1.1 Elements of an *Inverse Problem* . . . 2
  - 1.1.1 Injectivity and stability of the Measurement Operator . . . 2
  - 1.1.2 Noise model and Prior model . . . 3
  - 1.1.3 Numerical algorithms . . . 5
- 1.2 Examples of Measurement Operator . . . 7
- 1.3 IP and Modeling. Application to MRI . . . 9
- 1.4 Inverse Problems, Smoothing, and ill-posedness . . . 13
  - 1.4.1 Well-posed Problems and Lipschitz Stability. . . . 13
  - 1.4.2 Ill-posed Problems and Unbounded Operators. . . . 14
  - 1.4.3 Smoothing and Sobolev scale of Hilbert spaces. . . . 15

2 Integral Geometry. Radon transforms **19**
- 2.1 The Radon Transform . . . 19
  - 2.1.1 Transmission Tomography . . . 19
  - 2.1.2 Two dimensional X-ray (Radon) transform . . . 21
  - 2.1.3 Three dimensional Radon transform . . . 28
- 2.2 Attenuated Radon Transform . . . 29
  - 2.2.1 Single Photon Emission Computed Tomography . . . 30
  - 2.2.2 Riemann Hilbert problem . . . 31
  - 2.2.3 Inversion of the Attenuated Radon Transform . . . 32
  - 2.2.4 Step (i): The $\overline{\partial}$ problem, an elliptic equation . . . 33
  - 2.2.5 Step (ii): jump conditions . . . 35
  - 2.2.6 Step (iii): reconstruction formulas . . . 37

3 Integral geometry and FIO **39**
- 3.1 Radon transform and FIO . . . 39
- 3.2 Oscillatory integrals, symbols, and phases . . . 40
- 3.3 Schwartz kernels and Fourier integral operators . . . 43
- 3.4 Critical set and propagation of singular support . . . 44
- 3.5 Applications to ΨDO and Radon transform . . . 46
- 3.6 Propagation of singularities for FIO. . . . 48
  - 3.6.1 Wave Front Set and Distributions. . . . 48
  - 3.6.2 Propagation of singularities in linear transforms . . . 49
- 3.7 Critical sets and Lagrangian manifolds . . . 53
- 3.8 Local theory of FIO . . . 54

3.9 Elliptic FIO and Elliptic ΨDO . . . 57
3.10 Local tomography and partial data . . . 59

4 Integral Geometry. Generalized Ray Transform 63
4.1 Generalized Ray Transform in two dimensions. . . . 63
4.1.1 Family of curves. . . . 63
4.1.2 Generalized Ray Transform. . . . 64
4.1.3 Adjoint operator and rescaled Normal operator . . . 65
4.2 Pseudo-differential operators and GRT . . . 67
4.2.1 Change of phase $(x-y)\cdot\xi$ . . . 67
4.2.2 Choice of a parametrix. . . . 68
4.2.3 Change of symbol, smoothing and composition . . . 69
4.2.4 Continuity of ΨDO in the spaces $H^s(X)$ . . . 71
4.2.5 Left-parametrices and injectivity . . . 73
4.3 Kinematic Inverse Source Problem . . . 74
4.3.1 Transport equation . . . 74
4.3.2 Variational form and energy estimates . . . 75
4.3.3 Injectivity result . . . 76
4.4 Summary on GRT. . . . 76
4.5 Application to generalized Radon transforms . . . 77
4.6 Composition and continuity of FIO . . . 79

Appendices 81

A Remarks on composition of FIOs and Stationary phase 83
A.1 Remarks on composition of FIOs . . . 83
A.2 Method of stationary phase . . . 88

5 Inverse wave problems 91
5.1 One dimensional inverse scattering problem . . . 92
5.2 Linearized Inverse Scattering problem . . . 94
5.2.1 Setting and linearization . . . 94
5.2.2 Far field data and reconstruction . . . 96
5.2.3 Comparison to X-ray tomography . . . 99
5.3 Inverse source problem in PAT . . . 100
5.3.1 An explicit reconstruction formula for Σ the unit sphere . . . 100
5.3.2 An explicit reconstruction for detectors on a plane . . . 101
5.4 One dimensional inverse coefficient problem . . . 103

6 Inverse Kinematic and Inverse Transport Problems 107
6.1 Inverse Kinematic Problem . . . 108
6.1.1 Spherical symmetry . . . 109
6.1.2 Abel integral and Abel transform . . . 111
6.1.3 Kinematic velocity Inverse Problem . . . 112
6.2 Forward transport problem . . . 114
6.3 Inverse transport problem . . . 118
6.3.1 Decomposition of the albedo operator and uniqueness result . . . 118

6.3.2 Stability in inverse transport . . . 122

7 Inverse diffusion 125
7.1 Cauchy Problem and Electrocardiac potential . . . 125
7.2 Half Space Problem . . . 126
7.2.1 The well posed problem . . . 127
7.2.2 The electrocardiac application . . . 128
7.2.3 Prior bounds and stability estimates . . . 128
7.2.4 Analytic continuation . . . 130
7.3 General two dimensional case . . . 131
7.3.1 Laplace equation on an annulus . . . 131
7.3.2 Riemann mapping theorem . . . 132
7.4 Backward Heat Equation . . . 132
7.5 Carleman Estimates, UCP and Cauchy problem . . . 134

8 Calderón problem 141
8.1 Introduction . . . 141
8.2 Uniqueness and Stability . . . 142
8.2.1 Reduction to a Schödinger equation . . . 142
8.2.2 Proof of injectivity result . . . 144
8.2.3 Proof of the stability result . . . 145
8.3 Complex Geometric Optics Solutions . . . 147
8.4 The Optical Tomography setting . . . 150
8.5 Anisotropy and non-uniqueness . . . 152

9 Coupled-physics IP I: PAT and TE 155
9.1 Introduction to PAT and TE . . . 156
9.1.1 Modeling of photoacoustic tomography . . . 156
9.1.2 First step: Inverse wave source problem . . . 158
9.1.3 Second step: Inverse problems with internal functionals . . . 159
9.1.4 Reconstruction of one coefficient. . . . 160
9.1.5 Introduction to Transient Elastography . . . 161
9.2 Theory of quantitative PAT and TE . . . 161
9.2.1 Uniqueness and stability results in QPAT . . . 161
9.2.2 Application to Quantitative Transient Elastography . . . 167
9.3 *Well-chosen* illuminations in PAT and TE . . . 168
9.3.1 The two dimensional case . . . 168
9.3.2 The n dimensional case . . . 169
9.4 The case of anisotropic coefficients . . . 170
9.4.1 Linear algebra problem . . . 170
9.4.2 Explicit reconstructions . . . 171
9.4.3 Control problem . . . 173

10 Coupled-physics IP II: UMT 175
10.1 Ultrasound Modulation Tomography . . . 175
10.2 Inverse problems in ultrasound modulation. . . . 177
10.3 Eliminations and redundant systems of ODEs . . . 180

10.3.1 Elimination of F . . . 180
10.3.2 System of ODEs for S_j . . . 182
10.3.3 ODE solution and stability estimates . . . 184
10.4 Well-chosen illuminations . . . 184
10.4.1 The case $n = 2$ . . . 184
10.4.2 The case $n \geq 3$ . . . 184
10.5 Remarks on hybrid inverse problems . . . 186

11 Priors and Regularization **189**
11.1 Smoothness Regularization . . . 190
11.1.1 Ill-posed problems and compact operators . . . 190
11.1.2 Regularity assumptions and error bound . . . 191
11.1.3 Regularization methods . . . 194
11.2 Sparsity and other Regularization Priors . . . 201
11.2.1 Smoothness Prior and Minimizations . . . 201
11.2.2 Sparsity Prior and Minimizations . . . 202
11.3 Bayesian framework and regularization . . . 203
11.3.1 Penalization methods and Bayesian framework . . . 204
11.3.2 Computational and psychological costs of the Bayesian framework 205

12 Geometric Priors and Parameterizations **209**
12.1 Reconstructing the domain of inclusions . . . 209
12.1.1 Forward Problem . . . 210
12.1.2 Factorization method . . . 211
12.1.3 Reconstruction of Σ . . . 215
12.2 Reconstructing small inclusions . . . 216
12.2.1 First-order effects . . . 217
12.2.2 Stability of the reconstruction . . . 219

A Notation and results **221**
A.1 Fourier transform . . . 221

Bibliography **223**

Index **229**

Chapter 1

What is an Inverse Problem

Three essential ingredients define an inverse problem in this book. The central element is the *Measurement Operator* (MO), which maps objects of interest, called *parameters*, to information collected about these objects, called *measurements* or *data*. The main objective of inverse problem theory is to analyze such a MO, primarily its injectivity and stability properties. Injectivity of the MO means that acquired data *uniquely* characterize the parameters. Often, the inversion of the MO amplifies *errors* in the measurements, which we refer to as *noise*. *Stability estimates* characterize this amplification.

When the amplification is considered "too large" by the user, which is a subjective notion, then the inverse problem needs to be modified. How this should be done depends on the structure of *noise*. The second essential ingredient of an inverse problem is thus a *noise* model, for instance a statement about its size in a given metric, or, if available, its statistical properties.

Once a MO and a noise model are available, the "too large" effect of noise on the reconstruction is mitigated by imposing additional constraints on the *parameters* that render the inversion *well-posed*. These constraints take the form of a *prior model*, for instance assuming that the parameters live in a finite dimensional space, or that parameters are sparsely represented in an appropriate frame.

The three elements (MO, noise model, prior model) are described in more detail in section 1.1. Examples of MO are given in section 1.2.

The role of modeling is often crucial in the derivation of the triplet (MO, noise model, prior model). In section 1.3, three different models of MO are obtained in a simplified setting of Magnetic Resonance Imaging, one of the most successful medical imaging modality.

Once a MO has been selected and proved to be injective, we need to understand how its inversion amplifies *noise*. Typically, difficulties in inverse problems arise because such an amplification becomes larger for higher frequencies. This is related to the *smoothing* properties of the MO. We introduce a Hilbert scale of spaces in section 1.4 to quantify such an amplification for a restricted but pedagogically useful class of inverse problems.

These preliminary notations set the stage for the introductory analysis in later chapters of several (MO, noise model, prior model) that find applications in many inverse problems

1.1 Elements of an *Inverse Problem*

The definition of an *inverse problem* (IP) starts with that of a mapping between objects of interest, which we call *parameters*, and acquired information about these objects, which we call *data* or *measurements*. The mapping, or *forward problem*, is called the *measurement operator* (MO). We denote it by $\mathfrak{M}$.

The MO maps parameters in a functional space $\mathfrak{X}$, typically a Banach or Hilbert space, to the space of *data* $\mathfrak{Y}$, typically another Banach or Hilbert space. We write

$$\mathrm{y} = \mathfrak{M}(\mathrm{x}) \quad \text{for } \mathrm{x} \in \mathfrak{X} \text{ and } \mathrm{y} \in \mathfrak{Y}, \tag{1.1}$$

the correspondence between the parameter x and the data y. Solving the inverse problem amounts to finding point(s) $\mathrm{x} \in \mathfrak{X}$ from knowledge of the data $\mathrm{y} \in \mathfrak{Y}$ such that (1.1) or an approximation of (1.1) holds.

The MO describes our best effort to construct a *model* for the available data y, which we assume here depend only on the sought parameters x. The choice of $\mathfrak{X}$ describes our best effort to characterize the space where we believe the parameters belong.

1.1.1 Injectivity and stability of the Measurement Operator

The first question to ask about the MO is whether we have acquired enough data to uniquely reconstruct the parameters. In other words, whether the MO is *injective*. Injectivity is the following statement:

$$\mathfrak{M}(\mathrm{x}_1) = \mathfrak{M}(\mathrm{x}_2) \qquad \Longrightarrow \qquad \mathrm{x}_1 = \mathrm{x}_2 \text{ for all } \quad \mathrm{x}_1, \mathrm{x}_2 \in \mathfrak{X}. \tag{1.2}$$

Then the data y, if given in the range of $\mathfrak{M}$, uniquely characterize the parameter x.

Measurement operators used in practice are typically discretized and available data typically contain noise as we shall see below. Such measurement operators are often not (quite) injective. Yet, most practical measurement operators can be seen as approximations to measurement operators that are indeed injective.

When $\mathfrak{M}$ is injective, we can construct an inversion operator $\mathfrak{M}^{-1}$ mapping the range of $\mathfrak{M}$ to a uniquely defined element in $\mathfrak{X}$. This inverse operation is what is implemented numerically in practice. The main features of the inverse operator are captured by what are called *stability estimates*. Such estimates quantify how errors in the available measurements translate into errors in the reconstructions. They take the form:

$$\|\mathrm{x}_1 - \mathrm{x}_2\|_{\mathfrak{X}} \leq \omega(\|\mathfrak{M}(\mathrm{x}_1) - \mathfrak{M}(\mathrm{x}_2)\|_{\mathfrak{Y}}), \tag{1.3}$$

where $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing function, such that $\omega(0) = 0$, quantifying the modulus of continuity of the inversion operator $\mathfrak{M}^{-1}$. This function gives an estimate of the reconstruction error $\|\mathrm{x}_1 - \mathrm{x}_2\|_{\mathfrak{X}}$ based on what we believe is the error in the data acquisition $\|\mathfrak{M}(\mathrm{x}_1) - \mathfrak{M}(\mathrm{x}_2)\|_{\mathfrak{Y}}$.

When noise is not amplified too drastically so that the error on the reconstructed parameters is *acceptable*, for instance *typically* when $\omega(\mathrm{x}) = C\mathrm{x}$ for some constant C, then we say that the inverse problem is *well-posed*. When noise is strongly amplified and the reconstruction is contaminated by *too large* a noisy component, for instance

typically when $\omega(\mathrm{x}) = |\log|\mathrm{x}||^{-1}$ so that measurement errors of 10^{-10} translate into reconstruction errors of $\frac{1}{10}$, then we say that the inverse problem is *ill-posed*.

The notion of the ill-posedness of an inverse problem is thus to some extent a *subjective* notion. The modulus of continuity ω depends on the choices of $\mathfrak{X}$ and $\mathfrak{Y}$. Let us assume that the metric space $\mathfrak{Y}$ is endowed with a metric $d(\mathrm{y}_1, \mathrm{y}_2) = \omega(\|\mathrm{y}_1 - \mathrm{y}_2\|_{\mathfrak{Y}})$. Then the distance between the two reconstructions x_1 and x_2 is bounded by the distance between the two measurements y_1 and y_2, which seems to correspond to a "nice", well-posed, inverse problem. The stability estimates and corresponding moduli of continuity are thus *subjective*.

Note that we considered the difference between $\mathrm{y}_1 = \mathfrak{M}(\mathrm{x}_1)$ and $\mathrm{y}_2 = \mathfrak{M}(\mathrm{x}_2)$, that is to say, errors of measurements in the *range* of the measurement operator. This is legitimate since all that we have constructed so far is measurements in $\mathfrak{Y}$ that are in the range of the measurement operator $\mathfrak{M}$. In practice, however, noise in the data acquisition may cause the measurements to leave the natural set $\mathfrak{M}(\mathfrak{X})$ where measurements are currently defined. Data then need to be projected onto the range of the MO first, or an entirely new set of stability estimates need to be developed.

In spite of the aforementioned caveats, the notions of injectivity and stability of a measurement operator are very fruitful concepts that provide practically useful information about the structure of the inverse problem of interest. Most of this book is devoted to the analysis of injectivity of different MO and the derivation of (typically several) corresponding stability estimates. Most practical inverse problems are approximations to injective MOs with well-characterized stability properties. Most successful imaging modalities are modeled by injective MOs with *good* stability properties.

1.1.2 Noise model and Prior model

The MO is typically not sufficient to describe an IP *satisfactorily*. Often, "noisy" contributions need to be added to realistically model the available data, for instance contributions of parameters not included in x because we have no chance of reconstructing them, or random fluctuations in how detectors operate. Such errors may have *undesirable* effects on the reconstructions, which, in turn, requires that the user introduce prior constraints on the parameters. We first define what we mean by noise.

Modeling and Measurement Errors. Rather than modeling the IP as $\mathrm{y} = \mathfrak{M}(\mathrm{x})$, it is customary to define an IP as $\mathrm{y} = \mathfrak{M}(\mathrm{x}) + \mathrm{n}$, where n is "noise" and its definition is precisely

$$\mathrm{n} = \mathrm{n}(\mathrm{x}, \mathrm{y}) := \mathrm{y} - \mathfrak{M}(\mathrm{x}), \tag{1.4}$$

the *discrepancy* between the model $\mathfrak{M}(\mathrm{x})$ and the available data y. n typically has two contributions. One standard contribution is the *detector* noise, since measurements are performed by instruments that are imperfect. Often, an equally important or more important contribution is what we may call a *modeling* error, for instance reflecting that $\mathfrak{M}$ is an imperfect modeling of the physical underlying process mapping parameters x to data y.

Let us assume that we can project the available data $\tilde{\mathrm{y}}$ onto the range of $\mathfrak{M}$ and decompose it as $\mathrm{y} = \mathfrak{M}(\mathrm{x}_2)$ plus $\tilde{\mathrm{y}} - \mathfrak{M}(\mathrm{x}_2)$. We discard $\tilde{\mathrm{y}} - \mathrm{y}$ for the moment. Then

$n = \mathfrak{M}(x_2) - \mathfrak{M}(x)$ and the stability estimates provide a measure of the error $x - x_2$ in the reconstruction. Estimating the above error term is therefore an important part of solving an IP in practice. Several guiding principles to do so will be considered in Chapter 12. For the moment, let us simply mention that it often makes sense to model n as a "random" process. Indeed, if a simple (deterministic) model for n was available, that is a model involving well-characterized parameters, then we would modify the MO to account for said model.

The randomness comes from the aforementioned sources: the detector and modeling errors. What are the statistics of n, and in particular the ensemble average $m(x) = \mathbb{E}\{n(x)\}$ and the two point correlation function $c(x, y) = \{n(x)n(y)\}$ are practically important information whose knowledge can very significantly improve the solution to a given IP.

Prior model. We now have two items characterizing an IP: a MO and a noise model. The very reason "noise" is modeled is because it has an *undesirable* effect on the reconstruction, which is estimated by the stability estimates associated to the MO. One is then faced with essentially (a combination of) three strategies: (i) acquire more accurate data to lower the size of $\|n\|_{\mathcal{Y}}$ and hence of $\|x_1 - x_2\|_{\mathcal{X}}$. When the stability estimates imply a strong amplification of the noise level in the reconstruction, this strategy has limited value; (ii) change the MO and acquire different data. This ideal scenario may not always be feasible; and (iii) restrict the class in which the unknown parameters are sought. We now focus on (iii) and thus need a *prior model*. Several major methodologies to do so can be classified as follows:

1. *Penalization theory* is a deterministic methodology that restricts the domain of definition of the parameters. It includes two subcategories, one being *regularization theory*, which assumes that the parameters of interest are "sufficiently smooth", and the other one being *sparsity theory*, which assumes that the parameters are sparsely represented in a given, specific, basis.

2. The *Bayesian framework* is an alternative, very versatile, methodology to incorporate prior assumptions in a statistical fashion. A *prior* probability density describes the potential values that the parameters can take. A *posterior* probability density describes how these potential values are affected by the measurements.

3. Several *geometric constraints* give rise to simplified reconstructions (of small inclusions for instance) or qualitative methods of reconstructions (of support of inclusions for instance).

The structure (MO, noise model, prior model) completes the description of the inverse problem. Which prior model should be chosen strongly depends on the other two components (MO, noise model). Heuristically, when the inversion of the MO does not amplifies noise too drastically, then this limited lack of information in the available data can be compensated by *mild* prior models, typically those considered in item 1 above. However, when noise is strongly amplified, then more *drastic* prior models, such as those in items 2 and 3, need to be chosen in order to compensate for the fact that available data are not very informative. The analysis of an inverse problem therefore starts with

that of the stability properties of the MO. Most of the rest of the book is devoted to this task.

1.1.3 Numerical algorithms

The choice of the prior model has important consequences on the complexity of the numerical algorithms that are used to solve the inverse problem in practice. We briefly describe such complexity in cases that often appear in practice.

When the prior model takes the form of a penalization, then the computational cost is typically reasonable and addressed in items (a) and (b) below. In the more versatile Bayesian framework, used to compensate for the lack of information in the available data, the computational cost may be significantly higher as is explained in item (c) below. When the prior model consists of geometric constraints, the computational cost then wildly depends on the choice of geometric constraints; see item (d) below.

To simplify the presentation, we assume that $\mathfrak{M}$ is linear and consider an inverse problem of the form $\mathrm{y} = \mathfrak{M}\mathrm{x} + \mathrm{n}$. Nonlinear problems $\mathrm{y} = \mathfrak{M}(\mathrm{x}) + \mathrm{n}$ are typically linearized, for instance by iteratively solving linear problems in the form $\mathrm{y} - \mathfrak{M}(\mathrm{x}_0) = A(\mathrm{x}-\mathrm{x}_0)+\mathrm{n}$, where A is some differential of $\mathfrak{M}$ at x_0. The computational cost of solving the inverse problem modeled by $\mathrm{y} = \mathfrak{M}\mathrm{x} + \mathrm{n}$ typically falls into one of the following categories.

(a) Linear penalization theories such as regularization theories typically replace $\mathrm{y} = \mathfrak{M}\mathrm{x} + \mathrm{n}$ by

$$\mathfrak{M}^*\mathrm{y} = (\mathfrak{M}^*\mathfrak{M} + \delta B)\mathrm{x}_\delta, \tag{1.5}$$

where $\delta > 0$ is a regularization parameter and B is a positive definite operator such that $(\mathfrak{M}^*\mathfrak{M} + \delta B)$ is an invertible operator with bounded inverse. Here $\mathfrak{M}^*$ is the adjoint operator to $\mathfrak{M}$. The inverse problem then involves solving a *linear system* of equations.

(b) Penalization theories such as those based on sparsity constraints typically replace $\mathrm{y} = \mathfrak{M}\mathrm{x} + \mathrm{n}$ by

$$\mathrm{x}_\delta = \operatorname{argmin} \|\mathrm{y} - \mathfrak{M}\mathrm{x}\|_{\mathfrak{Y}_1} + \delta\|\mathrm{x}\|_{\mathfrak{X}_2}, \tag{1.6}$$

where $\mathfrak{Y}_1$ is typically a L^2 norm and $\mathfrak{X}_2$ for instance a L^1 norm to promote sparsity. Again, $\delta > 0$ is a small, regularizing parameter. Solving such a problem therefore requires solving an optimization (minimization) problem, which is algorithmically more challenging than the linear problem in (1.5).

(c) In the setting of IPs with poor stability estimates, the available data may not be very informative. The prior model then needs to compensate for this lack of information. The Bayesian framework offers a very versatile way to introduce prior models.

The premise of such a framework is that we *know* a *prior distribution* $\pi(\mathrm{x})$ which assigns a probability (density) to all potential candidates x before any data are acquired. We also assume that we *know* the *likelihood function*, which is the *conditional* distribution $\pi(\mathrm{y}|\mathrm{x})$ of the data y conditioned on knowing the parameter x. This is equivalent to knowing the distribution of the noise n. Bayes theorem then states that the *posterior distribution* is given by

$$\pi(\mathrm{x}|\mathrm{y}) = C\pi(\mathrm{y}|\mathrm{x})\pi(\mathrm{x}). \tag{1.7}$$

Here C is a normalizing constant so that $\pi(\mathrm{x}|\mathrm{y})$ is a probability density, i.e., so that $\int_{\mathfrak{X}} \pi(\mathrm{x}|\mathrm{y}) d\mu(\mathrm{x}) = 1$ for $d\mu(\mathrm{x})$ the measure of integration on $\mathfrak{X}$. In other words, what we know a priori, $\pi(\mathrm{y}|\mathrm{x})$, the noise model, and $\pi(\mathrm{x})$, the prior model, before acquiring any data, plus the additional knowledge obtained from measuring y, allows us to calculate $\pi(\mathrm{x}|\mathrm{y})$ the probability distribution of the unknown parameter x knowing the data y.

There are many advantages and caveats associated to this formalism, which will be presented in more detail in Chapter 11. However, from a computational view point, the Bayesian framework poses extraordinary challenges. If we know the distribution of the noise level n, then estimating $\pi(\mathrm{y}|\mathrm{x})$ requires solving a forward problem $\mathrm{x} \mapsto \mathfrak{M}(\mathrm{x})$. This calculation has to be performed for a large number of values of x in order to sample $\pi(\mathrm{x}|\mathrm{y})$. Moreover, sampling a distribution $\pi(\mathrm{x}|\mathrm{y})$ that does not necessarily admit a closed form expression is in itself computationally very intensive. Sampling (1.7) is thus computationally significantly more challenging than solving (1.5) or (1.6).

One of the main advantages of the Bayesian framework is that it insists on an important aspect of inverse problems in general that motivated our description of an inverse problem as the triplet (MO, noise model, prior model): data are often not sufficiently informative to provide accurate reconstructions of x. The Bayesian framework recognizes this fact by providing as an output a distribution of possible parameters x with probability $\pi(\mathrm{x}|\mathrm{y})$.

Once it is defined, this posterior probability density needs to be processed and presented in a way that we can analyze and understand. Two main methodologies are used to do so. The first one corresponds to estimating the *maximum likelihood* of the posterior distribution

$$\mathrm{x} = \operatorname{argmax} \pi(\mathrm{x}|\mathrm{y}) = \operatorname{argmin} \, -\pi(\mathrm{x}|\mathrm{y}). \tag{1.8}$$

This MAP (maximum a posteriori) estimation is a minimization problem that bears very strong resemblances with (1.6), including its computational cost. Most models of the form (1.6) can be recast as MAP estimators of a Bayesian posterior.

A second, more accurate, method consists of estimating several statistical moments of $\pi(\mathrm{x}|\mathrm{y})$, for instance

$$\bar{\mathrm{x}} = \int_{\mathfrak{X}} \mathrm{x} \pi(\mathrm{x}|\mathrm{y}) d\mu(\mathrm{x}), \qquad \mathrm{c} = \int_{\mathfrak{X}} (\mathrm{x} - \bar{\mathrm{x}}) \otimes (\mathrm{x} - \bar{\mathrm{x}}) \pi(\mathrm{x}|\mathrm{y}) \mathrm{d}\mu(\mathrm{x}), \tag{1.9}$$

with $\bar{\mathrm{x}}$ the ensemble average of x and c the correlation "matrix" of x. For $\mathrm{x} = (\mathrm{x}_i)$ a vector, think of $\mathrm{x} \otimes \tilde{\mathrm{x}}$ as the matrix with elements $\mathrm{x}_i \tilde{\mathrm{x}}_j$. Here, $d\mu(\mathrm{x})$ is a measure of integration on $\mathfrak{X}$. Other important moments, for instance for *uncertainty quantification* involve the estimation of the quantiles of x:

$$\varphi_\delta = \int_{\mathfrak{X}} H(\varphi(\mathrm{x}) - \delta) \pi(\mathrm{x}|\mathrm{y}) d\mu(\mathrm{x}). \tag{1.10}$$

Here φ is a functional from $\mathfrak{X}$ to $\mathbb{R}$, and could be for instance a norm of x, or the value of one component of a vector valued x, or the evaluation of the parameter x at a spatial point and/or a specific time. Also, $H(t)$ is the Heaviside function equal to $H(t) = 1$ for $t \geq 0$ and $H(t) = 0$ for $t < 0$. Then φ_δ above is the probability that $\varphi(\mathrm{x})$ be larger than a given parameter $\delta > 0$.

All of these moments require that we *sample* $\pi(\mathrm{x}|\mathrm{y})$. This is an extremely difficult and computationally expensive task. Standard methodologies to do so involve the Markov Chain Monte Carlo method that will be briefly presented in Chapter 11.

(d) Geometric constraints and qualitative methods, are different prior models. They severely restrict the set of potential parameters or try to recover not all of x but rather partial information about x, such as for instance its spatial support. Many geometric constraints and qualitative methods have been developed in the mathematical literature. Their main advantage is that they are computationally much more tractable than Bayesian reconstructions; see Chapter 12.

1.2 Examples of Measurement Operator

We now present several examples of measurement operators. The first three MO are mostly pedagogical, while the last two examples of MO find important applications in, e.g., medical imaging.

Example 1. Let $\mathfrak{X} = \mathcal{C}([0,1])$, the space of continuous functions. Let $\mathfrak{Y} = \mathfrak{X}$ and define

$$\mathfrak{M}(f)(x) = \int_0^x f(y)dy.$$

Here, a "point" x in $\mathfrak{X}$ is the function denoted by f, traditionally denoted by $f(x)$. The operator $\mathfrak{M}$ is certainly injective since the equality of data $\mathfrak{M}(f) = \mathfrak{M}(g)$ implies that $f = \frac{d}{dx}\mathfrak{M}(f) = \frac{d}{dx}\mathfrak{M}(g) = g$, i.e., the equality of parameters.

However, the operator $\mathfrak{M}$ is "smoothing" since $\mathfrak{M}(f)$ is one derivative more regular than f. So inverting the operator, as indicated above, involves differentiating the data. If "*noise*" is added to the data and "*noise*" is high frequency, the the derivative of "*noise*" will be large and may overwhelm the reconstruction of $f(x)$. The objective of penalization theory is precisely to make sure this *undesirable* fact does not happen.

Example 2. On $\mathfrak{X} = \mathcal{C}^1_0([0,1])$, the space of continuously differentiable functions with value 0 at $x = 0$ and $\mathfrak{Y} = \mathcal{C}([0,1])$, we can define

$$\mathfrak{M}(f)(x) = f'(x), \qquad \text{the derivative of } f.$$

Note that the derivative of a continuous function is not continuous (and may not exist as a function although it can be defined as a distribution). So here, we found it convenient to define the domain of definition of $\mathfrak{M}$ as a subset of the space of continuous functions.

Why do we also insist on $f(0) = 0$ in $\mathfrak{X}$? It is because otherwise, $\mathfrak{M}$ would *not* be injective since antiderivatives are defined up to a constant. In $\mathfrak{X}$, we have from the fundamental theory of calculus that

$$f(x) = f(0) + \int_0^x f'(y)dy = \int_0^x f'(y)dy = \int_0^x \mathfrak{M}(f)(y)dy.$$

Now obviously, $\mathfrak{M}(f) = \mathfrak{M}(g)$ implies $f = g$. From the point of view of inverse problems, this is a very favorable situation. If "*noise*" is added to $\mathfrak{M}(f)$, it will be *integrated* during the reconstruction, which is a very stable process.

Example 3. With the same setting as in Example 1, consider $\mathfrak{M}(f)(x) = \int_0^1 f(y)dy$. This operator is well defined. But it is clearly not injective. All we learn about f is its mean on $(0,1)$. This operator corresponds to data that are not very informative.

Example 4. Let $\mathfrak{X} = \mathcal{C}_c(\mathbb{R}^2)$ the space of continuous functions with compact support (functions supported in a bounded domain in $\mathbb{R}^2$, i.e., vanishing outside of that domain). Let $\mathfrak{Y} = \mathcal{C}(\mathbb{R} \times (0, 2\pi))$. We define $l(s,\theta)$ for $s \in \mathbb{R}$ and $\theta \in (0, 2\pi)$ as the *line* with direction perpendicular to $\vartheta = (\cos\theta, \sin\theta)$ and at a distance $|s|$ from the origin $(0,0)$. More precisely, let $\vartheta^\perp = (-\sin\theta, \cos\theta)$ the rotation of ϑ by $\frac{\pi}{2}$. Then

$$l(s,\theta) = \{x \in \mathbb{R}^2 \text{ such that } x = s\vartheta + t\vartheta^\perp \quad \text{for } t \in \mathbb{R}\}. \tag{1.11}$$

We define

$$\mathfrak{M}(f)(s,\theta) = \int_{l(s,\theta)} f(x)dl = \int_{\mathbb{R}} f(s\vartheta + t\vartheta^\perp)dt,$$

where dl is the line measure along $l(s,\theta)$. In other words, $\mathfrak{M}$ maps a function to the value of its integrals along *any* line. The operator $\mathfrak{M}$ is the **two-dimensional Radon Transform**. We shall see that $\mathfrak{M}$ is injective and admits an explicit inversion.

The Radon transform and its inversion form the mathematical back-bone of Computerized Tomography (CT), one of the most successful medical imaging techniques available to-date.

Example 5. Let us conclude with a more involved, though practically very relevant, example: the Calderón problem. We first introduce the following elliptic partial differential equation

$$\begin{aligned} -\nabla \cdot \gamma(x)\nabla u(x) &= 0, && x \in X \\ u(x) &= f(x) && x \in \partial X, \end{aligned} \tag{1.12}$$

where X is a smooth, bounded, open, domain in $\mathbb{R}^n$, ∂X its boundary, $\gamma(x)$ a smooth coefficient in X bounded above and below by positive constants, and $f(x)$ is a prescribed Dirichlet data for the elliptic problem. This equation is a standard elliptic problem and is known to admits a unique solution. Moreover, the outgoing current

$$j(x) = \gamma(x)\frac{\partial u}{\partial \nu}(x), \quad \text{with } \nu(x) \text{ the outward unit normal to } X \text{ at } x \in \partial X,$$

is a well defined function. This allows us to define the Dirichlet-to-Neumann (a.k.a. Poincaré-Steklov) operator

$$\Lambda_\gamma : \begin{array}{rcl} H^{\frac{1}{2}}(\partial X) & \to & H^{-\frac{1}{2}}(\partial X) \\ f(x) & \mapsto & \Lambda_\gamma[f](x) = j(x) = \gamma(x)\dfrac{\partial u}{\partial \nu}(x). \end{array} \tag{1.13}$$

Here, $H^s(\partial X)$ are standard Hilbert spaces of distributions defined at the domain's boundary.

Let now $\mathfrak{X} = \mathcal{C}^2(\bar{X})$ and $\mathfrak{Y} = \mathcal{L}(H^{\frac{1}{2}}(\partial X), H^{-\frac{1}{2}}(\partial X))$, where the space $\mathcal{L}(\mathfrak{X}_1, \mathfrak{X}_2)$ means the space of linear bounded (continuous) operators from $\mathfrak{X}_1$ to $\mathfrak{X}_2$. We define the measurement operator

$$\mathfrak{M} : \quad \mathfrak{X} \ni \gamma \mapsto \mathfrak{M}(\gamma) = \Lambda_\gamma \in \mathfrak{Y}. \tag{1.14}$$

So the measurement operator maps the unknown conductivity γ to the Dirichlet-to-Neumann operator Λ_γ, which is by construction a functional of γ since the solution u of (1.12) is a functional of γ. The measurement operator therefore lives in a "huge" (admittedly, this is subjective) space. Acquiring data means acquiring Λ_γ, which means for each and every function $g \in H^{\frac{1}{2}}(\partial X)$ at the domain's boundary, perform an experiment that measures $j(x)$.

As we shall see, the Calderón problem is a typical example of what is usually regarded as an *ill-posed* inverse problem. It finds important applications in Electrical Impedance Tomography and Optical Tomography, two medical imaging modalities.

1.3 Inverse Problems and Modeling: application to Magnetic Resonance Imaging

This section presents an extremely simplified version of the extremely successful medical imaging modality called Magnetic Resonance Imaging (MRI). While doing so, we observe that MRI reconstructions may be modeled by (at least) three different measurement operators, and hence three different inverse problems. This serves as an example of the importance of modeling when dealing with practical inverse problems.

MRI exploits the precession of the spin of protons in a magnetic field $\mathbf{H}(\mathbf{x})$, which is a vector in $\mathbb{R}^3$ for each position $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$. The axis of the precession is that of the magnetic field and the frequency of the precession is $\omega(\mathbf{x}) = \gamma|\mathbf{H}|(\mathbf{x})$, where $\gamma = e/(2m)$ is called the gyromagnetic ratio, e is the electric charge of the proton and m its mass.

In a nutshell, MRI works as follows. Assume first that we impose a strong static magnetic field $\mathbf{H}_0 = H_0\mathbf{e}_z$ along the z axis. All protons end up with their spin parallel to $\mathbf{H}_0$ and slightly more so in the direction $\mathbf{H}_0$ than in $-\mathbf{H}_0$. This difference is responsible for a macroscopic *magnetization* $\mathbf{M}$ pointing in the same direction as $\mathbf{H}_0$.

In a second step, we generate radio frequency magnetic waves at the Larmor frequency $\omega_0 = \gamma|\mathbf{H}_0|$. In clinical MRI, the frequency is typically between 15 and 80 MHz (for hydrogen imaging), which corresponds to wavelengths between 20 and 120 m (since $\omega = ck = 2\pi c/\lambda$ and $c \approx 3\,10^8$). So the wavelength is not what governs spatial resolution in MRI, which is as most successful medical imaging modalities, sub-millimetric. For instance the pulse (assumed to be independent of $\mathbf{x}$ to start with) may be of the form $\mathbf{H}_1(t) = 2H_1\cos(\omega_0 t)\mathbf{e}_x$ and turned on for a duration t_p. Because the field oscillates at the Larmor frequency, the spins of the protons are affected. The resulting effect on the macroscopic magnetization is that it precesses around the axis $\mathbf{e}_z$ at frequency ω_0. The spins make an angle with respect to the direction $\mathbf{e}_z$ given at time t_p by

$$\theta = \gamma H_1 t_p.$$

Generally, t_p is chosen such that $\theta = \pi/2$ or $\theta = \pi$. The corresponding pulses are called 90^0 and 180^0 pulses, respectively. Thus, after a 90^0 pulse, the magnetization oscillates in the xy plane and after a 180^0 pulse, the magnetization is pointing in the direction $-\mathbf{H}_0$.

Once the radio frequency is turned off (but not the static field $\mathbf{H}_0$), protons tend to realign with the static field $\mathbf{H}_0$. By doing so, they emit radio frequency waves at the

Larmor frequency ω_0 that can be *measured*. This wave is called the free induction decay (FID) signal. The FID signal after a 90^0 pulse will have the form

$$S(t) = \rho e^{i\omega_0 t} e^{-t/T_2}. \tag{1.15}$$

Here ρ is the density of the magnetic moments and T_2 is the spin-spin relaxation time. (There is also a spin-lattice relaxation time $T_1 \ll T_2$, which cannot be imaged with 90^0 pulses and which we ignore.) The main reason for doing all this is that the density ρ and the relaxation time T_2 depend on the tissue sample. We restrict ourselves to the reconstruction of ρ here, knowing that similar experiments can be devised to image T_2 (and T_1) as well. To simplify, we assume that measurements are performed over a period of time that is small compared to T_2 so that the exponential term e^{-t/T_2} can be neglected.

Now human tissues are not spatially homogeneous, which makes imaging a lot more useful. The density of magnetic moments $\rho = \rho(\mathbf{x})$ depends on type of tissue at $\mathbf{x} \in \mathbb{R}^3$. This is the parameter we wish to reconstruct.

The physical mechanism that allows for the good spatial resolution of MRI (sub-millimeter resolution for brain imaging) is that only tissue samples under a static magnetic field $\mathbf{H}$ such that $|\mathbf{H}| = \gamma\omega_0$ will be affected by the radio frequency pulse $\mathbf{H}_1(t) = 2H_1 \cos(\omega_0 t)\mathbf{e}_x$. We thus need to make sure that the static field has the correct amplitude in as small a spatial area as possible. To do so, we impose the static field

$$\mathbf{H}(z) = \mathbf{H}_0 + G_z z \mathbf{e}_z.$$

Only those protons in the slice with z is close to 0 will be affected by the pulse $\mathbf{H}_1$ since we have assumed that $|\mathbf{H}_0| = \gamma\omega_0$. As a consequence, the measured signal takes the form

$$S(t) = e^{i\omega_0 t} \int_{\mathbb{R}^2} \rho(x, y, 0) dx dy \quad \text{so that} \quad e^{-i\omega_0 t} S(t) = \int_{\mathbb{R}^2} \rho(x, y, 0) dx dy. \tag{1.16}$$

The above right-hand-side thus describes the average density in the plane $z = 0$. MRI is thus a tomographic technique (tomos meaning section or slice in Greek).

By changing $\mathbf{H}_0$ or ω_0, we can obtain the average density in the plane $z = z_0$ for all values of $z_0 \in \mathbb{R}$. Moreover, by rotating the generated magnetic field $\mathbf{H}(z)$, we are ideally able to obtain the average density in any plane in $\mathbb{R}^3$. Planes may be parameterized by their normal vector $\phi \in \mathbb{S}^2$, with $\mathbb{S}^2$ the unit sphere in $\mathbb{R}^3$, and their distance s to the origin $(0, 0, 0) \in \mathbb{R}^3$. Let $P(s, \phi)$ be the corresponding plane. Then what we have obtained is that MRI experiments allows us to obtain the plane integrals of the density

$$\mathsf{R}(s, \phi) = \int_{P(s,\phi)} \rho(\mathbf{x}) d\sigma(\mathbf{x}), \tag{1.17}$$

where $d\sigma(\mathbf{x})$ is the surface (Euclidean) measure on the plane $P(s, \phi)$. Here, $\mathsf{R}(s, \phi)$ is the **three-dimensional Radon transform** of $\rho(\mathbf{x})$. This is the first inverse problem we encounter. The measurement operator maps functions defined on $\mathbb{R}^3$ (for instance compactly supported continuous functions) to the Radon transform, which is a function (for instance compactly supported continuous function) of $(s, \phi) \in \mathbb{R} \times \mathbb{S}^2$. We thus have the Inverse Problem:

3D Radon Transform: Reconstruct density $\rho(\mathbf{x})$ from knowledge of $\mathsf{R}(s,\phi)$
Equivalently: Reconstruct a function from its plane integrals.

There are several issues with the above inverse problem. First of all, the Radon transform integrates over planes, which is a smoothing operation. This smoothing has to be undone in the reconstruction procedure. With more modeling, we will obtain another MO that does not involve any smoothing. More fundamentally from a practical point of view, rotating the whole magnetic field from the direction e_z to an arbitrary direction ϕ is very challenging technologically. One could also rotate the object of interest rather than the heavy magnet. However, for the imaging of human patients, this is not feasible either for rather obvious reasons. Additional modeling is therefore necessary.

So far, we have a vertical discrimination of the proton density. The transversal discrimination is obtained by imposing a static field linearly varying in the x and y directions. Remember that after the 90^0 pulse, the magnetization $\mathbf{M}(x,y,0)$ rotates with frequency ω_0 in the xy plane (i.e., is orthogonal to $\mathbf{e}_z$), and is actually independent of x and y. Let us now impose a static field $\mathbf{H}(y) = \mathbf{H}_0 + G_y y \mathbf{e}_z$ for a duration T. Since the frequency of precession is related to the magnetic field, the magnetization at position y will rotate with frequency $\omega(y) = \omega_0 + \gamma G_y y$. Therefore, compared to the magnetization at $z = 0$, the magnetization at z will accumulate a phase during the time T the field $G_y y \mathbf{e}_z$ is turned on given by $T(\omega(y) - \omega_0) = T\gamma G_y y$. Once the field $G_y y \mathbf{e}_z$ is turned off, the magnetization will again rotate everywhere with frequency ω_0. However, the phase depends on position y. This part of the process is call *phase encoding*. A measurement of the FID would then give us a radio frequency signal of the form

$$S(t;T) = e^{i\omega_0 t} \int_{\mathbb{R}^2} e^{i\gamma G_y T y} \rho(x,y,0) dx dy. \tag{1.18}$$

By varying the time T or the gradient G_y, we see that we can obtain the frequency content in y of the density $\rho(x,y,0)$. More precisely,

$$\int_{\mathbb{R}} \rho(x,y,0) dx = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ik_y y} S(t; \frac{k_y}{\gamma G_y}) dk_y. \tag{1.19}$$

This provides us with the *line integrals* of $\rho(x,y,0)$ at $z = 0$ for all the lines that are parallel to the x-axis. Note that the phase encoding was performed by using a field that was linear in the y variable. We can use a field that is linear in the variable $\cos\theta x + \sin\theta y$ instead. Denoting by $\vartheta = (\cos\theta, \sin\theta) \in \mathbb{S}^1$ a unit vector in $\mathbb{R}^2$, by $\vartheta^\perp = (-\sin\theta, \cos\theta)$ its rotation by $\frac{\pi}{2}$, and by $l(s,\theta)$ the line with normal ϑ at a distance s from the origin $(0,0) \in \mathbb{R}^2$ defined in (1.11), we are thus able to measure all line integrals of the function $\rho(x,y,0)$:

$$\mathsf{R}(s,\theta) = \int_{l(s,\theta)} \rho(x,y,0) dl(x,y) = \int_{\mathbb{R}} \rho(s\vartheta + t\vartheta^\perp, 0) dt, \tag{1.20}$$

where $dl(x,y)$ is the line (Euclidean) measure on $l(s,\theta)$. This is the second inverse problem we encounter: we wish to reconstruct $\rho(x,y)$ from knowledge of its line integrals $\mathsf{R}(s,\theta)$. This is the **two-dimensional Radon transform** of $\rho(x,y,0)$; see Example 4 above. The measurement operator maps functions defined on $\mathbb{R}^2$ (for instance compactly supported continuous functions) to the Radon transform, which is a function (for

instance compactly supported continuous function) of $(s, \phi) \in \mathbb{R} \times \mathbb{S}^1$. We thus have the Inverse Problem:

2D Radon Transform: Reconstruct density $\rho(x, y)$ from knowledge of $\mathsf{R}(s, \theta)$
Equivalently: Reconstruct a function from its line integrals.

The 2D Radon transform still involves smoothing (integration along lines), as does the three-dimensional Radon transform. Note, however, that there is no need to rotate the magnet or the patient to acquire the data. Although the inverse Radon transform is useful and used in practical MRI, the missing information in the x variable in measurements of the form (1.18) can in fact be obtained by additional modeling. Indeed, nothing prevents us from adding an $x-$dependent static field during the FID measurements. Let us assume that after time T (where we reset time to be $t = 0$), we impose a static field of the form $\mathbf{H}(x) = \mathbf{H}_0 + G_x x \mathbf{e}_z$. The magnetization will now precess around the z axis with $x-$dependent frequency $\omega(x) = \omega_0 + \gamma G_x x$. This implies that the measured signal will be of the form

$$S(t; T) = \int_{\mathbb{R}^2} e^{i\gamma G_y T y} e^{i(\omega_0 + \gamma G_x x)t} \rho(x, y, 0) dx dy. \tag{1.21}$$

We have thus access to the *measured data*

$$d(k_x, k_y) = e^{-i\omega_0 k_x/(\gamma G_x)} S(\frac{k_x}{\gamma G_x}; \frac{k_y}{\gamma G_y}) = \int_{\mathbb{R}^2} e^{ik_y y} e^{ik_x x} \rho(x, y, 0) dx dy. \tag{1.22}$$

By varying T (or G_y) and t and G_x, we can obtain the above information for essentially all values of k_x and k_y. This is our third Inverse Problem:

2D Fourier Transform: Reconstruct density $\rho(x, y, 0)$ from knowledge of $d(k_x, k_y)$
Equivalently: Reconstruct a function from its plane wave decomposition.

This is a well-known problem whose solution involves applying the *Inverse Fourier Transform*

$$\rho(x, y, 0) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-i(k_x x + k_y y)} d(k_x, k_y) dk_x dk_y. \tag{1.23}$$

Several approximations have been made to obtain this reconstruction formula. Within this framework, we see however that density reconstructions are relatively simple: all we have to do is to invert a Fourier transform. The above procedure can be repeated for all values of z providing the density $\rho(x, y, z)$ everywhere in the domain of interest.

We do not consider the difficulties of MRI further. The above derivation shows that MRI can be modeled by at least three different inverse problems. A first inverse problem, based on the three dimensional Radon transform, is not very practical. The second and third inverse problems, based on the inverse Radon transform (RT) and the inverse Fourier transform (FT), are used in practical reconstructions in MRI. Note that the IP based on the FT requires the acquisition of more data than the IP based on the RT. The reason for acquiring more data is that the IP based on the FT is *better posed* than than based on the FT. The next section introduces a simple Hilbert scale that allows one to quantify the notion of *well-* or *ill-*posedness in some simple cases.

1.4 Inverse Problems, Smoothing, and ill-posedness

The elements (noise model, prior model) were introduced in response to the "too large" amplification of noise during the inversion of the MO. Such MOs can be characterized as *ill-posed*. Functional analytic definitions of well- and ill-posedness are presented in sections 1.4.1 and 1.4.2.

The description of this subjective notion requires that we quantify and understand this amplification. The main reason why noise is amplified in the reconstruction is because the MO is a smoothing operator. The more smoothing (regularizing) the MO, the more attenuated are the high frequencies of the unknown parameters, and hence the more amplified are the high frequencies of the noise during the reconstruction. In this section, we introduce a Hilbert scale of functional spaces that allows us to quantify the aforementioned regularization for a restricted but pedagogically interesting class of inverse problems. This allows us to be more specific about the notions of well-posedness and ill-posedness of an inverse problem.

1.4.1 Well-posed Problems and Lipschitz Stability.

Let us revisit and make more explicit the notion of well-posedness and ill-posedness mentioned in section 1.1. We assume that $\mathfrak{M}$ is a linear operator A from $\mathfrak{X}$ to $\mathfrak{Y}$ for $\mathfrak{X}$ and $\mathfrak{Y}$ Banach spaces associated with their natural norms. For a given data $\mathrm{y} \in \mathfrak{Y}$, we would like to solve the linear problem

$$\text{Find x such that} \qquad A\mathrm{x} = \mathrm{y}. \tag{1.24}$$

As we mentioned earlier, a well-posed problem is a problem where data noise is not amplified too drastically during the reconstruction. Mathematically, this subjective notion may be described by the property that the (bounded) operator A is invertible ($A^{-1}\mathrm{y}$ is defined for all $\mathrm{y} \in \mathfrak{Y}$) and (hence by the open mapping theorem) of bounded inverse, i.e., $\|A^{-1}\mathrm{y}\|_{\mathfrak{X}} \leq C\|\mathrm{y}\|_{\mathfrak{Y}}$ for a constant C that depends on A but not on $\mathrm{y} \in \mathfrak{Y}$. The error between two solutions x_1 and x_2 corresponding to two data y_1 and y_2 satisfies that

$$\|\mathrm{x}_1 - \mathrm{x}_2\|_{\mathfrak{X}} \leq C\|\mathrm{y}_1 - \mathrm{y}_2\|_{\mathfrak{Y}}. \tag{1.25}$$

This stability estimate is consistent with the statement that "Noise" $\mathrm{y}_1 - \mathrm{y}_2$ is not amplified by more than a multiplicative factor C during the reconstruction. Moreover, when noise levels are reduced by a factor two, which may be performed by adding detectors or obtaining higher quality measurements, then (1.25) states that the in the worst possible case (since (1.25) for arbitrary $\mathrm{x}_1 - \mathrm{x}_2$, and hence C reflects the amplification of noise in the worst scenario), the error in the reconstruction will also be reduced by a factor two.

Note that the choice of the spaces $\mathfrak{X}$ and $\mathfrak{Y}$ and their norms $\|\cdot\|_{\mathfrak{X}}$ and $\|\cdot\|_{\mathfrak{Y}}$ matters. The definition and the boundedness of the operator A^{-1} obviously depends upon these choices and hence so does the validity of (1.25). An estimate of the form (1.25) is a *stability* estimate with Lipschitz constant C and we then say that the inverse problem (1.24) is *Lipschitz-stable*.

1.4.2 Ill-posed Problems and Unbounded Operators.

The Fourier transform recalled in section A.1 of the appendix is an example of a *well-posed* inverse problem from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. We will see in the following chapter that the two-dimensional and three-dimensional Radon transforms are also well posed with appropriate choices of $\mathfrak{X}$ and $\mathfrak{Y}$. Many inverse problems are, however, *considered* to be *ill-posed* for instance because the application of A^{-1} to the noisy data y produces a result that is deemed *inadequate* or because it is not possible to define an operator A^{-1}. Being ill-posed *does not* mean that a problem cannot be solved. However, it means that additional information needs to be incorporated into the inversion.

Let us attempt to mathematize these subjective notions. Typically, we can distinguish two notions of ill-posedness. The first one corresponds to operators A that are not injective. In this case, the data do not uniquely determine the parameters. This situation is typically remedied by acquiring additional information. We will not consider such settings much in these notes except to say that the Bayesian framework considered in Chapter 11 is adapted to such problems. Many practical inverse problems may be seen as discretization of injective operators.

The second notion of ill-posedness involves operators A that are injective but not surjective on the whole space $\mathfrak{Y}$ (i.e., the range of A defined by $\text{Range}(A) = A(\mathfrak{X})$ is a proper subset of $\mathfrak{Y}$; that is to say, is not equal to $\mathfrak{Y}$). Because A is injective, A^{-1} can be defined from $\text{Range}(A)$ to $\mathfrak{X}$. However, it is *not* a *bounded* operator for the norms of $\mathfrak{Y}$ and $\mathfrak{X}$ in the sense that a Lipschitz equality such as (1.25) does not hold. From a practical point of view, applying A^{-1} to the available (noisy) data y provides a results that the user *feels* is too different from the expected x.

Mathematically, the unbounded character of A^{-1} very much depends on the choice of functional space. The operator A^{-1} could very well be defined and bounded from the other space $\mathfrak{Y}'$ to the other space $\mathfrak{X}'$, in which case the same inverse problem based on the MO A could be *ill posed* from $\mathfrak{X}$ to $\mathfrak{Y}$ but *well-posed* from $\mathfrak{X}'$ to $\mathfrak{Y}'$. In other words, a user comfortable with the modeling of A from $\mathfrak{X}'$ to $\mathfrak{Y}'$ deals with a well-posed problem, whereas the user insisting on a modeling of A from $\mathfrak{X}$ to $\mathfrak{Y}$ needs to add information to the inversion to obtain a more satisfactory answer.

In spite of the subjectivity of the notion of ill-posedness, one of the main reasons why an inverse problem is deemed ill-posed *in practice* is because the MO A is "smoothing". Smoothing means that Ax is "more regular" than x, in the sense that details (small scale structures) are attenuated by the MO. Again, this does not mean that the details cannot be reconstructed. When the MO is injective, they can. This means, however, that the reconstruction has to undo this smoothing. As soon as the data are noisy (i.e., always in practice), and that the noise contribution has small scale structures (i.e., often in practice), then the deregularization process has the effect of amplifying the noise in a way that can potentially be very *harmful* (which is, as we already said, *subjective*).

The answer to ill-posedness is to impose prior assumptions on the parameters we wish to reconstruct. As we mentioned earlier in this chapter, the simplest example of such an assumption is to assume that the function is sufficiently smooth. In order to define what we mean by ill-posedness and quantify the degree of ill-posedness, we need a scale of spaces in which the "smoothing" of A can be measured. We will use what is probably the simplest scale of function spaces, namely the scale of Hilbert spaces.

1.4.3 Smoothing and Sobolev scale of Hilbert spaces.

Let $s \geq 0$ be a non-negative real-number. We define the scale of Hilbert spaces $H^s(\mathbb{R}^n)$ as the space of measurable functions $f(x)$ such that

$$\|f\|^2_{H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (1+|k|^2)^s |\mathcal{F}_{x\to k} f|^2(k) dk < \infty. \tag{1.26}$$

Here, $\mathcal{F}_{x\to k} f$ is the Fourier transform of f defined in section A.1 of the appendix. We verify that $H^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$ since the Fourier transform is an isometry. We also verify that

$$\{f \in H^1(\mathbb{R}^n)\} \iff \Big\{ f \in L^2(\mathbb{R}^n) \text{ and } \frac{\partial f}{\partial x_i} \in L^2(\mathbb{R}^n),\ 1 \leq i \leq n \Big\}. \tag{1.27}$$

This results from (A.7). More generally the space $H^m(\mathbb{R}^n)$ for $m \in \mathbb{N}$ is the space of functions such that all partial derivatives of f of order up to m are square integrable. The advantage of the definition (1.26) is that it holds for real values of s as well. So $H^{\frac{1}{2}}(\mathbb{R}^n)$ is the space of functions such that "half-derivatives" of f are square integrable. Notice also that s characterizes the degree of smoothness of a function $f(x)$. The larger s, the smoother the function $f \in H^s(\mathbb{R}^n)$, and the faster the decay of its Fourier transform $\hat{f}(k)$ as can be seen from the definition (1.26).

It is also useful to define the Hilbert scale for functions supported on subdomains of $\mathbb{R}^n$. Let X be a sufficiently smooth subdomain of $\mathbb{R}^n$. We define two scales. The first scale is $H^s_0(X)$, defined as the closure of $C^\infty_0(X)$, the space of functions of class C^∞ with support in X (so these functions and all their derivatives vanish at the boundary of X), for the norm in $H^s(\mathbb{R}^n)$. Thus, $f \in H^s_0(X)$ can be described as the limit of functions $f_n \in C^\infty_0(\mathbb{R})$ uniformly bounded in $H^s(\mathbb{R}^n)$. We also define $H^s(X)$ as the space of functions f on X that can be extended to functions f^* in $H^s(\mathbb{R}^n)$ (i.e., $f = f^* \chi_X$, where χ_X is the characteristic function of X) and $\|f\|_{H^s(X)}$ is the lower bound of $\|f\|_{H^s(\mathbb{R}^n)}$ over all possible extensions. The are several (sometimes not exactly equivalent) ways to define the scale of Hilbert spaces $H^s(X)$.

Finally, it is also convenient to define H^s for negative values of s. We define $H^{-s}(\mathbb{R}^n)$ for $s \geq 0$ as the subspace of $\mathcal{S}'(\mathbb{R}^n)$, the space of *tempered distributions*, such that (1.26) holds. For bounded domains we define $H^{-s}(X)$ as the dual to $H^s_0(X)$ equipped with the norm

$$\|f\|_{H^{-s}(X)} = \sup_{g \in H^s_0(X)} \frac{\int_X fg dx}{\|g\|_{H^s_0(X)}}. \tag{1.28}$$

We can verify that the two definitions agree when $X = \mathbb{R}^n$, in which case $H^s_0(\mathbb{R}^n) = H^s(\mathbb{R}^n)$.

A well-posed, ill-posed inverse problem. Let us illustrate on a simple example how the Hilbert scale can be used to understand the ill-posedness of inverse problems. Let $f(x)$ be a function in $L^2(\mathbb{R})$ and let u be the solution in $L^2(\mathbb{R})$ of the following ODE

$$-u'' + u = f, \qquad x \in \mathbb{R}. \tag{1.29}$$

There is a unique solution in $L^2(\mathbb{R})$ to the above equation given by

$$u(x) = \frac{1}{2}\int_{\mathbb{R}} e^{-|y-x|} f(y)dy = (g * f)(x), \qquad g(x) = \frac{1}{2}e^{-|x|},$$

as can be verified by inspection. In the Fourier domain, this is

$$(1 + k^2)\hat{u}(k) = \hat{f}(k).$$

This implies that when $f \in L^2(\mathbb{R})$, then u is not only in $L^2(\mathbb{R})$ but also in $H^2(\mathbb{R})$ as is easily verified.

Exercise 1.4.1 *Check this in detail.*

The problem is ill-posed... Let us define the operator A as follows

$$\begin{array}{rccc} A: & L^2(\mathbb{R}) & \to & L^2(\mathbb{R}) \\ & f & \mapsto & Af = u, \end{array} \tag{1.30}$$

where u is the solution to (1.29). As such, the operator A is not invertible on $L^2(\mathbb{R})$. Indeed the inverse of the operator A is formally defined by $A^{-1}u = -u'' + u$. However for $u \in L^2(\mathbb{R})$, $-u''$ is not a function in $L^2(\mathbb{R})$ but a distribution in $H^{-2}(\mathbb{R})$. The inverse problem consisting of reconstructing $f(x) \in L^2(\mathbb{R})$ from $u(x) \in L^2(\mathbb{R})$ is therefore *ill-posed*. The reason is that the operator A is regularizing.

... and is well-posed. However let us define the "same" operator

$$\begin{array}{rccc} \tilde{A}: & L^2(\mathbb{R}) & \to & H^2(\mathbb{R}) \\ & f & \mapsto & \tilde{A}f = u. \end{array} \tag{1.31}$$

Now $\tilde{A}$ is invertible with a bounded inverse from $H^2(\mathbb{R})$ to $L^2(\mathbb{R})$ given by $\tilde{A}^{-1}u = -u'' + u$. So $\tilde{A}$ is *well-posed* from $L^2(\mathbb{R})$ to $H^2(\mathbb{R})$ as can easily be verified. Yet a third instantiation of the "same" operator is

$$\begin{array}{rccc} \check{A}: & H^{-2}(\mathbb{R}) & \to & L^2(\mathbb{R}) \\ & f & \mapsto & \check{A}f = u. \end{array} \tag{1.32}$$

Now $\check{A}$ is invertible with a bounded inverse from $L^2(\mathbb{R})$ to $H^{-2}(\mathbb{R})$ given by $\check{A}^{-1}u = -u'' + u$ and thus $\tilde{A}$ is *well-posed* from $H^{-2}(\mathbb{R})$ to $L^2(\mathbb{R})$.

If we assume that noise (the error between measurement u_1 and measurement u_2) is small in the H^2-norm, so that $\|u_1 - u_2\|_{H^2(\mathbb{R})} \leq \delta$, and are comfortable with a small error in the parameter in the $L^2(\mathbb{R})$ sense, then there is "no problem" of ill-posedness. The reconstruction will be accurate in the sense that $\|f_1 - f_2\|_{L^2(\mathbb{R})} \leq C\delta$, where $f_j = \tilde{A}^{-1}u_j$, $j = 1, 2$.

The same occurs if we assume that noise is small in the L^2-norm and we are comfortable with a small error in the parameter in the $H^{-2}(\mathbb{R})$ sense. This would typically mean

that we are satisfied if spatial moments of u (i.e., quantities of the form $\int_{\mathbb{R}} u(x)\varphi(x)dx$ for $\varphi \in H^2(\mathbb{R})$) are accurate, rather than insisting that u be accurate in the L^2 sense.

However, in many instance, noise is not small in the strong norm $H^2(\mathbb{R})$, but rather in the weaker norm $L^2(\mathbb{R})$. At least this is our *perception*. Moreover, we do not want small errors in the $H^{-2}(\mathbb{R})$ sense, but rather insist that the reconstructions "look good" in some $L^2(\mathbb{R})$ sense. Then the problem is certainly ill-posed as A is not boundedly invertible for the spaces $\mathfrak{X} = \mathfrak{Y} = L^2(\mathbb{R})$.

A possible definition of ill-posedness. We now introduce a useful practical distinction among ill-posed problems. As heuristic and subjective as the notion may be, an ill-posed inverse problems that require relatively mild action from the user (in terms of the introduction of prior information) will be called *mildly ill-posed*. Some inverse problems may require much more stringent action from the user. They will be called *severely ill-posed* problems.

Using the Hilbert scale introduced earlier, a possible mathematical definition of *mildly* and *severely* ill-posed problems is as follows. We assume here that A is *injective*. It is not that the inverse of A does not exist that causes problems. It is because A is smoothing that action is necessary. When the smoothing is *mild*, then that action may be limited to an appropriate penalization such as those that will be described in Chapter 11. When the smoothing is *severe*, then typically much more drastic prior information needs to be introduced, for instance by using the Bayesian framework introduced in Chapter 11. We somewhat arbitrarily separate *mild* smoothing by a finite number of derivatives from *severe* smoothing by an "infinite" number of derivatives.

The problem (1.24) with $\mathfrak{X} = \mathfrak{Y} = L^2(\mathbb{R}^n)$ is said to be *mildly ill-posed* provided that there exists a positive constant C and $\alpha > 0$ such that

$$\|Af\|_{H^\alpha(\mathbb{R}^n)} \geq C\|f\|_{L^2(\mathbb{R}^n)}. \tag{1.33}$$

We define $\|Af\|_{H^\alpha(\mathbb{R}^n)} = +\infty$ if f does not belong to $H^\alpha(\mathbb{R}^n)$. We say that A is *mildly ill-posed of order* α if α is the smallest real number such that (1.33) holds for some $C = C(\alpha)$. Notice that we can choose any $\alpha \geq 2$ for $\tilde{A}$ so the operator that maps f to u solution of (1.29) is a mildly ill-posed problem of order 2. The operator $\tilde{A}$ in (1.31) essentially integrates twice the function f. Any injective operator that corresponds to a finite number of integrations is therefore mildly ill-posed.

We call the inversion a *severely ill-posed* problems when no such α exists. Unfortunately, there are many practical instances of such inverse problems. A typical example is the following operator

$$Bf(x) = \mathcal{F}^{-1}_{k\to x}[e^{-k^2}\mathcal{F}_{x\to k}f](x). \tag{1.34}$$

Physically this corresponds to solving the heat equation forward in time: a very smoothing operation. We easily verify that the operator B maps $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$. Hoverer it damps high frequencies exponentially strongly, and more strongly than any finite number of integrations (m integrations essentially multiply high frequencies by $|k|^{-m}$) so no $\alpha > 0$ in (1.33) exists for B. Note that it does not mean that B is never invertible. Indeed for sufficiently smooth functions $g(x)$ in the range of B (for instance for functions g such that $\hat{g}(k)$ has compact support), we can easily define the inverse operator

$$B^{-1}g(x) = \mathcal{F}^{-1}_{k\to x}[e^{k^2}\mathcal{F}_{x\to k}g](x).$$

Physically, this corresponds to solving the heat equation backwards in time. It is clear that on a space of sufficiently smooth functions, we have $BB^{-1} = B^{-1}B = Id$. Yet, if noise is present in the data, it will be amplified by e^{k^2} in the Fourier domain. Unless noise is low frequency, this has devastating effects on the reconstruction. It is instructive to numerically add the inverse Fourier transform of $\varepsilon\hat{n}(k)e^{k^2}$ to an image $f(x)$ for $\hat{n}(k) = (1+k^2)^{-1}$, say, so that noise is square integrable. Even when ε is machine precision, the image is quickly drowned by noise even if a relatively small number of Fourier modes is used to perform the Fourier transform.

Chapter 2

Integral Geometry. Radon transforms

This chapter and the next two are devoted to integral transforms, which involves the reconstruction of an object for instance described by a function from knowledge of integral quantities of the object such as its integrals along lines or planes.

In this chapter we consider the simplest example of integral geometry: the integration of a two-dimensional function along all possible lines in the plane, which is called the Radon transform, and the inversion of such a transform. This forms the mathematical backbone for one of the most successful medical imaging techniques: computed (or computerized) tomography (CT).

Later in the chapter, we consider the three dimensional Radon transform, which concerns the integral of functions over planes in three dimensions of space, as well as a specific example of a weighted two dimensional Radon transform, the attenuated Radon transform, which finds application in the important medical imaging modality called Single Photon Emission Computerized Tomography (SPECT).

2.1 The Radon Transform

This section presents the main theoretical results on the integration of functions on hyperplanes.

2.1.1 Transmission Tomography

In transmission tomography, objects to be imaged are probed with non-radiating sources such as X-rays. X-rays are composed of high energy photons (on the order of 60keV, which corresponds to a wavelength of about 0.02nm) that propagate through the object along straight lines unless they interact with the underlying medium and get absorbed. Let $x = (x_1, x_2)$ denote two-dimensional spatial position and $\vartheta \in \mathbb{S}^1$ orientation. We denote by $u(x, \vartheta)$ the density of X-rays with position x and orientation ϑ, and by $a(x)$ a linear attenuation coefficient. Velocity of X-rays is normalized to 1 so that locally the density $u(x, \vartheta)$ satisfies the following transport equation:

$$\vartheta \cdot \nabla_x u(x, \vartheta) + a(x)u(x, \vartheta) = 0, \qquad x \in X, \quad \vartheta \in \mathbb{S}^1. \tag{2.1}$$

Here X is the physical domain (assumed to be convex) of the object and $\mathbb{S}^1$ is the unit circle. We identify any point $\vartheta \in \mathbb{S}^1$ with the angle $\theta \in [0, 2\pi)$ such that $\vartheta = (\cos\theta, \sin\theta)$. The advection operator is given by

$$\vartheta \cdot \nabla_x = \cos\theta \frac{\partial}{\partial x_1} + \sin\theta \frac{\partial}{\partial x_2}$$

and models free transport of X-rays while $a(x)u(x,\vartheta)$ models the number of absorbed photons per unit distance at x.

The probing source is emitted at the boundary of the domain and takes the form

$$u(x,\vartheta) = \delta(x - x_0)\delta(\vartheta - \vartheta_0), \tag{2.2}$$

where $x_0 \in \mathbb{R}^2$, say $x_0 \in \partial X$ and ϑ_0 is *entering* the domain X, i.e., $\vartheta_0 \cdot n(x_0) < 0$ where n is the outward normal to X at $x_0 \in \partial X_0$. Above the "delta" functions mean that a unit amount of photons is created at (x_0, ϑ_0) in the sense that for any domain $(x_0, \vartheta_0) \in Y \subset \mathbb{R}^2 \times \mathbb{S}^1$, we have

$$\int_Y u(x,\vartheta) dx d\vartheta = 1.$$

The solution to (2.1)-(2.2), which is a first-order ODE in the appropriate variables, is given by

$$u(x + t\vartheta, \vartheta) \;=\; u(x,\vartheta) \exp\Big(- \int_0^t a(x + s\vartheta) ds\Big), \quad x \in \mathbb{R}^2, \quad \vartheta \in \mathbb{S}^1. \tag{2.3}$$

Indeed, write $v(t) = u(x + t\vartheta, \vartheta)$ and $b(t) = a(x + t\vartheta)$ so that $\dot v + bv = 0$ and integrate to obtain the above result. For our specific choice at (x_0, ϑ_0), we thus obtain that

$$u(x,\vartheta) = \delta(x - t\vartheta - x_0)\delta(\vartheta - \vartheta_0) \exp\Big(- \int_0^t a(x - s\vartheta) ds\Big).$$

In other words, on the half line $x_0 + t\vartheta_0$ for $t \geq 0$, there is a positive density of photons. Away from that line, the density of photons vanishes.

For $x_1 = x_0 + \tau\vartheta_0 \in \partial X$ different from x_0, if a detector collects the amount of photons reaching x_1 (without being absorbed), it will measure

$$\exp\Big(- \int_0^\tau a(x_1 - s\vartheta) ds\Big) = \exp\Big(- \int_0^\tau a(x_0 + s\vartheta) ds\Big).$$

The travel time (for a particle with rescaled speed equal to 1, so it is also a "distance") from one side of ∂X to the other side depends on (x_0, ϑ_0) and is denoted by $\tau(x_0, \vartheta_0) > 0$. By taking the logarithm of the measurements, we have thus access to

$$\int_0^{\tau(x_0,\vartheta_0)} a(x_0 + t\vartheta_0) dt.$$

This is the integral of a over the line segment (x_0, x_1). By varying the point x_0 and the direction of the incidence ϑ_0, we can have access to integrals of $a(x)$ over all possible segments (and since a can be extended by 0 outside X without changing the measurements, in fact over all possible lines) crossing the domain X.

The main question in transmission tomography is thus how one can recover a function $a(x)$ from its integration over all possible lines in the plane $\mathbb{R}^2$. This will be the object of subsequent sections. In practice, we need to consider integrations over a finite number of lines. How these lines are chosen is crucial to obtain a rapid and practical inversion algorithm. We do not consider the problems of discretization here and refer the reader to [45, 46].

2.1.2 Two dimensional X-ray (Radon) transform

We have seen that the problem of transmission tomography consisted of reconstructing a function from its integration along lines. We have considered the two-dimensional problem so far. Since X-rays do not scatter, the three dimensional problem can be treated by using the two-dimensional theory: it suffices to image the object slice by slice using the two dimensional reconstruction, as we did in MRI (Transmission Tomography is indeed a tomographic method since tomos means slice in Greek as we already know).

We need to represent (parameterize) lines in the plane in a more convenient way than by describing them as the line joining x_0 and x_1. This is done by defining an origin $\mathbf{0} = (0, 0)$, a direction $(\cos\theta, \sin\theta) = \vartheta \in S^1$, and a scalar s indicating the (signed) distance of the line to $\mathbf{0}$. The line is defined by the set of points x such that $x \cdot \vartheta^\perp = s$, where $\vartheta^\perp$ is the rotation of ϑ by $\frac{\pi}{2}$, i.e., the vector given by $\vartheta^\perp = (-\sin\theta, \cos\theta)$. More precisely, for a smooth function $f(x)$ on $\mathbb{R}^2$, we define the X-ray transform $Rf(s, \theta)$ for $(s, \theta) \in Z = \mathbb{R} \times (0, 2\pi)$ as

$$Rf(s,\theta) = \int_{\mathbb{R}} f(s\vartheta^\perp + t\vartheta)dt = \int_{\mathbb{R}^2} f(x)\delta(x \cdot \vartheta^\perp - s)dx. \tag{2.4}$$

Notice that the cylinder Z is a double covering of the space of lines in the real plane

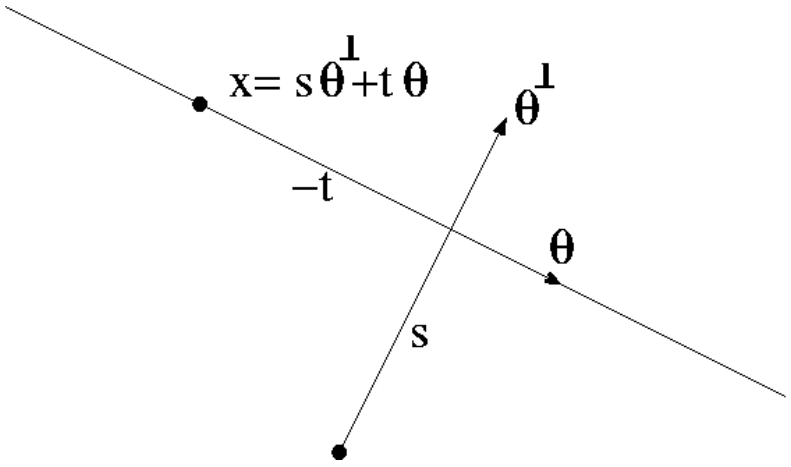

Figure 2.1: Geometry of the X-ray transform.

$\mathbb{R}^2$. Indeed one easily verifies that

$$Rf(s,\theta) = Rf(-s,\theta+\pi), \qquad \text{as} \quad \{x \cdot \vartheta^\perp = s\} = \{x \cdot (-\vartheta^\perp) = (-s)\}.$$

Thus there is a redundancy of order two in the parameterization of lines in the Radon transform.

Radon transform versus X-ray transform. We pause here to make a remark regarding notation. The notation introduced here for the X-ray transform Rf is not the same as the notation introduced from the Radon transform $\mathsf{R}f$ in Chapter 1 even though both transforms integrate functions along all lines. Let us explain the reason for this change of notation. A line in a two dimensional space is a hyperplane, as is a plane in a three dimensional space, a notion that is used in the definition of the three dimensional Radon transform that we consider below. So it is natural to parameterize hyperplanes by their uniquely defined (up to a sign) normal vector.

Now, we have just seen that the reason why line integrals appear in Computerized Tomography is because particles propagate along straight lines. This is true independent of dimension. And it is natural to parameterize lines by their main direction ϑ. This is the point of view of the *X-ray* transform (since X-rays roughly follow straight lines in medical imaging applications).

In three (or higher) dimensions, the three dimensional Radon transform $\mathsf{R}f(s,\vartheta)$ (along hyperplanes) and the three dimensional X-ray transform $Rf(x,\vartheta)$ (along rays) are not the same object (if only because there are many more lines than there are hyperplanes). In two dimensions, however, the X-ray and Radon transforms correspond to the same geometric object: functions are integrated along all possible lines. In Chapter 1, we emphasized the parameterization of hyperplanes and thus modeled lines by their normal vector ϑ. In this chapter, we emphasize the X-ray transform and parameterize lines by their main direction ϑ. One can go from one notation to the other by replacing ϑ by its 90 degree rotation $\vartheta^\perp$. The rest of this chapter uses the line parameterizations of the X-ray transform. However, in two dimensions, we still refer to the integration of a function along lines, independent of the chosen parameterization, as the Radon transform.

Some properties of the Radon transform. Let us derive some important properties of the Radon transform. We first define the operator

$$R_\theta f(s) = Rf(s,\theta). \tag{2.5}$$

This notation will often be useful in the sequel. The first important result on the Radon transform is the *Fourier slice theorem*:

Theorem 2.1.1 *Let $f(x)$ be a smooth function. Then for all $\vartheta \in S^1$, we have*

$$[\mathcal{F}_{s\to\varsigma} R_\theta f](\varsigma) = \widehat{R_\theta f}(\varsigma) = \hat{f}(\varsigma\vartheta^\perp), \qquad \varsigma \in \mathbb{R}. \tag{2.6}$$

Proof. We have that

$$\widehat{R_\theta f}(\varsigma) = \int_{\mathbb{R}} e^{-is\varsigma} \int_{\mathbb{R}^2} f(x)\delta(x\cdot\vartheta^\perp - s)dxds = \int_{\mathbb{R}^2} e^{-ix\cdot\vartheta^\perp\varsigma} f(x)dx.$$

This concludes the proof. □

This result should not be surprising. For a given value of ϑ, the Radon transform gives the integration of f over all lines parallel to ϑ. So obviously the oscillations in the direction ϑ are lost, but not the oscillations in the orthogonal direction $\vartheta^\perp$. The oscillations of f in the direction $\vartheta^\perp$ are precisely of the form $\hat{f}(\alpha\vartheta^\perp)$ for $\alpha \in \mathbb{R}$. It is therefore not surprising that the latter can be retrieved from the Radon transform $R_\theta f$.

Notice that this result also gives us a *reconstruction* procedure. Indeed, all we have to do is to take the Fourier transform of $R_\theta f$ in the variable s to get the Fourier transform $\hat{f}(\sigma\vartheta^\perp)$. It remains then to obtain the latter Fourier transform for all directions $\vartheta^\perp$ to end up with the full $\hat{f}(k)$ for all $k \in \mathbb{R}^2$. Then the object is simply reconstructed by using the fact that $f(x) = (\mathcal{F}^{-1}_{k\to x}\hat{f})(x)$. We will consider other (equivalent) reconstruction methods and explicit formulas later on.

Before doing so, we derive additional properties satisfied by Rf. From the Fourier slice theorem, we deduce that

$$R_\theta[\frac{\partial f}{\partial x_i}](s) = \vartheta_i^\perp \frac{d}{ds}(R_\theta f)(s). \tag{2.7}$$

Exercise 2.1.1 Verify (2.7).

This is the equivalent for Radon transforms of the property (A.7) for the Fourier transform.

Smoothing properties of the Radon and X-ray transforms. Let us now look at the regularizing properties of the Radon transform. To do so, we introduce a function $\chi(x)$ of class $C_0^\infty(\mathbb{R}^2)$ (i.e., χ is infinitely many times differentiable) and with compact support (i.e. there is a radius R such that $\chi(x) = 0$ for $|x| > R$). When we are interested in an object defined on a bounded domain X, we can choose $\chi(x) = 1$ for $x \in X$.

As we did for $\mathbb{R}^n$ in the previous chapter, let us now define the Hilbert scale for the cylinder Z as follows. We say that $g(s,\theta)$ belongs to $\mathrm{H}^s(Z)$ when

$$\|g\|^2_{\mathrm{H}^s(Z)} = \int_0^{2\pi}\int_{\mathbb{R}} (1+\sigma^2)^s |\mathcal{F}_{s\to\sigma} g(\sigma)|^2 d\sigma d\theta < \infty. \tag{2.8}$$

This is a constraint stipulating that the Fourier transform in the s variable decays sufficiently fast at infinity. No constraint is imposed on the directional variable other than being a square-integrable function. We have then the following result:

Theorem 2.1.2 *Let $f(x)$ be a distribution in $H^s(\mathbb{R}^2)$ for some $s \in \mathbb{R}$. Then we have the following bounds*

$$\begin{aligned} \sqrt{2}\|f\|_{H^s(\mathbb{R}^2)} &\leq \|Rf\|_{\mathrm{H}^{s+\frac{1}{2}}(Z)} \\ \|R(\chi f)\|_{\mathrm{H}^{s+\frac{1}{2}}(Z)} &\leq C_\chi \|\chi f\|_{H^s(\mathbb{R}^2)}. \end{aligned} \tag{2.9}$$

The constant C_χ depends on the function $\chi(x)$, and in particular on the size of its support.

Proof. From the Fourier slice theorem $\widehat{R_\theta w(\sigma)} = \widehat{w}(\sigma\vartheta^\perp)$, we deduce that

$$\begin{aligned} \int_Z |\widehat{Rw}(\sigma,\theta)|^2 (1+\sigma^2)^{s+1/2} d\sigma d\theta &= \int_Z |\widehat{w}(\sigma\vartheta^\perp)|^2 (1+\sigma^2)^{s+1/2} d\sigma d\theta \\ &= 2\int_{\mathbb{R}^2} |\widehat{w}(k)|^2 \frac{(1+|k|^2)^{s+1/2}}{|k|} dk, \end{aligned}$$

using the change of variables from polar to Cartesian coordinates so that $dk = \sigma d\sigma d\theta$ and recalling that $\hat{f}(\sigma\vartheta) = \hat{f}((-\sigma)(-\vartheta))$. The first inequality in (2.9) then follows from the fact that $|k|^{-1} \geq (1+|k|^2)^{-1/2}$ using $w(x) = f(x)$. The second inequality is slightly more difficult because of the presence of $|k|^{-1}$. We now choose $w(x) = f(x)\chi(x)$. Let us split the integral into $I_1 + I_2$, where I_1 accounts for the integration over $|k| > 1$ and I_2 for the integration over $|k| < 1$. Since $(1+|k|^2)^{1/2} \leq \sqrt{2}|k|$ for $|k| > 1$, we have that

$$I_1 \leq 2\sqrt{2} \int_{\mathbb{R}^2} |\widehat{\chi f}(k)|^2 (1+|k|^2)^s dk \leq 2\sqrt{2} \|\chi f\|^2_{H^s}.$$

It remains to deal with I_2. We observe that

$$I_2 \leq C \|\widehat{\chi f}\|^2_{L^\infty(\mathbb{R}^2)}.$$

Let $\psi \in C_0^\infty(\mathbb{R}^2)$ such that $\psi = 1$ on the support of χ so that $\psi\chi f = \chi f$. We define $\psi_k(x) = e^{-ix\cdot k}\psi(x)$. Upon using the definition (A.2), the Parseval relation (A.4) and the Cauchy Schwarz inequality $(f,g) \leq \|f\|\|g\|$, we deduce that

$$|\widehat{\chi f}|(k) = |\widehat{\psi\chi f}|(k) = \Big| \int_{\mathbb{R}^2} \psi_k(x)(\chi f)(x) dx \Big| \leq C \|\psi_k\|_{H^{-s}(\mathbb{R}^2)} \|\chi f\|_{H^s(\mathbb{R}^2)}.$$

Since $\psi(x)$ is smooth and compactly supported, ψ_k is in $H^{-s}(\mathbb{R}^2)$ uniformly in $|k| < 1$. This implies that

$$I_2 \leq C \|\widehat{\chi f}\|^2_{L^\infty(\mathbb{R}^2)} \leq C_\chi \|\chi f\|^2_{H^s(\mathbb{R}^2)},$$

where the constant C_χ depends on the support of χ. This concludes the proof. □

The theorem should be used as follows. For a function (or more generally a distribution) $f(x)$ of compact support, we choose χ of class C^∞, with compact support, and equal to 1 on the support of f. We then use the above theorem with $\chi f = f$. The constant C_χ depends then implicitly on the size of the support of $f(x)$.

The above inequalities show that R is a *smoothing* operator. This is not really surprising as an integration over lines is involved in the definition of the Radon transform. However, the result tells us that the Radon transform is smoothing by exactly *one half of a derivative*. The second inequality in (2.9) tells us that the factor $\frac{1}{2}$ is optimal, in the sense that the Radon transform does not regularize by more than one half of a derivative. Moreover this corresponds to (1.33) with $\alpha = \frac{1}{2}$, which shows that the inversion of the Radon transform in two dimensions is a mildly ill-posed problem of order $\alpha = \frac{1}{2}$: when we reconstruct f from Rf, a differentiation of order one half is involved.

At the same time, seen as an operator from $H^s(\mathbb{R}^2)$ to $\mathrm{H}^{s+\frac{1}{2}}(Z)$, the Radon transform is a well posed problem. For $s = -\frac{1}{2}$, small noise in the $L^2(Z)$ sense will generate small reconstruction errors in the $H^{-\frac{1}{2}}(\mathbb{R}^2)$ sense.

Filtered-backprojection inversions. Let us now consider such *explicit* reconstruction formulas. In order to do so, we need to introduce two new operators, the adjoint operator R^* and the Hilbert transform H. The adjoint operator R^* to R (with respect to the usual inner products $(\cdot,\cdot)_{\mathbb{R}^2}$ and $(\cdot,\cdot)_Z$ on $L^2(\mathbb{R})$ and $L^2(Z)$, respectively) is given for every smooth function $g(s,\theta)$ on Z by

$$(R^*g)(x) = \int_0^{2\pi} g(x\cdot\vartheta^\perp, \theta) d\theta, \qquad x \in \mathbb{R}^2. \tag{2.10}$$

That R^* is indeed the adjoint operator to R is verified as follows

$$\begin{aligned}(R^*g, f)_{\mathbb{R}^2} &= \int_{\mathbb{R}^2} f(x) \int_0^{2\pi} g(x\cdot\vartheta^\perp, \theta) d\theta dx \\ &= \int_{\mathbb{R}^2} f(x) \int_0^{2\pi}\int_{\mathbb{R}} \delta(s - x\cdot\vartheta^\perp) g(s,\theta) ds d\theta dx \\ &= \int_0^{2\pi}\int_{\mathbb{R}} g(s,\theta) \int_{\mathbb{R}^2} f(x)\delta(s-x\cdot\vartheta^\perp) dx ds d\theta \\ &= \int_0^{2\pi}\int_{\mathbb{R}} g(s,\theta)(Rf)(s,\theta) ds d\theta \ = (g, Rf)_Z.\end{aligned}$$

We also introduce the Hilbert transform defined for smooth functions $f(t)$ on $\mathbb{R}$ by

$$\mathsf{H}f(t) = \frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}} \frac{f(s)}{t-s} ds. \tag{2.11}$$

Here p.v. means that the integral is understood in the principal value sense, which in this context is equivalent to

$$\mathsf{H}f(t) = \lim_{\varepsilon\to 0}\int_{\mathbb{R}\setminus(t-\varepsilon,t+\varepsilon)} \frac{f(s)}{t-s} ds.$$

Both operators turn out to be *local* in the Fourier domain in the sense that they are multiplications in the Fourier domain. More precisely we have the following lemma.

Lemma 2.1.3 *We have in the Fourier domain the following relations:*

$$\begin{aligned}(\mathcal{F}_{x\to\xi} R^* g)(\xi) &= \frac{2\pi}{|\xi|}\Big((\mathcal{F}_{s\to\sigma} g)(-|\xi|, \hat\xi^\perp) + (\mathcal{F}_{s\to\sigma} g)(|\xi|, -\hat\xi^\perp)\Big) \\ (\mathcal{F}_{t\to\sigma}\mathsf{H}f)(\sigma) &= -i\,\mathrm{sign}(\sigma)\mathcal{F}_{\mathrm{t}\to\sigma}\mathrm{f}(\sigma).\end{aligned} \tag{2.12}$$

We have used the notation $\hat\xi = \xi/|\xi|$. For $\vartheta = (\cos\theta, \sin\theta)$ with $\vartheta \in S^1$ and $\theta \in (0, 2\pi)$, we also identify the functions $g(\theta) = g(\vartheta)$. Assuming that $g(s,\theta) = g(-s, \theta+\pi)$, which is the case in the image of the Radon transform (i.e., when there exists f such that $g = Rf$), and which implies that $\hat g(\sigma,\vartheta) = \hat g(-\sigma, -\vartheta)$ we have using shorter notation the equivalent statement:

$$\begin{aligned}\widehat{R^*g}(\xi) &= \frac{4\pi}{|\xi|}\,\hat g(|\xi|, -\hat\xi^\perp) \\ \widehat{\mathsf{H}f}(\sigma) &= -i\,\mathrm{sign}(\sigma)\hat{\mathrm{f}}(\sigma).\end{aligned} \tag{2.13}$$

Proof. Let us begin with R^*g. We compute

$$\begin{aligned}\widehat{R^*g}(\xi) &= \int e^{-ix\cdot\xi} g(x\cdot\vartheta^\perp,\theta) d\theta dx = \int e^{-is\xi\cdot\vartheta^\perp} g(s,\theta) ds d\theta e^{-it\xi\cdot\vartheta} dt \\ &= \int 2\pi\delta(|\xi|\hat\xi\cdot\vartheta)\hat g(\xi\cdot\vartheta^\perp,\theta) d\theta = \int \frac{2\pi}{|\xi|}\delta(\hat\xi\cdot\vartheta)\hat g(\xi\cdot\vartheta^\perp,\theta) d\theta \\ &= \frac{2\pi}{|\xi|}\Big(\hat g(-|\xi|,\hat\xi^\perp) + \hat g(|\xi|, -\hat\xi^\perp)\Big).\end{aligned}$$

In the proof we have used that $\delta(\alpha x) = \alpha^{-1}\delta(x)$ and the fact that there are two directions, namely $\hat{\xi}$ and $-\hat{\xi}$ on the unit circle, which are orthogonal to $\hat{\xi}^{\perp}$. When g is in the form $g = Rf$, we have $\hat{g}(-|\xi|, \hat{\xi}^{\perp}) = \hat{g}(|\xi|, -\hat{\xi}^{\perp})$, which explains the shorter notation (2.13).

The computation of the second operator goes as follows. We verify that

$$\mathsf{H}f(t) = \frac{1}{\pi}\Big(\frac{1}{x} * f(x)\Big)(t).$$

So in the Fourier domain we have

$$\widehat{\mathsf{H}f}(\sigma) = \frac{1}{\pi}\widehat{\frac{1}{x}}(\sigma)\hat{f}(\sigma) = -i\text{sign}(\sigma)\hat{f}(\sigma).$$

The latter is a result of the following calculation

$$\frac{1}{2}\text{sign}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{ix\xi}}{i\xi} d\xi.$$

This concludes the proof of the lemma. □

The above calculations also show that $\mathsf{H}^2 = \mathsf{H} \circ \mathsf{H} = -Id$, where Id is the identity operator, as can easily be seen in from its expression in the Fourier domain. This property is referred to as saying that the Hilbert transform is an *anti-involution*. We are now ready to introduce some reconstruction formulas.

Theorem 2.1.4 *Let $f(x)$ be a smooth function and let $g(s,\theta) = Rf(s,\theta)$ be its Radon transform. Then, f can explicitly be reconstructed from its Radon transform as follows:*

$$f(x) = \frac{1}{4\pi} R^*\Big(\frac{\partial}{\partial s}\mathsf{H}g(s,\theta)\Big)(x). \tag{2.14}$$

In the above formula, the Hilbert transform H acts on the s variable.

Proof. The simplest way to verify the inversion formula is to do it in the Fourier domain. Let us denote by

$$w(s,\theta) = \frac{\partial}{\partial s}\mathsf{H}g(s,\theta).$$

Since $g(-s, \theta+\pi) = g(s,\theta)$, we verify that the same property holds for w in the sense that $w(-s,\theta+\pi) = w(s,\theta)$. Therefore (2.14) is equivalent to the statement:

$$\hat{f}(\xi) = \frac{1}{|\xi|}\hat{w}(|\xi|, -\hat{\xi}^{\perp}), \tag{2.15}$$

according to (2.13). Notice that $\hat{w}$ is the Fourier transform of w in the *first variable* only.

Since in the Fourier domain, the derivation with respect to s is given by multiplication by $i\sigma$ and the Hilbert transform H is given by multiplication by $-i\text{sign}(\sigma)$, we obtain that

$$\mathcal{F}^{-1}_{\sigma\to s}\frac{\partial}{\partial s}\mathsf{H}\mathcal{F}_{s\to\sigma} = |\sigma|.$$

In other words, we have

$$\hat{w}(\sigma,\theta) = |\sigma|\hat{g}(\sigma,\theta).$$

Thus (2.15) is equivalent to

$$\hat{f}(\xi) = \hat{g}(|\xi|, -\hat{\xi}^{\perp}).$$

This, however, is nothing but the Fourier slice theorem stated in Theorem 2.1.1 since $(-\hat{\xi}^{\perp})^{\perp} = \hat{\xi}$ and $\xi = |\xi|\hat{\xi}$. This concludes the proof of the reconstruction. □
The theorem can equivalently be stated as

$$I = \frac{1}{4\pi} R^* \frac{\partial}{\partial s} \mathsf{H} R = \frac{1}{4\pi} R^* \mathsf{H} \frac{\partial}{\partial s} R. \tag{2.16}$$

The latter equality comes from the fact that H and ∂_s commute as can easily be observed in the Fourier domain (where they are both multiplications). Here, I is the identity operator, which maps a function $f(x)$ to itself $I(f) = f$.

Here is some additional useful notation in the manipulation of the Radon transform. Recall that $R_\theta f(s)$ is defined as in (2.5) by

$$R_\theta f(s) = Rf(s, \theta).$$

The adjoint R_θ (with respect to the inner products in $L^2_s(\mathbb{R})$ and $L^2_x(\mathbb{R}^2)$) is given by

$$(R^*_\theta g)(x) = g(x \cdot \vartheta^{\perp}). \tag{2.17}$$

Indeed (since ϑ is *frozen* here) we have

$$\int_{\mathbb{R}} (R_\theta f)(s) g(s) ds = \int_{\mathbb{R}^2} \int_{\mathbb{R}} f(x) \delta(s - x \cdot \vartheta^{\perp}) g(s) dx ds = \int_{\mathbb{R}^2} g(x \cdot \vartheta^{\perp}) f(x) dx,$$

showing that $(R_\theta f, g)_{L^2(\mathbb{R})} = (f, R^*_\theta g)_{L^2(\mathbb{R}^2)}$. We can then recast the inversion formula as

$$I = \frac{1}{4\pi} \int_0^{2\pi} \vartheta^{\perp} \cdot \nabla R^*_\theta \mathsf{H} R_\theta d\theta. \tag{2.18}$$

The only new item to prove here compared to previous formulas is that R^*_θ and the derivation commute, i.e., for any function $g(s)$ for $s \in \mathbb{R}$, we have

$$\vartheta^{\perp} \cdot \nabla (R^*_\theta g)(x) = (R^*_\theta \frac{\partial}{\partial s} g)(x).$$

This results from the fact that both terms are equal to $g'(x \cdot \vartheta^{\perp})$.

One remark on the smoothing properties of the Radon transform. We have seen that the Radon transform is a smoothing operator in the sense that the Radon transform is half of a derivative smoother than the original function. The adjoint operator R^* enjoys exactly the same property: it regularizes by half of a derivative. It is not surprising that these two half derivatives are exactly canceled by the appearance of a full derivation in the reconstruction formula. Notice that the Hilbert transform (which corresponds to multiplication by the smooth function $i\text{sign}(\sigma)$ in the Fourier domain) is a bounded operator with bounded inverse in $L^2(\mathbb{R})$ (since $\mathsf{H}^{-1} = -\mathsf{H}$).

Exercise 2.1.2 Show that

$$f(x) = \frac{1}{4\pi} \int_0^{2\pi} (\mathsf{H} g')(x \cdot \vartheta^{\perp}, \theta) d\theta.$$

Here g' means first derivative of g with respect to the s variable only.

Exercise 2.1.3 Show that

$$f(x) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_{\mathbb{R}} \frac{\frac{d}{ds} g(s,\theta)}{x \cdot \vartheta^{\perp} - s} ds d\theta.$$

This is Radon's original inversion formula.

Exercise 2.1.4 Starting from the definition

$$f(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ik\cdot x} \hat{f}(k) dk,$$

and writing it in polar coordinates (with change of measure $dk = |k|d|k|d\hat{k}$), deduce the above reconstruction formulas by using the Fourier slice theorem.

2.1.3 Three dimensional Radon transform

Let us briefly mention the case of the Radon transform in three dimensions (generalizations to higher dimensions being also possible). The Radon transform in three dimensions consists of integrating a function $f(x)$ over all possible planes. As we mentioned earlier, the Radon transform is therefore a distinct object from the X-ray transform, which integrates a function along all possible lines.

A plane $\mathcal{P}(s,\vartheta)$ in $\mathbb{R}^3$ is characterized by its direction $\vartheta \in S^2$, where S^2 is the unit sphere, and by its signed distance to the origin s. Notice again the double covering in the sense that $\mathcal{P}(s,\vartheta) = \mathcal{P}(-s,-\vartheta)$. The Radon transform is then defined as

$$\mathsf{R}f(s,\vartheta) = \int_{\mathbb{R}^3} f(x)\delta(x\cdot\vartheta - s)dx = \int_{\mathcal{P}(s,\theta)} f d\sigma. \tag{2.19}$$

Notice the change of notation compared to the two-dimensional case. The Fourier slice theorem still holds

$$\widehat{\mathsf{R}f}(\sigma,\vartheta) = \hat{f}(\sigma\vartheta), \tag{2.20}$$

as can be easily verified. We check that $\mathsf{R}f(s,\vartheta) = \mathsf{R}f(-s,-\vartheta)$. The reconstruction formula is then given by

$$f(x) = \frac{-1}{8\pi^2} \int_{S^2} g''(x\cdot\vartheta,\vartheta)d\vartheta. \tag{2.21}$$

Here $d\vartheta$ is the usual (Lebesgue) surface measure on the unit sphere.

The result can be obtained as follows. We denote by $S^2/2$ half of the unit sphere (for instance the vectors ϑ such that $\vartheta\cdot\mathbf{e}_z > 0$).

$$\begin{aligned} f(x) &= \frac{1}{(2\pi)^3} \int_{\frac{S^2}{2}} \int_{\mathbb{R}} \hat{f}(r\vartheta) e^{ir\vartheta\cdot x} |r|^2 dr d\vartheta = \frac{1}{(2\pi)^3} \int_{\frac{S^2}{2}} \int_{\mathbb{R}} \widehat{\mathsf{R}f}(r,\vartheta) e^{ir\vartheta\cdot x} |r|^2 dr d\vartheta \\ &= \frac{1}{(2\pi)^2} \int_{\frac{S^2}{2}} (-g)''(\vartheta\cdot x,\vartheta) d\vartheta = \frac{1}{2}\frac{-1}{(2\pi)^2} \int_{S^2} g''(\vartheta\cdot x,\vartheta) d\vartheta. \end{aligned}$$

Here we have used the fact that the inverse Fourier transform of $r^2\hat{f}$ is $-f''$.

Exercise 2.1.5 Generalize Theorem 2.1.2 and prove the following result:

Theorem 2.1.5 *There exists a constant C_χ independent of $f(x)$ such that*

$$\begin{aligned} \sqrt{2}\|f\|_{H^s(\mathbb{R}^3)} &\leq \|\mathsf{R}f\|_{\mathrm{H}^{s+1}(Z)} \\ \|\mathsf{R}(\chi f)\|_{\mathrm{H}^{s+1}(Z)} &\leq C_\chi \|\chi f\|_{H^s(\mathbb{R}^3)}, \end{aligned} \tag{2.22}$$

where $Z = \mathbb{R} \times S^2$ and $\mathrm{H}^s(Z)$ is defined in the spirit of (2.8).

The above result shows that the Radon transform is more smoothing in three dimensions than it is in two dimensions. In three dimensions, the Radon transform smoothes by a full derivative rather than a half derivative.

Notice however that the inversion of the three dimensional Radon transform (2.21) is *local*, whereas this is not the case in two dimensions. What is meant by local is the following: the reconstruction of $f(x)$ depends on $g(s, \vartheta)$ only for the planes $\mathcal{P}(s, \vartheta)$ that pass through x (and an infinitely small neighborhood so that the second derivative can be calculated). Indeed, we verify that $x \in \mathcal{P}(x \cdot \vartheta, \vartheta)$ and that all the planes passing by x are of the form $\mathcal{P}(x \cdot \vartheta, \vartheta)$. The two dimensional transform involves the Hilbert transform, which unlike differentiations, is a *non-local* operation. Thus the reconstruction of f at a point x requires knowledge of *all* line integrals $g(s, \theta)$, and not only for those lines passing through x.

Exercise 2.1.6 Calculate R^*, the adjoint operator to R (with respect to the usual L^2 inner products). Generalize the formula (2.16) to the three dimensional case.

2.2 Attenuated Radon Transform

In the previous sections, the integration over lines for the Radon transform was not weighted. We could more generally ask whether integrals of the form

$$R_\alpha f(s, \vartheta) := \int_{\mathbb{R}} f(s\vartheta^\perp + t\vartheta)\alpha(s\vartheta^\perp + t\vartheta, \vartheta)dt,$$

over all possible lines parameterized by $s \in \mathbb{R}$ and $\vartheta \in S^1$ and assuming that the (positive) weight $\alpha(x, \vartheta)$ is known, uniquely determine $f(x)$. This is a much more delicate question for which only partial answers are known.

The techniques developed in the next chapter allow one to prove that R_α is invertible up to a finite dimensional kernel by an application of the Fredholm theory of compact operators. Moreover, Jan Boman [22] constructed an example of a function $f \in C_0^\infty(\mathbb{R}^2)$ with compact support and a uniformly positive weight $\alpha(x, \vartheta)$ also of class $C^\infty(\mathbb{R}^2)$ such that $R_\alpha f(s, \vartheta) = 0$. The operator $R_\alpha f$ is therefore not always injective.

Proving that $R_\alpha f$ is injective, when it is indeed injective, is a hard problem. In the next chapter, we will see a methodology to prove injectivity based on energy estimates. The alternative to energy estimates is to use what we will refer to as analytic/unique continuation techniques. For the rest of this chapter, we focus on the inversion of one example of such weighted Radon transforms, namely the Attenuated Radon transform

(AtRT). The invertibility of the AtRT has been obtained recently by two different methods in [6] and [47]. Both rely on the continuation properties of analytic functions. We focus here on the second method, which recasts the inverse Radon transform and the inverse Attenuated Radon transform as a Riemann-Hilbert problem [1]. The rest of this section is significantly more technical than the inversion of the Radon transform. It is presented here as an example of the complications that often appear in the proof of injectivity of many transforms of integral geometry when they are indeed injective.

2.2.1 Single Photon Emission Computed Tomography

An important application for the attenuated Radon transform is SPECT, single photon emission computed tomography. The principle is the following: radioactive particles are injected in a domain. These particles emit then some radiation. The radiation propagates through the tissues and gets partially absorbed. The amount of radiation reaching the boundary of the domain can be measured. The imaging technique consists then of reconstructing the location of the radioactive particles from the boundary measurements.

We model the density of radiated photons by $u(x,\theta)$ and the source of radiated photons by $f(x)$. The absorption of photons (by the human tissues in the medical imaging application) is modeled by $\sigma(x)$. We assume that $\sigma(x)$ is *known* here. The absorption can be obtained, for instance, by transmission tomography as we saw in earlier sections. The density $u(x,\theta)$ satisfies then the following transport equation

$$\vartheta\cdot\nabla u(x,\theta)+\sigma(x)u(x,\theta)=f(x),\qquad x\in\mathbb{R}^2,\vartheta\in S^1. \tag{2.23}$$

We assume that $f(x)$ is compactly supported and impose that no radiation comes from infinity:

$$\lim_{s\to\infty}u(x-s\vartheta,\theta)=0. \tag{2.24}$$

The transport equation (2.23) with conditions (2.24) admits a unique solution that can be obtained by the method of characteristics. Let us define the following *symmetrized* beam transform

$$D_\theta\sigma(x)=\frac{1}{2}\int_0^\infty[\sigma(x-t\vartheta)-\sigma(x+t\vartheta)]dt=\frac{1}{2}\int_{\mathbb{R}}\operatorname{sign}(t)\sigma(x-t\vartheta)dt. \tag{2.25}$$

We verify that $\vartheta\cdot\nabla D_\theta\sigma(x)=\sigma(x)$ so that $e^{D_\theta\sigma(x)}$ is an integrating factor for (2.23) in the sense that

$$\vartheta\cdot\nabla(e^{D_\theta\sigma(x)}u(x,\theta))=(e^{D_\theta\sigma}f)(x,\theta).$$

Therefore the solution $u(x,\theta)$ is given by

$$e^{D_\theta\sigma(x)}u(x,\theta)=\int_0^\infty(e^{D_\theta\sigma}f)(x-t\vartheta,\theta)dt. \tag{2.26}$$

We recall that $\vartheta=(\cos\theta,\sin\theta)$ and that $\vartheta^\perp=(-\sin\theta,\cos\theta)$ and decompose $x=s\vartheta^\perp+t\vartheta$. We deduce from (2.26) that

$$\lim_{t\to+\infty}e^{D_\theta\sigma(s\vartheta^\perp+t\vartheta)}u(s\vartheta^\perp+t\vartheta,\theta)=\int_{\mathbb{R}}(e^{D_\theta\sigma}f)(s\vartheta^\perp+t\vartheta,\theta)dt.$$

In the above expression the left hand side is know from the measurements. Indeed $u(s\vartheta^\perp + t\vartheta, \theta)$ is the radiation outside of the domain to image and is thus measured and $e^{D_\theta \sigma(s\vartheta^\perp + t\vartheta)}$ involves the attenuation coefficient $\sigma(x)$ which we have assumed is known. The objective is thus to reconstruct $f(x)$ from the right hand side of the above relation, which we recast as

$$(R_\sigma f)(s,\theta) = (R_{\sigma,\theta} f)(s) = (R_\theta(e^{D_\theta \sigma} f))(s), \tag{2.27}$$

where R_θ is the Radon transform defined for a function of $f(x,\theta)$ as

$$R_\theta f(s) = \int_{\mathbb{R}} f(s\vartheta^\perp + t\vartheta, \theta) dt = \int_{\mathbb{R}^2} f(x,\theta)\delta(x\cdot\vartheta^\perp - s) dx.$$

When $\sigma \equiv 0$, we recover that the measurements involve the Radon transform of $f(x)$ as defined in (2.4). Thus in the absence of absorption, SPECT can be handled by inverting the Radon transform as we saw in earlier sections. When absorption is constant, an inversion formula has been known for quite some time [62]. The inversion formula for non-constant absorption is more recent and was obtained independently by two different techniques [6, 47]. We do not consider here the method of $A-$analytic functions developed in [6]. We will present the method developed in [47] based on the extension of the transport equation in the complex domain and on the solution of a Riemann Hilbert problem.

2.2.2 Riemann Hilbert problem

Riemann Hilbert problems find many applications in complex analysis. We consider here the simplest of Riemann Hilbert problems and refer the reader to [1] for more general cases and applications.

Let T be a smooth closed curve in the complex plane, which in our application will be the unit circle, i.e., the complex numbers λ such that $|\lambda| = 1$. The reason why we choose the notation λ to represent complex numbers will appear more clearly in the next section. We denote by D^+ the open bounded domain inside the curve T, i.e., in our application the unit disk $\{\lambda \in \mathbb{C}, |\lambda| < 1\}$, and by D^- the open unbounded domain outside of the curve T, i.e., in our application $\{\lambda \in \mathbb{C}, |\lambda| > 1\}$. The orientation of the curve T is chosen so that D^+ is on the "left" of the curve T.

For a smooth function $\phi(\lambda)$ defined on $D^+ \cup D^-$, we denote by $\phi^+(t)$ and $\phi^-(t)$ the traces of ϕ on T from D^+ and D^-, respectively. So in the case where T is the unit circle, we have

$$\phi^+(t) = \lim_{0<\varepsilon\to 0} \phi((1-\varepsilon)t), \qquad \phi^-(t) = \lim_{0<\varepsilon\to 0} \phi((1+\varepsilon)t).$$

We define $\varphi(t)$ on T as the jump of ϕ, i.e.,

$$\varphi(t) = \phi^+(t) - \phi^-(t). \tag{2.28}$$

Let $\varphi(t)$ be a smooth function defined on T. The Riemann Hilbert problem is stated as follows. Find a function $\phi(\lambda)$ on $D^+ \cup D^-$ such that

1. $\phi(\lambda)$ is analytic on D^+ and analytic on D^-

2. $\lambda\phi(\lambda)$ is bounded as $|\lambda| \to \infty$ on D^-

3. the jump of ϕ is given by $\varphi(t) = \phi^+(t) - \phi^-(t)$.

The solution to the above Riemann Hilbert problem is *unique* and is given by the Cauchy formula

$$\phi(\lambda) = \frac{1}{2\pi i}\int_T \frac{\varphi(t)}{t-\lambda}dt, \qquad \lambda \in \mathbb{C}\backslash T = D^+ \cup D^-. \tag{2.29}$$

This is the form of the Riemann Hilbert problem we will use in the sequel. We refer the reader to [1] for the theory.

2.2.3 Inversion of the Attenuated Radon Transform

We now want to apply the theory of Riemann Hilbert problems to invert the attenuated Radon transform (AtRT). The first step is to extend the transport equation to the complex domain as follows. We parameterize the unit circle in the complex plane as

$$\lambda = e^{i\theta}, \qquad \theta \in (0, 2\pi). \tag{2.30}$$

The new parameter takes values on the unit circle T for $\theta \in (0, 2\pi)$. It can also be seen more generally as an arbitrary complex number $\lambda \in \mathbb{C}$. With the notation $x = (x_1, x_2)$, the transport equation (2.23) can be recast as

$$\Big(\frac{\lambda + \lambda^{-1}}{2}\frac{\partial}{\partial x_1} + \frac{\lambda - \lambda^{-1}}{2i}\frac{\partial}{\partial x_2} + \sigma(x)\Big)u(x,\lambda) = f(x), \qquad x \in \mathbb{R}^2, \quad \lambda \in T. \tag{2.31}$$

We can simplify the above equation by identifying x with $z = x + iy$ and by defining

$$\frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\Big), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\Big). \tag{2.32}$$

The transport equation (2.31) is then equivalent to

$$\Big(\lambda\frac{\partial}{\partial z} + \lambda^{-1}\frac{\partial}{\partial \bar{z}} + \sigma(z)\Big)u(z,\lambda) = f(z), \qquad z \in \mathbb{C}, \quad \lambda \in T. \tag{2.33}$$

The same boundary conditions (2.24) that no information comes from infinity need to be added in the new variables as well.

The above equation can also be generalized to $\lambda \in \mathbb{C}$ instead of T. It is in this framework that the Riemann Hilbert problem theory is used to invert the attenuated Radon transform. This will be done in three steps

(i) We show that $u(z,\lambda)$ is *analytic* in $D^+ \cup D^- = \mathbb{C}\backslash T$ and that $\lambda u(z,\lambda)$ is bounded as $\lambda \to \infty$.

(ii) We verify that $\varphi(x,\theta) = u^+(x,\theta) - u^-(x,\theta)$, the jump of u at $\lambda = e^{i\theta}$ can be written as a function of the measured data $R_a f(s,\theta)$.

(iii) We solve the Riemann Hilbert problem using (2.29) and evaluate (2.33) at $\lambda = 0$ to obtain a reconstruction formula for $f(z) = f(x)$.

2.2.4 Step (i): The $\overline{\partial}$ problem, an elliptic equation

Let us now analyze (2.33). In the absence of absorption the fundamental solution of (2.33) solves the following equation

$$\Big(\lambda\frac{\partial}{\partial z}+\lambda^{-1}\frac{\partial}{\partial\bar z}\Big)G(z,\lambda)=\delta(z),\qquad |G(z,\lambda)|\to 0 \text{ as } |z|\to\infty, \tag{2.34}$$

for $\lambda\in\mathbb{C}\backslash(T\cup\{0\})$.

Lemma 2.2.1 *The unique solution to (2.34) is given by*

$$G(z,\lambda)=\frac{\operatorname{sign}(|\lambda|-1)}{\pi(\lambda\bar z-\lambda^{-1}z)},\qquad \lambda\notin(T\cup\{0\}). \tag{2.35}$$

Proof. The formula can be verified by inspection. A more constructive derivation is the following. Let us define the change of variables

$$\zeta=\lambda^{-1}z-\lambda\bar z,\qquad\qquad \bar\zeta=\overline{\lambda^{-1}}\bar z-\bar\lambda z. \tag{2.36}$$

Let us assume that $|\lambda|>1$. The Jacobian of the transformation is $|\lambda|^2-|\lambda|^{-2}$. We verify that

$$\lambda\frac{\partial}{\partial z}+\lambda^{-1}\frac{\partial}{\partial\bar z}=(|\lambda|^{-2}-|\lambda|^2)\frac{\partial}{\partial\bar\zeta}.$$

The change of variables (2.36) has been precisely tailored so that the above holds. Denoting $\tilde G(\zeta)=G(z)$, we thus obtain

$$\frac{\partial}{\partial\bar\zeta}\tilde G(\zeta)=\frac{1}{|\lambda|^{-2}-|\lambda|^2}\delta(z(\zeta))=-\delta(\zeta).$$

So $-\tilde G(\zeta)$ is the fundamental solution of the $\overline{\partial}$ operator $\dfrac{\partial}{\partial\bar\zeta}$. We verify that

$$\frac{\partial}{\partial\bar\zeta}\frac{1}{\zeta}=\pi\delta(\zeta). \tag{2.37}$$

Indeed let $\psi(z)$ be a smooth test function in $C_0^\infty(\mathbb{R}^2)$ and let $d\mu(\zeta)$ be the Lebesgue measure $dxdy$ in $\mathbb{C}\sim\mathbb{R}^2$. Then

$$\begin{aligned}\int_{\mathbb{C}}\psi(\zeta)\frac{\partial}{\partial\bar\zeta}\frac{1}{\zeta}d\mu(\zeta)&=-\int_{\mathbb{C}}\frac{\partial\psi}{\partial\bar\zeta}\frac{1}{\zeta}d\mu(\zeta)=-\lim_{\varepsilon\to0}\int_{\mathbb{C}\backslash\{|\zeta|<\varepsilon\}}\frac{\partial\psi}{\partial\bar\zeta}\frac{1}{\zeta}d\mu(\zeta)\\&=-\lim_{\varepsilon\to0}\int_{\mathbb{C}\backslash\{|\zeta|<\varepsilon\}}\frac{\partial\zeta^{-1}\psi}{\partial\bar\zeta}d\mu(\zeta)=\frac{1}{2i}\int_{|\zeta|=\varepsilon}\frac{\psi}{\zeta}d\mu(\zeta),\end{aligned}$$

by the Green formula with complex variables:

$$\int_{\partial X}udz=\int_{\partial X}(udx+iudy)=\int_X(i\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y})dxdy=2i\int_X\frac{\partial u}{\partial\bar z}d\mu(z).$$

Sending ε to 0, we find in the limit

$$\int_{\mathbb{C}}\psi(\zeta)\frac{\partial}{\partial\bar\zeta}\frac{1}{\zeta}d\mu(\zeta)=\frac{1}{2i}2\pi i\psi(0)=\pi\int_{\mathbb{R}^2}\psi(\zeta)\delta(\zeta)d\mu(\zeta).$$

This implies that $\tilde{G}(\zeta) = (-\pi\zeta)^{-1}$, hence $G(z) = (-\pi\zeta)^{-1} = (\pi(\lambda\bar{z} - \lambda^{-1}z))^{-1}$. This is (2.35) for $|\lambda| > 1$. For $|\lambda| < 1$, we verify that the Jacobian of the transformation $z \to \zeta(z)$ becomes $|\lambda|^{-2} - |\lambda|^2$ so that

$$\frac{\partial}{\partial\bar{\zeta}}\tilde{G}(\zeta) = \frac{1}{|\lambda|^{-2} - |\lambda|^2}\delta(z(\zeta)) = \delta(\zeta).$$

This yields (2.35) for $|\lambda| < 1$. □

The above proof shows that $(\pi z)^{-1}$ is the fundamental solution of the $\bar{\partial} = \frac{\partial}{\partial\bar{z}}$ operator. This implies that the solution to the following $\bar{\partial}$ problem

$$\frac{\partial}{\partial\bar{z}}f(z) = g(z), \qquad z \in \mathbb{C}, \tag{2.38}$$

such that $f(z)$ vanishes at infinity is given by convolution by the fundamental solution, i.e.,

$$f(z) = \frac{1}{\pi}\int_{\mathbb{C}}\frac{g(\zeta)}{z-\zeta}d\mu(\zeta) = \frac{1}{2\pi i}\int_{\mathbb{C}}\frac{g(\zeta)}{z-\zeta}d\zeta \wedge d\bar{\zeta}. \tag{2.39}$$

Here we have used that $dz \wedge d\bar{z} = (dx + idy) \wedge (dx - idy) = 2idx \wedge dy = 2id\mu(z)$, whence the change of 2-form in the above integrations.

Notice that the Green function $G(z, \lambda)$ tends to 0 as $z \to \infty$ for $\lambda \notin T$. This is clearly not true when $\lambda \in T$, where $G(z, \lambda) = \delta(l_\theta(z))$, where $l_\theta(z)$ is the segment $\{t\vartheta, t > 0\}$. The reason is that for $\lambda \notin (T \cup \{0\})$,

$$\lambda\frac{\partial}{\partial z} + \lambda^{-1}\frac{\partial}{\partial\bar{z}} \qquad \text{and} \qquad \frac{\partial}{\partial\bar{z}},$$

are elliptic operators, in the sense that in the Fourier domain, their symbol given by $\lambda k_z + \lambda^{-1}\overline{k_z}$ and $\overline{k_z}$, respectively, are positive provided that k_z is not 0. Indeed we verify that

$$\lambda k_z + \lambda^{-1}\overline{k_z} = 0 \quad \text{implies} \quad |\lambda|^2 = 1,$$

when $k_z \neq 0$ since $|k_z| = |\overline{k_z}| \neq 0$.

Let us now define $h(z, \lambda)$ as the solution to

$$\Big(\lambda\frac{\partial}{\partial z} + \lambda^{-1}\frac{\partial}{\partial\bar{z}}\Big)h(z, \lambda) = \sigma(z), \qquad |h(z, \lambda)| \to 0 \text{ as } |z| \to \infty, \tag{2.40}$$

for $\lambda \notin (T \cup \{0\})$. The solution is given by

$$h(z, \lambda) = \int_{\mathbb{R}^2} G(z - \zeta, \lambda)\sigma(\zeta)d\mu(\zeta). \tag{2.41}$$

We now verify that

$$\Big(\lambda\frac{\partial}{\partial z} + \lambda^{-1}\frac{\partial}{\partial\bar{z}}\Big)(e^{h(z,\lambda)}u(z, \lambda)) = e^{h(z,\lambda)}f(z),$$

so that for $\lambda \notin (T \cup \{0\})$, the solution of (2.33) is given by

$$u(z, \lambda) = e^{-h(z,\lambda)}\int_{\mathbb{R}^2} G(z - \zeta, \lambda)e^{h(\zeta,\lambda)}f(\zeta)d\mu(\zeta). \tag{2.42}$$

We verify that $G(z,\lambda)$, $h(z,\lambda)$ and $u(z,\lambda)$ are defined by continuity at $z=0$ since $G(z,\lambda)=0$ by continuity. We now verify that $G(z,\lambda)$ is analytic in D^+ (including at $z=0$) and in D^-. Assuming that $a(z)$ and $f(z)$ are smooth functions, this is also therefore the case for $h(z,\lambda)$ and $u(z,\lambda)$. Moreover we easily deduce from (2.42) that $\lambda u(z,\lambda)$ is bounded on D^-. The solution $u(z,\lambda)$ of the transport equation extended to the complex plane is therefore a good candidate to apply the Riemann Hilbert theorem.

2.2.5 Step (ii): jump conditions

We now want to find the limit of $u(z,\lambda)$ as λ approaches T from above (in D^-) and below (in D^+). Let us write $\lambda = re^{i\theta}$ and let us send $r-1$ to ∓ 0 on $D^\pm$. The Green function behaves according to the following result

Lemma 2.2.2 *As $r-1\to\mp 0$, the Green function $G(x,\lambda)$ tends to*

$$G_\pm(x,\theta)=\frac{\pm 1}{2\pi i(\vartheta^\perp\cdot x\mp i0\,\mathrm{sign}(\vartheta\cdot x))}. \tag{2.43}$$

Proof. Let us assume that $|\lambda|>1$, i.e., $r=1+\varepsilon$ with $\varepsilon>0$. We then find

$$\begin{aligned}G(z,re^{i\theta})&=\frac{1}{\pi}\frac{1}{re^{i\theta}\bar z-\frac{e^{-i\theta}}{r}z}=\frac{1}{\pi}\frac{1}{(1+\varepsilon)e^{i\theta}\bar z-e^{-i\theta}(1-\varepsilon)z+o(\varepsilon)}\\&=\frac{1}{2\pi}\frac{1}{-i\mathcal{I}(e^{-i\theta}z)+\varepsilon\mathcal{R}(e^{-i\theta}z)+o(\varepsilon)}=\frac{1}{2i\pi}\frac{-1}{\vartheta^\perp\cdot x+i\varepsilon(\vartheta\cdot x)+o(\varepsilon)}.\end{aligned}$$

Passing to the limit $\varepsilon\to 0$, we obtain

$$G_-(x,\theta)=\frac{1}{2i\pi}\frac{-1}{\vartheta^\perp\cdot x+i0\mathrm{sign}(\vartheta\cdot x)}.$$

Here by convention ± 0 is the limit of $\pm\varepsilon$ as $0<\varepsilon\to 0$. The limit on D^+ is treated similarly. □

We have chosen to define $G_\pm(x,\theta)$ as functions of $\theta\in(0,2\pi)$ instead of functions of $e^{i\theta}$. We have also identified $x=(x,y)$ with $z=x+iy$. The above lemma gives us a convergence in the sense of distributions. We can equivalently say that for all smooth function $\psi(x)$, we have

$$\int_{\mathbb{R}^2}G_\pm(x-y,\theta)\psi(y)dy=\pm\frac{1}{2i}(\mathsf{H}R_\theta\psi)(x\cdot\vartheta^\perp)+(D_\theta\psi)(x). \tag{2.44}$$

We recall that the Hilbert transform H is defined in (2.11) and the Radon transform in (2.4)-(2.5).

Proof. The derivation is based on the following result. For any $f(x)\in C_0^\infty(\mathbb{R})$, we have

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}}\frac{f(x)}{ix+\varepsilon}dx=-i\mathrm{p.v.}\int_{\mathbb{R}}\frac{f(x)}{x}dx+\mathrm{sign}(\varepsilon)\pi f(0). \tag{2.45}$$

Exercise 2.2.1 Prove the above limit called Plemelj's formula.

Let us denote $x = \varsigma\vartheta^\perp + \tau\vartheta$ and $y = s\vartheta^\perp + t\vartheta$. We have

$$\begin{aligned}
&\int_{\mathbb{R}^2} G_+(y)\psi(x-y)dy = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{\psi((\varsigma-s)\vartheta^\perp + (\tau-t)\vartheta)}{is + 0\text{sign}(t)}dsdt \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \text{p.v.} \int_{\mathbb{R}} \frac{\psi((\varsigma-s)\vartheta^\perp + (\tau-t)\vartheta)}{is}dsdt + \frac{1}{2}\int_{\mathbb{R}} \text{sign}(t)\psi(\varsigma\vartheta^\perp + (\tau-t)\vartheta)dt \\
&= \frac{1}{2\pi}\text{p.v.} \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\psi((\varsigma-s)\vartheta^\perp + (\tau-t)\vartheta)}{is}dtds + \frac{1}{2}\int_{\mathbb{R}} \text{sign}(t)\psi(x-t\vartheta)dt \\
&= \frac{1}{2i}(HR_\theta\psi)(x\cdot\vartheta^\perp) + (D_\theta\psi)(x).
\end{aligned}$$

A similar derivation yields the limit on D^-. □

We deduce that the function $h(z,\lambda)$ defined in (2.40) admits the limits

$$h_\pm(x,\theta) = \pm\frac{1}{2i}(\mathsf{H}R_\theta\sigma)(x\cdot\vartheta^\perp) + (D_\theta\sigma)(x). \tag{2.46}$$

Notice that R_θ and D_θ involve integrations in the direction ϑ only so that

$$R_\theta[u(x)v(x\cdot\vartheta^\perp)](s) = v(s)R_\theta[u](s), \quad D_\theta[u(x)v(x\cdot\vartheta^\perp)](x) = v(x\cdot\vartheta^\perp)D_\theta[u](x).$$

Using this result and (2.44), we deduce that the limits of the solution $u(z,\lambda)$ to (2.31) are given by

$$\begin{aligned}
u^\pm(x,\theta) = e^{-D_\theta\sigma}e^{\frac{\mp1}{2i}(\mathsf{H}R_\theta\sigma)(x\cdot\vartheta^\perp)}\frac{\pm1}{2i}\mathsf{H}\Big(e^{\frac{\pm1}{2i}(\mathsf{H}R_\theta\sigma)(s)}R_\theta(e^{D_\theta\sigma}f)\Big)(x\cdot\vartheta^\perp) \\
+e^{-D_\theta\sigma}D_\theta(e^{D_\theta\sigma}f)(x).
\end{aligned} \tag{2.47}$$

We recall that $R_\theta(e^{D_\theta\sigma}f) = R_{\sigma,\theta}f(s)$ are our measurements. So whereas u^+ and u^- do not depend only on the measurements (they depend on $D_\theta(e^{D_\theta\sigma}f)(x)$ which is not measured), the difference $u^+ - u^-$ depends *only* on the measurements. This is the property that allows us to invert the attenuated Radon transform. More precisely, let us define

$$\varphi(x,\theta) = (u^+ - u^-)(x,\theta). \tag{2.48}$$

Using (2.47), we deduce that

$$i\varphi(x,\theta) = R^*_{-\sigma,\theta}\mathsf{H}_\sigma R_{\sigma,\theta}f(x), \tag{2.49}$$

where we have defined the following operators

$$\begin{aligned}
&R^*_{\sigma,\theta}g(x) = e^{D_\theta\sigma(x)}g(x\cdot\vartheta^\perp), \qquad \mathsf{H}_\sigma = C_c\mathsf{H}C_c + C_s\mathsf{H}C_s \\
&C_cg(s,\theta) = g(s,\theta)\cos(\frac{\mathsf{H}R_\theta\sigma(s)}{2}), \quad C_sg(s,\theta) = g(s,\theta)\sin(\frac{\mathsf{H}R_\theta\sigma(s)}{2}).
\end{aligned} \tag{2.50}$$

Here $R^*_{\sigma,\theta}$ is the formal adjoint operator to $R_{\sigma,\theta}$.

The above derivation shows that $i\varphi(x,\theta)$ is real-valued and of the form $e^{-D_\theta a(x)}M(x\cdot\vartheta^\perp,\theta)$ for some function M. We deduce therefore that

$$\vartheta\cdot\nabla\varphi(x,\theta) + \sigma\varphi(x,\theta) = 0. \tag{2.51}$$

2.2.6 Step (iii): reconstruction formulas

We have seen that $u(z,\lambda)$ is analytic in λ on $D^+ \cup D^-$ and is of order $O(z^{-1})$ at infinity. Moreover the jump of $u(z,\lambda)$ across T is given by $\varphi(x,\theta)$ for $0 \leq \theta < 2\pi$. We thus deduce from the Cauchy formula (2.29) that

$$u(x,\lambda) = \frac{1}{2\pi i}\int_T \frac{\varphi(x,t)}{t-\lambda}dt, \qquad \lambda \in D^+ \cup D^-, \tag{2.52}$$

where we identify $\varphi(x,t)$ with $\varphi(x,\theta)$ for $t = e^{i\theta}$. We now deduce from (2.33) in the vicinity of $\lambda = 0$ that

$$f(x) = \lim_{\lambda\to 0} \lambda^{-1}\frac{\partial}{\partial\bar{z}}u(x,\lambda). \tag{2.53}$$

Indeed we verify that $u(z,\lambda) = O(\lambda)$ on D^+ so that $\sigma(x)u(x,\lambda) \to 0$ as $\lambda \to 0$. Since $u(x,\lambda)$ is known thanks to (2.52) in terms of the boundary measurements $R_{\sigma,\theta}f(s)$, this is our reconstruction formula. Let us be more specific. We verify that

$$u(x,\lambda) = \frac{1}{2\pi i}\int_T \frac{\varphi(x,t)}{t}dt + \lambda\frac{1}{2\pi i}\int_T \frac{\varphi(x,t)}{t^2}dt + O(\lambda^2). \tag{2.54}$$

We thus deduce from (2.53) and the fact that $u(x,\lambda) = O(\lambda)$ on D^+ that

$$0 = \frac{1}{2\pi i}\int_T \frac{\varphi(x,t)}{t}dt \quad \text{and} \quad f(x) = \frac{1}{2\pi i}\int_T \frac{\partial\varphi}{\partial\bar{z}}(x,t)\frac{1}{t^2}dt. \tag{2.55}$$

The second equality is the reconstruction formula we were looking for since φ is defined in (2.49) in terms of the measurements $R_{a,\theta}f(s)$. The first equality is a *compatibility* conditions that $i\varphi$ must satisfy in order for the data to be the attenuated Radon transform of a function $f(x)$. This compatibility condition is similar to the condition $g(s,\theta) = g(-s,\theta+\pi)$ satisfied by the Radon transform in the absence of absorption. These compatibility conditions are much more difficult to visualize when absorption does not vanish because the integrals along the line $\{s\vartheta^\perp + t\vartheta; t \in \mathbb{R}\}$ differ depending on the direction of integration.

Let us recast the reconstruction formula so that it only involves real-valued quantities.

Exercise 2.2.2 Using $t = e^{i\theta}$ and $dt = ie^{i\theta}$, deduce that

$$\begin{aligned}
\frac{1}{2\pi i}\int_T \frac{\varphi(x,t)}{t}dt &= \frac{1}{2\pi}\int_0^{2\pi}\varphi(x,\theta)d\theta \\
\frac{1}{2\pi i}\int_T \frac{\partial\varphi}{\partial\bar{z}}(x,t)\frac{1}{t^2}dt &= \frac{1}{4\pi}\int_0^{2\pi}\vartheta^\perp\cdot\nabla(i\varphi)(x,\theta)d\theta + \frac{1}{4\pi}\int_0^{2\pi}\vartheta\cdot\nabla\varphi(x,\theta)d\theta.
\end{aligned}$$

Use (2.51) and (2.55) to show that

$$f(x) = \frac{1}{4\pi}\int_0^{2\pi}\vartheta^\perp\cdot\nabla(i\varphi)(x,\theta)d\theta. \tag{2.56}$$

Let us denote by $g_\sigma(s,\theta) = R_\sigma f(s,\theta)$ the SPECT measurements. From the above results we recast (2.56) as

$$f(x) = [\mathcal{N}g](x) \equiv \frac{1}{4\pi}\int_0^{2\pi}\vartheta^\perp\cdot\nabla(R^*_{-\sigma,\theta}\mathsf{H}_\sigma g)(x,\theta)d\theta. \tag{2.57}$$

Exercise 2.2.3 Show that (2.57) simplifies to (2.14) when $\sigma \equiv 0$.

Exercise 2.2.4 (Reconstruction with constant absorption.) We assume that $f(x) = 0$ for $|x| \geq 1$ and that $\sigma(x) = \mu$ for $|x| < 1$. This corresponds thus to the case of a constant absorption coefficient on the support of the source term $f(x)$.
(i) Show that

$$e^{D_\theta \sigma(x)} = e^{\mu x \cdot \vartheta}, \qquad |x| < 1.$$

Deduce that

$$\vartheta^\perp \cdot \nabla (e^{D_\theta \sigma(x)} g(x \cdot \vartheta^\perp, \theta)) = e^{\mu x \cdot \vartheta} \frac{\partial g}{\partial s}(x \cdot \vartheta^\perp).$$

(ii) Verify that the operator H_μ defined by $H_\mu = H_a$ for a constant is diagonal in the Fourier domain and that

$$\widehat{\mathsf{H}_\mu u}(\sigma) = -i \mathrm{sign}_\mu(\sigma) \hat{u}(\sigma), \qquad \mathrm{sign}_\mu(\sigma) = \begin{cases} \mathrm{sign}(\sigma) & |\sigma| \geq \mu, \\ 0 & |\sigma| < \mu. \end{cases}$$

(iii) Show that

$$\begin{aligned} g_\mu(s, \theta) &= R_\theta(e^{\mu x \cdot \vartheta} f)(s), \\ f(x) &= \frac{1}{4\pi} \int_0^{2\pi} e^{-\mu x \cdot \vartheta} (\mathsf{H}_\mu \frac{\partial}{\partial s} g_\mu)(x \cdot \vartheta^\perp, \theta) d\theta. \end{aligned} \tag{2.58}$$

Verify that in the limit $\mu \to 0$, we recover the inversion for the Radon transform.

Chapter 3

Integral geometry and FIO

We recast the Radon transform as a Fourier Integral Operator (FIO) and look at propagation of singularities for such operators.

3.1 Radon transform and FIO

Let us consider the Radon transform in dimension n

$$\mathsf{R}f(s,\vartheta) = \int_{\mathbb{R}^n} \delta(s - x\cdot\vartheta) f(x) dx. \tag{3.1}$$

Since the above delta function is one dimensional, we may formally recast it as

$$\delta(s) = \int_{\mathbb{R}} e^{i\sigma s} \frac{d\sigma}{2\pi}. \tag{3.2}$$

We thus observe that the Radon transform may be written as

$$\mathsf{R}f(s,\vartheta) = \int e^{i\varphi(s,\vartheta,x,\sigma)} a(s,\vartheta,x,\theta) f(x) d\sigma dx, \tag{3.3}$$

where the *phase* φ and the *amplitude* a are given by

$$\varphi(s,\vartheta,x,\sigma) = \sigma(s - x\cdot\vartheta), \qquad a(s,\vartheta,x,\sigma) = \frac{1}{2\pi},$$

respectively.

All integrals above, starting with (3.2), involve integrands that are not integrable in the sense of Lebesgue. They will be properly defined in the next section. We will see in this chapter and the next that the representation of problems in integral geometry as oscillatory integrals as in (3.3) provides extremely fruitful. In particular, it is the right language to understand how *singularities propagate*, which is the main criterion we should use to understand how ill-posed an inverse problem really is.

3.2 Oscillatory integrals, symbols, and phases

We now introduce necessary notation and material for the definition of operators of the form (3.3). The presentation essentially follows those in [33, 36].

We first define the space of symbols on a domain $X \subset \mathbb{R}^n$ and for some $n \in \dot{\mathbb{N}}$, where $\dot{\mathbb{N}} = \mathbb{N}\backslash\{0\}$. We also define $\dot{\mathbb{R}}^N = \mathbb{R}^N\backslash\{0\}$.

Definition 3.2.1 (Symbols) *We define $S^m(X \times \mathbb{R}^N)$ as the space of functions $a \in C^\infty(X \times \mathbb{R}^N)$ such that for all compact $K \subset X$ and all $\alpha \in \mathbb{N}^n$ and $\beta \in \mathbb{N}^N$, there is a constant $C = C_{K,\alpha,\beta}(a)$ such that*

$$|\partial_x^\alpha \partial_\theta^\beta a(x,\theta)| \leq C(1+|\theta|)^{m-|\beta|}, \quad (x,\theta) \in K \times \mathbb{R}^N. \tag{3.4}$$

In the above definition, we used the multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta - N)$. We now list several properties of the space of symbols, whose proofs may be found in [33].

(i) The space $S^m(X \times \mathbb{R}^N)$ is a Fréchet space with seminorm $C_{K,\alpha,\beta}(a)$ chosen as the smallest constants in (3.4). This is the topology we now consider.

(ii) The operator $\partial_x^\alpha \partial_\theta^\beta$ is continuous from $S^m(X \times \mathbb{R}^n)$ to $S^{m-|\beta|}(X \times \mathbb{R}^n)$.

(iii) If $m < m'$, then $S^m(X \times \mathbb{R}^N) \subset S^{m'}(X \times \mathbb{R}^N)$. [1]

(iv) The intersection of all $S^m(X \times \mathbb{R}^n)$ for $m \in \mathbb{R}$ (or $m \in \mathbb{N}$) is called $S^{-\infty}(X \times \mathbb{R}^N)$; this is also a Fréchet space.

(v) If $a_j \in S^{m_j}(X \times \mathbb{R}^N)$ for $j = 1, 2$ then $a_1 a_2 \in S^{m_1+m_2}(X \times \mathbb{R}^N)$ and the bilinear map thus defined is continuous.

(vi) If $\{a_j\}_{j\geq 1}$ is a sequence of symbols in S^m converging at each point $(x,\theta) \in X \times \mathbb{R}^N$ to $a(x,\theta)$, then $a \in S^m(X, \mathbb{R}^N)$ and $a_j \to a$ in the topology generated by the above seminorms in $S^{m'}(X \times \mathbb{R}^N)$ for all $m' > m$.

(vii) $S^{-\infty}(X, \mathbb{R}^N)$ is dense in $S^m(X, \mathbb{R}^N)$ for the topology generated by the above seminorms in $S^{m'}(X \times \mathbb{R}^N)$ for $m' > m$.

(viii) Let $a_j \in S^{m_j}(X \times \mathbb{R}^N)$ for all $j \in \mathbb{N}$ with m_j strictly decreasing to $-\infty$ as j increases to ∞. Then there exists a unique $a \in S^{m_0}(X \times \mathbb{R}^N)$ up to the addition of an element in $S^{-\infty}(X \times \mathbb{R}^N)$ such that $a - \sum_{j=0}^J a_j \in S^{m_J}(X \times \mathbb{R}^N)$ for all $k \in \mathbb{N}$. We then write $a \sim \sum_j a_j$.

Exercise 3.2.1 *Show that for $a(x,\theta) \in C^\infty(X \times \mathbb{R}^N)$ such that $a(x,\lambda\theta) = \lambda^m a(x,\theta)$ for $|\theta| > 1$ and $\lambda > 1$, we have $a \in S^m(X \times \mathbb{R}^N)$.*

Definition 3.2.2 (Phase) *A function $\varphi(x,\theta) \in C^\infty(X \times \dot{\mathbb{R}}^N)$ is called a* phase function *if for all $(x,\theta) \in X \times \dot{\mathbb{R}}^N$, we have*

(i) $\Im\varphi(x,\theta) \geq 0$,

(ii) $\varphi(x,\lambda\theta) = \lambda\varphi(x,\theta)$ *for all* $\lambda > 0$,

(iii) $d\varphi = \varphi'_x dx + \varphi'_\theta d\theta \neq 0$; *in other words* $|\varphi'_x|^2 + |\theta|^2|\varphi'_\theta|^2 > 0$, *on* $X \times \dot{\mathbb{R}}^N$.

Here φ'_x denotes the vector of the n partial derivatives $\partial_{x_j}\varphi$ while φ'_θ denotes the N partial derivatives $\partial_{\theta_j}\varphi$. Property (ii) means that φ is a positively homogeneous function of degree *one*. Property (i) implies that $|e^{i\varphi(x,\theta)}| \leq 1$.

[1] Hilbert spaces H^m becomes smaller as m increases. Spaces of symbols S^m become larger.

For $X \subset \mathbb{R} \times \mathbb{S}^{n-1} \times \mathbb{R}^n$ and $N = 1$, we find that

$$\varphi(s, \vartheta, x, \sigma) = \sigma(s - x \cdot \vartheta),$$

is a phase as (i) its imaginary part vanishes, (ii) it is positively homogeneous of degree one in σ, and (iii) its differential is given by

$$d\varphi = \sigma ds - \sigma x \cdot \vartheta^{\perp}(\vartheta) dg(\vartheta) - \sigma \vartheta dx + (s - x \cdot \vartheta) d\sigma,$$

with $d\varphi = 0$ implying that $\sigma = 0$ so that $d\varphi \neq 0$ on $X \times \dot{\mathbb{R}}^N$. Here, $\vartheta^{\perp}(\vartheta)$ is a parameterization of $T_{\vartheta}\mathbb{S}^{n-1}$ the tangent space to $\mathbb{S}^{n-1}$ at ϑ embedded in $\mathbb{R}^n$ and $dg(\vartheta)$ is the differential at that point. This generalizes $\vartheta^{\perp}(\vartheta) = (-\sin\tilde{\theta}, \cos\tilde{\theta})$ for $\vartheta = (\cos\tilde{\theta}, \sin\tilde{\theta})$ and $g(\vartheta) = \tilde{\theta}$ in dimension $n = 2$.

We are now ready to define general oscillatory integrals of the form

$$I_{a,\varphi}(x) := \int e^{i\varphi(x,\theta)} a(x, \theta) d\theta. \tag{3.5}$$

The above integration is implicitly performed over $\mathbb{R}^N$.

By definition of the phase and the symbol, the above integration over any ball of the form $|\theta| < R < \infty$ is defined in the usual (Lebesgue) sense. It is the integration in the vicinity of $|\theta| \sim \infty$ that difficulties may arise.

For a phase $a \in S^m(X \times \mathbb{R}^N)$ with $m + k < -N$, we obtain that $I_{a,\varphi}(x) \in C^k(X)$. Indeed, for $k = 0$, we obtain that $|e^{i\varphi(x,\theta)} a(x, \theta)| \leq C(1 + |\theta|)^m$, which is integrable on $\mathbb{R}^N$ for $-m > N$.

Exercise 3.2.2 *Verify that the above function $I_{a,\varphi}(x) \in C^k(X)$ when $m + k < -N$.*

When m is larger than $-N$, then the above integral is not necessarily defined in the sense of Lebesgue. For $m = 0$ and $N = 1$, the calculation in (3.2) shows that $I_{a,\varphi}(x)$, once properly defined, is not necessarily a function but rather a distribution in $\mathcal{D}'(X)$.

The definition of the integral (3.5) proceeds by integrations by parts. Each differentiation of $a(x, \theta)$ in the θ variable lowers the order of the resulting symbol by one; more precisely $\partial_{\theta}^{\beta} a(x, \theta)$ belongs to $S^{m-|\beta|}(X \times \mathbb{R}^N)$ if a is of order m. When $m - |\beta| < -N$, then the integral is defined in the usual sense. Such integrations by parts require that $e^{i\varphi(x,\theta)}$ be written as the partial derivative of another function. However, in order for these integrations to be defined generally, the integrations by parts need to be performed with respect to both variables (x, θ). Then, we have the following precise statement.

Define

$$\Phi(x, \theta) = |\varphi'_x|^2 + |\theta|^2 |\varphi'_{\theta}|^2, \tag{3.6}$$

which is positively homogeneous of degree 2, i.e., $\Phi(x, \lambda\theta) = \lambda^2 \Phi(x, \theta)$ for $\lambda > 0$ for $(x, \theta) \in X \times \dot{\mathbb{R}}^N$. Because Φ and φ are not defined at $\theta = 0$ (or may not be regular there), we introduce a cut-off function $\chi(\theta) \in C_0^{\infty}(\mathbb{R}^N)$ with compact support such that $\chi(\theta) = 1$ in a neighborhood of $\theta = 0$. We then define the (transpose) first-order operator

$$L^t = \frac{1 - \chi(\theta)}{i\Phi(x, \theta)} \Big(|\theta|^2 \bar{\varphi}'_{\theta} \partial_{\theta} + \bar{\varphi}'_x \partial_x \Big) + \chi(\theta) = d' \partial_{\theta} + b' \partial_x + c', \tag{3.7}$$

with d' and b' with components $(d'_j)_{1\leq j\leq N}$ and $(b'_j)_{1\leq j\leq N}$ such that $d'_j \in S^0$, $b'_j \in S^{-1}$ and $c' \in S^{-\infty}$ for $1 \leq j \leq N$. Here and below, we use the shorter notation S^m instead of $S^m(X \times \mathbb{R}^N)$.

It is then not difficult to verify that $L^t(e^{i\varphi}) = e^{i\varphi}$.

The adjoint operator L is given by

$$L = d\partial_\theta + b\partial_x + c, \qquad d = -d', \quad b = -b', \quad c = c' + \mathrm{div}_\theta d' + \mathrm{div}_x b'. \tag{3.8}$$

Here, we define $\mathrm{div}_\theta d' = \sum_j \partial_{\theta_j} d'_j$ and $\mathrm{div}_x b' = \sum_j \partial_{x_j} b'_j$. This shows that $d_j \in S^0$, $b_j \in S^{-1}$ and $c \in S^{-1}$ for $1 \leq j \leq N$.

Let $u \in C_0^\infty(X)$ and $a \in S^{-\infty}$. We thus obtain after k integrations by parts:

$$\langle I_{a,\varphi}, u\rangle = \int (L^t)^k e^{i\varphi(x,\theta)} a(x,\theta)u(x)dxd\theta = \int e^{i\varphi(x,\theta)} L^k\big(a(x,\theta)u(x)\big)dxd\theta. \tag{3.9}$$

Note that no terms appear on the boundary ∂X by choice of the test function u. No 'boundary' term appears as $|\theta| \to \infty$ either by choice of a.

For $a \in S^{-\infty}$, we *define* the distribution $I_{a,\varphi} \in \mathcal{D}'(X)$ by the above right-hand side, which is defined in the usual (Lebesgue) sense when $m-k < -N$ since $L^k(a(x,\theta)u(x)) \in S^{m-k}$ and is independent of k.

Exercise 3.2.3 Prove the latter statement (using the symbol properties of the coefficients (d, b, c)).

Moreover, for m and k fixed, the above map $a \mapsto I_{a,\varphi}$ is continuous as a distribution in $\mathcal{D}'^{(k)}(X)$ acting on test functions in $C_0^k(X)$. Since $S^{-\infty}$ is dense in S^m and since (3.9) is independent of k and uniquely defined for $a \in S^{-\infty}$, we obtain that the above definition is independent of k with $m - k < -N$ for $a \in S^m$ and uniquely defined.

We have the result ([33])

Theorem 3.2.3 (Oscillatory integrals) *Let φ be a phase function on $X \times \dot{\mathbb{R}}^N$ and $a \in S^m(X \times \mathbb{R}^N)$. Then $I_{a,\varphi} \in \mathcal{D}'(X)$ in (3.5) is defined in a unique way by (3.9) for any k such that $m - k < -N$. Moreover, the map $a \to I_{a,\varphi}$ is continuous from S^m to $\mathcal{D}'^{(k)}(X)$.*

We have thus defined distributions of the form $I_{a,\varphi}(x) = \int e^{i\varphi(x,\theta)} a(x,\theta)d\theta$. By the continuity argument, we observe that

$$I_{a,\varphi}(x) = \lim_{\varepsilon\to 0} \int e^{i\varphi(x,\theta)} a(x,\theta)\chi(\varepsilon\theta)d\theta \tag{3.10}$$

for any $\chi \in \mathcal{S}(\mathbb{R}^N)$ with $\chi(0) = 1$ (here limit is in the sense of distributions).

Remark 3.2.4 (Continuous dependence on parameters and differentiation) If $t \mapsto a(x,\theta;t)$ and $t \mapsto \varphi(x,\theta;t)$ are continuous, then $t \mapsto I_{a(\cdot;t)\varphi(\cdot,t)}$ is also continuously defined as a distribution thanks to (3.10) and the dominated Lebesgue convergence theorem.

For instance, the distribution defined with $a(x+h,\theta)$ and $\varphi(x+h,\theta)$ for $h \in \mathbb{R}^n$ is differentiable with respect to h. For $h = h_j e_j$ with $(e_1, \ldots, e_n)$ a basis of $\mathbb{R}^n$, we find that

$$\partial_{x_j} I_{a,\varphi}(x) = \int e^{i\varphi(x,\theta)}\big(i\partial_{x_j}\varphi(x,\theta)a(x,\theta) + \partial_j a(x,\theta)\big)d\theta.$$

If $a \in S^m(X\times\mathbb{R}^N)$, we observe that $i\partial_{x_j}\varphi(x,\theta)a(x,\theta)+\partial_j a(x,\theta)$ is a symbol in $S^{m+1}(X\times\mathbb{R}^N)$ since $\partial_{x_j}\varphi(x,\theta)$ is homogeneous of degree 1 in θ.

More generally, for each $I_{a,\varphi}(x)$ define with $a \in S^m(X\times\mathbb{R}^N)$ and a phase $\varphi \in C^\infty(X\times\dot{\mathbb{R}}^N)$, and for each $P(D)$ a differential operator of order k, then $P(D)I_{a,\varphi}(x)$ is a distribution in $\mathcal{D}'(X)$ that can be written in the form (3.5) with the same phase φ and a symbol $a_P(x,\theta) \in S^{m+k}(X\times\mathbb{R}^N)$.

3.3 Schwartz kernels and Fourier integral operators

We are now ready to define Fourier integral and pseudo-differential operators. Let $X \subset \mathbb{R}^{n_X}$ and $Y \subset \mathbb{R}^{n_Y}$ be open sets. Then $Z = X\times Y \subset \mathbb{R}^{n_X}\times\mathbb{R}^{n_Y} = \mathbb{R}^{n_X+n_Y}$ plays the role of the open set X in the preceding section while $z = (x,y)$ plays the role of the variable x. We wish to see the distributions $I_{a,\varphi}$ constructed above as the kernels of operators mapping functions in the Y variable to functions (in fact distributions) in the X variables.

The *Schwartz kernel theorem* states that there is a bijection between the distributions $K \in \mathcal{D}'(X\times Y)$ and the bounded linear operators $A : C_0^\infty(Y) \to \mathcal{D}'(X)$, with the correspondence given by

$$\langle Au, v\rangle_X = \langle K, v\otimes u\rangle_{X\times Y} = \int_Z K(x,y)u(y)v(x)dxdy, \quad u \in C_0^\infty(Y),\ v \in C_0^\infty(X). \tag{3.11}$$

We then say that K is the distribution kernel of the operator A.

Let now $\varphi(x,y,\theta)$ be a phase function in $X\times Y\times\dot{\mathbb{R}}^N$ and $a(x,y,\theta)$ a symbol in $S^m(X\times Y\times\mathbb{R}^N)$. Then $K = I_{a,\varphi} \in \mathcal{D}'(X\times Y)$ defined in the preceding section is the distribution kernel of the operator

$$Au(x) = \int e^{i\varphi(x,y,\theta)}a(x,y,\theta)u(y)dyd\theta, \qquad u \in C_0^\infty(Y). \tag{3.12}$$

Such an operator is called a **Fourier integral operator (FIO)**. When $\varphi(x,y,\theta) = (x-y)\cdot\theta$, then the operator is called a **Pseudo-differential operator (ΨDO)**.

The operators A are more regular than stated above and are defined on a larger class of functions (and distributions) when the phase satisfies additional properties.

Let us assume that φ is a phase function in the (y,θ) variables, i.e., such that $d_{y,\theta}\varphi \neq 0$ for all $x \in X$ (this is obviously more restrictive than $d_{x,y,\theta}\varphi \neq 0$ as we had assumed earlier). Then as we did in the preceding section, we can construct a vector field L^t

$$L^t = L^t(x,y,\theta,\partial_y,\partial_\theta) = d'\partial_y + b'\partial_\theta + c',$$

such that $L^t e^{i\varphi(x,y,\theta)} = e^{i\varphi(x,y,\theta)}$. Denoting by L its adjoint (in the usual way in the (y,θ) variables), we find that

$$Au(x) = \int e^{i\varphi(x,y,\theta)}L^k\big(a(x,y,\theta)u(y)\big)dyd\theta, \qquad u \in C_0^\infty(Y), \tag{3.13}$$

for $k = N+m+j+1$, say, for $j \in \mathbb{N}$. The integral is defined in the sense of Lebesgue and is a function in $C^j(X)$ by the dominated Lebesgue convergence theorem. Since

this holds for all $j \in \mathbb{N}$, we thus obtain that A is continuously defined from $C_0^\infty(Y)$ to $C^\infty(X)$. More precisely, we find that A is continuously defined from $C_0^k(Y)$ to $C^j(X)$ when $m + N + j < k$.

Let us now assume that φ is a phase in the (x, θ) variables. We then define the adjoint operator A^t to A by

$$\langle u, A^t v\rangle = \langle Au, v\rangle = \langle K, v \otimes u\rangle \quad u \in C_0^\infty(Y),\ v \in C_0^\infty(X). \tag{3.14}$$

We have just seen that A^t was continuously defined from $C_0^\infty(X)$ to $C^\infty(Y)$. By duality, we may then define Au for all $u \in \mathcal{E}'(Y)$ by $\langle Au, v\rangle = \langle u, A^t v\rangle$ for all $v \in C_0^\infty(X)$. This defines $Au \in \mathcal{D}'(X)$. This extension is by construction continuous and uniquely defined by density of $C_0^\infty(Y)$ in $\mathcal{E}'(Y)$. By duality, we also obtain that A maps $\mathcal{E}'^j(Y)$ (the dual space to $C^j(Y)$) to $\mathcal{D}'^k(X)$ (the dual space to $C_0^k(X)$) continuously.

We summarize these results in the

Theorem 3.3.1 (Regularity property) *Let A be given by (3.12) for $\varphi(x, y, \theta)$ a phase function in $X \times Y \times \dot{\mathbb{R}}^N$ and $a(x, y, \theta) \in S^m(X \times Y \times \mathbb{R}^N)$.*

If $\varphi(x, y, \theta)$ is a phase function in the variables (y, θ) for all $x \in X$, then A is continuous from $C_0^k(Y)$ to $C^j(X)$ when $k > m + N + j$. In particular, A is continuous from $C_0^\infty(Y)$ to $C^\infty(X)$.

If $\varphi(x, y, \theta)$ is a phase function in the variables (x, θ) for all $y \in Y$, then A is continuous from $\mathcal{E}'^j(Y)$ to $\mathcal{D}'^k(X)$ when $k > m + N + j$. In particular, A is continuous from $\mathcal{E}'(Y)$ to $\mathcal{D}'(X)$.

The phase function $\varphi(x, y, \theta) = (x - y) \cdot \theta$ is a phase function in the (y, θ) and (x, θ) variables. We thus obtain that ΨDO are continuous from $\mathcal{C}_0^\infty(Y)$ to $\mathcal{C}^\infty(X)$ and have a continuous extension from $\mathcal{E}'(Y)$ to $\mathcal{D}'(X)$. We will also obtain that ΨDO of order m are continuous from $H^s(Y)$ to $H^{s+m}(X)$. This property does not necessarily hold for FIO, although it will be shown to hold for a restricted class of FIO that find many applications in inverse problems theory.

Exercise 3.3.1 Let A given by (3.12) satisfy the hypotheses of Theorem 3.3.1 (with a symbol $a(x, y, \theta) \in S^m(X \times Y \times \mathbb{R}^N)$) and let $P(D)$ be a differential operator of order k. As in Remark 3.2.4, show that $P(D)A$ (the operator which to $u \in \mathcal{C}_0^\infty(Y)$ applies the differential operator $P(D)$ to the function $Au(x)$) is a FIO with the same phase function as A and with a symbol of class $S^{m+k}(X \times Y \times \mathbb{R}^N)$.

Remark 3.3.2 (Case of regular Schwartz kernels) *We observe that for $a \in S^m(X \times Y \times \dot{\mathbb{R}}^N)$ with $m < -N - k$, then $K_A(x, y)$ is a function in $C^k(X \times Y)$. As a consequence, A maps continuously $\mathcal{E}'^k(Y) = (\mathcal{C}_0^k(Y))'$ to $\mathcal{C}^k(X)$.*

3.4 Critical set and propagation of singular support

Assuming that u is not C^∞ in the vicinity of a point $y \in Y$, we wish to find sufficient conditions under which Au is in C^∞ in the vicinity of a point $x \in Y$. To do so, we analyze the regularity of the distribution $I_{a,\varphi}$.

Definition 3.4.1 (Critical set of a phase) *Let $\varphi(x,\theta)$ a phase function on $X\times\dot{\mathbb{R}}^N$. We define the* critical set *of φ as*

$$C_\varphi := \big\{(x,\theta)\in X\times\dot{\mathbb{R}}^N \quad \textit{such that} \quad \varphi'_\theta = 0\big\}. \tag{3.15}$$

Note that $(x,\theta)\in C_\varphi$ implies that $(x,\lambda\theta)\in C_\varphi$ for $\lambda>0$. In other words, C_φ is a cone. For $\varphi(x,y,\theta)=(x-y)\cdot\theta$, we find that $C_\varphi=\{(x,y,\theta);x=y\}$.

Let us assume that $a(x,\theta)$ is a symbol of order m that vanishes in a conical neighborhood of C_φ, i.e., for all (x,θ) such that the (Euclidean) distance from $(x,\frac{\theta}{|\theta|})$ to C_φ is less than a given positive number.

As in (3.7), we construct the transpose first-order operator as

$$L^t = \frac{1-\chi(\theta)}{i(|\varphi'_\theta|^2+\beta)}\varphi'_\theta\partial_\theta + \chi(\theta), \tag{3.16}$$

with $0\le\beta(x,\theta)\in C^\infty(X\times\dot{\mathbb{R}}^N)$ equal to 1 in the vicinity of C_φ and such that $\operatorname{supp}\beta\cap\operatorname{supp}a=\emptyset$. Then we find again that $L^te^{i\varphi}=e^{i\varphi}$ on the support of a so that

$$I_{a,\varphi}(x) = \int e^{i\varphi(x,\theta)}L^k(a(x,\theta))d\theta, \tag{3.17}$$

for $k=N+m+1$. As in (3.13), we deduce that $I_{a,\varphi}(x)\in C^\infty(X)$.

This provides information on the *singular support* of $I_{a,\varphi}(x)$. The singular support of a distribution u defined on X, denoted by sing supp u, is defined as follows. Let S^c_u be the set of $x_0\in X$ such that $\varphi u\in C^\infty(X)$ for some function $\varphi(x)\in C^\infty_0(X)$ such that $\varphi(x_0)=1$. Then x_0 belongs to the complement set of the singular support of u, i.e., sing supp $u=S_u=X\backslash S^c_u$.

Let now Π denote the natural projection $\Pi: X\times\mathbb{R}^N\ni(x,\theta)\mapsto\Pi(x,\theta)=x\in X$. Then the above result shows that sing supp $I_{a,\varphi}\subset\Pi(C_\varphi)$. Indeed, let $x_0\notin\Pi(C_\varphi)$ and let V be a neighborhood of x_0 such that $V\cap\Pi(C_\varphi)=\emptyset$. Let $b\in C^\infty_0(X)$ such that $b=1$ on V and $b=0$ on $\Pi(C_\varphi)$. Then for each $u\in C^\infty_0(V)$, we find that $\langle I_{a,\varphi},u\rangle=\langle I_{ba,\varphi},u\rangle$. In other words, $I_{a,\varphi}=I_{ba,\varphi}$ in $\mathcal{D}'(V)$ and we have just seen that $I_{ba,\varphi}\in C^\infty(X)$.

We summarize this as:

Theorem 3.4.2 (Regularity of $I_{a,\varphi}$)
(i) If $a\in S^m(X\times\mathbb{R}^N)$ vanishes in a conical neighborhood of C_φ, then $I_{a,\varphi}(x)\in C^\infty(X)$.
(ii) sing supp $I_{a,\varphi}\subset\Pi(C_\varphi)=\{x\in X$ *such that $\varphi'_\theta(x,\theta)=0$ for some $\theta\in\dot{\mathbb{R}}^N\}$.*

The above result was obtained for a phase in the x variable. We now extend these results to an understanding of Au defined in (3.12). Let $Z=X\times Y$ as before and $\varphi(x,y,\theta)$ a phase. Define C_φ as the closed subset of $X\times Y\times\dot{\mathbb{R}}^N$ where the N constraints $\varphi'_\theta(x,y,\theta)=0$ hold. We define Z_φ as the open set of $X\times Y$ where $\varphi'_\theta(x,y,\theta)\neq0$ for all $\theta\in\dot{\mathbb{R}}^N$. Then for $a(x,y,\theta)\in S^m$, the above theorem shows that $I_{a,\varphi}(x,y)\in C^\infty(Z_\varphi)$.

Let now $x_0\in X$ such that $x_0\times Y\subset Z_\varphi$ so that $I_{a,\varphi}(x,y)$ is C^∞ in the vicinity of x_0. Define again a function $b(x)\in C^\infty_0(X)$ such that $\operatorname{supp}b\times Y\subset Z_\varphi$ and $b(x)=1$ in a neighborhood V of x_0. Then $I_{a,\varphi}=I_{ba,\varphi}$ in the vicinity of x_0. Since the phase φ does not have any critical point on Z_φ, we deduce from Theorem 3.3.1 that A defined

in (3.12) maps $C_0^\infty(Y)$ to $C^\infty(V)$ (and has a continuous extension mapping $\mathcal{E}'(Y)$ to $\mathcal{D}'(V)$). As a consequence, we find that

$$\text{sing supp } Au \subset \Pi_X C_\varphi = \{x \in X, \quad (x,y,\theta) \in C_\varphi \text{ for some } (y,\theta) \in Y \times \dot{\mathbb{R}}^N\}, \quad (3.18)$$

for any smooth function $u \in C_0^\infty(Y)$.

Let us now assume that $u \in \mathcal{E}'(Y)$. We wish to find a constraint on the singular support of Au in terms of that of u. This is an interesting question when Au is smooth as soon as u is smooth (unlike (3.18)) and we therefore restrict ourselves to the setting where $d_{y,\theta}\varphi(x,y,\theta) \neq 0$ and $d_{x,\theta}\varphi(x,y,\theta) \neq 0$ so that by Theorem 3.3.1, A defined in (3.12) maps $C_0^\infty(Y)$ to $C^\infty(X)$ and has a continuous extension mapping $\mathcal{E}'(Y)$ to $\mathcal{D}'(X)$.

Let $x_0 \in X$ be such that $(x_0,y,\theta) \notin C_\varphi$ for all $y \in \text{sing supp } u$ and all $\theta \in \dot{\mathbb{R}}^N$. Let W be a sufficiently small neighborhood of x_0 and $d \in C_0^\infty(W)$ with $d = 1$ in the vicinity of x_0. Let us finally take $c \in C_0^\infty(Y)$ be such that $c(y) = 1$ in the vicinity Y_c of sing supp u but with a sufficiently small support so that $(x,y,\theta) \notin C_\varphi$ for all $x \in W$, $y \in \text{supp } c$ and $\theta \in \dot{\mathbb{R}}^N$. Then $I_{acd,\varphi}$ is in $C^\infty(W \times Y_C)$.

As a consequence $I_{acd,\varphi}$ extends as a continuous operator from $\mathcal{E}'(Y_c) \to C^\infty(W)$. We now decompose $u = v + w$ with $v \in C^\infty(Y)$ and w supported in Y_c. We know from Theorem 3.3.1 that $Av \in C^\infty(X)$ since $d_{y,\theta}\varphi(x,y,\theta) \neq 0$. We have also just seen that $\langle I_{a,\varphi}, w\rangle_Y = \langle I_{ac,\varphi}, w\rangle_Y = \langle I_{acd,\varphi}, w\rangle_Y \in C^\infty(W)$. This proves that $Au(x)$ is of class C^∞ in the vicinity of x_0.

Collecting the preceding derivation, we obtain the

Theorem 3.4.3 (Propagation of singular support) *Let $a \in S^m(X \times Y \times \mathbb{R}^N)$ and φ a phase on $X \times Y \times \dot{\mathbb{R}}^N$ with singular set $C_\varphi = \{(x,y,\theta) \in X \times Y \times \dot{\mathbb{R}}^N$ s.t. $\varphi'_\theta(x,y,\theta) = 0\}$. Then for $u \in C_0^\infty(Y)$, we find that*

$$\text{sing supp } Au \subset \Pi_X C_\varphi = \{x \in X, \quad (x,y,\theta) \in C_\varphi \text{ for some } (y,\theta) \in Y \times \dot{\mathbb{R}}^N\}. \quad (3.19)$$

Assume moreover that $d_{y,\theta}\varphi(x,y,\theta) \neq 0$ and $d_{x,\theta}\varphi(x,y,\theta) \neq 0$ on $X \times Y \times \dot{\mathbb{R}}^N$. Then for $u \in \mathcal{E}'(Y)$ and $S_u :=$ sing supp u, we have that

$$\text{sing supp } Au \subset \Pi_X C_\varphi(S_u) = \{x \in X, (x,y,\theta) \in C_\varphi \text{ for some } (y,\theta) \in S_u \times \dot{\mathbb{R}}^N\}. \quad (3.20)$$

In other words, when $d_{y,\theta}\varphi \neq 0$ and $d_{x,\theta}\varphi \neq 0$ on $X \times Y \times \dot{\mathbb{R}}^N$, then the singular support of u is propagated by the relation C_φ and then projected onto the first component $x \in X$. When only $d_{x,y,\theta}\varphi \neq 0$ is available, then all we can deduce is that Au is smooth away from the projection of C_φ onto the first component $x \in X$ independent of the singularities of u. Of course, Y in (3.19) may be replaced by supp u with additional obvious improvements if a vanishes on parts of Y.

3.5 Applications to ΨDO and Radon transform

For a ΨDO, $\varphi(x,y,\theta) = (x-y)\cdot\theta$ with $C_\varphi = \{x = y\}$ and $d_{x,\theta}\varphi \neq 0$ and $d_{y,\theta}\varphi \neq 0$ on $X \times Y \times \dot{\mathbb{R}}^n$ with $n = n_X = n_Y$. There is no real point in having Y different from X

since all singularities occur for $x = y$ and we thus assume that $X = Y$. Then Theorem 3.4.3 applies and we find that for P a ΨDO,

$$\text{sing supp } Pu \subset \text{sing supp } u \quad \text{for all } u \in \mathcal{E}'(X). \tag{3.21}$$

For the Radon transform in two dimensions (with obvious extensions in higher dimensions) we have $\varphi(s, \theta, x, \sigma) = \sigma(s - x \cdot \vartheta)$ so that $C_\varphi = \{s = x \cdot \vartheta\} \subset \mathbb{R}^2 \times \mathbb{R}^2 \times \dot{\mathbb{R}}$. We verify that $d_{s,\theta,\sigma}\varphi \neq 0$ and $d_{x,\sigma}\varphi \neq 0$ since $\partial_s\varphi = \sigma \neq 0$ and $\partial_x\varphi = -\sigma\theta^\perp \neq 0$ so that (3.20) applies. We thus have for the Radon transform that

$$\text{sing supp } \mathsf{R}f \subset \{(s, \theta) \in \mathbb{R} \times [0, 2\pi) \text{ s.t. } s = x \cdot \vartheta \text{ for some } x \in \text{sing supp } f\}. \tag{3.22}$$

This result holds for any $u \in \mathcal{E}'(\mathbb{R}^2)$.

Exercise 3.5.1 *Prove that in dimension $n \geq 2$ and for all $u \in \mathcal{E}'(\mathbb{R}^n)$, we have:*

$$\text{sing supp } \mathsf{R}f \subset \{(s, \vartheta) \in \mathbb{R} \times \mathbb{S}^{n-1} \text{ s.t. } s = x \cdot \vartheta \text{ for some } x \in \text{sing supp } f\}. \tag{3.23}$$

For the adjoint Radon transform $\mathsf{R}^*g(x)$, defined as

$$\mathsf{R}^*g(x) = \int_{\mathbb{R}\times\mathbb{S}^{n-1}} \delta(s - x \cdot \vartheta) g(s, \vartheta) ds d\vartheta = \int_{\mathbb{R}^2\times\mathbb{S}^{n-1}} e^{-i\sigma(s-x\cdot\vartheta)} \frac{1}{2\pi} g(s, \vartheta) d\sigma ds d\vartheta, \tag{3.24}$$

we have that $\varphi'(x, s, \theta, \sigma) = -\varphi(\theta, s, x, \sigma) = -\sigma(s - x \cdot \vartheta)$ with again $C'_\varphi = \{s = x \cdot \vartheta\}$. We thus obtain that in dimension $n \geq 2$ and for all $u \in \mathcal{E}'(\mathbb{R}^n)$, we have:

$$\text{sing supp } \mathsf{R}^*g \subset \{x \in \mathbb{R}^n \text{ s.t. } s = x \cdot \vartheta \text{ for some } (s, \vartheta) \in \text{sing supp } g\}. \tag{3.25}$$

How about $\mathsf{R}^*\mathsf{R}$? We recall the definitions

$$\mathsf{R}f(s, \vartheta) = \int_{\mathbb{R}^n} f(x)\delta(s - x \cdot \vartheta) dx, \quad \mathsf{R}^*g(x) = \int_{\mathbb{R}\times\mathbb{S}^{n-1}} g(s, \theta)\delta(s - x \cdot \vartheta) ds d\vartheta.$$

Formally, we thus obtain

$$\begin{aligned} \mathsf{R}^*\mathsf{R}f(x) &= \int \delta(s - x \cdot \vartheta)\delta(s - y \cdot \vartheta) f(y) dy ds d\vartheta \\ &= \frac{1}{2\pi} \int e^{i(x-y)\cdot\vartheta\sigma} f(y) dy d\sigma d\vartheta = \frac{1}{\pi} \int e^{i(x-y)\cdot\xi} f(y) dy \frac{1}{|\xi|^{n-1}} d\xi. \end{aligned}$$

The last equality comes from the change of variables $\xi = \sigma\vartheta$ in polar coordinates and the change of variables $\sigma \to -\sigma$. In other words $\pi\mathsf{R}^*\mathsf{R}$ is a ΨDO with amplitude $\tilde{a}(x, y, \xi) = |\xi|^{1-n}$. The latter function is not smooth at $\xi = 0$. However, the above integral is defined in the sense of Lebesgue in the vicinity of $\xi = 0$. We may thus write $|\xi|^{1-n} = \chi(\xi)|\xi|^{1-n} + a(\xi)$ with $\chi(\xi) \in C_0^\infty(\mathbb{R}^n)$ so that the corresponding integral is a $C^\infty(\mathbb{R}^n)$ function in the x variable, and with $a \in S^{1-n}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$.

Note that $c_n\mathsf{R}^*\mathsf{R} = \mathcal{F}^{-1}|\xi|^{1-n}\mathcal{F}$ for $c_n = \pi/(2\pi)^n$, which is therefore an invertible operator with inverse $(c_n\mathsf{R}^*\mathsf{R})^{-1} = \mathcal{F}^{-1}|\xi|^{n-1}\mathcal{F}$. This inverse operator may also be written as a ΨDO with amplitude $\tilde{b}(x, y, \xi) = |\xi|^{n-1}$. As for a, we may write $\tilde{b} = |\xi|^{n-1}\chi(\xi) + b(\xi)$ with a first contribution providing a $C^\infty(\mathbb{R}^n)$ function in the x variable and $b \in S^{n-1}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$.

Modulo C^∞ contributions coming from integrations in the vicinity of $\xi = 0$, we thus obtain that $\mathsf{R}^*\mathsf{R}$ is a ΨDO of order $1-n$ and that its inverse $(\mathsf{R}^*\mathsf{R})^{-1}$ is a ΨDO of order $n-1$. Thus (3.21) applies and we find that

$$\text{sing supp } (\mathsf{R}^*\mathsf{R})^k f = \text{sing supp } f \quad \text{for all } u \in \mathcal{E}'(X), \tag{3.26}$$

and all $k \in \mathbb{N}$. The equality, say for $k = 1$ comes from the fact that $f = (\mathsf{R}^*\mathsf{R})^{-1} g$ for $g = \mathsf{R}^*\mathsf{R} f$ so that the singular support of f is included in the singular support of g, as well as the reciprocal.

Remark 3.5.1 Note that we obtained the above result by writing $\mathsf{R}^*\mathsf{R}$ as a ΨDO. We may also consider $\mathsf{R}^*\mathsf{R}$ as a composition of the two FIO R^* and R. By doing so and using (3.22) and (3.24), we would find that

$$\begin{aligned}
&\text{sing supp } \mathsf{R}^*\mathsf{R} f \\
\subset\ & \Pi_x C'_\varphi \text{sing supp } \mathsf{R} f = \{x \in \mathbb{R}^2 \text{ s.t. } s = x \cdot \vartheta \text{ for } (s,\theta) \in \text{sing supp } \mathsf{R} f\} \\
\subset\ & \Pi_x C'_\varphi \Pi_{(s,\theta)} C_\varphi \text{sing supp } f = \{x \in \mathbb{R}^2 \text{ s.t. } s = x \cdot \vartheta \ \&\ s = y \cdot \vartheta \text{ for } y \in \text{sing supp } f\} \\
\subset\ & \mathbb{R}^2, \qquad \text{when sing supp } f \neq \emptyset.
\end{aligned}$$

Indeed, as soon as $y \in \text{sing supp } f$, then there exists a line passing through y and any other point $x \in \mathbb{R}^2$. This result is therefore far from the optimal result in (3.26). It also shows the limitations of the propagation of singularities obtained by using the notion of singular support. The latter is not sufficiently rich to account for the propagation of singularities. We now revisit this problem with the notion of wavefront set.

3.6 Propagation of singularities for FIO.

3.6.1 Wave Front Set and Distributions.

As we have seen in Remark 3.5.1, the notion of singular support is not sufficient to describe the propagation of singularities for a FIO such as the Radon transform or the generalized ray transform. For a large class of FIO A with adjoint A^*, we expect A^*A to be a ΨDO as is the case for the Radon transform. The propagation of singular supports by the relations $\Pi_X C_\varphi$ and $\Pi_Y C'_\varphi$ as in Remark 3.5.1 does not provide the identity map. Some cancellations are therefore missed by this composition since we know that the singular support of A^*Af is included in that of f.

One way to solve this problem is to refine the notion of singular support to a support that is defined in the space of positions and directions. Mathematically, the appropriate space to model positions and directions is the *cotangent bundle* of the manifold of interest. For us, the spatial domain is an open set $X \subset \mathbb{R}^n$ so that the cotangent bundle T^*X may be identified with $X \times \mathbb{R}^n$, which in coordinates is often denoted as (x,ξ). Note that only the direction $\hat{\xi} = \frac{\xi}{|\xi|} \in \mathbb{S}^{n-1}$ is of interest. By homogeneity in ξ, we thus mostly deal with $(x,\xi) \in X \times \dot{\mathbb{R}}^n$. Once we have the cotangent bundle T^*X, we need a notion of phase-space singular support (called microlocal singular support, or μsupp) for distributions.

Definition 3.6.1 *For $u \in \mathcal{D}'(X)$ and $X \subset \mathbb{R}^n$, we define the Wave Front Set of u denoted by $WF(u) \subset X \times \dot{\mathbb{R}}^n$ as follows. We say that $(x_0, \xi_0) \notin WF(u)$ if and only if there exists a function $\phi \in C_0^\infty(X)$ with $\phi(x_0) \neq 0$ such that the Fourier transform $\xi \to \widehat{\phi u}(\xi)$ is rapidly decreasing uniformly in a conic neighborhood of the half ray with direction ξ_0, i.e., for ξ such that $\hat{\xi} - \hat{\xi}_0$ is sufficiently small. This means that for each $k \geq 0$, we have C_k such that $|\widehat{\phi u}(\xi)| \leq C_k |\xi|^{-k}$ uniformly in ξ in the conic neighborhood of ξ_0.*

Exercise 3.6.1 *Show that the Wave Front Set of $\delta(x)$ is the set $(0, \xi \neq 0)$.*

Let X be an open domain with smooth boundary and let $\nu(x)$ be the outward unit normal to X at $x \in \partial X$. Let u be the function equal to 1 on X and 0 otherwise. Show that $WF(u) = \{(x, \lambda\nu(x));\ x \in \partial X,\ \lambda \neq 0\}$.

The main result on Wave Front Sets of the distributions defined as oscillatory integrals in (3.5) is then as follows:

Theorem 3.6.2 *Let $X \in \mathbb{R}^n$, Γ an open cone in $X \times \dot{\mathbb{R}}^N$ and φ a real-valued phase function in Γ. If $a \in S^m(X \times \mathbb{R}^N)$, vanishes near the zero section $X \times \{0\}$ and* cone supp $a \subset \Gamma$, *then for the distribution $I_{a,\varphi}$ defined in (3.5), we have*

$$WF(I_{a,\varphi}) \subset \{(x, \varphi'_x);\ (x, \theta) \in \text{cone supp}\, a, \quad \varphi'_\theta(x, \theta) = 0\}. \tag{3.27}$$

Here, cone supp a is defined as the set of $(x, \lambda\theta)$ for (x, θ) in the support of a and $\lambda > 0$.

Proof. The proof of this result goes as follows. Let $\phi(x)$ be a function concentrating near a point x_0. Then

$$\widehat{I_{a,\varphi}\phi}(\xi) = \int_{\mathbb{R}^N \times \mathbb{R}^n} e^{i\varphi(x,\theta) - ix\cdot\xi} a(x, \theta)\phi(x) d\theta dx.$$

Let $\psi(x, \theta) = \varphi(x, \theta) - x \cdot \xi$. Then

$$d\psi = \varphi'_\theta d\theta + (\varphi'_x - \xi)dx.$$

This means that for ξ in a cone away from $\xi_0 = \varphi'_x$, we can define a smooth differential operator of the form $L = b\partial_x + c$ for $b \in S^{-1}(\Gamma)$ so that $L^t e^{i\psi(x,\xi)} = e^{i\psi(x,\xi)}$. The reason is that since ξ is away from $\xi_0 = \varphi'_x$, which is homogeneous of degree one in θ, then

$$|\varphi'_x(x, \theta) - \xi| \geq C(|\theta| + |\xi|).$$

The usual integrations by parts then give us that $|\widehat{I_{a,\varphi}\phi}(\xi)| \leq C|\xi|^{-k}$ for all $k \in \mathbb{N}$. This proves that $(x_0, \xi_0) \notin WF(I_{a,\varphi})$ and concludes the proof of the result. □

3.6.2 Propagation of singularities in linear transforms

Let $K(x, y) \in \mathcal{D}'(X \times Y)$ be a Schwartz kernel for a linear operator from $X \subset \mathbb{R}^{n_X}$ to $Y \subset \mathbb{R}^{N_Y}$. We have seen the notion of $WF(K)$, a subset in $T^*(X \times Y) \sim (X \times Y) \times \dot{\mathbb{R}}^{n_X + n_Y}$. We recast the latter as $X \times \mathbb{R}^{n_X} \times Y \times \mathbb{R}^{n_Y} \backslash X \times Y \times \{0\}_{\mathbb{R}^{n_X + n_Y}}$ and we parametrize it by $(x, \xi; y, \zeta)$ rather than (x, y, ξ, ζ). It is defined for $(\xi, \zeta) \neq 0$ in $\mathbb{R}^{n_X + n_Y}$.

We define the following singular sets:

$$\begin{aligned} WF'(K) &= \big\{(x,\xi;y,-\zeta)\in X\times Y\times\dot{\mathbb{R}}^{n_X+n_Y} \text{ such that } (x,\xi:y,\zeta)\in WF(K)\big\}\\ WF_X(K) &= \big\{(x,\xi)\in X\times\dot{\mathbb{R}}^{n_X} \text{ such that } (x,\xi;y,0)\in WF(K)\big\}\\ WF'_Y(K) &= \big\{(y,-\zeta)\in Y\times\dot{\mathbb{R}}^{n_Y} \text{ such that } (x,0;y,\zeta)\in WF(K)\big\}. \end{aligned}$$

Let us consider the case of the kernel of a FIO $K(x,y)=I_{a,\varphi}(x,y)$ for $a\in S^m(X\times Y\times\dot{\mathbb{R}}^N)$ and $\varphi(x,y,\theta)$ a phase in the (x,y,θ) variables. From Theorem 3.6.2, we deduce that

$$\begin{aligned} WF'(I_{a,\varphi}) &\subset \big\{\big(x,\varphi'_x(x,y,\theta);y,-\varphi'_y(x,y,\theta)\big) \text{ s.t. } \varphi'_\theta=0,\ \theta\neq 0\big\}\\ WF_X(I_{a,\varphi}) &\subset \big\{\big(x,\varphi'_x(x,y,\theta)\big) \text{ s.t. } \big(x,\varphi'_x(x,y,\theta);y,0\big)\in WF(I_{a,\varphi})\big\}\\ WF'_Y(I_{a,\varphi}) &\subset \big\{\big(y,-\varphi'_y(x,y,\theta)\big) \text{ s.t. } \big(x,0;y,-\varphi'_y(x,y,\theta)\big)\in WF(I_{a,\varphi})\big\}. \end{aligned}$$

When φ is a phase in the (y,θ) variables, i.e., when $d_{y,\theta}\varphi\neq 0$, then $\varphi'_\theta=0$ and $\varphi'_y(x,\theta)=0$ together is not possible for $\theta\neq 0$. We thus deduce that $WF_X(I_{a,\varphi})=\emptyset$.

Similarly, when φ is a phase in the (x,θ) variables, i.e., when $d_{x,\theta}\varphi\neq 0$, then $\varphi'_\theta=0$ and $\varphi'_x(x,\theta)=0$ together is not possible for $\theta\neq 0$. We then deduce that $WF'_Y(I_{a,\varphi})=\emptyset$.

In the latter case, we have then seen in Theorem 3.3.1 that A mapped $\mathcal{E}'^j(Y)$ to $\mathcal{D}'^k(X)$ so that $Au(x)$ was defined in $\mathcal{D}'(X)$ for all $u\in\mathcal{E}'(Y)$.

Theorem 3.6.3 *Let $a\in S^m(X\times Y\times\mathbb{R}^N)$ and $\varphi\in C_0^\infty(X\times Y\times\dot{\mathbb{R}}^N)$ a phase in the (x,θ) variables, i.e., $d_{x,\theta}\varphi\neq 0$ for all $y\in Y$. Let A be the FIO defined in (3.12) with Schwartz kernel $K(x,y)=I_{a,\varphi}(x,y)$. Then for $u\in\mathcal{E}'(Y)$, we have $Au\in\mathcal{D}'(X)$ and*

$$WF(Au)\subset WF'(K)\big(WF(u)\big)\cup WF_X(K)=WF'(K)\big(WF(u)\ \cup O_Y\big). \tag{3.28}$$

Proof. By assumption φ is a phase in the (x,θ) variables for all y so that Au is defined in $\mathcal{D}'(X)$ for all $u\in\mathcal{E}'(Y)$. Once $Au(x)$ is defined as a distribution, we wish to understand its microlocal support. We want to prove that

$$WF(Au)\subset WF'(K)\big(WF(u)\big)\cup WF_X(K)=WF'(K)\big(WF(u)\ \cup O_Y\big),$$

the latter set being given by

$$\Gamma_{Au}=\{(x,\xi);\ (x,\xi,y,-\zeta)\in WF(K) \text{ and } (y,\zeta)\in\big(WF(u)\cup O_Y\big)\}.$$

Let $(x_0,\xi_0)\notin\Gamma_{Au}$. We want to show that $(x_0,\xi_0)\notin WF(Au)$. Let $\phi(x)\in C_0^\infty(X)$ of sufficiently small support in the vicinity of x_0 with $\phi(x_0)=1$. Then, using the Parseval identity,

$$\phi(x)Au(x)=\int\phi(x)K(x,y)u(y)dy,\quad \widehat{\phi Au}(\xi)=\frac{1}{(2\pi)^{n_X}}\int\widehat{\phi K}(\xi,-\zeta)\hat{u}(\zeta)d\zeta.$$

We want to analyze the decay properties of $\widehat{\phi Au}(\xi)$ as $|\xi|\to\infty$ in a conic vicinity of ξ_0. Such estimates require that we localize the y variable. We introduce a (locally finite) partition of unity $\phi_j(y)$ such that $\sum_j\phi_j^2(y)=1$ and obtain that

$$\phi Au(x)=\sum_j\int K_j(x,y)u_j(y)dy,\quad K_j(x,y)=\phi(x)K(x,y)\phi_j(y),\ u_j(x,y)=u(y)\phi_j(y).$$

The choice of the functions ϕ_j and ϕ is as follows.

Since $(x_0,\xi_0)\notin\Gamma_{Au}$, we have that $\Gamma_K(x_0,\xi_0,y)\cap\Gamma_u(y)=\emptyset$ for all $y\in Y$. The result remains true, i.e., $\Gamma_K(x,\xi,y)\cap\Gamma_u(y)=\emptyset$, for all $y\in Y$ and all (x,ξ) in a conic vicinity of (x_0,ξ_0). We assume that $\phi(x)$ is supported in that vicinity. Now for $\zeta\in\Gamma_u(y_j)$, we deduce from the fact that $WF(K)$ is closed that $\zeta\notin\Gamma_K(x,\xi,y)$ for (x,ξ,y) in a sufficiently small neighborhood of (x_0,ξ_0,y_j). For Y compact, we find a finite number of points y_j and neighborhoods of y_j where the functions ϕ_j are supported.

We denote by $\tilde\Gamma_j$ the set of (ξ,ζ) for ξ in the above conic vicinity of ξ_0 and ζ such that $(y,\zeta)\in WF(u)\cup O_Y$ for y in the above vicinity of y_j. By construction, $\tilde\Gamma_j$ is in the vicinity of $\xi_0\times(WF(u)_{|y=y_j}\times\{0\})$. We denote by Γ_j the set of (ξ,ζ) such that there is a x and y in the vicinities of x_0 and y_j, respectively, so that $(x,\xi,y,-\zeta)\in WF(K)$. By construction, Γ_j is in the vicinity of $WF'(K)_{|x=x_0,y=y_j}$. Since $\Gamma_K(x_0,\xi_0,y_j)\cap\Gamma_u(y_j)=\emptyset$, the above construction shows that $\Gamma_j\cap\tilde\Gamma_j=\emptyset$.

We then have by the Parseval identity

$$\widehat{\phi Au}(\xi)=\frac{1}{(2\pi)^{n_X}}\sum_j\int\hat K_j(\xi,-\zeta)\hat u_j(\zeta)d\zeta.$$

Each term is treated in the same way. We consider ξ in the above vicinity of ξ_0. Define $p_j\in C^\infty(\dot{\mathbb{R}}^{n_Y})$ a function homogeneous of degree 0 (i.e., a function $p_j(\zeta)=p_j(\frac{\zeta}{|\zeta|})$) such that $p_j(\zeta)=1$ for $\zeta\in WF(u)_{y\in\text{supp }\phi_j}$ and $p_j(\zeta)=0$ for $(\xi,-\zeta)\in WF(K_j)_{(x,y)\in\text{supp }\phi\times\text{supp }\phi_j}$. Then we find that

$$\Big|\int\hat K_j(\xi,-\zeta)\hat u_j(\zeta)p_j(\zeta)d\zeta\Big|\le C\int\langle\xi\rangle^{-k}\langle\zeta\rangle^{-k}\langle\zeta\rangle^q d\zeta\le C\langle\xi\rangle^{-k},$$

where k is arbitrary large enough for some fixed q that exists for the distribution u_j.

Define $v_j(\zeta)=(1-p_j(\zeta))\hat u_j(\zeta)$, which is a function that is bounded by $C_q\langle\zeta\rangle^{-q}$ for any q by construction of p_j. Since $(\xi,0)\notin WF(K_j)_{(x,y)\in\text{supp }\phi\times\text{supp }\phi_j}$ by construction, we find that for ε sufficiently small and all $k>0$,

$$|\hat K_j(\xi,-\zeta)|\le C_k\langle\xi\rangle^{-k},\qquad\text{on }|\zeta|<\varepsilon|\xi|.$$

For the distribution K_j, we also have the uniform bound $|\hat K_j(\xi,-\zeta)|\le(\langle\xi\rangle+\langle\zeta\rangle)^Q$. We thus find that

$$\Big|\int\hat K_j(\xi,-\zeta)v_j(\zeta)d\zeta\Big|\le C_k\langle\xi\rangle^{-k}+C\int_{|\zeta|>\varepsilon|\xi|}(\langle\xi\rangle+\langle\zeta\rangle)^Q|v_j(\zeta)|d\zeta.$$

Since $|v_j(\zeta)|\le C_q\langle\zeta\rangle^{-q}$ for any q, we obtain a bound of the form $C_k\langle\xi\rangle^{-k}$ for all k. This shows that for this choice of ξ, we have $|\widehat{\phi Au}(\xi)|\le C_k\langle\xi\rangle^{-k}$ for all k, which implies that $(x_0,\xi_0)\notin WF(Au)$. □

Remark 3.6.4 The above theorem is written for an FIO of the form (3.12). However, the proof extends to any operator A with Schwartz kernel K satisfying the hypotheses of the theorem. The main additional difficulty for more general operators is to prove that Au is indeed a distribution when u is a distribution satisfying appropriate constraints. In the above theorem, this issue was solved by assuming that φ was a phase in the variables (x,θ) so that Theorem 3.3.1 applies.

Remark 3.6.5 When φ is a phase in the variables (y,θ), then $WF_X(K)=\emptyset$, so that

$$WF(Au)\subset WF'(K)(WF(u))=\big\{(x,\xi);\ (x,\xi,y,-\zeta)\in WF(K) \text{ and } (y,\zeta)\in WF(u)\big\}.$$

This is one of the main results of the chapter. It describes the possible set of singularities of $WF(Au)$. Now, operators of the form A may not always preserve all singularities. The operators that do so are the *elliptic* ones. We are still several steps away from being able to obtain such a result.

When A is a ΨDO , then $\varphi(x,y,\theta)=(x-y)\cdot\theta$ and we find that

$$WF(K_A)\subset N^*\Delta:=\{(x,\xi;x,-\xi),\ x\in X,\ \xi\in\dot{\mathbb{R}}^n\}. \tag{3.29}$$

Since clearly $WF'_Y(K_A)=\emptyset$ and $WF_X(K_A)=\emptyset$, we find that for $u\in\mathcal{D}'(X)$, we have

$$WF(Au)\subset WF(u). \tag{3.30}$$

This important result shows that ΨDO do not propagate μlocal singularities.

Let us conclude with a few remarks on the propagation of singularities. In all above estimates, we obtain that $WF(Au)$ is *included* in $WF'(A)WF(u)$, not that it is *equal* to it as a set. Inclusion is sometimes (often) strict, even for ΨDO . For instance, consider the ΨDO $P=-\partial_{x_1}\partial_{x_2}$ in two space dimensions. Consider the function $f(x)=H(x_1)H(x_2)$, where H is the Heaviside function. Verify that

$$WF(f)=\{(0,x_2;\xi_1\neq 0,0)\}\cup\{(x_1,0;0,\xi_2\neq 0)\}\cup\{(0,0;\xi\neq 0)\}$$

whereas $WF(Pf)$ is given by only that last component of the union. Singularities are therefore possibly destroyed by operators. Incidentally, P is a wave operator in the diagonal variables so that the Cartesian axes form the light cone associated to that operator (Exercise!).

Now, let us consider an operator A with a relation $WF'(A)$ and let us assume the existence of a parametrix Q, i.e., a FIO such that $QA=I+R$ with R a smoothing operator, and let us assume that we can verify that $WF'(Q)$ is an inverse relation of $WF'(A)$ in the sense that $WF'(Q)WF'(A)=WF'(A)WF'(Q)=I$. Then,

$$WF(u)=WF(QAu)\subset WF'(Q)WF(Au)\subset WF'(Q)WF'(A)WF(u)=WF(u)$$

so that all inclusions are equalities. It remains to apply $WF'(A)$ to obtain that

$$WF(Au)=WF'(A)WF(u). \tag{3.31}$$

For the Radon transform, we know that an inverse exists, and we can write it as a FIO whose canonical relation is clearly the inverse of that of the Radon transform. The above relation then holds. In the case of a generalized Radon transform considered in the next chapter, we will construct such a parametrix Q and show that the canonical relations are indeed inverse to each-other. In general, obtaining such a result is not straightforward. The remainder of this chapter consider (relatively simple) situations where such a statement can be made.

3.7 Critical sets and Lagrangian manifolds

The critical set C_φ plays an important role in the preceding propagation of singularities. In order to propagate such singularities back to their original domain of definition, we need to apply additional transforms to Au, for instance an adjoint transform. We then would like the product of two FIO to be a FIO. We impose some constraints so that a reasonable such calculus can be developed.

The first item is to understand the geometry of C_φ better. Let $g(x,\theta) = \{g_j(x,\theta)\}$ with $g_j = \partial_{\theta_j}\varphi$ for $1 \leq j \leq N$. C_φ is thus defined as $g^{-1}(0)$. We would like it to be a reasonable geometric object, i.e., a sub-manifold of $X \times \dot{\mathbb{R}}^N$. We know this to be the case when dg, the derivative of g, is surjective onto its image space, which is equivalent to the derivatives dg_j to be linearly independent. We introduce

Definition 3.7.1 (Non-degenerate phase function) *Let Γ be a cone in $X \times \dot{\mathbb{R}}^N$. We say that $\varphi(x,\theta)$ is* non-degenerate *on Γ when*

$$\Big(d\frac{\partial\varphi}{\partial\theta_j}\Big)_{1\leq j\leq N} \quad \textit{are linearly independent on } C_\varphi \cap \Gamma.$$

This implies that C_φ restricted to Γ is a sub-manifold of Γ of co-dimension N and hence of dimension n, the dimension of X. It turns out that the main player in the theory of FIO is not quite C_φ but rather, as is apparent in (3.27), the following manifold.

Definition 3.7.2 (Lagrangian manifold) *Consider the map*

$$C_\varphi \ni (x,\theta) \mapsto T_\varphi(x,\theta) = (x,\varphi'_x) \in T^*X\backslash\{0\}.$$

We call $\Lambda_\varphi = T_\varphi(C_\varphi)$ the Lagrangian manifold.

We know that φ'_x cannot vanish on C_φ since φ is a phase. We verify that when φ is non-degenerate, then Λ_φ is a submanifold in $T^*X\backslash 0$ for exactly the same reasons that C_φ is a submanifold of $X \times \dot{\mathbb{R}}^N$ (Exercise!).

Moreover, Λ_φ is indeed Lagrangian, which means the following. Consider the one form $\alpha = \xi dx$ on T^*X. On C_φ, we have $\varphi'_\theta = 0$ so that by homogeneity $\varphi = \theta\cdot\varphi'_\theta = 0$. Thus, restricted to C_φ, $d\varphi = 0 = \varphi'_x dx + \varphi'_\theta d\theta = \xi dx = 0$. Therefore, on Λ_φ, we have $\alpha = \xi dx = 0$ so that the symplectic form $-\omega = d\alpha = d\xi \wedge dx$ itself vanishes on Λ_φ. A submanifold of T^*X of dimension n such that $\omega_{|\Lambda_\varphi} = 0$ is called Lagrangian.

The following result, which we state without proof or even derivation, justifies the introduction of the Lagrangian manifold. Let $\varphi(x,\theta)$ and $\tilde\varphi(x,\tilde\theta)$ be non-degenerate phase functions in the vicinity of $(x_0,\theta_0) \in X \times \dot{\mathbb{R}}^N$ and $(x_0,\tilde\theta_0) \in X \times \dot{\mathbb{R}}^{\tilde N}$. Note that N and $\tilde N$ may differ. Let C_φ and $C_{\tilde\varphi}$ be the corresponding critical sets and Λ_φ and $\Lambda_{\tilde\varphi}$ the corresponding Lagrangian, all manifolds of the same dimension n as X. Let $\Lambda_\varphi = \Lambda_{\tilde\varphi}$, which means that the two phases generate the same sub-manifold of T^*X. Let now $a(x,\theta)$ be in $S^\mu(X \times \mathbb{R}^N)$ with support close to (x_0,θ_0). Then there is an amplitude $\tilde a(x,\tilde\theta)$ supported in the vicinity of $(x_0,\tilde\theta_0)$ such that $I_{a,\varphi} = I_{\tilde a,\tilde\varphi}$. Moreover, $\tilde a \in S^{\mu+\frac{1}{2}(N-\tilde N)}$.

This result is based on an application of the stationary phase (and not the non-stationary phase results we have obtained so far by introducing various operators L^t and

integrations by parts). It has two consequences. First, it is the Lagrangian manifold $\Lambda \equiv \Lambda_\varphi$ that is the main player in the propagation of singularities, something that was already apparent in (3.27), which states that $WF(I_{a,\varphi}) \subset \Lambda$. Second, it is not the order of the symbol a that directly indicates the smoothing properties of $I_{a,\varphi}$ since the former depends on N while the latter should only depend on φ via Λ.

3.8 Local theory of FIO

Let us now come back to the definition of FIO A with kernel in $\mathcal{D}'(X \times Y)$. More precisely, let $\varphi(x, y, \theta)$ be a non-degenerate phase function in (a conic open subset of) $X \times Y \times \mathbb{R}^N$ and $a(x, y, \theta)$ an amplitude in (a corresponding conic open subset of) $S^\mu(X \times Y \times \mathbb{R}^N)$. Let $\Lambda = T_\varphi(C_\varphi)$ be the associated Lagrangian manifold, given by

$$\Lambda = \{(x, \xi; y, \zeta) \in T^*X \times T^*Y \backslash 0,\ (x, y, \theta) \in C_\varphi,\ \xi = \varphi'_x(x, y, \theta),\ \zeta = \varphi'_y(x, y, \theta)\}$$

(or its restriction to an open conic subset Γ of $T^*X \times T^*Y \backslash 0$). Both C_φ and Λ are manifolds of dimension $n_X + n_Y$ since φ is non-degenerate.

We denote by Λ' the set of points $(x, \xi; y, \zeta)$ such that $(x, \xi; y, -\zeta) \in \Lambda$. Note that Λ is a Lagrange manifold in the sense that $\omega_x \otimes -\omega_y$ is a symplectic form in $T^*X \times T^*Y \backslash 0$ that vanishes when restricted to Λ (Exercice!); such an object is also called a *homogeneous canonical relation* from T^*Y to T^*X. This relation is not necessarily a graph, in the sense that (y, ζ) is not necessarily in relation with only one element (x, ξ).

We are now ready to define our classes of FIO.

Definition 3.8.1 (Classes of FIO) *Let $\varphi(x, y, \theta)$ be a non-degenerate phase function on $X \times Y \times \dot{\mathbb{R}}^N$ and $a(x, y, \theta)$ an amplitude in (a corresponding conic open subset of) $S^\mu(X \times Y \times \mathbb{R}^N)$ with $\mu = m + \frac{n_X + n_Y}{4} - \frac{N}{2}$ for $m \in \mathbb{N}$. Let Λ be the associated Lagrangian manifold. We define $I^m(X \times Y; \Lambda')$ as the FIO from $\mathcal{E}'(Y)$ to $\mathcal{D}'(X)$ with Schwartz kernel*

$$I_{a,\varphi} = \int e^{i\varphi(x,y,\theta)} a(x, y, \theta) d\theta.$$

Here, n_X and n_Y are the dimensions of X and Y, respectively.

We have seen that the presence of $m - \frac{N}{2}$ in μ was dictated by the fact that a Lagrangian may be represented by phases with different numbers of phase variables. In the case of a ΨDO , where $n_X = n_Y = N$, we observe that $m = \mu$ so that m indeed represents the number of derivatives of the operator A. For Radon transforms, $n_X = n_Y = n$ and $N = 1$ so that $\mu = m + \frac{n-1}{2}$. Since $m = \frac{1-n}{2}$ corresponds to the correct regularization by $\frac{n-1}{2}$ derivatives, we do obtain that $\mu = 0$.

Now that we have our class of FIO, we can consider composing them, so that objects such as A^*A may be identified as FIO (or better yet ΨDO possibly). A first issue is that operators so far map compactly supported distributions to arbitrary distributions. We thus assume that all operators are *properly supported*. What this means is that, up to a modification in $S^{-\infty}$, our FIO maps compacty supported distributions to compactly supported distributions. A necessary condition is that the phase be properly supported in the following sense. For any compact K in X and $x \in K$, there is a compact K' such that for all $(x, y, \theta) \in C_\varphi$, then $y \in K'$. Assume the same reverting the roles of X and

Y. Seeing C_φ as a relation from X to Y or vice-versa, we essentially impose that C_φ^{-1} of a compact set is compact, or heuristically that infinity is mapped to infinity. These notions of proper support allow us to assume that A and its adjoint A^* are properly supported, i.e. map arbitrary distributions to arbitrary distributions and compactly supported distributions to compactly supported distributions. From now on, we assume all operators to be properly supported.

We can now state the main result on the composition of FIO.

Theorem 3.8.2 (Local composition of FIO) *Let X_j be open domains of dimension n_j for $j = 1, 2, 3$ and let $A_j \in I^{m_j}(X_j, X_{j+1}, \Lambda'_j)$ for $j = 1, 2$ properly supported. We see Λ'_j as homogeneous canonical relations from T^*X_{j+1} to T^*X_j. The FIO A_j are defined by non-degenerate phase functions φ_j in a cone $\Gamma_j \subset X_j \times X_{j+1} \times \dot{\mathbb{R}}^{N_j}$ and amplitudes $a_j \in S^{\mu_j}(X_j \times X_{j+1} \times \mathbb{R}^{N_j})$ supported in Γ_j, where $\mu_j = m_j + \frac{1}{4}(n_j + n_{j+1} - 2N_j)$.*

Let us define $\varphi_3(x_1, x_3; x_2, \theta_1, \theta_2) = \varphi_1(x_1, \frac{x_2}{|\theta|}, \theta_1) + \varphi_2(\frac{x_2}{|\theta|}, x_3, \theta_2)$ where we have defined $|\theta| = \sqrt{|\theta_1|^2 + |\theta_2|^2}$.

We assume: (i) that φ_3 is a phase, which is equivalent to the fact that $\Lambda'_1 \circ \Lambda'_2 \in T^*(X_1) \times T^*(X_3)\backslash 0$ (the $\backslash 0$ is what matters to show that $d\varphi_3 \neq 0$).

We assume: (ii) that φ_3 is a non-degenerate phase function, which means that the differentials $d(\varphi_3)'_{y,\theta,\sigma}$ are of maximal rank $N_1 + N_2 + n_2$ (i.e. linearly independent). Geometrically, this is equivalent to proving that $\Lambda'_2 \times \Lambda'_1$ intersects the manifold $T^*X_3 \times (\mathrm{Diag}T^*(X_2)) \times T^*X_1$ transversally (in the very large $T^*X_3 \times T^*X_2 \times T^*X_2 \times T^*X_1$).

Then, the operator $A_1 \circ A_2$ is a FIO in $I^{m_1+m_2}(X_1, X_3, \Lambda'_1 \circ \Lambda'_2)$ associated to the phase φ_3 for an amplitude a_3 in $S^{\mu_1+\mu_2-n_2}(X_1 \times X_3 \times \mathbb{R}^{N_1+N_2+n_2})$. The latter amplitude has a leading term (of maximal order) given by

$$a_{30}(x_1, x_3, x_2, \theta_1, \theta_2) = c a_1(x_1, \frac{x_2}{|\theta|}, \theta_1) a_2(\frac{x_2}{|\theta|}, x_3, \theta_2) |\theta|^{-n_Y},$$

with $c = c(n_1, n_2, n_3, N_1, N_2)$ a normalizing constant.

We say that two sub-manifolds M and N of a manifold P intersect transversally if at each point x in $M \cap N$, the span of the tangent spaces T_xM and T_xN fills the whole of T_xP. Verifying this assumption in practice, i.e., that φ_3 is a non-degenerate phase, is often not straightforward. Luckily, this verification simplifies in a number of interesting cases we will consider shortly.

Before doing so, let us make a few remarks. The normalization in $x_2/|\theta|$ occurs so that φ_3 becomes homogeneous of degree 1 in $(x_2, \theta_1, \theta_2)$. That $(x_2, \theta_1, \theta_2)$ play the role of phase variables is natural in the formal composition of two FIOs. However, $N_1 + N_2 + n_2$ is often not the smallest possible dimension of phase variables to describe the canonical relation $\Lambda'_1 \circ \Lambda'_2$. Indeed, the composition of two ΨDO leads to a ΨDO as we will indicate shortly. The form of the amplitude for a_{30} results from an application of stationary phase; see a little more detail in the appendix.

We now consider cases where the verification of the above assumptions tremendously simplifies.

Theorem 3.8.3 (Composition with ΨDO) *Let A_1 and A_2 be two FIO as described above with one of them being a ΨDO . Note that a ΨDO is a FIO associated with a*

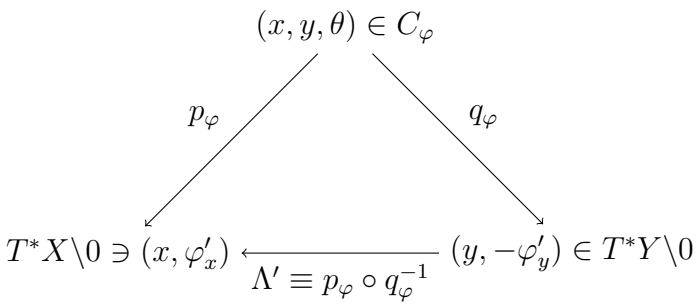

Figure 3.1: Projections from the critical set $C_\varphi = \{\varphi'_\theta = 0\}$ to the cotangent spaces of the homogeneous canonical relation Λ' (with q_φ^{-1} seen as a relation, not necessarily a map unless (3.32) holds).

diagonal canonical relation $\Lambda' = \{(x, \xi; y, \zeta),\ y = x,\ \zeta = \xi\}$. *The above transversality condition is then always satisfied and the canonical relation of the product* $A_1 A_2$ *or* $A_2 A_1$ *is always that of the FIO.*

This is an important result, which may proved in an easier way than applying the above composition essentially by means of stationary phase and appropriate Taylor expansions; see appendix. We thus observe that the composition of ΨDO is a ΨDO as well. This gives an example where the phase of the composition $A_1 \circ A_2$ is described by n variables and not $N_1 + N_2 + n_2$. When the symbols of $FIOs$ do not depend on the middle variable x_2, it is then straightforward to see that $2n$ variables simplify since the integral of $e^{ix_2 \cdot \xi_2}$ in both variables x_2 and ξ_2 results in a constant. When the symbols depend on x_2, applying stationary phase becomes necessary.

The last type of results we wish to consider aim to form the product A^*A and prove that the latter is a ΨDO for $A \in I^m(X, Y, \Lambda')$. This is simply incorrect in general, especially when the dimensions n_X and n_Y do not agree. However, when $n_X = n_Y$, we may hope for the relation Λ' mapping T^*Y to T^*X to be a bijection, at least locally. This is again not always verified for arbitrary (non-degenerate) phases $\varphi(x, y, \theta)$. However, it holds for phases such that appropriate mappings from C_φ to T^*X and T^*Y are (at least locally) bijective. Indeed, all manifolds are now of dimension $2n$. That they are locally bijective may then come from an application of the implicit function theorem.

More precisely, Let us come back to the definition of C_φ as the set where $\varphi'_\theta = 0$. Consider the map q_φ from C_φ to T^*Y associating the point $(y, \zeta = -\varphi'_y)$. Defining g as the map from (x, y, θ) to $(y, \varphi'_y, \varphi'_\theta)$, we wish to show that the equation $g(x, y, \theta) = (y, 0, \zeta)$ is uniquely solvable, which holds locally if the Jacobian of the transformation is invertible, in other words if the determinant of the Jacobian $\frac{\partial(\varphi'_y, \varphi'_\theta)}{\partial(x, \theta)}$ has non-vanishing determinant. This means

$$\mathrm{Det} \begin{pmatrix} \varphi''_{xy} & \varphi''_{x\theta} \\ \varphi''_{y\theta} & \varphi''_{\theta\theta} \end{pmatrix} \neq 0. \tag{3.32}$$

When the above constraint holds for all (x, y, θ) of interest, we thus obtain that q_φ is locally invertible, or in other words that (y, ζ) may be used as coordinates on the $2n$-dimensional manifold C_φ. The same contraint shows that the map p_φ from C_φ to T^*X

mapping $(x, y, \theta) \in C_\varphi$ to $(x, \xi = \varphi'_x)$ is invertible since the above determinant is also that of the Jacobian $\frac{\partial(\varphi'_x, \varphi'_\theta)}{\partial(y,\theta)}$.

Note that the above diffeomorphisms are clearly local diffeomorphisms. Locally, they imply that the canonical relation Λ' from T^*Y to T^*X is given by the map $p_\varphi \circ q_\varphi^{-1}$. We then say that the canonical relation is a *canonical graph* (see Fig. 3.1) and have the following:

Theorem 3.8.4 (Composition with adjoints; canonical graphs) *Let $A = I_{a,\varphi} \in I^m(X, Y, \Lambda')$ properly supported with Λ' given globally (after possibly shrinking X and Y) by a canonical graph $p_\varphi \circ q_\varphi^{-1}$. Let $\Lambda^{-'} = (\Lambda^{-1})'$ be the inverse canonical graph $q_\varphi \circ p_\varphi^{-1}$. Then A^* represented locally by the phase $-\varphi(y, x, \theta)$ and the amplitude $a^*(y, x, \theta)$ is a FIO in $I^m(Y, X, \Lambda^{-'})$. Moreover, A^*A is a ΨDO in $I^{2m}(Y, Y, (N^*\Delta)')$ where $N^*\Delta = \{(y, \zeta; y, -\zeta\}$. Moreover, A is continuous from $H^s(Y)$ to $H^{s-m}(X)$ while the normal operator A^*A is continuous from $H^s(Y)$ to $H^{s-2m}(Y)$.*

To complete our analysis of the Radon transform or similar transforms, we need one last result on the analysis of ΨDO and their approximate inverses.

3.9 Elliptic FIO and Elliptic ΨDO

The main results we need from the theory of Elliptic ΨDO is the following.

Recall that a ΨDO is given by

$$Au(x) = \int e^{i(x-y)\cdot\theta} a(x, y, \theta) u(y) dy d\theta$$

for an amplitude $a \in S^m(X \times X \times \mathbb{R}^n)$. Let us still denote by $I^m(X)$ the space of such operators. As before, we assume that the operators are properly supported so that composition of such operators are well defined.

A first result allows one to simplify the symbol by eliminating the $y-$ (or $x-$) dependence.

Theorem. Let $A \in I^m(X)$ properly supported. Define $b(x, \xi) = e^{-ix\cdot\xi} A(e^{ix\cdot\xi})$, the conjugation of A with the exponential function $e^{ix\cdot\xi}$. Then $b \in S^m(X)$, is given asymptotically by

$$b(x, \xi) \sim \sum_{\alpha \in \mathbb{N}^n} \frac{i^{-|\alpha|}}{\alpha!} (\partial_\xi^\alpha \partial_y^\alpha a(x, y, \xi))_{|y=x}.$$

Moreover, A is a ΨDO with symbol $b(x, \xi)$ (instead of $a(x, y, \xi)$).

Let now A and B be two ΨDO in I^p and I^q, respectively. Then AB is a ΨDO in I^{p+q} and we have the following relation. Assume that the symbols of A and B are given by $a(x, \xi)$ and $b(x, \xi)$, respectively. Then the symbol of AB is $c(x, \xi) \in S^{p+q}(X)$ and we have

$$c(x, \xi) \sim (a\#b)(x, \xi) \sim \sum_{\alpha \in \mathbb{N}^n} \frac{i^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha a(x, \xi) \partial_x^\alpha b(x, \xi).$$

This is the definition of $a\#b$, which we observe is a unique symbol in $S^{p+q}(X)$ up to an element in $S^{-\infty}(X)$.

Let us now introduce the notion of ellipticity.

Definition 3.9.1 (Elliptic ΨDO) *We say that a ΨDO $A \in S^m(X)$ is* elliptic *if its symbol $a = a(x,\xi)$ verifies*

$$|\xi|^{-m}|a(x,\xi)| \geq C > 0 \quad \text{uniformly in } x \text{ and } |\xi| > R$$

for some $R \geq 0$.

Note that we can always modify $a(x,\xi)$ on $|\xi| < 2R$ such that $a(x,\xi)$ remains in $S^m(X)$ and $a(x,\xi)$ satisfies the above condition with $R = 0$. The modification involves a correction in $S^{-\infty}(X)$.

Then we have:

Theorem. Let $A \in I^m(X)$ be an elliptic ΨDO . Then there exists an elliptic ΨDO B such that $BA \sim I$, i.e., $BA = I - R$ with an operator in $I^{-\infty}(X)$.

The operator B is called a *parametrix* for A. It serves as an inverse for A as far as singularities are concerned.

let us now return to general FIO. Let $A \in I^m(X, Y, \Lambda')$ with symbol $a(x, y, \theta)$ and let us assume again that A is elliptic, which means:

Definition 3.9.2 (Elliptic FIO) *We say that a FIO $A \in I^m(X, Y, \Lambda')$ is* elliptic *if its symbol $a = a(x, y, \theta) \in S^\mu(X, Y, \mathbb{R}^N$ for $\mu = m + \frac{1}{4}(n_X + n_Y - 2N)$ verifies*

$$|\theta|^{-\mu}|a(x,y,\theta)| \geq C > 0 \quad \text{uniformly in } (x,y) \in X \times Y \text{ and } |\theta| > R$$

for some $R \geq 0$.

As for ΨDO , we may modify a so that the order of A does not change and a satisfies the above constraint with $R = 0$.

Let us now assume that the canonical relation Λ' is a graph, i.e., φ is a non-degenerate phase in (x, θ) and (y, θ) and (3.32) holds.

The using the preceding results, we find that A^*A is well defined and is a ΨDO . When a is elliptic, then the symbol of A^*A is also elliptic by application of the stationary phase (this is a theorem).

We therefore obtain the existence of a parametrix Q such that $QA^*A \sim I$. This provides an inversion for A^* up to a smoothing term. As for the propagation of singularities, we are finally in a position to answer the question of inclusions.

First, consider an elliptic ΨDO P with parametrix Q. Then

$$WF(u) = WF(QPu) \subset WF(Pu) \subset WF(u), \quad \text{so that} \quad WF(Pu) = WF(u).$$

Now, let A be an elliptic FIO with Λ' as a (global) canonical graph. Then

$$WF(u) = WF(QA^*Au) = WF(A^*Au) \subset \Lambda^{-'}WF(Au) \subset \Lambda^{-'}\Lambda' WF(u) = WF(u).$$

This clearly implies that

$$WF(Au) = \Lambda' WF(u).$$

This is our final and most useful result. Let us assume that Λ' is a global canonical graph (which may always be achieved from the local result provided X and Y are appropriately restricted). Then for any elliptic FIO A, we have that the μlocal singularities

$WF(Au)$ are exactly the mapping of the μlocal singularities $WF(u)$ propagated by the canonical relation (a graph) Λ'.

The Radon transform is an example of such a transform, with a global canonical graph (provided we quotient by the $2-1$ maps appearing because of the double covering of hyperplanes appearing naturally in the parametrization in (s,θ) (Exercise!)). The singularities (x,ξ) of u are therefore exactly mapped to the singularities $(s,\theta,ds,d\theta)$ obtained from the canonical relation.

3.10 Local tomography and partial data

Let S be the square root of the Laplacian defined by

$$\mathsf{S} = \mathcal{F}^{-1}|\xi|\mathcal{F}. \tag{3.33}$$

Then we observe that $(c_n\mathsf{R}^*\mathsf{R})^{-1} = \mathsf{S}^{n-1}$, or in other words,

$$I = \mathsf{S}^{n-1}c_n\mathsf{R}^*\mathsf{R}, \qquad c_n = \frac{\pi}{(2\pi)^n}. \tag{3.34}$$

This provides an alternative formula to the inversion of the Radon transform with the inverse operator $\mathsf{R}^{-1} = c_n\mathsf{S}^{n-1}\mathsf{R}^*$.

Note that the application of the operator R^* is *local* in this sense: to calculate R^*g at x in (3.24), we need to know $g(s,\vartheta)$ for all (s,ϑ) corresponding to hyperplanes passing through that specific point x. In odd dimensions $n = 1+2k$, the operator $\mathsf{S}^{n-1} = (-\Delta)^k$ is also local as a differential operator. In even dimension, and in the practical case of dimension $n = 2$, then S^{n-1} is not local. However, S^{n-2} and S^n are local operators.

Λ-tomography, or local tomography consists of reconstructing functions by applying only *local* operators in the sense described above. Let us assume $n = 2$ for concreteness. The two main functions that are reconstructed in practice in local tomography from knowledge of $\mathsf{R}f$ are

$$\mathsf{S}f = c_n\mathsf{S}^2\mathsf{R}^*\mathsf{R}f, \qquad \mathsf{S}^{-1}f = c_n\mathsf{R}^*\mathsf{R}f.$$

Since both S and S^{-1} are ΨDO (up to regularization in the vicinity of $\xi = 0$), we find that

$$\text{sing supp } \mathsf{S}f = \text{sing supp } \mathsf{S}^{-1}f = \text{sing supp } f.$$

In other words, all the singularities of f are captured by both $\mathsf{S}f$ and $\mathsf{S}^{-1}f$. Since S is a ΨDO or order 1 (with its harmless singular amplitude at $\xi = 0$), they have been amplified in $\mathsf{S}f$ (by one derivative) and attenuated in $\mathsf{S}^{-1}f = c_n\mathsf{R}^*\mathsf{R}f$ (also by one derivative).

When $\mathsf{R}f(s,\theta)$ is available on $\mathbb{R}\times\mathbb{S}^1$, then local tomography may not be very useful as the exact f may be reconstructed by either (2.16) or (3.34). However, it becomes a very powerful tool in the presence of limited data. Moreover, when $\mathsf{R}f(s,\theta)$ is available for only a subset of lines (s,θ), then the explicit formulas may no longer be the most desirable as they involve unknown information.

We now present some conditions under which the singular support of f may be reconstructed. Let $\chi(s,\theta)\in C_0^\infty(\mathbb{R}\times\mathbb{S}^1)$ be a function equal to 1 on the set of singularities of $\mathsf{R}f$, in other words, from (3.23)

$$\chi(s,\theta) = 1 \quad \text{on} \quad \{(s,\theta)\in\mathbb{R}\times\mathbb{S}^1 \ \text{ s.t. } \ s = x\cdot\vartheta \text{ for some } \ x\in \text{sing supp } f\}.$$

We now have for all $k \in \mathbb{Z}$,

$$\mathsf{S}^k c_n \mathsf{R}^* \chi \mathsf{R} f = \mathsf{S}^k c_n \mathsf{R}^* \mathsf{R} f + \mathsf{S}^k c_n \mathsf{R}^* (1-\chi) \mathsf{R} f = \mathsf{S}^{k-1} f + \mathsf{S}^k c_n \mathsf{R}^* (1-\chi) \mathsf{R} f$$

Since $\chi = 1$ on the singular support of $\mathsf{R}f$, we deduce that $(1-\chi)\mathsf{R}f \in C^\infty(\mathbb{R} \times \mathbb{S}^1)$, in other words, we have that

$$\text{sing supp } \mathsf{S}^k \mathsf{R}^* \chi \mathsf{R} f = \text{sing supp } f. \tag{3.35}$$

Choosing k even provides a local reconstruction since $\mathsf{S}^k \mathsf{R}^* \chi$ is a local operator using only available data and providing the exact local for all the singularities of the original function f.

The tools based on the propagation of singular support are *not* sufficiently precise to address the following problems. Let us assume that f has a given singular support, or even more precisely a given microlocal support. What are the minimal conditions on the set of available data so that f may be reconstructed up to a smooth term? Let us consider a similar problem. Assume that $f = f_1 + f_2$ with f_1 and f_2 having disjoint microlocal support and that the set where $\mathsf{R}f$ is measured captures the singularities of Rf_1 as prescribed by the above propagation of singularity results. Can we find an algorithm that reconstructs the singularities of f_1 and none of f_2?

The answer to both problems can be satisfactorily obtained by the results on propagation of wave front sets. Note that the notion of singular support is not sufficient because the singular support of $\mathsf{R}f_1$ and $\mathsf{R}f_2$ may overlap even if those of f_1 and f_2 do not: it is clear that there are lines (s, θ) such that $s = x_1 \cdot \vartheta$ and $s = x_2 \cdot \vartheta$ so that singularities at disjoint places x_1 and x_2 will propagate to singularities at the same place (s, θ).

Let us recall that the phase and amplitudes for the Radon transform are given by $\varphi = (s - x \cdot \vartheta)\sigma$ and $a = (2\pi)^{-1}$. The WF of the kernel $I_{a,\varphi}$ is thus given by

$$WF'(I_{a,\varphi}) = \{(x \cdot \vartheta, \vartheta, \sigma, -\Pi_\vartheta x \sigma; x, \sigma\vartheta)\} \tag{3.36}$$

using the parametrization of C_φ in $(\vartheta, x; \sigma)$ knowing that $s = x \cdot \vartheta$. Here, Π_ϑ represents the projection onto the hyperplane orthogonal to ϑ.

Using the parametrization of C_φ using the variables $(x, \xi) \in T^*\mathbb{R}^2$, we find

$$WF'(I_{a,\varphi}) = \{(x \cdot \hat{\xi}, \hat{\xi}, |\xi|, -\Pi_{\hat{\xi}} x |\xi|; x, \xi)\}.$$

We know that the canonical relation is a local graph. Identifying (s, ϑ) with $(-s, -\vartheta)$ on Z, we obtain that the canonical relation is a global graph in the variables (x, ξ) (so that the above reflects $p_\varphi \circ q_\varphi^{-1}$). For completeness, let us look at $q_\varphi \circ p_\varphi^{-1}$) and parametrize the manifold by $(s, \vartheta; \sigma, \zeta)$ to get (with $\sigma > 0$)

$$WF'(I_{a,\varphi}) = \{(s, \vartheta, \sigma, \zeta; s\vartheta - \sigma^{-1}\zeta, \sigma\vartheta)\}.$$

All the information about the propagation of micro-local support in the Radon transform is encoded in the above relations. Now, let f be a given source with wavefront set $WF(f)$. Let S_f be an open set in Z that includes all points $(x \cdot \hat{\xi}, \hat{\xi})$ such that $(x, \xi) \in WF(f)$, i.e., an open set including $\Pi_{(s,\vartheta)} WF'(I_{a,\varphi}) WF(f)$, the projection on Z

of $WF'(I_{a,\varphi})WF(f)$. Then we claim that the above set of measurements is essentially optimal for a given $WF(f)$. Indeed, assuming that S_f does not include all such points (s,θ). Then the singularity of f at a point (x,ξ) is *not visible* in the available data set and therefore *cannot* be reconstructed.

Now, for such S_f, we can construct an other open set $\tilde{S}_f$ still including the points $\Pi_{(s,\vartheta)}WF'(I_{a,\varphi})WF(f)$ and such that its closure belongs to S_f (by Urysohn's lemma, there is always room between an open set and a closed subset). Such gimmicks are necessary to guarantee that we can introduce smooth cut-offs. More precisely, still by Urysohn, we can construct a smooth function $p(s,\vartheta)$ such that equal to 1 on $\tilde{S}_f$ and equal to 0 on the complement of S_f. Therefore, our data set allows us to construct $p(s,\vartheta)\mathsf{R}f$. Now

$$\mathsf{R}^{-1}p(s,\vartheta)\mathsf{R}f = f + \mathsf{R}^{-1}(1-p(s,\vartheta))\mathsf{R}f.$$

The latter term is clearly smooth since the singular support of $\mathsf{R}f$ is included $\tilde{S}_f$. So, we have achieved our goal: find the smallest set of measurements (modulo the fact that it needs to be open) cinlduing the projection onto Z of the propagation of singularities in the micro-local support of f. All we have to do then is to apply the inverse Radon transform. The reconstructed image $f+g$ will be very different form f. However, all singularities of f will be reconstructed procisely (both in location and in amplitude) since g is smooth.

Note that we may combine local data and local tomography since

$$\mathsf{S}^k c_n \mathsf{R}^* p(s,\vartheta)\mathsf{R}f = \mathsf{S}^{k-1}f + \mathsf{S}^k c_n(1-p(s,\vartheta))\mathsf{R}f,$$

with S^k local for appropriate values of k tand the second term on the right-hand side still smooth. The mocro-local support of the singularities of f will be reconstructed, though obviously not with the right amplitude.

Let us consider a refinement of the preceding ideas. Assume now that $f = f_1 + f_2$ and that $WF(f_1)\cap WF(f_2) = \emptyset$. We wish to reconstruct the singularities of f_1 and not those of f_2. We also assume that the available data set includes all singularities of f_1. We cannot introduce a local cut-off $p(s,\vartheta)$ since the singular supports of f_1 and f_2 may intersect (think of f_j as indicatrix functions of domains Ω_j such that their boundaries intersect transversally, i.e., at each point of intersection of the boundaries, the normal unit vectors are not colinear).

However, such a separation exists in the phase space (cotangent bundle) by assumption. Define $T_j = WF'(I_{a,\varphi})WFf_j$ the two propagated wavefront sets, which by assumption do not meet. Let us parametrize T^*Z by (y,ζ) (this can be done using two charts for $\mathbb{S}^{n-1}$ and a partition of unity). In these variables, we can now find an amplitude $a(y,\zeta)\in S^0$ such that $a=1$ in T_1 and $a=0$ in T_2 (again by Urysohn). Let P be the microlocal localization operator given by the ΨDO with kernel

$$\int_{\mathbb{R}^n} e^{i(z-y)\cdot\zeta}a(y,\zeta)d\zeta.$$

Let us now compute

$$c_n\mathsf{S}^k\mathsf{R}^*P\mathsf{R}f = \mathsf{S}^{k-1}f_1 + c_n\mathsf{S}^k\mathsf{R}^*P\mathsf{R}f_2 + c_n\mathsf{S}^k\mathsf{R}^*(I-P)\mathsf{R}f_1$$

The second and third terms are smooth since $P\mathsf{R}f_2$ and $(I-P)\mathsf{R}f_1$ are smooth by construction. WIth $k=1$, we reconstruct f_1 up to the addition of a smooth term. Note that the singularities of f_2 are entirely gone. We could make the game a little more complex by choosing $a(y,\zeta)$ a more general function of ζ amplifiying or damping some singularities at will.

Exercise 3.10.1 *Show that provided that the singularites of f_1 propagate to the set of available measurements, we can achieve a reconstruction of $\mathsf{S}^{k-1}f_1$ up to smooth terms by applying $c_nQ\mathsf{S}^l\mathsf{R}^*p(s,\vartheta)$ to the available data, for an appropriate ΨDO Q in the image space (space of positions x).*

It is a general result, called the Egorov theorem, that for A and FIO with canonical relation a graph and P a ΨDO , we can find a ΨDO Q such that $AP-QA$ is a smoothing operator. Moreover, the symbols of P and Q are related to each-other by the canonical relation associated to A; see [33, 36].

Chapter 4

Integral Geometry. Generalized Ray Transform

This chapter focuses on the (weighted) integration of functions along a family of two dimensional curves. This integral geometry problem, which may be seen as a generalization of the Radon transform and the attenuated Radon transform is described in section 4.1. Proving the injectivity of such integral transforms is a difficult problem. We saw in Chapter 2 how the method of unique/analytic continuation was used to obtain the injectivity of the Attenuated Radon transform. In that chapter, we mentioned Jan Boman's example of a weighted Radon transform that was *not* injective. In section 4.3 of this chapter, we present another method based on energy estimates and due to Mukhometov [44] (see also [54]) that shows that the integral transform is injective for a wide class of families of curves.

The integral transforms will be written in the language of the oscillatory integrals and Fourier Integral Operators introduced in the preceding chapter. Because of the explicit structure of the transforms, however, a lot of the theoretical difficulties presented in the preceding chapter without proof can be bypassed. In particular, the composition of FIO will turn out to be explicitly computable. We will also bypass the calculus of elliptic ΨDO by developing the tools we need. We will also revisit the notions of local tomography and partial data as we did for the Radon transform.

4.1 Generalized Ray Transform in two dimensions.

4.1.1 Family of curves.

We consider the following family of curves

$$\mathbb{R} \ni t \mapsto x = \gamma(t, s, \vartheta) \in \mathbb{R}^2, \qquad (s, \vartheta) \in \mathbb{R} \times \mathbb{S}^1. \tag{4.1}$$

We assume that curves are traveled along with unit speed so that $|\frac{d\gamma}{dt}| = 1$. We will make several assumptions on the curves as we proceed.

Define $y = (t, s)$ so that $\gamma = \gamma(y, \vartheta)$. We *assume* that the map $y \mapsto \gamma(y, \vartheta)$ is globally invertible for all values of ϑ with inverse $\tilde{\gamma}$ so that

$$\gamma(\tilde{\gamma}(x, \vartheta), \vartheta) = x, \quad \tilde{\gamma}(\gamma(y, \vartheta), \vartheta) = y.$$

We denote the inverse function

$$\tilde{F}(x,\vartheta) = (t(x,\vartheta), s(x,\vartheta)). \tag{4.2}$$

The function $s(x,\vartheta)$ provides the unique curve to which x belongs for a fixed ϑ, i.e., $x \in F(\mathbb{R}, s(x,\vartheta),\vartheta)$.

For the Radon transform corresponding to integration along straight lines, we have

$$F(t,s,\vartheta) = s\vartheta^{\perp} + t\vartheta, \qquad s(x,\vartheta) = x\cdot\vartheta^{\perp}.$$

This corresponds to a parameterization of the set of lines where ϑ is the vector tangent to the line and $\vartheta^{\perp} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\vartheta$ is the vector ϑ rotated by $\frac{\pi}{2}$. This notation generalizes to the multi-dimensional cases, where ϑ still parameterizes the direction of the lines.

4.1.2 Generalized Ray Transform.

The generalized Ray transform (GRT) is then the integral of functions over curves $(s,\vartheta) \mapsto F(t,s,\vartheta)$:

$$\begin{aligned} Rf(s,\vartheta) &= \int_{\mathbb{R}} f(F(t,s,\vartheta))dt = \int_{\mathbb{R}^2} f(F(t,s_0,\vartheta))\delta(s-s_0)ds_0dt \\ &= \int_{\mathbb{R}^2} f(x)\delta(s-s(x,\vartheta))J(x,\vartheta)dx, \end{aligned} \tag{4.3}$$

where $J(x,\vartheta)$ is the (uniformly positive) Jacobian of the transformation $x \to \tilde{F}(x,\vartheta)$ at fixed ϑ:

$$J(x,\vartheta) := \Big|\frac{d\tilde{F}}{dx}\Big|(x,\vartheta) = \Big|\frac{ds_0dt}{dx}\Big|(x,\vartheta). \tag{4.4}$$

Exercise 4.1.1 *Check the change of variables in detail.*

More generally, we considered weighted GRT of the form

$$R_w f(s,\vartheta) = \int_{\mathbb{R}} f(F(t,s,\vartheta))w(t,s,\vartheta)dt, \tag{4.5}$$

where $w(y,\vartheta)$ is a given, positive, weight. Such integrals are of the same form as before:

$$\begin{aligned} R_w f(s,\vartheta) &= \int_{\mathbb{R}} f(F(t,s,\vartheta))w(t,s,\vartheta)dt = \int_{\mathbb{R}^2} f(F(t,s_0,\vartheta))\delta(s-s_0)w(t,s_0,\vartheta)ds_0dt \\ &= \int_{\mathbb{R}^2} f(x)\delta(s-s(x,\vartheta))J_w(x,\vartheta)dx, \end{aligned}$$

with a different expression for $J(x,\vartheta)$:

$$J(x,\vartheta) \equiv J_w(x,\vartheta) := \Big|\frac{d\tilde{F}}{dx}\Big|(x,\vartheta)w(\tilde{F}(x,\vartheta),\vartheta). \tag{4.6}$$

To simplify, we shall use the notation $J(x,\vartheta)$ rather than $J_w(x,\vartheta)$ and R_J rather than R_w. Thus generally, we consider an operator of the form

$$R_J f(s,\vartheta) = \int_{\mathbb{R}^2} f(x)\delta(s - s(x,\vartheta))J(x,\vartheta)dx, \tag{4.7}$$

where $J(x,\vartheta)$ is a smooth, uniformly positive, and bounded weight.

The objective of this section is to obtain a parametrix for the weighted GRT R_J. A (left) parametrix is an opeartor P_J such that $P_J R_J = I - T_J$, where T_J is a compact operator. Provided that $+1$ is not an eigenvalue of K_J, then the Fredholm alternatives guarantees that $(I - T_J)$ is invertible. This provides the inversion procedure

$$f = (I - T_J)^{-1} P_J (R_J f).$$

A complete characterization of when $+1$ is not an eigenvalue of T_J is not known in general, and Boman's counter-example mentioned in Chapter 2 shows that R_J, and hence $I - T_J$, may not be injective in some situations.

4.1.3 Adjoint operator and rescaled Normal operator

When $J \equiv 1$ and $s(x,\vartheta) = x \cdot \vartheta^{\perp}$, then R_J is the standard Radon transform in two dimensions. We have obtained in Chapter 2 the inversion formula:

$$I = \frac{1}{4\pi} R_J^* \Lambda R_J, \qquad \Lambda = H\frac{\partial}{\partial s}.$$

We thus see the need to introduce the Riesz operator Λ and the adjoint operator R_J^*. In curved geometries, however, no explicit formula such as the one given above can be obtained in general. We no longer have access to the Fourier slice theorem, which uses the invariance by translation of the geometry in the standard Radon transform.

The adjoint operator (AGRT) for the L^2 inner product on $\mathbb{R} \times \mathbb{S}^1$ is defined as

$$R_K^* g(x) = \int_{\mathbb{S}^1} g(s(x,\vartheta),\vartheta)K(x,\vartheta)d\vartheta = \int_{\mathbb{R}\times\mathbb{S}^1} g(s,\vartheta)K(x,\vartheta)\delta(s - s(x,\vartheta))d\vartheta ds. \tag{4.8}$$

The "normal" operator is thus given by

$$R_K^* R_J f(x) = \int_{\mathbb{R}^2\times\mathbb{S}^1} f(y)K(x,\vartheta)J(y,\vartheta)\delta(s(x,\vartheta) - s(y,\vartheta))dyd\vartheta. \tag{4.9}$$

Exercise 4.1.2 *Check this.*

We need to introduce $H\partial_s$ to make the operator invertible from L^2 to L^2 as in the case of the standard Radon transform. A simple way to do so is to recast the operators as Fourier integral operators (FIOs) as follows. We formally recast the GRT and AGRT as the following oscillatory integrals

$$\begin{aligned} R_J f(s,\vartheta) &= \int_{\mathbb{R}^2\times\mathbb{R}} f(x)e^{i(s-s(x,\vartheta))\sigma}J(x,\vartheta)\frac{dxd\sigma}{2\pi} \\ R_K^* g(x) &= \int_{\mathbb{R}\times\mathbb{S}^1\times\mathbb{R}} g(s,\vartheta)e^{-i(s-s(x,\vartheta))\sigma}K(x,\vartheta)\frac{dsd\vartheta d\sigma}{2\pi}. \end{aligned} \tag{4.10}$$

We then introduce the Riesz operator $\Lambda = H\partial_s$ given by the Fourier multiplier $|\sigma|$ in the Fourier domain:

$$\Lambda f(s) = H\partial_s f(s) = \mathcal{F}^{-1}_{\sigma\to s}|\sigma|\mathcal{F}_{s\to\sigma}f(s).$$

We thus have

$$\Lambda R_J f(s,\vartheta) = \int_{\mathbb{R}^2\times\mathbb{R}} f(x)|\sigma|e^{i(s-s(x,\vartheta))\sigma}J(x,\vartheta)\frac{dxd\sigma}{2\pi}.$$

Exercise 4.1.3 *Check this.*

The "normal" operator for the weights J and K is then defined as

$$Ff(x) := R^*_K\Lambda R_J f(x) = \int_{\mathbb{R}^2\times\mathbb{S}^1\times\mathbb{R}} f(y)|\sigma|e^{i(s(y,\vartheta)-s(x,\vartheta))\sigma}K(x,\vartheta)J(y,\vartheta)\frac{dyd\sigma d\vartheta}{2\pi}. \quad (4.11)$$

It remains to analyze such an operator. We shall have two main objectives: (i) find the appropriate value for $K(x,\vartheta)$ that makes F an approximation of identity; and (ii) see how R_J can be inverted in some specific cases.

We first observe that $Ff(x)$ is real valued if we choose J and K to be real-valued. The contribution from $\sigma > 0$ is the same as that from $\sigma < 0$ by complex-conjugating (4.11). Thus,

$$Ff(x) = \int_{\mathbb{R}^2\times\mathbb{S}^1\times\mathbb{R}_+} f(y)e^{i(s(y,\vartheta)-s(x,\vartheta))\sigma}K(x,\vartheta)J(y,\vartheta)\frac{dy\,\sigma d\sigma d\vartheta}{\pi}. \quad (4.12)$$

The variables (ϑ,σ) may be recast as $\zeta = \sigma\vartheta$ in $\mathbb{R}^2$ so that $\hat{\zeta} = \vartheta$ and $|\zeta| = \sigma$. We then recast the above operator as

$$Ff(x) = \int_{\mathbb{R}^2\times\mathbb{R}^2} f(y)e^{i(s(y,\hat{\zeta})-s(x,\hat{\zeta}))|\zeta|}K(x,\hat{\zeta})J(y,\hat{\zeta})\frac{dyd\zeta}{\pi}. \quad (4.13)$$

The phase and the amplitude are then defined as

$$\varphi(x,y,\zeta) = \big(s(x,\hat{\zeta}) - s(y,\hat{\zeta})\big)|\zeta|, \qquad a(x,y,\hat{\zeta}) = \frac{1}{\pi}K(x,\hat{\zeta})J(y,\hat{\zeta}). \quad (4.14)$$

The phase is homogeneous of degree 1 in ζ since $\phi(x,y,t\zeta) = t\phi(x,y,\zeta)$ for $t > 0$. The amplitude is here homogeneous of degree 0. Let us assume that $f(y)$ is compactly supported in the open domain $X \subset \mathbb{R}^n$ for $n = 2$ and without loss of generality that $Ff(x)$ is also reconstructed on X

We may thus recast the above operator as

$$Ff(x) = \int_{X\times\mathbb{R}^n} e^{i\varphi(x,y,\zeta)}a(x,y,\zeta)f(y)dyd\zeta, \qquad \varphi(x,y,\zeta) = \phi(x,\zeta) - \phi(y,\zeta) \quad (4.15)$$

for $\phi(x,\zeta)$ a phase function and $a(x,y,\zeta) \in S^m(X\times X\times\mathbb{R}^n)$ with $m = 0$ in the application to GRT.

Our first objective is to prove that for an appropriate choice of $K(x,\vartheta)$, then F may be decomposed as $F = I - T$ where T is a compact, smoothing, operator. This provides an approximate inversion to the generalized Radon transform and an iterative exact reconstruction procedure in some cases. In general, however, it is difficult to show that T does not admit 1 as an eigenvalue. This property is expected to hold generically. But it may not hold or it may not be known how to prove it for specific transforms of interest.

A second objective is therefore to look at the operator $N = R_J^* \Lambda R_J$, which is a self-adjoint (normal) operator. We shall show that N is injective in some cases of interest (using the Mukhometov technique) and that $QN = I - T$ for some operator Q and compact operator T. This will allow us to prove that N is an invertible operator in L^2. The generalized transform can then be inverted by means of, e.g., a conjugate gradient algorithm. This provides an explicit reconstruction procedure that is guaranteed to provide the correct inverse.

4.2 Pseudo-differential operators and GRT

We now analyze the operator F appearing in the analysis of the GRT. Note that the operator is defined by means of the oscillatory integrals studied in section 3.1. We will use the machinery developed in that section and expand it as necessary. The first objective is to show that F is in fact a ΨDO up to a smooth perturbation for the specific phase φ given in (4.14).

4.2.1 Change of phase $(x-y)\cdot\xi$

We start by showing that singularities of $f(x)$ at a point x_0 do not propagate to singularities in $Ff(x)$ at any other point than x_0 under general assumptions on the phase.

More specifically, let us *assume* that

$$\varphi'_\zeta(x,y,\zeta) = 0 \quad \textit{implies that} \quad x = y. \tag{4.16}$$

Exercise 4.2.1 *Show that the above constraint is satisfied for the Radon transform. Show that this constraint is still satisfied for curves Γ sufficiently close to the straight lines and for weights w sufficiently close to* 1.

Let then $\chi_0(x,y) = 1$ when $x = y$ and supported in the vicinity of $x = y$ in the sense that $\chi_0(x,y) = 0$ when $|x-y| > \delta$ for δ to be chosen arbitrarily small. Also we assume that $\chi_0 \in C^\infty(X \times Y)$. We define

$$F_0 f(x) = \int_{X\times\mathbb{R}^n} f(y) e^{i\varphi(x,y,\zeta)} a(x,y,\zeta)\chi_0(x,y)dyd\zeta, \qquad F_1 := F - F_0. \tag{4.17}$$

Since the critical set of ϕ is the diagonal $\{x = y\}$, we deduce from (3.20) in Theorem 3.4.3 that F_1 maps $C_0^\infty(X)$ to $C^\infty(X)$.

We now want to show that F_0 is a pseudo-differential operator, namely that after an appropriate change of variables, the phase $\varphi(x,y,\zeta)$ can in fact be recast as $(x-y)\cdot\xi$. Let us consider the identity

$$\phi(x,\zeta)-\phi(y,\zeta)=(x-y)\cdot\int_0^1\phi'_x\big(tx+(1-t)y,\zeta\big)dt.$$

For $x\in X$ given and $y\in X$ sufficiently close to x, we thus wish to define the change of variables

$$\xi=\xi(\zeta;x,y)=\int_0^1\phi'_x\big(tx+(1-t)y,\zeta\big)dt\quad\text{ defined from }\dot{\mathbb{R}}^n\text{ to }\dot{\mathbb{R}}^n. \tag{4.18}$$

We *assume* that

$$\Big|\det\frac{\partial^2\phi}{\partial x\partial\zeta}(x,x,\zeta)\Big|>0\text{ for all }\zeta\in\dot{\mathbb{R}}^n\text{ and }x\in X. \tag{4.19}$$

Exercise 4.2.2 *For $\phi(x,\zeta)=s(x,\hat\zeta)|\zeta|$, show that the above assumption is verified when the family of lines $\mathcal{F}$ is sufficiently close to the straight lines.*

Then by continuity and for y sufficiently close to x, we also obtain that

$$L(x,y,\zeta):=\Big|\det\frac{\partial\xi(\zeta;x,y)}{\partial\zeta}\Big|>0\text{ for all }\zeta\in\dot{\mathbb{R}}^n. \tag{4.20}$$

This implies that $\zeta\mapsto\xi(\zeta)$ is a diffeomorphism (global change of variables with continuous inverse) on $\dot{\mathbb{R}}^n$ with inverse $\xi\mapsto\zeta(\xi)$. By construction, $\xi(\zeta)$ is homogeneous of degree one so that $L(x,y,\zeta)$ is homogeneous of degree 0 in ζ.

Exercise 4.2.3 *Check that $\xi=\zeta^\perp$ and that $L(x,y,\zeta)=1$ when $\mathcal{F}(t,x,\vartheta)=s\vartheta^\perp+t\vartheta$.*

With this we recast the operator F_0 as

$$F_0f(x)=\int_{X\times\mathbb{R}^n}f(y)e^{i(x-y)\cdot\xi}M(x,y,\xi)dyd\xi, \tag{4.21}$$

with

$$M(x,y,\xi)=a(x,y,\zeta(\xi))L(x,y,\zeta(\xi))\chi_0(x,y).$$

Note that $M(x,y,\xi)$ is a symbol of class $S^m(X\times X\times\mathbb{R}^n)$ when a is. We thus obtain that F_0 is a ΨDO and that F may be written as a ΨDO modulo a smooth operator.

4.2.2 Choice of a parametrix.

Let us come back to the operator F_0 as it appear in GRT. We then find that

$$M(x,y,\xi)=\frac{1}{\pi}K(x,\hat\zeta(\xi))J(y,\hat\zeta(\xi))L(x,y,\zeta(\xi)),$$

which is a symbol homogeneous of degree 0 in ξ. Recall that $J(x,\vartheta)$ is given by the problem of interest while $K=K(x,\vartheta)$ is a kernel that we can choose. A natural way to

address the inversion of R_J is to choose $K(x,\vartheta)$ so that the above operator is as close to the identity operator as possible.

We have already shown that all singularities of the kernel of the above oscillatory integral were located on the diagonal $x=y$. Since K depends only on x and not on (x,y), we thus define

$$K(x,\hat{\zeta}) = \frac{1}{4\pi J(x,\hat{\zeta})L(x,x,\hat{\zeta})}, \tag{4.22}$$

since $L(x,x,\zeta)$ is homogeneous of degree 0 in ζ. With this definition, we obtain that $M(x,x,\hat{\xi}) = (2\pi)^{-2}$. In other words, we have

$$\begin{aligned} F_0 f(x) &= f(x) - T_0 f(x), \\ T_0 f(x) &:= \int_{X\times\mathbb{R}^2} f(y) e^{i(x-y)\cdot\xi}\big(M(x,x,\hat{\xi}) - M(x,y,\hat{\xi})\big)dyd\xi. \end{aligned} \tag{4.23}$$

We will prove in detail in the next section that T_0 is a compact operator. This will show that $R_K^*\Lambda$ is an inverse of R_J up to a remainder that is a compact operator, in other words that $R_K^*\Lambda R_J = I - T$ where T is a compact operator since we already know that F_1 is a compact operator. This means that $R_K^*\Lambda$ is a *left-parametrix* for the generalized Radon transform R_J.

4.2.3 Change of symbol, smoothing and composition

We will prove that T_0 is a compact operator in L^2, and in fact an operator mapping $L^2(X)$ to $H^1(X)$. This will be proved by showing that T_0 and $\partial_{x_j} T_0$ for $j=1,2$ are bounded operators from $L^2(X)$ to $L^2(X)$.

Such results hold in fact for arbitrary ΨDO and we consider these results in detail now. Let

$$Pf(x) = \int_{X\times\mathbb{R}^n} e^{i(x-y)\cdot\xi} a(x,y,\xi) f(y) dyd\xi, \tag{4.24}$$

with $a \in S^m(X\times X\times\mathbb{R}^n)$. Define P_x as

$$P_x f(x) = \int_{X\times\mathbb{R}^n} e^{i(x-y)\cdot\xi} a(x,x,\xi) f(y) dyd\xi. \tag{4.25}$$

Then we have the following result

Lemma 4.2.1 *Let P and P_x be the ΨDO defined above for $a \in S^m(X\times X\times\mathbb{R}^n)$. Then $P - P_x$ is a ΨDO with a symbol of order $m-1$.*

Proof. We write

$$\begin{aligned} (P-P_x)f(x) &= \sum_{j=1}^n \int e^{i(x-y)\cdot\xi}(-i(x-y)_j)\Big(i\int_0^1 \partial_{y_j} a(x,tx+(1-t)y,\xi)dt\Big) f(y)dyd\xi \\ &= \sum_{j=1}^n \int e^{i(x-y)\cdot\xi}\Big(i\int_0^1 \partial^2_{y_j\xi_j} a(x,tx+(1-t)y,\xi)dt\Big) f(y)dyd\xi \end{aligned}$$

Define $\alpha(x,y,\xi) = \left(i \int_0^1 \partial^2_{y_j \xi_j} a(x, tx + (1-t)y, \xi) dt \right)$. We observe that it is a symbol of degree $m-1$, which proves our result. □

The definition of P_x may be replaced by that of P_y with a symbol $a(y,y,\xi)$ independent of x. The same proof as above shows that $P - P_y$ is also a ΨDO of order $m-1$. Iterating the process k times, we obtain the following result:

Corollary 4.2.2 *Let P be a ΨDO with $a \in S^m(X \times X \times \mathbb{R}^n)$ for a bounded domain $X \subset \mathbb{R}^n$. Then for each $k \in \mathbb{N}$, there exist ΨDO P_X and P_Y of order m with symbols $b_X, b_Y \in S^m(X \times X \times \mathbb{R}^n)$ with b_X independent of the variable y and b_Y independent of the variable x, such that $P - P_X$ and $P - P_Y$ are of order $m-k$.*

When $m - k < -n - j$, then we observe that Pf maps $C_0^0(X)$ to $C^j(X)$, or $L^2(X)$ to $H_0^j(X)$, the completion of $C_0^\infty(X)$ for the H^j norm. By duality Pf also maps $\mathcal{E}'^j(X)$ to $\mathcal{D}'^0(X)$ and $H^{-j}(X)$ to $L^2(X)$. This proves the

Corollary 4.2.3 *Let P be a ΨDO with $a \in S^m(X \times X \times \mathbb{R}^n)$ for a bounded domain $X \subset \mathbb{R}^n$. Then for each $j \in \mathbb{N}$, there exist ΨDO P_X and P_Y of order m with symbols $b_X, b_Y \in S^m(X \times X \times \mathbb{R}^n)$ with $b_X = b_X(x,\theta)$ independent of the variable y and $b_Y = b_Y(y,\theta)$ independent of the variable x, such that $P - P_X$ and $P - P_Y$ map $C_0^0(X)$ to $C^j(X)$ and $\mathcal{E}'^j(X)$ to $\mathcal{D}'^0(X)$.*

Exercise 4.2.4 Let P and P_X be defined as in (4.24) with $a \in S^m(X \times X \times \mathbb{R}^n)$ and (4.25). Using Remark 3.2.4 and Exercise 3.3.1, show that the operator $\partial_{x_j}(P - P_X)$ is a ΨDO of order m.

Prove this result directly by modifying the proof of Lemma 4.2.1.

We now come to the notion of composition of operators, here principally of pseudo-differential operators. This obviously requires that the range of the first operator be in the domain of definition of the second operator. One way to ensure this is to require that all operators but one in a chain of composition be *properly supported*.

Definition 4.2.4 (Properly supported operator) *An operator $A: C_0^\infty(Y) \to \mathcal{D}'(X)$ is properly supported if the support of its Schwartz kernel* supp K_A *is* proper. *A closed set in $X \times Y$ is proper if the two projections $(x,y) \mapsto x$ and $(x,y) \mapsto y$ are proper maps. A map is proper if the inverse image of every compact subset is compact.*

Heuristically, a map A is proper if for a function u with compact support in Y, there is a vicinity of the boundary of ∂X not in the support of Au. In that case, the operator $A: C_0^\infty(Y) \to \mathcal{E}'(X)$ is then continuous. Similarly, a properly supported map continuous from $C_0^k(Y) \to C^j(X)$ can be extended as a continuous operator from $C_0^k(Y) \to C_0^j(X)$ and $C^k(Y) \to C^j(X)$; a properly supported map continuous from $\mathcal{E}'(Y) \to \mathcal{D}'(X)$ can be extended as a continuous operator from $\mathcal{E}'(Y) \to \mathcal{E}'(X)$ and $\mathcal{D}'(Y) \to \mathcal{D}'(X)$.

If a ΨDO is properly supported, then it is continuous as an operator from $C_0^\infty(X) \to C_0^\infty(X)$, from $C^\infty(X) \to C^\infty(X)$, from $\mathcal{E}'(X) \to \mathcal{E}'(X)$, and from $\mathcal{D}'(X) \to \mathcal{D}'(X)$. If A and B are two ΨDO and one is properly supported, then the compositions AB and BA are well-defined continuous operators from $C_0^\infty(X) \to C^\infty(X)$ and $\mathcal{E}'(X) \to \mathcal{D}'(X)$. The main question is whether AB and BA are ΨDO . It turns out that they are modulo a smooth perturbation [33]. What we show is a slightly weaker statement, which is however sufficient in most practical situations.

Theorem 4.2.5 (Composition of ΨDO) *Let A and B be two ΨDO of order m_A and m_B, respectively, with one of them properly supported. Then for each $(j,k) \in \mathbb{N}^2$, there are operators R, R' continuous from $\mathcal{E}'^j(X)$ to $C^k(X)$ such that $AB = C + R$ and $BA = C' + R'$ with C and C' ΨDO of order $m_A + m_B$.*

Proof. We consider the case AB for concreteness with B properly supported. According to Corollary 4.2.2, there are ΨDO A_X and B_Y such that $A - A_X$ and $B - B_Y$ are ΨDO with order n_A and n_B as negative as we want. Following the results of Remark 3.3.2, we thus obtain for $n_A, n_B < -N - l$ that $A - A_X$ and $B - B_Y$ map $\mathcal{E}'^l(X)$ to $C^l(X)$. Moreover A_X has a symbol $a_X(x,\xi)$ independent of y and B_Y has a symbol $b_Y(y,\xi)$ independent of x. Thus A_X and B_Y are properly supported. We write

$$AB = A_X B_Y + (A - A_X)B + A_X(B - B_Y) = A_X B_Y + A(B - B_Y) + (A - A_X)B_Y,$$

and use the first equality for B properly supported, where all products are defined in appropriate spaces. Both A_X and B are ΨDO mapping $\mathcal{E}'^p(X)$ to $\mathcal{E}'^q(X)$ according to Theorem 3.3.1 for $q > \sup(m_A, m_B) + N + p$ since they are properly supported. As a consequence, choosing l large enough, we obtain that $R = (A - A_X)B + A_X(B - B_Y)$ is continuous from $\mathcal{E}'^j(X)$ to $C^k(X)$.

It remains to show that $C = A_X B_Y$ is a ΨDO of order $m_A + m_B$. Since a_X is independent of y, we have

$$A_X u(x) = \int e^{ix\cdot\xi} a_X(x,\xi)\hat{u}(\xi)d\xi, \qquad \widehat{B_Y u}(\xi) = \int e^{-iy\cdot\xi} b_Y(y,\xi)u(y)dy.$$

As a consequence,

$$A_X B_Y u(x) = \int e^{i(x-y)\cdot\xi} a_X(x,\xi) b_Y(y,\xi)u(y)dyd\xi. \tag{4.26}$$

This shows that C is a ΨDO with a symbol $a_X(x,\xi)\ b_Y(y,\xi)$ of order $m_A + m_B$. □

4.2.4 Continuity of ΨDO in the spaces $H^s(X)$

Consider a ΨDO with symbol of order 0. We prove the continuity result:

Theorem 4.2.6 *Let P be the ΨDO defined in (4.24) with $a(x,y,\hat{\xi}) \in S^0(X \times X \times \mathbb{R}^n)$ with X bounded. Then P is continuous from $L^2(X)$ to $L^2(X)$ with a constant C such that*

$$\|Pf\|_{L^2(X)} \leq C\|f\|_{L^2(X)}. \tag{4.27}$$

Proof. We first note that $P - P_X$ is an operator with a smooth kernel on the bounded domain X and is therefore bounded in $L^2(X)$. We may therefore consider that the symbol $a = a(x,\xi)$. Let us extend f by 0 outside of X. Then we may write

$$Pf(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi)\hat{f}(\xi)d\xi.$$

The objective is now to replace a by functions that are independent of x following [55]. Consider the case $n = 2$, with obvious generalizations to higher dimensions. Then we write

$$a(x,\xi) = a(0,\xi) + \int_0^{x_1} \partial_{x_1} a(t_1,0,\xi)dt_1 + \int_0^{x_2} \partial_{x_2} a(0,t_2,\xi)dt_2 + \int_0^{x_1}\int_0^{x_2} \partial^2_{x_1x_2} a(t_1,t_2,\xi)dt_1dt_2.$$

Consider the operator

$$P_1 f(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} \int_0^{x_1} \partial_{x_1} a(t_1, 0, \xi) dt_1 \hat{f}(\xi) d\xi = \int_0^{x_1} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \partial_{x_1} a(t_1, 0, \xi) \hat{f}(\xi) d\xi dt_1,$$

by Fubini and (3.10). Then by the Minkowski inequality, we find

$$\|P_1 f\| \leq \int_0^{x_1} \Big\| \int_{\mathbb{R}^n} e^{ix\cdot\xi} \partial_{x_1} a(t_1, 0, \xi) \hat{f}(\xi) d\xi \Big\| dt_1.$$

With $Q(\xi) = \partial_{x_1} a(t_1, 0, \xi)$ or any other term appearing in the expansion of $a(x, \xi)$ above, which are all symbols of order 0, it thus remains to show that

$$\Big\| \int_{\mathbb{R}^n} e^{ix\cdot\xi} Q(\xi) \hat{f}(\xi) d\xi \Big\|_{L^2_x(\mathbb{R}^n)} \leq C \|\hat{f}\|_{L^2_\xi(\mathbb{R}^n)} = C(2\pi)^n \|f\|_{L^2(X)}.$$

By Plancherel and the uniform bound of $Q(\xi) \in S^0(\mathbb{R}^n)$, we find

$$\Big\| \int_{\mathbb{R}^n} e^{ix\cdot\xi} Q(\xi) \hat{f}(\xi) d\xi \Big\|_{L^2_x(\mathbb{R}^n)} = \|Q(\xi)\hat{f}(\xi)\|_{L^2_\xi(\mathbb{R}^n)} \leq C \|\hat{f}(\xi)\|_{L^2_\xi(\mathbb{R}^n)},$$

which concludes the proof. □

Corollary 4.2.7 *Let P be the ΨDO defined in (4.24) with $a(x, y, \hat{\xi}) \in S^m(X \times X \times \mathbb{R}^n)$. We assume that P is properly supported. Then P is continuous from $H^s_{\text{loc}}(X) \to H^{s+m}_{\text{loc}}(X)$ and from $H^s_{\text{comp}}(X) \to H^{s+m}_{\text{comp}}(X)$ for all $s \in \mathbb{R}$.*

Proof. We follow [33]. We prove the continuity from $H^s_{\text{loc}}(X) \to H^{s+m}_{\text{loc}}(X)$. The continuity from $H^s_{\text{comp}}(X) \to H^{s+m}_{\text{comp}}(X)$ follows since P is properly supported. We thus have to prove that for $\varphi \in C_0^\infty(X)$, then φP is continuous from $H^s_{\text{loc}}(X) \to H^{s+m}_{\text{loc}}(\mathbb{R}^n)$. Let now $\psi \in C_0^\infty(X)$ be such that $\psi = 1$ in the vicinity of the points x such that there is a $y \in \text{supp } \varphi$ and $(x, y) \in \text{supp K(x, y)}$, the Schwartz kernel of P. Then $\varphi P = \varphi P \psi$ and it suffices to prove that the latter operator is bounded from $H^s(\mathbb{R}^n) \to H^{s+m}(\mathbb{R}^n)$.

Treating then $R = \varphi P \psi$ as in the proof of Theorem 4.2.6, we replace it by R_X with $R - R_X$ bounded from $H^s(\mathbb{R}^n) \to H^{s+m}(\mathbb{R}^n)$ since its Schwartz kernel is smooth and compactly supported. Next, the symbol of R_X is decomposed as $a(x, \xi)$ is in that proof, and the result then follows from estimates of the form

$$B_Q = \Big\| \int_{\mathbb{R}^n} e^{ix\cdot\xi} Q(\xi) \hat{f}(\xi) d\xi \Big\|^2_{H^s(\mathbb{R}^n)} \leq C \|f\|^2_{H^{s+m}(\mathbb{R}^n)}, \tag{4.28}$$

for a symbol $Q(\xi) \in S^m(\mathbb{R}^n)$ and hence so that $|Q(\xi)| \leq C(1 + |\xi|^2)^{\frac{m}{2}}$. We find

$$B_Q = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (1+|\xi|^2)^s |Q(\xi)\hat{f}(\xi)|^2 d\xi \leq \frac{C}{(2\pi)^n} \int_{\mathbb{R}^n} (1+|\xi|^2)^{s+m} |\hat{f}(\xi)|^2 d\xi = C\|f\|^2_{H^{s+m}(\mathbb{R}^n)}.$$

This proves the result. □

Coming back to the generalized Radon transform, we have shown that T_0 was bounded from $L^2(X)$ to $H^1(X)$ since T^0 is a ΨDO with symbol in $S^{-1}(X \times X \times \mathbb{R}^n)$ according to Lemma 4.2.1 and $\partial_{x_j} T^0$ is a ΨDO with symbol in $S^0(X \times X \times \mathbb{R}^n)$ according to Exercise 3.3.1 so that Theorem 4.2.6 applies. This also shows that $F = I - T_0 + F_1$, with F_1 a smoothing operator, is bounded from $L^2(X)$ to $L^2(X)$. We summarize these results as follows:

$$\|T_0 f(x)\|_{H^1(X)} + \|Ff(x)\|_{L^2(X)} \leq C\|f(x)\|_{L^2(X)}. \tag{4.29}$$

4.2.5 Left-parametrices and injectivity

The above result states that $F = I - T$ with T compact from $L^2(X)$ to $L^2(X)$ since the injection $i : H^1(X) \to L^2(X)$ is compact. We may therefore regard $R_K^* \Lambda$ in (4.11) as a *left-parametrix*, i.e., an inversion of R_J up to a compact perturbation.

We are in fact in a position to obtain a left-parametrix up to a compact perturbation that is as regularizing as we wish. For this, we introduce the notion of *elliptic* ΨDO .

Definition 4.2.8 *We say that a ΨDO P with symbol $a(x, y, \xi) \in S^m(X \times X \times \mathbb{R}^n)$ is elliptic if there exist two positive constants C_1 and C_2 such that $|a(x, y, \xi)| \geq C_1 |\xi|^m$ uniformly in $(x, y) \in X \times X$ for $|\xi| > C_2$.*

Then we have the following result:

Theorem 4.2.9 (Parametrix for ΨDO) *Let A be an elliptic properly supported ΨDO of order m. Then for all $(j, k) \in \mathbb{N}^2$, there is a properly supported ΨDO B of order $-m$ such that $BA = I + R$ with R bounded from $\mathcal{E}'^j(X)$ to $C^k(X)$. In fact, $AB = I + R'$ with R' bounded from $\mathcal{E}'^j(X)$ to $C^k(X)$ so that B is a* parametrix *of A.*

Proof. Since A is elliptic, up to a smooth perturbation, we may assume that $|a(x, y, \xi)| \geq C_1 |\xi|^m$ $(x, y, \xi) \in X \times X \times \mathbb{R}^n$. We still call A the resulting operator. We then define the properly supported operator B with symbol $b(x, y, \xi) = (a_Y(y, \xi))^{-1} \in S^{-m}(X \times X \times \mathbb{R}^n)$, where $a_Y(y, \xi) = a(y, y, \xi)$ is the symbol of the operator A_Y defined as in the proof of Theorem 4.2.5 so that $A - A_X$ maps $\mathcal{E}'^l(X)$ to $C^l(X)$ for l sufficiently large. We obtain that $BA_X = I$ and that $R = B(A - A_X)$ is as smooth as indicated for l chosen large enough. □

For the GRT, the operator F defined in (4.11) is an elliptic ΨDO of degree 0 so long as $K(x, \theta) J(y, \theta) \geq C$ uniformly on $X \times X \times \mathbb{R}^2$. This is the case for the specific choice of K in (4.22), in which case we have seen that $F = I - T$ with T a compact operator. The result in Theorem 4.2.9 is more precise as it implies the existence of a ΨDO Q such that $QF = I - R$ with R a compact perturbation mapping $L^2(X)$ to any $H^s(X)$ for $s \geq 0$.

Other choices for K may also be possible. An interesting choice is $K(x, \hat{\zeta}) = J(x, \hat{\zeta})$, in which case the operator $N = R_J^* \Lambda R_J$ is self-adjoint since $\Lambda = \Lambda^{\frac{1}{2}} \Lambda^{\frac{1}{2}}$ with $\Lambda^{\frac{1}{2}}$ a self-adjoint operator. Theorem 4.2.9 again guarantees the existence of a left-parametrix M, again a ΨDO of order 0, so that $MN = I - S$ with S a smoothing compact perturbation.

However, what these results do *not* provide is an injectivity result, namely a result stating that $R_J f$ uniquely characterizes f. All we get is the existence of (many) left-parametrices Q_J so that $Q_J R_J = I - T_J$ with T_J a compact perturbation that obviously depends on the choice of Q_J. All we know is that F is injective if 1 is not an eigenvalue of T. Even if 1 is an eigenvalue of T, the operator F can be inverted on the complement of a finite dimensional linear space corresponding to the singular value 1 (the eigenvectors associated to 1 of the operator T^m for m sufficiently large).

It is not know how to prove that F is invertible for all (reasonable) choices of $\mathcal{F}(t, s, \vartheta)$. Of course, we know that $T_J = 0$ for the Radon transform for the right choice of Q_J. Therefore, when $\mathcal{F}(t, s, \vartheta)$ is close to $s\vartheta^\perp + t\vartheta$, then by continuity of all involved parameters with respect to changes in $\mathcal{F}(t, s, \vartheta)$, T_J is of norm less than 1 and then $(I - T_J)^{-1} Q_J$ is a bounded left-inverse for R_J (we do not need T_J to be compact).

It is in fact known that for $J(x,\theta)$ (real) analytic in its variables, as is the case for $\mathcal{F}(t,s,\vartheta) = s\vartheta^{\perp} + t\vartheta$, then R_J is indeed injective [56].

4.3 Kinematic Inverse Source Problem

We now present an entirely different technique of energy estimates developed by Mukhometov showing that $R = R_J$ defined above is injective in the case where the weight $w \equiv 1$ and the curves are parameterized so that $|\dot{\mathcal{F}}| = 1$, i.e., curves are traveled along with speed equal to 1.

4.3.1 Transport equation

We consider a bounded domain $X \subset \mathbb{R}^2$ with smooth surface ∂X parameterized by $0 \leq \tau \leq T$ and points $x = S(\tau)$ with $S(0) = S(T)$ and $|\dot{S}(\tau)| = 1$.

For a point x in $\bar{X}$ and $0 \leq \tau \leq T$, we denote by $\tilde{\mathcal{F}}(x,\tau)$ the unique curve joining x and $S(\tau)$. For a function f supported in X, we define the curve integrals

$$g(\tau_1,\tau_2) = \int_{\tilde{\mathcal{F}}(S(\tau_1),\tau_2)} f\,dt, \tag{4.30}$$

where $dt = \sqrt{dx^2 + dy^2}$ is the Lebesgue distance measure along the curve. We thus travel along the curve with speed equal to 1.

We assume $g(\tau_1,\tau_2)$ known for all $0 \leq \tau_1, \tau_2 \leq T$, which corresponds to the curve integrals of f for all possible curves in the family passing through X. We then have the following result:

Theorem 4.3.1 *Under the above hypotheses for the family of curves $\mathcal{F}$, a function $f \in C^2(\overline{X})$ is uniquely determined by its integrals $g(\tau_1,\tau_2)$ given by (4.30) along the curves of $\mathcal{F}$. Moreover we have the stability estimate*

$$\|f\|_{L^2(X)} \leq C\|\frac{\partial g(\tau_1,\tau_2)}{\partial \tau_1}\|_{L^2((0,T)\times(0,T))}. \tag{4.31}$$

The rest of the section is devoted to the proof of this theorem. The proof of injectivity of the reconstruction of f from knowledge of g is based on analyzing the following transport equation. We introduce the function

$$u(x,\tau) = \int_{\tilde{\mathcal{F}}(x,\tau)} f\,dt \tag{4.32}$$

for $x \in \bar{X}$. We denote by $\vartheta(x,\tau)$ the unit tangent vector to the curve $\tilde{\mathcal{F}}(x,\tau)$ at x and orientated such that

$$\vartheta(x,\tau)\cdot\nabla u(x,\tau) = f(x), \qquad \vartheta(x,\tau) = \begin{pmatrix}\cos\phi(x,\tau)\\ \sin\phi(x,\tau)\end{pmatrix}. \tag{4.33}$$

The latter relation is obtained by differentiating (4.32) with respect to arc length.

Exercise 4.3.1 *Check this.*

4.3.2 Variational form and energy estimates

We now differentiate the above with respect to τ and obtain

$$\frac{\partial}{\partial\tau}\big(\vartheta\cdot\nabla u\big)=0. \tag{4.34}$$

We find that

$$\frac{\partial}{\partial\tau}\vartheta=\phi_\tau J\vartheta,\qquad J=\begin{pmatrix}0&-1\\1&0\end{pmatrix},\qquad \phi_\tau:=\partial_\tau\phi.$$

We calculate

$$J\vartheta\cdot\nabla u\frac{\partial}{\partial\tau}\vartheta\cdot\nabla u=\phi_\tau(J\vartheta\cdot\nabla u)^2+J\vartheta\cdot\nabla u\,\vartheta\cdot\nabla u_\tau$$

with $u_\tau=\partial_\tau u$. Similarly, we have

$$-\vartheta\cdot\nabla u\frac{\partial}{\partial\tau}J\vartheta\cdot\nabla u=\phi_\tau(\vartheta\cdot\nabla u)^2-\vartheta\cdot\nabla u\,J\vartheta\cdot\nabla u_\tau.$$

Upon adding these two identities and using (4.34), we obtain

$$\begin{aligned}-\frac{\partial}{\partial\tau}\Big(J\vartheta\cdot\nabla u\,\vartheta\cdot\nabla u\Big)&=\phi_\tau|\nabla u|^2+J\vartheta\cdot\nabla u\,\vartheta\cdot\nabla u_\tau-\vartheta\cdot\nabla u\,J\vartheta\cdot\nabla u_\tau\\&=\phi_\tau|\nabla u|^2+\nabla\cdot(J\nabla u\,u_\tau).\end{aligned}$$

Indeed, denoting by $R=(\vartheta|J\vartheta)$ the rotation matrix and $T=(\nabla u|\nabla u_\tau):=(a|b)$, we find that

$$\vartheta\cdot aJ\vartheta\cdot b-J\vartheta\cdot a\vartheta\cdot b=\det(R^tT)=\det(R)\mathrm{det}\mathrm{T}=\mathrm{det}\mathrm{T}=Ja\cdot b,$$

independent of ϑ. This little miracle occurs in dimension $n=2$. For $a=\nabla u$ and $b=\nabla u_\tau$, this gives $J\nabla u\cdot\nabla u_\tau=\nabla\cdot(J\nabla u\,u_\tau)$ since $\nabla\cdot J\nabla=0$. It remains to integrate over $X\times(0,T)$ and use the fact that $S(0)=S(T)$ on the surface of X to obtain that

$$\int_0^T\!\!\int_X\phi_\tau|\nabla u|^2dxd\tau=\int_0^T\!\!\int_{\partial X}\nabla u\cdot Jn(x,\tau)u_\tau(x,\tau)d\Sigma(x)d\tau, \tag{4.35}$$

where n is the outward unit normal to X at $x\in\partial X$ and $d\Sigma(x)$ the surface (length) measure on ∂X. Now, $S(\tau')$ at the surface has tangent vector $\dot S(\tau')d\tau'=-Jn(\tau')d\Sigma(x)$ assuming the parameterization $S(\tau)$ counter-clock-wise. Since $u(S(\tau'),\tau)=g(\tau',\tau)$, we find that $\nabla u\cdot\dot S(\tau')=\partial_{\tau'}g(\tau',\tau)$ and $u_\tau(x,\tau)=\partial_\tau g(\tau',\tau)$ so that eventually,

$$\int_0^T\!\!\int_X\phi_\tau(x,\tau)|\nabla u|^2(x,\tau)dxd\tau=-\int_0^T\!\!\int_0^T\frac{\partial}{\partial\tau_1}g(\tau_1,\tau_2)\frac{\partial}{\partial\tau_2}g(\tau_1,\tau_2)d\tau_1d\tau_2. \tag{4.36}$$

From the definition of τ and ϕ_τ, we observe that

$$\det(\vartheta|\partial_\tau\vartheta)=\phi_\tau\det(\vartheta|J\vartheta)=\phi_\tau.$$

The *assumption* we make on the family of curves is such that the vector $\partial_\tau\vartheta$ cannot vanish and cannot be parallel to ϑ. In the choice of orientation of S, we find that

$$\phi_\tau>0. \tag{4.37}$$

Note that this is a non-local assumption on the curves. It states that the curves passing by a point x separate sufficiently rapidly in the sense that ϕ increases sufficiently rapidly as the boundary parameter τ increases.

4.3.3 Injectivity result

Since $|f(x)| \leq |\nabla u(x,\tau)|$ from the definition of the transport equation and ϕ_τ integrates to 2π in τ, we find that

$$2\pi \int_X |f(x)|^2 dx = \int_0^T \int_X \phi_\tau |f(x)|^2 dx d\tau \leq -\int_0^T \int_0^T \frac{\partial}{\partial \tau_1} g(\tau_1,\tau_2) \frac{\partial}{\partial \tau_2} g(\tau_1,\tau_2) d\tau_1 d\tau_2. \tag{4.38}$$

Since $g(\tau,\tau') = g(\tau',\tau)$, this shows that

$$\|f\|_{L^2(X)} \leq \frac{1}{\sqrt{2\pi}} \big\|\partial_\tau g\big\|_{L^2((0,T)\times(0,T))}. \tag{4.39}$$

This concludes the proof of Theorem 4.3.1.

When two measurements g are equal so that their difference and hence the difference of their differentials vanishes, then the difference of sources f also vanishes. This provides the injectivity of the transform $R_J f(s,\vartheta)$ for f supported on a compact domain. Indeed, if $g = 0$, the $\partial_\tau g = 0$ and hence $f = 0$. This also provides an injectivity result for $N = R_J^* \Lambda R_J = (\Lambda^{\frac{1}{2}} R_J)^* \Lambda^{\frac{1}{2}} R_J$. Indeed $Nf = 0$ implies $(Nf, f) = 0 = \|\Lambda^{\frac{1}{2}} R_J f\|$ so that $\Lambda^{\frac{1}{2}} R_J f = 0$ and hence $f = 0$ since R_J is injective.

4.4 Summary on GRT.

What have we done so far? We have defined a generalized ray transform $R_J f(s,\vartheta)$. We have then quickly brought our functions back into the space of positions by applying a rescaled adjoint operator $R_K^* \Lambda$. This lead to the definition of the operators $F = R_K^* \Lambda R_J$ and $N = R_J^* \Lambda R_J$. We have seen that by an appropriate choice of K, then $F = I - T$ where T is a compact operator mapping $L^2(X)$ to $H^1(X)$. However, we do not know that F is invertible in general although by continuity we know that it is so when the curves are close to the straight lines and the weight w is close to 1. Since T is compact, we know that the space of functions such that $T\psi = \psi$ is finite dimensional. But we do not know whether it is trivial.

We have then changed gears slightly and have looked at the normal operator $N = R_J^* \Lambda R_J$. Such an operator, like F, is a pseudo-differential operator. Moreover, it is invertible up to compact perturbations in the sense that $QN = I - T$ for T compact and Q another pseudo-differential operator of order 0. Here again, we do not know that 1 is not an eigenvalue of T nor that Q is invertible. However, we have seen that in some situations, R_J, and hence N, was injective by using a transport equation. This allowed us to show that N was in fact an invertible operator in $L^2(X)$.

This provides a reasonable theory for the reconstruction of functions from full data, i.e., from knowledge of $R_J f(s,\vartheta)$ for all $(s,\vartheta) \in \mathbb{R} \times \mathbb{S}^1$. In many practical problems, such data are not available. It then remains a very difficult problem to prove injectivity of the transform. In the case of the Radon transform, the Fourier slice theorem shows that the Radon transform is injective as soon as an open set of values of ϑ is available (and all values of s). This is because the Fourier transform of a compactly supported function is an analytic function in the Fourier variable and that an analytic function known on an arbitrarily small open set is known everywhere. For the generalized ray

transform, no such results are available. However, it is interesting to understand which singularities of the function $f(x)$ may be reconstructed from available measurements. This requires that we understand how singularities propagate when we apply the Radon transform and the adjoint of the Radon transform. We can apply the propagation of singularity theory to the GRT as we now do.

4.5 Application to generalized Radon transforms

We revisit the propagation of singularities that we considered in the preceding chapter for the Radon transform. We now consider the GRT and observe that there are essentially no differences between the two.

Consider the operator R_J defined earlier in the chapter. Then we have the phase $\phi(s,\vartheta,x,\sigma)=(s-s(x,\vartheta))\sigma$ and the distribution kernel

$$K_J(s,\vartheta,x)=\int_{\mathbb{R}} e^{i(s-s(x,\vartheta))\sigma}\frac{J(x,\vartheta)}{2\pi}d\sigma.$$

The phase stations at $s=s(x,\vartheta)$, the set where $\phi'_\sigma=0$. Away from that set, the distribution kernel is smooth. So all the action takes place on the set

$$WF(K_J)=\Big\{\big(s,\vartheta,\sigma,-\sigma\partial_\vartheta s(x,\vartheta);x,-\sigma\nabla_x s(x,\vartheta)\big),\ s=s(x,\vartheta)\Big\}.$$

This implies that

$$WF'(K_J)=\Big\{\big(s,\vartheta,\sigma,-\sigma\partial_\vartheta s(x,\vartheta);x,\sigma\nabla_x s(x,\vartheta)\big),\ s=s(x,\vartheta)\Big\}.$$

Since ϕ is a phase in the (s,θ,σ) as well as the (x,σ) variables, we deduce that $WF'_Y(K_J)=\emptyset$ and $WF_X(K_J)=\emptyset$.

Let us assume that $(x,\xi)\in T^*\mathbb{R}^2$ is known and let us look at the points $(s,\vartheta,\xi_s,\xi_\theta)$ in correspondence with (x,ξ) through $WF'(K_J)$, i.e., the points $(s,\vartheta,\xi_s,\xi_\theta)$ such that $(s,\vartheta,\xi_s,\xi_\theta;x,\xi)\in WF'(K_J)$.

When $s(x,\theta)=x\cdot\vartheta^\perp$, we obtain that $\xi=\sigma\nabla_x s$ provides θ up to $\pm n\pi$ and then $\sigma=\pm|\xi|$. For the family of curve $\boldsymbol{F}$, we *assume* that knowledge of $\xi=\sigma\nabla_x s$ determines two angles $\theta_{1,2}$. Once θ_j is chosen, then σ is defined uniquely. Then $s=s(x,\vartheta)$ and $\xi_\theta=-\sigma\partial_\vartheta s(x,\vartheta)$ are also uniquely defined. The relation $WF'(K_J)$ is therefore a 1-to-2 relation.

Let us now consider the adjoint operator with kernel

$$K_J^*(x,s,\vartheta)=\int_{\mathbb{R}} e^{-i(s-s(x,\vartheta))\sigma}\frac{J(x,\vartheta)}{2\pi}d\sigma.$$

We obtain that

$$WF'(K_J^*)=\Big\{\big(x,\sigma\nabla_x s(x,\vartheta);s,\vartheta,\sigma,-\sigma\partial_\vartheta s(x,\vartheta)\big),\ s=s(x,\vartheta)\Big\}.$$

This is a 2-to-1 relation, which to the points $(s_j,\vartheta_j,\xi_{s,j},\xi_{\theta,j})$ associates back the unique corresponding point (x,ξ).

From these correspondences, it is clear that

$$WF'(K_J^*)(WF'(K_J)) = I,$$

is the identity on T^*X. In other words, we obtain that

$$WF(R_J^* R_J u) \subset WF'(K_J^*)(WF'(K_J)(WF(u))) = WF(u).$$

We have obtained that $R_J^* R_J$ was a ΨDO , for which the above result holds. We thus observe that the notion of wavefront set and propagation of wavefront sets by FIO provides an optimal description of the propagation of (microlocal) singularities for the GRT, unlike what we observed for the propagation of singular supports in Remark 3.5.1.

The notion of wavefront set also provides a very precise description of what singularities can or cannot be reconstructed in the presence of partial data. If $R_J f(s,\theta)$ is available only for a subset of point (s,θ), then the correspondence $WF'(K_J)$ precisely describes which singularities in $WF(f)$ may be reconstructed.

More precisely, assume that f_1 and f_2 are two sources with $WF(f_1) \cap WF(f_2) = \emptyset$. We wish to reconstruct the singularities of f_1 but not those of f_2 from knowledge of $R_J f$ with $f = f_1 + f_2$. Let $\Gamma_j = WF'(K_J)(WF(f_j)) \subset T^*Z$ for $j = 1, 2$ and $Z = (\mathbb{R} \times \mathbb{S}^1)$. We then define a symbol $p(z,\xi) \in C^\infty(T^*Z)$ for $z = (s,\theta)$ such that $p = 1$ on Γ_1 and $p = 0$ on Γ_2. We then define the ΨDO P

$$Pg(z) = \int_{Z \times \mathbb{R}^2} e^{i(z-z')\cdot\xi} p(z,\xi) g(z') d\xi dz'.$$

Remark 4.5.1 The expression of P above is somewhat imprecise since $Z = \mathbb{R} \times \mathbb{S}^1$ is not a subset of $\mathbb{R}^2$ and the Fourier transform on Z is not defined. We could define functions on $\mathbb{S}^1$ as $2\pi-$periodic functions defined on $\mathbb{R}$, in which case ξ_θ would be the discrete index in Fourier series. Rather, it is more convenient to write a partition of unity $\phi_1 + \phi_2 = 1$ on $\mathbb{S}^1$ with ϕ_1 compactly supported in $(0, 2\pi)$. For a function u on $\mathbb{S}^1$, we may then write $u_j = \phi_j u$ as compactly supported functions in $\mathbb{R}$, where the Fourier transform is defined. The Fourier transform then depends on the choice of the partition of unity. However, the singularities, which are local notions, do not. We therefore define wavefront set of u as the union of those of the u_j. Similarly, the above operator P is really defined by applying the operator to $g_j = \phi_j g$ and then sum to get the application of the operator on $g = g_1 + g_2$. The operator P thus also depends on the choice of the partition of unity, although its properties of propagation of singularities do not. In the rest of the section, we will write expressions as if Z was a subset of $\mathbb{R}^2$.

The role of the operator P is to capture the singularities of $R_J f_1$ and not those of $R_J f_2$. Using the results obtained earlier in this section, we obtain that

$$WF(PR_J f_2) = \emptyset, \qquad WF(PR_J f_1) = WF(R_J f_1) \subset WF'(R_J)(WF(f_1)).$$

We recall that R_J admits a left parametrix given for instance by

$$Q_J R_J^* R_J = I - T,$$

where Q_J is a ΨDO of order 1 and T is arbitrarily smooth.

We write

$$\begin{aligned} Q_J R_J^* P R_J f &= Q_J R_J^* R_J f_1 + Q_J R_J^*(I-P) R_J f_1 + Q_J R_J^* P R_F f_2 \\ &= f_1 - T f_1 + Q_J R_J^*(I-P) R_J f_1 + Q_J R_J^* P R_F f_2. \end{aligned}$$

Since $I-P$ is a ΨDO with symbol vanishing on the wavefront set of $R_J f_1$ and P is a ΨDO with symbol vanishing on the wavefront set of $R_J f_2$, we deduce that the above expression is equal to f_1 plus a smooth function.

We thus obtain a very precise description of the propagation of singularities in the application of R_J and R_J^*: based on available data, we know exactly which singularities of a given source may be reconstructed or not. Moreover, by application of an appropriate ΨDO , we are able to select which singularities we want to reconstruct and which ones we prefer to smooth out.

Note that such a result cannot be obtained by replacing P by the multiplication by a cut-off function $\chi(z)$. Indeed, consider two delta sources located at two different points. The line passing through these two points is necessary to reconstruct the first delta source and clearly involves the singularities of the second delta source. The separation of the singularities has to be done microlocally.

Exercise 4.5.1 *Generalize the above construction to the $n-$dimensional Radon transform $\mathsf{R}f(x,\vartheta)$.*

4.6 Composition and continuity of FIO

Let us consider the operator R_J defined above. Let $\Lambda^{\frac{1}{2}}$ be the ΨDO from $L^2(\mathbb{R}^2)$ to $H^{-\frac{1}{2}}(\mathbb{R}^2)$ given by the symbol $\langle\xi\rangle^{\frac{1}{2}}$ (this may be seen as the $\frac{1}{4}$th power of $(I-\Delta)$). We consider the operator $R_J\Lambda^{\frac{1}{2}}$, which is well defined on $\mathcal{D}'(X)$.

The adjoint operator is $\Lambda^{\frac{1}{2}} R_J^*$. The normal operator is

$$N = \Lambda^{\frac{1}{2}} R_J^* R_J \Lambda^{\frac{1}{2}}.$$

The theory developed above shows that N is a ΨDO given by $\Lambda^{-\frac{1}{2}} Q_J \Lambda^{-\frac{1}{2}}$ and is therefore bounded in $L^2(X)$. This implies that

$$\|R_J\Lambda^{\frac{1}{2}} u\|^2_{L^2(Z)} = (R_J\Lambda^{\frac{1}{2}} u, R_J\Lambda^{\frac{1}{2}} u)_Z = (N^2 u, u)_X \leq C\|u\|^2_{L^2(X)}$$

In other words, $R_J\Lambda^{\frac{1}{2}}$ is bounded from $L^2(X)$ to $L^2(Z)$, or in other words R_J is bounded from $H^{-\frac{1}{2}}(X)$ to $L^2(Z)$, or more generally, using standard ΨDO calculus, from $H^s(X)$ to $H^{s+\frac{1}{2}}(Z)$, where the latter space is the space of functions with $s+\frac{1}{2}$ derivatives bounded in the variables (s,θ) (and not only in the variable s as in the definition of $\mathrm{H}^s(Z)$). We summarize this as

Theorem 4.6.1 (Continuity of the GRT) *Let R_J as defined above. Then there is a constant C_s such that for all $s \in \mathbb{R}$ and $u \in H^s(X)$, we have*

$$\|R_J u\|_{H^{s+\frac{1}{2}}(Z)} \leq C_s \|u\|_{H^s(X)}. \tag{4.40}$$

Such continuity results do not apply to all FIO defined in (3.12). However, they apply for a reasonably large class where the relation $WF'(I_{a,\varphi})$ is a bijection as we saw in the preceding chapter.

Appendices

Appendix A

Remarks on composition of FIOs and Stationary phase

A.1 Remarks on composition of FIOs

FIO and canonical relations. The FIO defined in (3.12) defines a map from $\mathcal{E}'(Y)$ to $\mathcal{D}'(X)$ when φ is a phase in the (y,θ) variables. Assuming it is a phase in the (x,θ) variables as well, we associate the Schwartz kernel $I_{a,\varphi}(x,y)$ to the canonical relation C defined earlier. We recall the definition of the homogeneous canonical relation Λ' given by

$$C_\varphi \ni (x,y,\theta) \mapsto (x,\varphi'_x; y, -\varphi'_y).$$

Definition A.1.1 (Class of FIO $I^m(X\times Y; C')$) *For $\varphi \in C^\infty(X\times Y\times \dot{\mathbb{R}}^N)$ a phase function with singular set C_φ and the above definition of Λ', we define the class of FIO $I^m(X\times Y;\Lambda')$ as the operators from $\mathcal{E}'(Y)$ to $\mathcal{D}'(X)$ with a Schwartz kernel given by*

$$I_{a,\varphi}(x,y) = \int e^{i\varphi(x,y,\theta)} a(x,y,\theta) d\theta,$$

for $a(x,y,\theta) \in S^{m+\frac{n_X+n_Y}{4}-\frac{N}{2}}(X\times Y\times \mathbb{R}^N)$.

What appears as a strange normalization of the order of the symbol a above is explained by the fact that we wish $A \in I^m$ to be corresponding to an operator that *differentiates* m times, as a ΨDO with symbol in S^m. In explaining this normalization, we also partially explain why the class of operators is parameterized by C' and not by φ. It turns out that I^m is indeed not modified, modulo operators in $S^{-\infty}$, by a change of phase that preserves the manifold Λ (or equivalently the relation Λ').

We first observe how we can increase the number of phase variables without changing the Schwartz kernel. Let us define

$$\phi(x,y,\theta,\psi) = \varphi(x,y,\theta) + \frac{|\psi|^2}{2|\theta|},$$

where $\psi \in \mathbb{R}^{N'}$. We observe that $\phi'_{x,y} = \varphi'_{x,y}$, $\phi'_\theta = \varphi'_\theta - \frac{|\psi|^2}{2|\theta|^2}\theta$, and $\phi'_\psi = \frac{\psi}{|\theta|}$. It is clear that ϕ is a phase when φ is. Moreover, $\phi'_{\theta,\psi} = 0$ implies that $\psi = 0$ so that $\Lambda_\varphi = \Lambda_\psi$.

The singular sets agree. The canonical relations for φ and ψ are therefore the same. Now consider the distribution

$$I_{b,\phi}(x,y) = \int e^{i\phi(x,y,\theta,\psi)} b(x,y,\theta) d\theta d\psi = \int e^{i\varphi(x,y,\theta,\psi)} b(x,y,\theta) d\theta e^{i\frac{|\psi|^2}{2|\theta|}} d\psi.$$

We obtain that

$$\int_{\mathbb{R}^{N'}} e^{i\frac{|\psi|^2}{2|\theta|}} d\psi = (i|\theta|)^{\frac{N'}{2}}.$$

We thus define

$$b(x,y,\theta) = \frac{1-\chi(\theta)}{(i|\theta|)^{\frac{N'}{2}}} a(x,y,\theta),$$

with $\chi(\theta)$ a compactly supported function with $\chi(0)=1$ which is introduced to remove the singularity at $\theta = 0$. Therefore, up to a $C^\infty(X\times Y)$ Schwartz kernel, we obtain that $I_{b,\phi} = I_{a,\varphi}$. Moreover, we clearly have that $b \in S^{m+\frac{n_X+n_Y}{4}-\frac{N+N'}{2}}(X\times Y\times \mathbb{R}^{N+N'})$. It shows that the order of the symbol has to be the above functional of the phase dimension N. Now, for ΨDO the order m of regularity of the operator A and that of the symbol S is the same. As a consequence, when $n_X = n_Y = N$, this justifies the presence of the term $\frac{n_X+n_Y}{4}$ in the definition, at least in the case $n_X = n_Y$, which is the case of interest here.

As we may increase the number of phase variables, in some cases, we can also reduce it, for instance to go back from ϕ to φ. The procedure to obtain a phase with the smallest dimension is as follows; see [33, 36] for derivations. Let us consider the phase $\varphi(x,y,\theta)$ and the rank k of the Hessian $\varphi''_{\theta\theta}(x,y,\theta)$. Up to relabeling of the phase variables, we may call $\theta' = (\theta_1,\dots\theta_{N-k})$ and $\theta'' = (\theta_{N-k+1},\dots,\theta_N)$ the variables so that $\det \varphi''_{\theta''\theta''} \neq 0$. On the singular set $\varphi'_\theta = 0$, the latter property implies that θ" is implicitly determined by a C^∞ function $\psi(x,y,\theta')$. The $\varphi_1(x,y,\theta) = \varphi(x,y,\theta',\psi(x,y,\theta'))$ happens to be a phase function of dimension $N-k$, which has the same critical set C_φ as φ.

A rather involved proof not reproduced here [36, Sections 3.1 & 3.2] implies that $\varphi(x,\theta)$ is equivalent to a phase of the form $\varphi_1(x,\theta) = \varphi(x,\theta',0) + \frac{Q(\theta",\theta")}{2|\theta'|}$ for a quadratic form Q. Here "equivalent" phases φ and φ_1 means that $I_{a,\varphi} = I_{a_1,\varphi_1}$ up to a C^∞ function for an appropriate choice of a_1 for a given a with a and a_1 symbols of the same order.

Then we have

$$\int_{\mathbb{R}^{N-k}} e^{i\varphi(x,\theta',0)} \Big(\int_{\mathbb{R}^k} e^{i\frac{Q(\theta",\theta")}{2|\theta'|}} a(x,y,\theta',\theta") d\theta" \Big) d\theta' =: \int_{\mathbb{R}^{N-k}} e^{i\varphi(x,\theta',0)} b(x,y,\theta') d\theta'.$$

This defines $b(x,y,\theta')$ and by the method of *stationary phase* recalled in Appendix A.2, we obtain that the order of the latter symbol is that of a plus $\frac{k}{2}$.

The above procedure thus allows us to replace a phase of order N by a phase of order $N-k$, where k is the rank of the Hessian $\varphi''_{\theta\theta}(x,y,\theta)$. Note that for the Radon transform, $N=1$ and the phase being linear in σ, the rank of the Hessian is 0. The number of phase variables to describe $\{s = x\cdot\vartheta\}$ thus cannot be reduced. For the case $\varphi(x,y\xi) = (x-y)\cdot\xi$, we similarly obtain that the Hessian vanishes so that $N=n$ is the smallest dimension for a phase corresponding to the canonical relation $\{x=y\}$.

Composition of a ΨDO with a FIO. We consider the composition of a properly supported FIO A with phase $\varphi(x,y,\theta)$ and symbol $a(x,y,\theta)$ for $(x,y)\in X\times Y$ with a ΨDO B with amplitude $b(z,x,\xi)$ for $z,x\in X$, with X and Y both subsets of $\mathbb{R}^n$. The composition is the operator, which we know to be well defined from $\mathcal{E}'(Y)$ to $\mathcal{E}'(X)$, is the operator defined by

$$BAf(z)=\int e^{i(z-x)\cdot\xi}b(z,x,\xi)e^{i\varphi(x,y,\theta)}a(x,y,\theta)f(y)d\theta d\xi dxdy.$$

Consider the phase

$$\psi(z,y;x,\xi,\theta)=(z-x)\cdot\xi+\varphi(x,y,\theta)=(z-x)\cdot\xi+\varphi(x,y,\theta)-\varphi(z,y,\theta)+\varphi(z,y,\theta),$$

so that

$$\psi(z,y;x,\xi,\theta)=\frac{\zeta\cdot\xi'}{|\theta|}+\varphi(z,y,\theta),\quad \zeta=(z-x)|\theta|,\quad \xi'=\xi-\int_0^1\varphi'_x(tx+(1-t)z,y,\theta)dt.$$

It is straightforward to verify that the determinant of the change of variables $(x,\xi)\to(\zeta,\xi')$ is $|\theta|^n$. Moreover, the stationary points of ψ are the same as those of φ in θ and the stationary points in ζ and ξ' are $\zeta=\xi'=0$, which implies $x=z$ and $\xi=\varphi'_x(z,y,\theta)$ on C_ψ. We obtain

$$\int e^{i\frac{\zeta\cdot\xi'}{|\theta|}}e^{i\varphi(z,y,\theta)}b(z,x,\xi(\xi',\zeta,x,y,\theta))a(x,y,\theta)|\theta|^{-n}f(y)d\theta d\xi' d\zeta dy.$$

Up to a smoother term which is also a FIO with a lower-order symbol, we thus have to analyze the operator

$$\int e^{i\frac{\zeta\cdot\xi'}{|\theta|}}e^{i\varphi(z,y,\theta)}b(z,z,\varphi'_x(z,y,\theta))a(z,y,\theta)|\theta|^{-n}f(y)d\theta d\xi' d\zeta dy.$$

Using the stationary phase recalled in the appendix with the zero-signature Q such that $((\zeta,\xi'),Q(\zeta,\xi'))=\zeta\cdot\xi'$, we find

$$\int e^{i\frac{\zeta\cdot\xi'}{|\theta|}}|\theta|^{-n}d\xi' d\zeta=(2\pi)^n.$$

Thus the above operator is a FIO with phase $\varphi(x,y,\theta)$ and amplitude

$$c(z,y,\theta)=(2\pi)^n b(z,z,\varphi'_x(z,y,\theta))a(z,y,\theta).$$

These calculations show that the FIO PA is a FIO with the same phase as A and with a symbol of order equal to the sum of the orders of the symbols of P and A. We can similarly show that AP is a FIO with phase φ and amplitude given by

$$d(z,y,\theta)=(2\pi)^n a(z,y,\theta)b(y,y,\varphi'_y(z,y,\theta)).$$

Note that when φ generates a canonical relation C, we find that $(z,\varphi'_x(z,y,\theta))=\chi(y,\varphi'(z,y,\theta))$ on C_φ. Let us consider ΨDO P and Q such that $PA-AQ$ is a smooth operator. Then we have found that (at least at the level of the principal symbol) that the symbols p of P and q of Q satisfy that $p=q\circ\chi$. This is Egorov's theorem, at least written for the principal symbol. The rest of the symbol can be shown to follow the same rule.

Composition of a FIO with its adjoint. We now consider the composition of a properly supported FIO A with phase $\varphi(x,y,\theta)$ and symbol $a(x,y,\theta)$ for $(x,y)\in X\times Y$ with its adjoint A^* with phase $-\varphi(y,x,\theta)$ and symbol $a^*(y,x,\theta)$. We assume that the canonical relation C' is a canonical graph.

As we did in the composition of a ΨDO with a FIO, we have

$$A^*Au(z)=\int e^{i\psi(z,y;x,\theta,\theta')}a^*(x,z,\theta')a(x,y,\theta)u(y)dy(dxd\theta d\theta'),$$

with $\psi(z,y;x,\theta,\theta')=\varphi(x,y,\theta)-\varphi(x,z,\theta')$. We then introduce

$$\begin{aligned}\psi(z,y;x,\theta,\theta') &= \varphi(x,y,\theta)-\varphi(x,z,\theta)+\varphi(x,z,\theta)-\varphi(x,z,\theta')\\ &= (z-y)\cdot\xi+(\theta-\theta')\cdot\frac{\tau}{|\xi|}\\ \xi &= -\int_0^1\varphi'_y(x,ty+(1-t)z,\theta)dt\\ \tau &= |\xi|\int_0^1\varphi'_\theta(x,z,t\theta+(1-t)\theta')dt.\end{aligned}$$

We observe that the phase ψ (seen as a phase in the (x,θ,θ') variables) is critical when $y=z$ and $\theta=\theta'$. To do so, we have to analyze the critical set of ψ given by

$$C_\psi=\{\varphi'_\theta(x,y,\theta)=0,\ \varphi'_\theta(x,z,\theta')=0,\ \varphi'_x(x,y,\theta)=\varphi'_x(x,z,\theta')\}.$$

We first have to deal with the fact that the phase ψ is not homogeneous of degree 1 in x. We therefore introduce the variable $\omega=\sqrt{|\theta|^2+|\theta'|^2}x$, and observe that the critical set of C_ψ in the new variables (ω,θ,θ') is not modified and that ψ in these variables is homogeneous of degree 1. Moreover, we can up to a smooth modification assume that $a(x,z,\theta)$ vanishes for $|\theta|<1$ so that the change of variables from (x,θ,θ') to (ω,θ,θ') is smooth. Up to a smooth perturbation, we obtain that A^*A is not modified if the integration in the variables (x,θ,θ') is performed in the vicinity of the above critical set C_ψ.

Let us denote $\xi=\varphi'_x(x,y,\theta)$. Since $\varphi'_\theta(x,y,\theta)=0$, on the critical set of φ, knowledge of (x,ξ) uniquely characterizes y since the canonical graph χ introduced earlier is a bijection. But z satisfies the same constraints and as a consequence, we obtain that $y=z$. But there is also a bijection between $C_\varphi\ni(x,y,\theta)$ and $(x,\xi=\varphi'_x)$ so that θ is uniquely determined. Since θ and θ' satisfy that same constraint, we deduce that $\theta=\theta'$ as announced. We can therefore assume that $y-z$ and $\theta-\theta'$ are small in the definition of Au modulo a smooth term.

We now consider the change of variables $(x,\theta,\theta')\mapsto(\xi,\tau,\theta-\theta')$. We find that the

Jacobian of the transformation is given by

$$\begin{aligned}\Big|\frac{\partial(\xi,\tau,\theta-\theta')}{\partial(\theta,\theta',x)}\Big| &= |\xi|^N \det\begin{pmatrix}\int_0^1 \varphi''_{\theta y}(v_{yz})dt & 0 & \int_0^1 \varphi''_{yx}(v_{yz})dt\\ \int_0^1 \varphi''_{\theta\theta}(v_{\theta\theta'})tdt & \int_0^1 \varphi''_{\theta\theta}(v_{\theta\theta'})(1-t)dt & \int_0^1 \varphi''_{\theta x}(v_{\theta\theta'})dt\\ 1 & -1 & 0\end{pmatrix}\\ &= |\xi|^N \det\begin{pmatrix}\int_0^1 \varphi''_{\theta y}(v_{yz})dt & \int_0^1 \varphi''_{yx}(v_{yz})dt\\ \int_0^1 \varphi''_{\theta\theta}(v_{\theta\theta'})dt & \int_0^1 \varphi''_{\theta x}(v_{\theta\theta'})dt\end{pmatrix}\end{aligned}$$

with $v_{yz} = (x, ty + (1-t)z, \theta)$ and $v_{\theta\theta'} = (x, z, t\theta + (1-t)\theta')$.

At $y = z$ and $\theta = \theta'$, the above determinant is given by

$$\Big|\frac{\partial(\xi,\tau,\theta-\theta')}{\partial(\theta,\theta',y)}\Big|(y,y;x,\theta,\theta) = |\xi|^N \det\begin{pmatrix}\varphi''_{\theta y}(x,y,\theta) & \varphi''_{yx}(x,y,\theta)\\ \varphi''_{\theta\theta}(x,y,\theta) & \varphi''_{\theta x}(x,y,\theta)\end{pmatrix} =: |\xi|^N D(y,x,\theta)$$

That the latter determinant does not vanish is equivalent to the fact that the canonical relation is a canonical graph [36, Prop. 4.1.3]. The latter determinant remains positive for $y - z$ and $\theta - \theta'$ small, i.e., in the (conical) vicinity of C_ψ. We thus recast A^*A, up to a smoothing perturbation, as

$$\begin{aligned}A^*Au(z) &\equiv \int e^{i(z-y)\cdot\xi} b(y,z,\xi)u(y)dyd\xi\\ b(y,z,\xi) &= \int a^*(x,z,\theta')a(x,y,\theta)e^{i\frac{(\theta-\theta')\cdot\tau}{|\xi|}}|\xi|^{-N}D^{-1}(y,x,\theta)d(\theta-\theta')d\tau\end{aligned}$$

Here, (x,θ,θ') are explicit expressions of $(\xi,\tau,\theta-\theta')$. Moreover the homogeneity of D^{-1} in ξ seems to be ξ^{N-n} (has to be checked in detail; this comes from changing x to $\omega = \sqrt{|\theta|^2+|\theta'|^2}x$. We get ξ^{-n} from the change of measure and then ξ^N from the derivatives in x in the above determinant. Notation has to be modified). In other words, A^*A is a ΨDO and the symbol b given above is in $S^{2m}(X\times X\times\mathbb{R}^n)$. Indeed, from the method of stationary phase recalled in the appendix, we obtain as earlier that $e^{i\frac{\theta\cdot\tau}{|\xi|}}$ increases the the order of the symbol by $|\xi|^N$. As a consequence, the order of the symbol b is given by

$$m + \frac{n-N}{2} + m + \frac{n-N}{2} - n + N = 2m.$$

Stability estimates for FIO Now let us consider the operator $A\Lambda^{-m}$. The above calculation shows that $A\Lambda^{-m}$ is a FIO of order $\frac{n-N}{2}$ and that $\Lambda^{-m}A^*A\Lambda^{-m} = (A\Lambda^{-m})^*A\Lambda^{-m}$ is a ΨDO of order 0. As a consequence, we obtain at long last that

$$\|A\Lambda^{-m}u\|^2_{L^2(X)} = ((A\Lambda^{-m})^*A\Lambda^{-m}u,u)_Y \le C\|u\|^2_{L^2(Y)}.$$

This shows that A is an operator of order m in the sense that for all $s\in\mathbb{R}$, we have

$$\|Au\|_{H^s(X)} \le C\|u\|_{H^{s+m}(Y)}. \tag{A.1}$$

This estimate generalizes the one obtained for the Radon transform where $m = -\frac{1}{2}$.

A.2 Method of stationary phase

We collect here some results on the method of stationary phase following [33].

Let $X \subset \mathbb{R}^n$ be an open set and $\varphi \in C^\infty(X)$ a real-valued function such that $d\varphi = \sum_j \frac{\partial \varphi}{\partial x_j} dx_j = \varphi' dx \neq 0$. We define the integral for $u \in C_0^\infty(X)$:

$$I(\lambda) = \int e^{i\lambda\varphi(x)} u(x) dx. \tag{A.2}$$

Non-stationary phase. Define the first-order operator

$$L^t u = \frac{1}{\lambda} \frac{1}{i|\varphi'|^2} \varphi' \partial_x u, \qquad Lu(x) = \frac{1}{\lambda} \partial_x \Big(\frac{1}{i|\varphi'|^2} \varphi' u \Big) = \frac{1}{\lambda} \sum_j \frac{\partial}{\partial x_j} \Big(\frac{1}{i|\varphi'|^2} \frac{\partial \varphi}{\partial x_j} u \Big)$$

Then we observe that $L^t e^{i\lambda\varphi} = e^{i\lambda\varphi}$ so that

$$I(\lambda) = \int L^t \Big(e^{i\lambda\varphi(x)} \Big) u(x) dx = \int e^{i\lambda\varphi(x)} Lu(x) dx = \int e^{i\lambda\varphi(x)} L^k u(x) dx.$$

This shows that for all compact $K \subset X$ and every $k \in \mathbb{N}$, we have

$$|I(\lambda)| \leq \frac{C_{k,K}}{\lambda^k} \|u\|_{C^k(K)}, \tag{A.3}$$

for all $u \in C_0^\infty(X)$ with supp u $\subset$ K and for all $\lambda > 1$. The above estimate is a result of *non-stationary* phase, and applies for phases $\varphi(x)$ with no critical point.

The main objective of stationary phase is to obtain and expression for $I(\lambda)$ as $|\lambda| \to \infty$ when φ has non-degenerate critical points. This means that there are points x_0 such that $\varphi'(x_0) = 0$ with det $\varphi"(x_0) \neq 0$ with $\varphi"(x)$ the Hessian of φ at x. This non-degeneracy condition implies that $\varphi'(x) \neq 0$ for $|x - x_0|$ sufficiently small and $x \neq x_0$. We call x_0 a non-degenerate critical point.

Morse Lemma. The Morse lemma shows that in the vicinity of x_0, a change of variables makes the phase a quadratic function.

Lemma A.2.1 (Morse Lemma) *Let $\varphi \in C^\infty(X)$ a real-valued, non-degenerate function in the vicinity of x_0, where $\varphi'(x_0) = 0$. Then there are neighborhoods U of $0 \in \mathbb{R}^n$ and V of x_0 and a diffeomorphism $\mathcal{H} : V \to U$ such that*

$$\mathcal{H}_* \varphi(y) := \varphi \circ \mathcal{H}^{-1}(y) = \varphi(x_0) + \frac{1}{2}(y_1^2 + \ldots y_r^2 - y_{r+1}^2 - \ldots - y_n^2), \tag{A.4}$$

where $(r, n-r)$ is the signature of the Hessian $\varphi"(x_0)$ (which has r and $n-r$ positive and negative eigenvalues, respectively).

After such a change of variables, it remains to integrate terms of the form (A.2) with φ a non-degenerate quadratic form. In one variable of space, we have the following Fourier transforms

$$\mathcal{F}_{x \to \xi} e^{-\frac{1}{2}\alpha x^2}(\xi) = \Big(\frac{2\pi}{\alpha} \Big)^{\frac{1}{2}} e^{-\frac{1}{2\alpha}\xi^2}, \qquad \mathcal{F}_{x \to \xi} e^{\frac{\pm i}{2} x^2}(\xi) = (2\pi)^{\frac{1}{2}} e^{\pm \frac{1}{4} i\pi} e^{\mp \frac{i}{2}\xi^2},$$

from which we deduce in n dimensions of space:

$$\mathcal{F}_{x\to\xi}e^{\frac{i}{2}\lambda(x,Qx)}(\xi)=\Big(\frac{2\pi}{\lambda}\Big)^{\frac{n}{2}}\frac{1}{|\mathrm{det}Q|^{\frac{1}{2}}}e^{i\frac{\pi}{4}\mathrm{sgn}\ Q}e^{-i\frac{1}{2\lambda}(\xi,Q^{-1}\xi)},$$

where sgn Q is the signature of Q equal to the number of positive eigenvalues of Q (counting multiplicities) minus the number of negative eigenvalues, i.e., $r-(n-r)=2r-n$ for the matrix in Lemma A.2.1. For $u\in C_0^\infty(\mathbb{R}^n)$, we deduce from the Parseval relation that

$$\int e^{\frac{i}{2}\lambda(x,Qx)}u(x)dx=\Big(\frac{1}{2\pi\lambda}\Big)^{\frac{n}{2}}\frac{1}{|\mathrm{det}Q|^{\frac{1}{2}}}e^{i\frac{\pi}{4}\mathrm{sgn}\ Q}\int e^{-i\frac{1}{2\lambda}(\xi,Q^{-1}\xi)}\hat{u}(\xi)d\xi.$$

From the Taylor expansion of e^{it}, we obtain that

$$e^{-i\frac{1}{2\lambda}(\xi,Q^{-1}\xi)}=\sum_{k=0}^{N-1}\frac{1}{k!}\Big(\frac{1}{2i\lambda}(\xi,Q^{-1}\xi)\Big)^k+R_N(\xi,\lambda),\quad |R_N(\xi,\lambda)|\le\frac{C^N|\xi|^{2N}}{\lambda^N N!}.$$

From the expression of differential operators in the Fourier domain, we deduce that

$$\int(\xi,Q^{-1}\xi)^k\hat{u}(\xi)d\xi=(2\pi)^n\big((D_x,Q^{-1}D_x)^ku\big)(0),$$

with $D_xu=(\partial_{x_1}u,\dots,\partial_{x_n}u)^t$. From this, we deduce that

$$\int e^{\frac{i}{2}\lambda(x,Qx)}u(x)dx=\sum_{k=0}^{N-1}\frac{(2\pi)^{\frac{n}{2}}e^{i\frac{\pi}{4}\mathrm{sgn}\ Q}}{k!|\mathrm{det}Q|^{\frac{1}{2}}\lambda^{k+\frac{n}{2}}}\big(\frac{1}{2i}(D_x,Q^{-1}D_x)\big)^k u\ (0)+S_N(u,\lambda),\qquad(A.5)$$

with

$$|S_N(u,\lambda)|\le\tilde{C}\frac{1}{N!\lambda^{N+\frac{n}{2}}}\int\Big|\frac{1}{2}(\xi,Q^{-1}\xi)^N\hat{u}(\xi)\Big|d\xi,$$

so that we have the estimates:

$$|S_N(u,\lambda)|\le C\frac{1}{N!\lambda^{N+\frac{n}{2}}}\|u\|_{W^{2N+n+1,1}(\mathbb{R}^n)}\wedge C_\varepsilon\frac{1}{N!\lambda^{N+\frac{n}{2}}}\|u\|_{H^{2N+\frac{n}{2}+\varepsilon}(\mathbb{R}^n)},\qquad(A.6)$$

for any $\varepsilon>0$. The first norm above means that all derivatives of order less than or equal to $2N+n+1$ are bounded in $L^1(\mathbb{R}^n)$. The second norm implies that all derivatives of order less than or equal to $2N+\frac{n}{2}+\varepsilon$ are bounded in $L^2(\mathbb{R}^n)$.

Stationary phase expansion. Let us now define

$$\tilde{u}=\mathcal{H}_*u:=u\circ\mathcal{H}^{-1}.\qquad(A.7)$$

Using the change of variables $y=\mathcal{H}(x)$, we find for $u\in C_0^\infty(V)$ with V defined in the above lemma that

$$I(\lambda)=e^{i\lambda\varphi(x_0)}\int_U e^{i\lambda\frac{1}{2}(y,Q(y))}\tilde{u}(y)|\mathrm{det}\ \mathcal{H}^{-1}(y)|dy,$$

where Q is the diagonal matrix with the r first diagonal elements equal to 1 and the last $n-r$ elements equal to -1. An application of (A.5) thus shows that

$$I(\lambda) = e^{i\lambda\varphi(x_0)} \sum_{k=0}^{N-1} \frac{(2\pi)^{\frac{n}{2}} e^{i\frac{\pi}{4}\text{sgn } Q}}{k!\lambda^{k+\frac{n}{2}}} \big(\frac{1}{2i}(D_y, Q^{-1}D_y)\big)^k (\tilde{u}|\det \mathcal{H}^{-1}|)\ (0) + S_N(u,\lambda),$$

with $S_N(u,\lambda)$ bounded as in (A.6). Writing the above expression in the original x coordinates and using (A.6), we prove the

Proposition A.2.2 *Let $\varphi \in C^\infty(X)$ a real-valued function with a non-degenerate critical point at x_0 and such that $\varphi'(x) \neq 0$ for $x \neq x_0$. Then there are differential operators $A_{2k}(D)$ of order $2k$ such that for every compact $K \subset X$ and $u \in C_0^\infty(X)$ with support in K, we have*

$$\Big| I(\lambda) - e^{i\lambda\varphi(x_0)} \sum_{k=0}^{N-1} \frac{1}{\lambda^{k+\frac{n}{2}}} \big(A_{2k}(D)u\big)(x_0) \Big| \leq \frac{C}{\lambda^{N+\frac{n}{2}}} \|u\|_{C^{2N+n+1}(K)}. \tag{A.8}$$

Moreover, the leading term in the expansion is

$$e^{i\lambda\varphi(x_0)} \frac{1}{\lambda^{\frac{n}{2}}} \frac{(2\pi)^{\frac{n}{2}} e^{i\frac{\pi}{4}\ \text{sgn}\ \varphi"(x_0)}}{|\det\ \varphi"(x_0)|^{\frac{1}{2}}}\ u(0). \tag{A.9}$$

The only term that requires justification is the presence of $|\det\ \varphi"(x_0)|^{\frac{1}{2}}$ in the last expression. In the vicinity of x_0, we have that $\varphi(x) = \varphi(x_0) + ((x-x_0), \varphi"(x_0)(x-x_0)) + O(\|x_0\|^3)$ so that $|\det\ \mathcal{H}|(0) = |\det\ \varphi"(x_0)|^{\frac{1}{2}}$ and (A.9) follows.

Chapter 5

Inverse wave problems

In the preceding two chapters, the probing mechanisms were based on particles: propagation of X-rays in CT and propagation of particles in media with varying indices of refraction in the generalized Radon transform. All these inverse problems were seen to display good stability properties. We have seen that reconstructions based on M.R.I. also enjoyed good stability properties. The third major way of probing unknown domains with good stability reconstructions is to use *waves*. In some sense, particles may be seen as high frequency wave packets with a frequency that is much larger than any other scale in the problem. There is also considerable interest in considering waves with lower, but still large, frequencies. Unsurprisingly, the propagation of such waves is modeled by wave equations. Three typical partial differential equations are then used. A scalar wave equation models the propagation of acoustic waves, the system of Maxwell's equations models the propagation of electromagnetic waves, and the system of elasticity models the propagation of elastic waves. In this chapter, we restrict ourselves to simplified scalar models for the three types of waves.

As waves propagate, they interact with the medium of interest because of variations in a spatially varying parameter called the sound speed (or light speed, or speed of elastic waves; but since we restrict ourselves to a scalar model, we call such a "speed" *sound speed*). The objective of inverse wave problems is therefore typically the reconstruction of such a sound speed from available measurements at the boundary of the domain. This reconstruction can be performed from time-dependent measurements or from frequency measurements. The theories of reconstructions of sound speeds, called inverse scattering theories, can be fairly involved mathematically. In this chapter, we consider several relatively simple, but still representative, inverse wave problems.

The first problem is a linearization of the inverse scattering problem in one dimension of space with time-dependent measurements. The second inverse problem is a linearization of the inverse scattering problem with measurements at the domain's boundary for one (sufficiently high) frequency. The third problem is an inverse source problem consisting of reconstructing the initial condition in a wave equation from spatio-temporal measurements of wave fields at the boundary of a domain enclosing the support of the initial condition. This inverse source problem finds applications in the medical imaging modality Photo-acoustic Tomography, which will be further analyzed in a later chapter. The fourth and final problem is a nonlinear inverse coefficient problem in one dimension of space.

This selection of inverse problems aims to demonstrate the following statement: Inverse problems based on (sufficiently high-frequency) waves that propagate in not-too-scattering environments involve measurement operators with good stability properties. Reconstructions for such inverse problems are therefore typically high resolution. Although such features remain true in higher spatial dimensions and for more general non-linear inverse problems, the geometric descriptions and mathematical analyses can become extremely complex. For general references on inverse scattering and related inverse problems, we refer the reader to, e.g., [26, 28, 38].

5.1 One dimensional inverse scattering problem

In this section, we consider an example of a well-posed inverse problem, i.e., an inverse problem with a Lipschitz stability in the L^2 sense. Inverse problems related to the wave equation are often of this nature. Indeed, the wave equation "propagates" singularities without dampening them, as does the Fourier transform for instance. Here, we consider a simple one dimensional wave equation and a linearization of the inverse scattering problem.

Let us consider the one dimensional wave equation

$$\frac{1}{c_s^2(x)}\frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial x^2} = \delta(t)\delta(x - x_s), \qquad t \in \mathbb{R}, \quad x \in \mathbb{R}, \tag{5.1}$$

with delta source term at time $t = 0$ and position $x = x_s$. Here, c_s is the unknown sound speed, which takes a constant value $c = c_s$ for $|x| \geq R > 0$. We assume causality so that $p(x, t; x_s) = 0$ for $t < 0$ and assume that p is bounded. We measure $p(x_s, t; x_s)$ as the domain's boundary as a function of time and want to reconstruct the unknown profile $c_s(x)$.

It is convenient to analyze the problem in the frequency domain. Let us define $u(x, \omega; x_s)$ the *causal Fourier transform* of $p(x, t; x_s)$ in the time variable

$$u(x, \omega; x_s) = \int_0^\infty p(x, t; x_s) e^{i\omega t} dt. \tag{5.2}$$

This transform can be inverted as follows:

$$p(x, t; x_s) = \frac{1}{2\pi} \int_{-\infty}^\infty u(x, \omega; x_s) e^{-i\omega t} d\omega. \tag{5.3}$$

The equation for $u(x, \omega; x_s)$ is the well-known Helmholtz equation

$$\frac{d^2 u}{dx^2} + \frac{\omega^2}{c_s^2(x)} u = -\delta(x - x_s), \qquad \omega \in \mathbb{R}, \quad x \in \mathbb{R}, \tag{5.4}$$

augmented with the following *radiation conditions*

$$\frac{du}{dx} \mp \frac{i\omega}{c} u \;\to\; 0, \qquad \text{as} \qquad x \;\to\; \pm\infty. \tag{5.5}$$

Since $p(x_s, t; x_s)$ is measured, then $u(x_s, \omega; x_s)$ is known by taking the Fourier transform of the available data.

Let us make a few assumptions. We assume that $c_s(x)$ is known on $(-\infty, x_s)$ (in Earth profile reconstructions, one is interested in positive depths only) and that $c_s(x)$ is close to the background sound speed c on (x_s, ∞) in the sense that

$$\frac{1}{c_s^2(x)} = \frac{1}{c^2}\big(1 - \alpha(x)\big), \tag{5.6}$$

where $\alpha(x)$ is small compared to 1. In effect we *linearize* the problem of the reconstruction of $c_s(x)$ from the scattering measurements $u(x_s, \omega; x_s)$.

The advantage is that the resulting problem is linear, relatively straightforward to invert and admits an explicit solution for small $\alpha(x)$. Let us define by u_i (i for incident) the solution of the unperturbed problem

$$\frac{d^2u_i}{dx^2} + \frac{\omega^2}{c^2}u_i = \delta(x - x_s), \qquad \frac{du_i}{dx} \mp \frac{i\omega}{c}u_i \;\to\; 0, \quad \text{as} \quad x \;\to\; \pm\infty. \tag{5.7}$$

The solution to the above problem is nothing but the *Green's function* of the Helmholtz equation with constant coefficients. It is given explicitly by

$$u_i(x, \omega; x_s) = g(x - x_s, \omega) = \frac{ce^{i\frac{\omega}{c}|x - x_s|}}{2i\omega}. \tag{5.8}$$

Exercise 5.1.1 *Verify the above formula for the Green's function (and verify that the radiation condition is satisfied).*

Let us now decompose the Helmholtz solution as the superposition of the incident field and the *scattered field*:

$$u(x, \omega; x_s) = u_i(x, \omega; x_s) + u_s(x, \omega; x_s).$$

From the equations for u and u_i, we verify that u_s satisfies the following equation

$$\frac{d^2u_s}{dx^2} + \frac{\omega^2}{c^2}u_s = \frac{\omega^2}{c^2}\alpha(x)(u_i + u_s), \tag{5.9}$$

with appropriate radiation conditions. By the principle of superposition, this implies that

$$u_s(x, \omega; x_s) = \omega^2 \int_{x_s}^{\infty} \frac{\alpha(y)}{c^2}(u_s + u_i)(y, \omega; x_s)g(x - y, \omega)dy. \tag{5.10}$$

So far, we have not used the assumption that $\alpha(x)$ was small. This approximation, called the Born approximation, allows us to deduce from the above equation that u_s is also of order α. This implies that αu_s is of order α^2, hence much smaller than the other contributions in (5.10). So neglecting u_s on the right hand side of (5.10) and replacing u_i and g by their expression in (5.8), we deduce that a good approximation of u_s is

$$u_s(x_s, \frac{ck}{2}; x_s) = \int_{\mathbb{R}} \frac{\alpha(x)}{4}e^{ikx}dx, \qquad k \in \mathbb{R}. \tag{5.11}$$

This implies that the scattering data $u_s(x_s, \omega; x_s)$ uniquely determines the fluctuation α and that the reconstruction is *stable*: all we have to do is to take the inverse Fourier transform of u_s to obtain $\alpha(x)$. Namely, we have

$$\alpha(x) = \frac{2}{\pi} \int_{\mathbb{R}} e^{-ikx} u_s(x_s, \frac{ck}{2}; x_s) dk. \tag{5.12}$$

Several assumptions have been made to arrive at this result. However as was the case with the MRI problem, we obtain in the end a very simple reconstruction procedure: all we have to do is to compute an inverse Fourier transform.

We can summarize the above result as follows.

Theorem 5.1.1 *Let u_i be given by (5.8) and u_s be the solution of*

$$\frac{d^2 u_s}{dx^2} + \frac{\omega^2}{c^2} u_s = \frac{\omega^2}{c^2} \alpha(x) u_i, \tag{5.13}$$

for α supported in (x_s, R) and such that the radiation conditions (5.5) hold. Consider the measurement operator

$$\alpha \mapsto \mathfrak{M}(\alpha) = u_s(x_s, \cdot; x_s), \tag{5.14}$$

mapping $\alpha(x) \in L^2(x_s, R)$ to $\omega \mapsto u_s(x_s, \omega; x_s) \in L^2(\mathbb{R})$. Then the measurement operator uniquely determines α by means of the explicit inversion formula (5.12). The inversion is Lipschitz stable in the sense that

$$\|\alpha - \tilde{\alpha}\|_{L^2(x_s,R)} \leq C \|\mathfrak{M}(\alpha) - \mathfrak{M}(\tilde{\alpha})\|_{L^2(\mathbb{R})}, \tag{5.15}$$

for a constant $C > 0$.

Exercise 5.1.2 *Write the proof of the above theorem in detail. State a similar theorem for a linearization of the wave equation in the time domain (5.1).*

5.2 Linearized Inverse Scattering problem

In the preceding section, we analyzed the linearization of an inverse scattering problem in one spatial dimension. In this section, we consider extensions to two and three dimensions of space. The model is again a wave equation written in the frequency domain (a Helmholtz equation). Unlike the one dimensional setting, in dimension two or more, plane waves with a given frequency are sufficient to uniquely determine (the low frequency component of) a sound speed. This is the setting that we present here.

5.2.1 Setting and linearization

Let us consider the wave equation in dimension $n = 2, 3$ given by

$$\frac{1}{c_s^2(x)} \frac{\partial^2 p}{\partial t^2} - \Delta p = 0, \qquad x \in \mathbb{R}^n, \quad t > 0, \tag{5.16}$$

with appropriate initial conditions. The velocity $c_s(x)$ is the unknown parameter. Let us assume that $c_s(x) = c$ for $|x| > R$ for a given radius $R > 0$. We assume that $p = 0$ for $t < 0$ and as in the preceding section, pass to the frequency domain by introducing

$$u(x,\omega) = \int_0^\infty e^{i\omega t} p(x,t) dt, \qquad p(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\omega t} u(x,\omega) d\omega. \tag{5.17}$$

The equation for u is then the following *Helmholtz* equation

$$\begin{aligned} &(\Delta + \frac{\omega^2}{c_s^2(x)}) u(x,\omega) = 0, \qquad x \in \mathbb{R}^n, \quad \omega \in \mathbb{R}, \\ &\hat{x} \cdot \nabla u(x,\omega) - i\frac{\omega}{c} u(x,\omega) = o(|x|^{-(n-1)/2}). \end{aligned} \tag{5.18}$$

As usual $\hat{x} = \frac{x}{|x|}$ and the second equation is the *radiation* condition, which ensures that no energy comes from infinity (only waves radiating *out* are allowed at infinity). The notation $o(x)$ means a quantity such that $o(x)/x \to 0$ as $0 < x \to 0$. So the decay at infinity should be faster than $|x|^{-1/2}$ in two dimensions and faster than $|x|^{-1}$ in three dimensions.

Let us now introduce the following linearization for the velocity and the frequency:

$$\frac{1}{c_s^2(x)} = \frac{1}{c^2}\big(1 - \alpha(x)\big), \qquad k = \frac{\omega}{c}. \tag{5.19}$$

We recast the Helmholtz equation as

$$\begin{aligned} &(\Delta + k^2) u(x,\omega) = \alpha(x) k^2 u(x,\omega), \\ &\hat{x} \cdot \nabla u(x,\omega) - iku(x,\omega) = o(|x|^{-(n-1)/2}). \end{aligned} \tag{5.20}$$

Let $\vartheta \in \mathbb{S}^n$ be a unit vector. We verify that

$$(\Delta + k^2) u_i(x,\omega;\vartheta) = 0, \qquad \text{where} \quad u_i(x,\omega;\vartheta) = e^{ik\vartheta\cdot x}. \tag{5.21}$$

Thus plane waves with the right wavenumber $k = |k\vartheta|$ are solutions of the homogeneous Helmholtz equation. Notice however that they *do not* satisfy the radiation conditions (they do radiate out in the direction ϑ but certainly not in the direction $-\vartheta$ since they *come* from infinity in that direction).

The forward problem we are interested in is the following: we have a probing plane wave coming from infinity and want to find a solution $u_s(x,\omega)$ modeling the response of the system to the probing. Therefore we impose that u_s does not radiate at infinity (i.e., satisfies the radiation condition) and that the whole field $u = u_i + u_s$ satisfies the Helmholtz equation. We thus end up with the following *scattering* problem

$$\begin{aligned} &(\Delta + k^2) u_s(x,\omega) = \alpha(x) k^2 (u_s + u_i)(x,\omega), \\ &\hat{x} \cdot \nabla u_s(x,\omega) - iku_s(x,\omega) = o(|x|^{-(n-1)/2}). \end{aligned} \tag{5.22}$$

In the above equation we have used (5.21). Under general assumptions on $\alpha(x)$, the above equation admits a unique solution [28]. The *inverse scattering* problem consists

then of reconstructing $\alpha(x)$ from measurements of u_s at infinity in all possible directions $\hat{x}$ for all possible incoming plane waves $\vartheta \in S^n$. We will not be concerned with this general problem and refer the reader to [28] for more details.

Instead, we concentrate on the linearization of the inverse scattering problem about the constant velocity profile $c_s(x) = c$. Let us assume that α is small (in some appropriate sense that we do not describe further). As a consequence, u_s is also small as can be seen from (5.22). We therefore neglect the term αu_s, which is second-order in α. This approximation, as in the preceding section, is called the Born approximation. It also has the advantage of linearizing the problem of reconstructing $\alpha(x)$ from scattering measurements, which will be described in detail below. We are thus now concerned with the linearized problem

$$\begin{aligned} &(\Delta + k^2)u_s(x,\omega) = \alpha(x)k^2 u_i(x,\omega), \\ &\hat{x}\cdot\nabla u_s(x,\omega) - iku_s(x,\omega) = o(|x|^{-(n-1)/2}). \end{aligned} \tag{5.23}$$

This equation can be solved explicitly as

$$u_s(x,\omega) = k^2 \int_{\mathbb{R}^n} \alpha(y) u_i(y,\omega) g_n(x-y)dy, \tag{5.24}$$

where g_n is the Green function, solution of the following equation

$$\begin{aligned} &(\Delta + k^2)g_n(x) = \delta(x) \\ &\hat{x}\cdot\nabla u_s(x,\omega) - iku_s(x,\omega) = o(|x|^{-(n-1)/2}), \end{aligned} \tag{5.25}$$

and is given for $n = 2, 3$ by

$$g_2(x) = \frac{i}{4}H_0(k|x|), \qquad g_3(x) = \frac{e^{ik|x|}}{4\pi|x|}. \tag{5.26}$$

Here, H_0 is the 0th order Hankel function of the first kind, given by

$$H_0(k|x|) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{1}{\sqrt{k^2-p^2}} e^{i(px+\sqrt{k^2-p^2}y)}dp, \tag{5.27}$$

where we have decomposed $x = (x, y)$ in Cartesian coordinates.

5.2.2 Far field data and reconstruction

The measurements we consider in this section are the far field scattering data. They correspond to the scattered waves propagating outwards at infinity. This simplification amounts to saying that the other component of the radiating field, composed of the *evanescent* waves, is not measured. Mathematically, we consider the asymptotic limit of u_s as $|x| \to \infty$. Let us consider the three dimensional case first. Since x goes to infinity, then $|x-y|$ is equal to $|x|$ plus a smaller order correction. So we have

$$u_s(x,\omega) = \frac{k^2}{4\pi|x|}\int_{\mathbb{R}^3} \alpha(y) e^{ik\vartheta\cdot y} e^{ik|x-y|}dy \ + \ \text{l.o.t.} \ .$$

Upon using the following approximation

$$|x-y| = |x||\hat{x} - \frac{y}{|x|}| = |x|\Big(1 + \frac{|y|^2}{|x|^2} - 2\frac{\hat{x}\cdot y}{|x|}\Big)^{\frac{1}{2}} = |x| - \hat{x}\cdot y + \text{ l.o.t.},$$

we obtain

$$u_s(x,\omega) = \frac{k^2 e^{ik|x|}}{4\pi|x|}\int_{\mathbb{R}^3} \alpha(y) e^{ik(\vartheta-\hat{x})\cdot y} dy \ + \ \text{l.o.t.} \ .$$

We thus observe that

$$\begin{aligned} u_s(x,\omega) &= \frac{k^2 e^{ik|x|}}{4\pi|x|} A(\hat{x}) + o\Big(\frac{1}{|x|}\Big), \\ A(\hat{x}) &= \hat{\alpha}\big(k(\hat{x}-\vartheta)\big) = \int_{\mathbb{R}^3} \alpha(y) e^{ik(\vartheta-\hat{x})\cdot y} dy. \end{aligned} \tag{5.28}$$

Recall that $\omega = ck$. So for a plane wave at a given frequency ω, i.e., at a given wavenumber k, and direction ϑ, the far field measurement is $A(\hat{x}) = A(\hat{x}; k, \vartheta)$ in the direction $\hat{x}$ (obtained by multiplying the measured signal by $4\pi|x|e^{-ik|x|}k^{-2}$).

In two space dimensions ($n = 2$), the final result is similar in the sense that u_s is proportional to $|x|^{-\frac{1}{2}}$ at infinity with a coefficient of proportionality $A(\hat{x})$ taking the same expression as given in (5.28).

The measurement operator we consider in this section thus takes the following form

$$L^\infty(B(0,R)) \ni \alpha \mapsto \mathfrak{M}[\alpha] = A(\cdot; k, \cdot) \in L^\infty(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}), \tag{5.29}$$

which to a (say bounded) sound speed fluctuation $\alpha(x)$ associates the far field measurement $(\hat{x}, \vartheta) \mapsto A(\hat{x}; k, \vartheta)$, which is also bounded as the Fourier transform of a bounded, compactly supported, function. Note that α lives in the n-dimensional space $B(0,R)$ whereas A for a fixed frequency k lives in the $2(n-1)$-dimensional space $\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}$. The latter dimensions agree when $n = 2$ whereas in dimensions $n \geq 3$, the far field data are *richer* than the unknown object α since $2(n-1) > n$ then. Note that within the (Born) approximation of linearization, the measurement operator again provides information about α in the Fourier domain since

$$\mathfrak{M}[\alpha](\hat{x}, \vartheta) = \hat{\alpha}(k(\vartheta - \hat{x})). \tag{5.30}$$

Each measurement $(\hat{x}, \vartheta)$ provides different information about the Fourier transform of the velocity fluctuation $\alpha(x)$. We distinguish two types of measurements. The first ones correspond to directions of measurements x such that $x\cdot\vartheta > 0$. These measurements are called *transmission* measurements since they correspond to the radiated wave that have passed through the object we wish to image. The second ones correspond to the directions such that $x\cdot\vartheta < 0$. They are called *reflection* measurements.

Transmission inverse scattering. Let us consider transmission measurements first, with $\hat{\alpha}(k(\hat{x}-\vartheta))$ known for $\hat{x}\cdot\vartheta > 0$. In particular we obtain for $\hat{x} = \hat{\vartheta}$ the value $\hat{\alpha}(0)$, which is the average of the fluctuation $\alpha(x)$ over the whole domain. More generally as $\hat{x}$ varies in $\mathbb{S}^1$ such that $\hat{x}\cdot\vartheta > 0$, we obtain $\hat{\alpha}(\xi)$ over a half-circle passing through 0, of radius k and symmetric about the axis ϑ. As ϑ varies on the unit circle, we observe

that $\hat{\alpha}(k(\hat{x}-\vartheta))$ fills the disk of radius $\sqrt{2}k$. At a fixed value of k, this is therefore all we can get: $\hat{\alpha}(\xi)$ for ξ such that $|\xi| \leq \sqrt{2}k$.

The picture in three dimensions is very similar: for a given $\vartheta \in S^2$, we have access to $\hat{\alpha}(k)$ for k on a half-sphere of radius $\sqrt{2}k$ passing through 0 and invariant by rotation about ϑ. As ϑ varies over the sphere S^2, we thus get $\hat{\alpha}(k)$ for all k such that $|k| \leq \sqrt{2}k$, as in the two-dimensional case.

The diffraction inverse problem is therefore *not injective*. All we can reconstruct from the measured data is a low-pass filter of the object $\alpha(x)$. The high frequencies are not measured. The high frequencies of α are encoded in the radiation field u_s. However they are the evanescent part of the waves. They decay therefore much more rapidly than $|x|^{-1}$ (when $n = 3$), actually exponentially, and thus cannot be measured accurately in practice.

Let us now consider reconstruction formulas. Since frequencies above $\sqrt{2}k$ cannot be reconstructed, we assume that

$$\alpha(x) = (\mathcal{F}^{-1}_{\xi \to x} \chi_{\sqrt{2}k}(\xi) \mathcal{F}_{x \to \xi} \alpha)(x), \tag{5.31}$$

where $\chi_{\sqrt{2}k}(\xi) = 1$ when $|\xi| < \sqrt{2}k$ and 0 otherwise, i.e. α does not have high wavenumbers. Note that this assumption is inconsistent with our earlier assumption that α was supported in $B(0, R)$. We do not deal with this minor technical difficulty here. Then the reconstruction is obviously unique according to what we just saw. Let us consider the two-dimensional case. We want to reconstruct $\alpha(x)$ from $\hat{\alpha}(k(\hat{x}-\vartheta))$, where $\hat{x}$ and ϑ run over the unit circle S^1. The inverse Fourier transform tells us that

$$\alpha(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ix\cdot\xi} \hat{\alpha}(\xi) d\xi = \frac{k^2}{(2\pi)^2} \int_0^{2\pi} \int_0^{\sqrt{2}} e^{ik\rho x\cdot\vartheta} \hat{\alpha}(k\rho\vartheta) \rho d\rho d\theta.$$

Observe that as ϑ covers the unit circle, all points of the disk $|\xi| < \sqrt{2}k$ are covered twice as $\hat{x}$ varies, once for a point such that $\hat{\xi} \cdot \vartheta^\perp > 0$ and once for a point such that $\hat{\xi} \cdot \vartheta^\perp < 0$. Therefore the information corresponding to $\hat{\xi} \cdot \vartheta^\perp > 0$ is sufficient. This information is parameterized as follows: for a given ϑ we write $\hat{x}$ as

$$\hat{x}(\phi, \vartheta) = \sin\phi\vartheta + \cos\phi\vartheta^\perp, \qquad 0 \leq \phi \leq \frac{\pi}{2}. \tag{5.32}$$

We thus obtain that

$$\hat{\alpha}(k(\hat{x}-\vartheta)) = \hat{\alpha}\Big(k\rho(\phi)\big(\frac{\sin\phi - 1}{\rho(\phi)}\vartheta + \frac{\cos\phi}{\rho(\phi)}\vartheta^\perp\big)\Big) = \hat{\alpha}\big(k\rho(\phi)\mathcal{R}(\phi)\vartheta\big),$$

with $\rho(\phi) = \sqrt{2}\sqrt{1-\cos\phi}$ and $\mathcal{R}(\phi)$ an explicitly defined rotation depending on ϕ. Here is the rest of the reconstruction:

$$\begin{aligned}\alpha(x) &= \frac{k^2}{(2\pi)^2} \int_0^{2\pi} \int_0^{\pi/2} e^{ik\rho(\phi)x\cdot\vartheta} \hat{\alpha}(k\rho(\phi)\vartheta) \rho(\phi) \frac{d\rho(\phi)}{d\phi} d\phi d\theta \\ &= \frac{k^2}{(2\pi)^2} \int_0^{2\pi} \int_0^{\pi/2} e^{ik\rho(\phi)x\cdot\mathcal{R}(\phi)\vartheta} \hat{\alpha}(k\rho(\phi)\mathcal{R}(\phi)\vartheta) \frac{1}{2} \frac{d\rho^2(\phi)}{d\phi} d\phi d\theta,\end{aligned}$$

so that finally

$$\alpha(x) = \frac{2k^2}{(2\pi)^2} \int_0^{2\pi} \int_0^{\pi/2} e^{ik\rho(\phi)x\cdot\mathcal{R}(\phi)\vartheta} \hat{\alpha}(k|\hat{x}(\phi,\vartheta) - \vartheta|) \sin\phi d\phi d\theta. \tag{5.33}$$

This is the reconstruction formula we were after.

Reflection inverse scattering. Let us conclude the section with a few words about reflection tomography. In that case, we only measure data in directions $\hat{x}$ such that $\hat{x} \cdot \vartheta < 0$. Following the same techniques as above, we see that we can reconstruct wavenumbers of $\alpha(x)$ in the corona of wavenumbers ξ such that $\sqrt{2}k < |\xi| < 2k$. The reconstruction from reflection data is therefore by no means unique. We cannot reconstruct low-frequency components of α and cannot reconstruct very-high frequencies either. Assuming that the wavenumber content of $\alpha(x)$ is in the above corona, then the reconstruction is unique. A reconstruction formula similar to what we just obtained can also be derived. Notice that when both the transmission and reflection data can be measured, we can reconstruct all wavenumbers ξ of $\alpha(x)$ such that $|\xi| < 2k$.

All these result are in sharp contrast to the one-dimensional example we saw in the preceding section. There, a given wavenumber k allows us to reconstruct one wavenumber of $\alpha(x)$. All wavenumbers are thus required (i.e. measurements for all frequencies ω) to reconstruct $\alpha(x)$. Here $\alpha(x)$ is also uniquely determined by measurements obtained for all values of k (since each value of k allows us to reconstruct all wavenumbers $|\xi| < 2k$). However because of the multidimensional nature of the measurements (the variable $\hat{x}$ is discrete in one-dimension instead of living on the unit sphere $\mathbb{S}^n$), measurements for all values of k is quite redundant: once we have obtained measurements at a given value of k_0, all measurements obtained for wavenumbers $k < k_0$ are redundant.

5.2.3 Comparison to X-ray tomography

Let us consider the case of transmission data in two space dimensions. We have seen that wavenumbers of $\alpha(x)$ up to $\sqrt{2}k$ could be reconstructed. However as k tends to ∞, this essentially means that all wavenumbers of $\alpha(x)$ can be reconstructed.

Indeed in that limit we observe that the half circle of radius k becomes the full line orthogonal to ϑ. That is, as $k \to \infty$, the measurements tend to

$$\hat{\alpha}(\sigma\vartheta^{\perp}) = R\alpha\Big(\sigma, \theta + \frac{\pi}{2}\Big).$$

Exercise 5.2.1 Show that the reconstruction formula (5.33) indeed converges to the inverse Radon transform as $k \to \infty$.

In the limit of infinite frequency, we therefore obtain that the transmission measurements tend to the Radon transform of α. We have seen in Chapter 2 that the knowledge of $R_\alpha(\sigma, \theta + \pi/2)$ for all values of σ and θ was sufficient to uniquely reconstruct the fluctuation $\alpha(x)$.

So how should we consider the inverse diffraction problem? How ill-posed is it? As we already mentioned, the first problem with diffraction tomography is that for a fixed frequency ω, the function $\alpha(x)$ cannot uniquely be reconstructed. Only the wavenumbers below $\sqrt{2}k$ (below $2k$) in the case of transmission (transmission and reflection) measurements can be reconstructed. However in the class of functions $\alpha(x) \in L^2(\mathbb{R}^2)$ such that (5.31) holds, we have uniqueness of the reconstruction. In this class we can perform a similar analysis to what we obtained in Theorem 2.1.2.

Let us consider the measurements $d(\phi, \vartheta) = \hat{\alpha}(k(\hat{x} - \vartheta))$ for $\vartheta \in S^1$ and $0 \leq \phi \leq \pi/2$ using (5.32). We verify that $1 \leq \rho'(\phi) \leq \sqrt{2}$ for $0 \leq \phi \leq \pi/2$.

Let us assume that the error we make is of the same order for every angle ϕ and every angle ϑ. An estimate of the total error will thus involve

$$\begin{aligned}
&\int_{S^1}\int_0^{\pi/2} |d(\phi,\vartheta)|^2 d\phi d\vartheta = \int_{S^1}\int_0^{\pi/2} |\hat{\alpha}(k\rho(\phi)\mathcal{R}(\phi)\vartheta)|^2 d\phi d\vartheta \\
&= \int_{S^1}\int_0^{\sqrt{2}} |\hat{\alpha}(k\rho\vartheta)|^2 (\rho')^{-1} d\rho d\vartheta \sim \int_{S^1}\int_0^{\sqrt{2}} |\hat{\alpha}(k\rho\vartheta)|^2 d\rho d\vartheta \\
&\sim \frac{1}{k}\int_{S^1}\int_0^{\sqrt{2}k} |\hat{\alpha}(u\vartheta)|^2 du d\vartheta \sim \frac{1}{k}\int_{S^1}\int_0^{\sqrt{2}k} \frac{|\hat{\alpha}(u\vartheta)|^2 u\ du\ d\vartheta}{u} \\
&\geq \frac{1}{k}\|\alpha\|^2_{H^{-\frac{1}{2}}(\mathbb{R}^2)}.
\end{aligned}$$

In some sense, the above formula also shows that the data $d(\phi,\vartheta)$ are more regular than the function $\alpha(x)$ by half of a derivative. This is consistent with the Radon transform in the limit as $k \to \infty$. To be more consistent with the Radon transform, notice in the limit $k \to \infty$ that $k\cos\phi \sim \sigma$ so that $k\sin\phi d\phi \sim d\sigma$ as the half circle converges to the real line. Since $\sin\phi \sim 1$ for most of the wavenumbers σ as $k \to \infty$, this implies that $kd\phi \sim d\sigma$. Therefore a total error in the angular measurements in diffraction tomography consistent with the measurement errors for the Radon transform is given by

$$\int_{S^1}\int_0^{\pi/2} |d(\phi,\vartheta)|^2 k d\phi d\vartheta \geq \|\alpha\|^2_{H^{-1/2}(\mathbb{R}^2)}.$$

We recover in this limit that the measurements in diffraction tomography regularize the function α by half of a derivative.

Note that we see here again that the ill-posedness of a problem very much depends on the norm in which the error on the data is measured.

5.3 Inverse source problem in PAT

Consider the following wave equation

$$\begin{aligned}
\frac{\partial^2 p}{\partial t^2} - \Delta p &= 0, && t > 0,\ x \in \mathbb{R}^n \\
p(0,x) &= f(x), && x \in \mathbb{R}^n \\
\partial_t p(0,x) &= 0, && x \in \mathbb{R}^n.
\end{aligned} \tag{5.34}$$

The inverse wave source problem consists of reconstructing the initial condition f supported in a open, convex, bounded domain X from knowledge of $p(t,x)$ for $t > 0$ and $x \in \partial X$. This inverse problem finds application in Photo-acoustic tomography (PAT). We will come back to the modeling of this imaging modality in Chapter 9.

5.3.1 An explicit reconstruction formula for Σ the unit sphere

Explicit inversion formulas are known when $\Sigma = \partial X$ is a simple geometry. Explicit reconstruction procedures based on time reversal are also known for general surfaces Σ enclosing the support of the source $f(x)$, which we assume is compactly supported.

All explicit inversion formulas are based one way or another on the Fourier transform.

Let us assume that Σ is the unit sphere $|x| = 1$. Then an explicit reconstruction formula by L. Kunyanski shows that in dimension $n = 3$, we have

$$f(x) = \frac{1}{8\pi^2}\nabla_x \cdot \int_{|y|=1} \nu(y)\Big(\frac{1}{t}\frac{d}{dt}\frac{p(y,t)}{t}\Big)_{t=|y-x|} dS_y. \tag{5.35}$$

The above formula generalizes to arbitrary dimension. We shall not delve into the details of this inversion formula. Like the formula based on the Radon transform, it uses the symmetries of the Fourier transform. However, it is based on Fourier transforms on the sphere (spherical harmonics) that we do not present here. The above reconstruction shows that the source term $f(x)$ can uniquely and stably be reconstructed from knowledge of u on the surface Σ.

5.3.2 An explicit reconstruction for detectors on a plane

In this section, we present an inversion procedure for point-wise detectors based on the use of the Fourier transform. We assume that f is compactly supported in the unit ball $B(0,1)$ and assume that the measurements are performed at each point of the hyperplane $x_n = 1$. The reconstruction procedure is essentially independent of dimension $n \geq 2$ and is therefore presented for arbitrary dimensions. We denote by $x' = (x_1, \ldots, x_{n-1})$ and by $z = x_n$. We want to reconstruct $f(x)$ from knowledge of $p(t, x', z = 1)$ for all $t \geq 0$ and $x' \in \mathbb{R}^{n-1}$ knowing that

$$\Big(\frac{\partial^2}{\partial t^2} - \Delta\Big)p = \delta_0'(t)f(x), \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^n.$$

We denote by $u(\omega, \xi', z)$ the partial Fourier transform of p given by

$$u(\omega, \xi', z) = \int_{\mathbb{R}^n} e^{-i(\omega t + \xi' \cdot x')} p(t, x', z) dt dx',$$

and by $\tilde{f}(\xi', z) = \int_{\mathbb{R}^{n-1}} e^{-i\xi' \cdot x'} f(x', z) dx'$.

Since differentiation in t corresponds to multiplication by $i\omega$ in the Fourier domain, with a similar expression in the x' variable, we find that

$$\Big(\omega^2 - |\xi'|^2 + \frac{\partial^2}{\partial z^2}\Big)u = -i\omega\tilde{f} := s(\omega, \xi', z).$$

For the reconstructions, it turns out that we need only the subset $\omega > |\xi'|$ and define $\theta = \sqrt{\omega^2 - |\xi'|^2}$. The above equation is a second-order non-homogeneous equation similar to the forced harmonic oscillator. Its solution is of the form

$$u(\omega, \xi', z) = A(\omega, \xi')e^{i\theta z} + B(\omega, \xi')e^{-i\theta z} + \int_0^z s(z')\frac{\sin\theta(z - z')}{\theta} dz',$$

which we recast as

$$u(\omega, \xi', z) = \Big(A + \int_0^z s(z')\frac{e^{-i\theta z'}}{2i\theta} dz'\Big)e^{i\theta z} + \Big(B - \int_0^z s(z')\frac{e^{i\theta z'}}{2i\theta} dz'\Big)e^{-i\theta z}.$$

The support of f and hence of s is a subset of $(-1,1)$ in the z variable. As $z \to \infty$, we want only outgoing waves (no incoming waves), which imposes that

$$B(\omega,\xi') = \int_0^1 s(z')\frac{e^{i\theta z'}}{2i\theta}dz'.$$

Similarly, as $z \to -\infty$, the radiation condition imposes that

$$A(\omega,\xi') = \int_{-1}^0 s(z')\frac{e^{-i\theta z'}}{2i\theta}dz'.$$

At $z=1$, where information is available, this means,

$$u(\omega,\xi',1) = e^{i\theta}\int_{-1}^1 s(\omega,\xi',z)\frac{e^{-i\theta z'}}{2i\theta}dz' = -\frac{\omega}{2\theta}e^{i\theta}\int_{\mathbb{R}} e^{-i\theta z}\tilde{f}(\xi',z)dz.$$

Since f is real valued, then $\tilde{f}^*(-\xi',z) = \tilde{f}(\xi',z)$. Let us therefore define

$$v(\theta,\xi') = \begin{cases} \dfrac{-2\theta}{\sqrt{\theta^2+|\xi'|^2}}u(\sqrt{\theta^2+|\xi'|^2},\xi',1)e^{-i\theta} & \theta > 0 \\ \dfrac{2\theta}{\sqrt{\theta^2+|\xi'|^2}}u(-\sqrt{\theta^2+|\xi'|^2},-\xi',1)e^{-i\theta} & \theta < 0. \end{cases}$$

Then we find that

$$v(\theta,\xi') = \int_{\mathbb{R}} e^{-i\theta z}\tilde{f}(\xi',z)dz := \hat{f}(\xi)$$

is the Fourier transform of f in all variables.

Note that only the frequencies $|\omega| > |\xi'|$ are used in the reconstruction. This corresponds to the propagating modes. The frequencies $|\omega| < |\xi'|$ are still measured but are not necessary in the reconstruction. These evanescent modes decay exponentially in z and their use might in fact yield unstable reconstructions.

Using the Parseval relation and the fact that $|\theta|d\theta = |\omega|d\omega$, we have

$$\|f\|_{L^2(B(0,1))} = c\|\hat{f}\|_{L^2(\mathbb{R}^n)} = c\|v\|_{L^2(\mathbb{R}\times\mathbb{R}^{n-1})} \leq C\|u_{|z=1}\frac{\theta}{\omega}\|_{L^2(\mathbb{R}\times\mathbb{R}^{n-1})} \leq C\|u_{|z=1}\|_{L^2(\mathbb{R}\times\mathbb{R}^{n-1})}$$

since $|\frac{\theta}{\omega}| \leq 1$. Consequently, we find that

$$\|f\|_{L^2(B(0,1))} \leq C\|p(t,x',1)\|_{\mathbb{R}\times\mathbb{R}^{n-1}}.$$

The inverse wave problem is therefore injective and well posed in the L^2 sense. This means that there exists a constant $C > 0$ such that

$$\|f\|_{L^2(B(0,1))} \leq C\|\mathfrak{M}f\|_{L^2(\mathbb{R}^n)}.$$

The above inverse wave problem is therefore stable. Measurement errors are not amplified too drastically during the reconstruction and the inverse wave source problem is consequently a high resolution inverse problem.

5.4 One dimensional inverse coefficient problem

The three inverse problems considered so far in this chapter were linear inverse problems. The first two inverse scattering problems were defined as the linearization of nonlinear inverse scattering problems. The last inverse source problem is linear without any approximation. In this section, we consider a one-dimensional inverse coefficient problem, which is nonlinear and treated as such. This nonlinear inverse problem enjoys the same properties as their linear counterparts: Wave equations propagate singularities and thus many measurement operators based on solutions of wave equations are Lipschitz stable.

Consider the equation

$$\begin{aligned} &\frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial x^2} - q(x)p = 0, && t > 0,\ x > 0 \\ &p(t,0) = f(t), && t > 0, \\ &p(0,x) = \partial_t p(0,x) = 0 && x > 0. \end{aligned} \tag{5.36}$$

This problem can equivalently be posed for $t \in \mathbb{R}$ by assuming that the source term $f(t)$ is supported in $t \in \mathbb{R}_+$. The solution p then vanishes for $t \leq 0$ by finite speed of propagation. For the same reason, the solution is supported in the domain $t \geq x \geq 0$.

Here, $q(x)$ is an unknown bounded potential. We then assume that $g(t) = \partial_x p(t,0) + f'(t)$ is measured at the domain's boundary. This defines the measurement operator

$$L^\infty(0,L) \ni q \mapsto \mathfrak{M}_f(q) = g \in L^\infty(0,2L). \tag{5.37}$$

The objective is to show that $g(t)$ for $t \in (0,2L)$ uniquely determines $q(x)$ on $(0,L)$ under some conditions on $f(t)$. Moreover, depending on the structure of $f(t)$ at $t = 0$, we obtain different stability estimates showing that the reconstruction of $q(x)$ is a well posed problem in appropriate topologies.

Cauchy problems at $x = 0$ and $t = 0$. Let us define $p_1(t,x) = p(t,x) - f(t-x)$, where $f(t-x)$ is seen to be the solution of the above problem when $q \equiv 0$. Then we find that

$$\begin{aligned} &\frac{\partial^2 p_1}{\partial t^2} - \frac{\partial^2 p_1}{\partial x^2} = q(x)p(t,x) := s(t,x) && t \in \mathbb{R},\ x > 0 \\ &p_1(t,0) = 0, \quad \frac{\partial p_1}{\partial x}(t,0) = g(t), && t \in \mathbb{R} \\ &p_1(0,x) = \frac{\partial p_1}{\partial t}(0,x) = 0 && x > 0. \end{aligned} \tag{5.38}$$

The above problem may be seen in two ways. It can either be seen as a Cauchy problem for $x > 0$ and $t \in \mathbb{R}$ with Cauchy data at $x = 0$. The solution is then given by

$$p_1(t,x) = \frac{1}{2}\int_{t-x}^{t+x} g(\tau)d\tau + \frac{1}{2}\int_{\Delta(t,x)} q(y)p(\tau,y)d\tau dy, \tag{5.39}$$

where we have defined the triangular-shaped area:

$$\Delta(t,x) = \big\{(\tau,y), \quad 0 < y < x, \quad t-(x-y) < \tau < t+(x-y)\big\}. \tag{5.40}$$

Alternatively, we can write the above problem as a Cauchy problem with data at $t = 0$. We need to ensure that the constraint $p_1(t, 0) = 0$ is satisfied. In order to do so, let us define $f(t, x) = f(t - x)$ for $t > 0$ and $x > 0$ and $f(t, x) = -f(t + x)$ for $t > 0$ and $x < 0$ and let us extend the potential so that $q(x) := q(-x)$ for $x < 0$. We then recast the wave equation for p_1 as

$$\begin{aligned} &\frac{\partial^2 p_1}{\partial t^2} - \frac{\partial^2 p_1}{\partial x^2} = q(x)\big(f(t,x) + p_1(t,x)\big) && t > 0,\ x \in \mathbb{R} \\ &p_1(0,x) = \frac{\partial p_1}{\partial t}(0,x) = 0 && x \in \mathbb{R}. \end{aligned} \tag{5.41}$$

The solution to that equation then satisfies the constraint $p(t, -x) = -p(t, x)$ and with $s(t, x) := q(x)(f(t, x) + p_1(t, x))$ is given explicitly by

$$p_1(t,x) = -\frac{1}{2}\int_{\tilde{\Delta}(t,x)} s(\tau, y)dyd\tau, \tag{5.42}$$

where $\tilde{\Delta}(t, x) = \{(\tau, y),\ 0 < \tau < t,\ x - (t - \tau) < y < x + (t - \tau)\}$.

Exercise 5.4.1 *Check the above two formulas for $p_1(t, x)$.*

We thus obtain again that

$$p_1(t,x) = -\frac{1}{2}\int_{\tilde{\Delta}(t,x)} q(y)\big[f(\tau, y) + p_1(\tau, y)\big]d\tau dy. \tag{5.43}$$

For $f(t) = \delta(t)$, we observe that $p_1(t, x)$ is a bounded function. This implies that $g(t)$ is also a bounded function. We define here $f(t) = \delta(0)$ as the limit of $f_\varepsilon(t)$ with $f_\varepsilon(t) = \frac{1}{\varepsilon}$ for $t \in (0, \varepsilon)$ and $f_\varepsilon(t) = 0$ otherwise. We also write p and p_1 the corresponding limits as $\varepsilon \to 0$.

Exercise 5.4.2 *Show that p_1 is indeed a bounded function. Find that the right-hand side in (5.43) with p_1 set to 0 is indeed bounded. Then use an argument based on a Gronwall lemma as indicated below to conclude.*

Nonlinear problem for $q(x)$. Let us come back to (5.39) and evaluate it at $t = 0$ for $x > 0$. Then we find

$$p_1(0,x) = \frac{1}{2}\int_{-x}^{x} g(\tau)d\tau + \frac{1}{2}\int_0^x \int_{-(x-y)}^{x-y} s(\tau, y)d\tau dy.$$

Differentiating with respect to x yields

$$g(x) + \int_0^x q(y)p(x - y, y)dy = 0. \tag{5.44}$$

This may be seen as a nonlinear equation for q since p depends on q.

The same equation may also be obtained from (5.43). Indeed, differentiating in x yields

$$\partial_x p_1(t,x) = -\frac{1}{2}\int_0^t (s(\tau, x + (t - \tau)) - s(\tau, x - (t - \tau)))\tau = -\int_0^t s(\tau, x + (t - \tau))d\tau.$$

Evaluated at $x = 0$, this yields

$$\partial_x p_1(t,0) = g(t) = -\int_0^t q(\tau)p(t-\tau,\tau)d\tau, \tag{5.45}$$

which is equivalent to (5.44). The latter may be recast as

$$g(x) + \int_0^x q(y)f(x-2y)dy + \int_0^x q(y)p_1(x-y,y)dy = 0.$$

Let us now assume that $f(t) = \delta(t)$ as the limit indicated above. Then we find that

$$g(x) + \frac{1}{2}q(\frac{x}{2}) + \int_0^x q(y)p_1(x-y,y)dy = 0.$$

However, by the finite speed of propagation, $p(t,x)$ and $p_1(t,x)$ are supported on $t \geq x$. This means that $p_1(x-y,y)$ is supported on $y \leq \frac{x}{2}$. As a consequence, changing x to $2x$, we find that for all $x > 0$,

$$q(x) = -2g(2x) - 2\int_0^x q(y)p_1(2x-y,y)dy. \tag{5.46}$$

This is a nonlinear equation for $q(x)$ of Fredholm type. We define $\tilde{g}(x) = -2g(2x)$.

Error estimates and Gronwall lemma. Let us consider $\tilde{g} = \mathfrak{M}_f(\tilde{q})$ and $g = \mathfrak{M}_f(q)$. We define $\delta q = q - \tilde{q}$ and $\delta p = p - \tilde{p}$ as well as $\delta\check{g}$ with obvious notation. Then we find that

$$\begin{aligned} \delta q(x) &= \delta\check{g}(x) - 2\int_0^x \delta q(y)p_1(2x-y,y)dy - 2\int_0^x \tilde{q}(y)\delta p_1(2x-y,y)dy \\ \delta p_1(t,x) &= \frac{1}{2}\int_{t-x}^{t+x} \delta g(\tau)d\tau + \frac{1}{2}\int_{\Delta(t,x)} \big[\delta q(y)p(\tau,y) + \tilde{q}(y)\delta p_1(\tau,y)\big]d\tau dy. \end{aligned} \tag{5.47}$$

Here, we used (5.39) for the expression of p_1.

Let us define

$$\Gamma(T) = \{(x,t) \in \mathbb{R}^2,\ 0 \leq t \leq T,\ x = t\} \cup \{(x,t) \in \mathbb{R}^2,\ 0 \leq t \leq T,\ x = 2T - t\}, \tag{5.48}$$

the part of the boundary of $\Delta(T,T)$ that does not include the segment $(0, 0 < t < T)$. We next define

$$\delta P(T) = \sup_{\Gamma(T)} |\delta p_1(t,x)|, \qquad \delta Q(T) = \sup_{t\in[0,T]} |\delta q|(t). \tag{5.49}$$

Then we deduce from (5.47) that

$$\begin{aligned} \delta Q(T) &\leq \|\delta g\|_{L^\infty(0,2T)} + C\int_0^T \delta Q(\tau)d\tau + C\int_0^T \delta P(\tau d\tau \\ \delta P(T) &\leq T\|\delta g\|_{L^\infty(0,2T)} + C\int_0^T \delta Q(\tau)d\tau + C\int_0^T \delta P(\tau)d\tau. \end{aligned}$$

Exercise 5.4.3 *Verify the above expression. The main difficulty involves the analysis of the term*

$$\int_{\Delta(t,x)} \delta q(y)\delta(\tau - y)d\tau dy,$$

which has to be seen as the limit with source term f_ε as $\varepsilon \to 0$. It is bounded by $\int_0^T |\delta q|(\tau)d\tau$ and hence by $\int_0^T \delta Q(\tau)d\tau$.

We deduce that

$$(\delta P + \delta Q)(T) \leq (1+T)\|\delta g\|_{L^\infty(0,2T)} + C\int_0^T (\delta P + \delta Q)(\tau)d\tau$$

with constant C uniform in T. As an application of the Gronwall lemma, we deduce that

$$(\delta P + \delta Q)(T) \leq Ce^{CT}\|\delta g\|_{L^\infty(0,2T)},$$

for some positive constant C also independent of T.

Exercise 5.4.4 *Verify that the Gronwall lemma applies to the above problem and yields the latter inequality.*

This proves the following result:

Theorem 5.4.1 *Let $f(t) = \delta(t)$ and q and $\tilde{q}$ be bounded potentials. Then $\mathfrak{M}_\delta(q) = \mathfrak{M}_\delta(\tilde{q})$ implies that $q = \tilde{q}$ and we have the following error estimate*

$$\|q - \tilde{q}\|_{L^\infty(0,T)} \leq Ce^{CT}\|\mathfrak{M}_\delta(q) - \mathfrak{M}_\delta(\tilde{q})\|_{L^\infty(0,2T)}. \tag{5.50}$$

The constant C is uniform in $T > 0$.

Let us now assume that the initial condition is not $f(t) = \delta(t)$ but rather $f_n(t) = \frac{1}{n!}t^n$ for $n \geq 0$. Then obviously $f_n^{(n+1)}(t) = \delta(t)$. Moreover, since q is independent of t, then $\partial_t^{n+1}u$ solves the same equation as u with f_n replaced by $\delta(t)$. As a consequence the measurements $g_n(t)$ with $f = f_n$ are such that $g_n^{(n+1)}(t) = g(t)$, the measurements corresponding to $f = \delta(t)$. This yields

Corollary 5.4.2 *Let $f_n(t) = \frac{1}{n!}t^n$ and q and $\tilde{q}$ be bounded potentials. Then $\mathfrak{M}_{f_n}(q) = \mathfrak{M}_{f_n}(\tilde{q})$ implies that $q = \tilde{q}$ and we have the following error estimate*

$$\|q - \tilde{q}\|_{L^\infty(0,T)} \leq Ce^{CT}\|\mathfrak{M}_{f_n}(q) - \mathfrak{M}_{f_n}(\tilde{q})\|_{W^{n+1,\infty}(0,2T)}. \tag{5.51}$$

The constant C is uniform in $T > 0$.

This straightforward corollary indicates that the stability estimate strongly depends on the probing mechanism. The smoother the probing mechanism, the worse is the stability estimate.

Intuitively, the above result is clear. A signal emitted at $t = 0$ propagates to position x during time $t = x$ and then back to $x = 0$ at time $t = 2x$. Only that signal provides information about q at position x. When multiple signals are emitted, then superpositions of signals create some blurring. The above corollary quantifies such a blurring.

Chapter 6

Inverse Kinematic and Inverse Transport Problems

Several problems of Integral Geometry model the propagation of particles interacting with the underlying medium. This is the case for the Radon transform seen as a transmission tomography problem in (2.1) and for the Attenuated Radon transform in (2.23). Another transport equation was briefly encountered in (4.33) in the analysis of the Generalized Ray transform. These source problems belong to a larger class of inverse kinematic and inverse transport problems that we analyze in this chapter.

The propagation of particles essentially follows two dynamics: a Hamiltonian (of classical mechanics) describes the "free" propagation of particles in the absence of interactions with the underlying structure; a scattering operator models how particles are absorbed and scattered when they interact with the underlying medium. The inverse kinematic and inverse transport problems aim to reconstruct the structures of the Hamiltonian and the scattering operator from measurements of particle densities.

The first part of this chapter considers the reconstruction of a simple Hamiltonian of the form $H(x,k) = c(x)|k|$ from travel time boundary measurements. The second part of the chapter assumes the Hamiltonian known with c a constant and considers the reconstruction of the absorption and scattering coefficients from boundary measurements of particle densities.

Scattering: a transition to ill-posedness. Before addressing these inverse problems in more detail, let us comment on the influence of scattering on the structure of an inverse problem and the stability properties of a measurement operator.

The inverse problems we have encountered in the preceding three chapters, to which we can add the simplified M.R.I. description of the first chapter, were "well-posed" inverse problems in the sense that they involved a stable inversion operator from a Hilbert space to another Hilbert space. Wave propagation (as in diffraction tomography) or particle propagation along straight lines (as in computerized tomography) all generate well-posed inverse problems.

What causes then an inverse problem to be ill-posed? As we mentioned in the introduction, an inverse problem is ill-posed when the measurement operator is a highly smoothing/regularizing operator. We have seen that solving the heat equation forward was a highly regularizing process. The main mechanism explaining its regularizing

property is *scattering*: using a kinematic description of the diffusion equation, particles represented by Brownian motions scatterer infinitely often before they exit the domain of interest.

Scattering is one of the main mechanisms that regularizes solutions to PDEs and hence renders inverse problems ill-posed. The linear transport equation, also known as the linear Boltzmann equation or the radiative transfer equation, offers an ideal example of a transition from non-scattering to scattering environments.

6.1 Inverse Kinematic Problem

In the absence of scattering or absorption, which we assume in this section, the propagation of particles is modeled by the following Hamiltonian dynamics

$$\begin{aligned} \frac{dx}{dt} &= \dot{x} = \nabla_k H(x(t), k(t)) \qquad x(0) = x_0, \\ \frac{dk}{dt} &= \dot{k} = -\nabla_x H(x(t), k(t)) \qquad k(0) = k_0. \end{aligned} \tag{6.1}$$

As we did in the preceding chapter, we consider the simplest example of a Hamiltonian

$$H(x, k) = c(x)|k|. \tag{6.2}$$

This Hamiltonian models the propagation of high frequency acoustic waves in a medium with a spatially varying sound speed $c(x)$. It also models the propagation of light in media with varying indices of refraction when polarization effects are neglected. The index of refraction $n(x)$ is defined as $n(x) = \frac{c}{c(x)}$ with c light speed in a vacuum. The same Hamiltonian may be used to model the propagation of seismic (shear) waves in the Earth when pressure waves and polarization properties of shear waves are neglected.

The equation (6.1) models the propagation of a single particle knowing its information (x_0, k_0) at time 0. The density $u(t, x, k)$ of an ensemble of such particles then solves the corresponding Liouville equation

$$\frac{\partial u}{\partial t} + \{H, u\} = 0, \qquad t > 0, \qquad u(0, x, k) = u_0(x, k), \tag{6.3}$$

where $u_0(x, k)$ is the density of particles at time $t = 0$ and where

$$\{H, u\} := \nabla_k H \cdot \nabla_x u - \nabla_x H \cdot \nabla_k u = c(x)\hat{k} \cdot \nabla_x u - |k|\nabla c(x) \cdot \nabla_k u, \tag{6.4}$$

is the Poisson bracket of the Hamiltonian H and the density u. The inverse kinematic problem thus concerns the reconstruction of the Hamiltonian H from measurements of the density u at the boundary of a domain of interest.

In the following section, we first consider a one-dimensional version of the inverse kinematic problem in a geometry with spherical symmetry. This inverse kinematic problem was solved by Herghlotz in 1905 as a means to understand the inner structure of the Earth from travel time measurements. We then revisit the Mukhometov method to solve the inverse kinematic problem in two dimensions of space.

6.1.1 Spherical symmetry

We wish to reconstruct the velocity field $c(r)$ inside the Earth assuming spherical symmetry of the Hamiltonian:

$$H(x,k) = c(r)|k| \qquad \text{with} \qquad r = |x|.$$

Our data is the time it takes from a particle following (6.1) and emitted at the surface of a sphere with a given direction to come back to the surface of that sphere. To simplify we assume that the Earth radius is normalized to 1. We also assume that $c(1)$ is known. We want to reconstruct $c(r)$ from the time it takes to travel along all possible geodesics (the trajectories of (6.1)). Because the geodesics depend on $c(r)$, the travel time is a *nonlinear* functional of the velocity profile $c(r)$.

Let us denote $x = r\hat{x}$ and $k = |k|\vartheta$. The Hamiltonian dynamics take the form

$$\dot{x} = c(r)\vartheta, \qquad \dot{k} = -c'(r)|k|\hat{x}. \tag{6.5}$$

We are interested in calculating the travel times between points at the boundary of the domain $r = |x| < 1$. This implies integrating dt along particle trajectories. Since we want to reconstruct $c(r)$, we perform a change of variables from dt to dr. This will allow us to obtain integrals of the velocity $c(r)$ along curves. The objective will then be to obtain a reconstruction formula for $c(r)$.

In order to perform the change of variables from dt to dr, we need to know where the particles are. Indeed the change of variables should only involve position r and no longer time t. This implies to solve the problem $t \mapsto r(t)$. As usual it is useful to find invariants of the dynamical system. The first invariant is as always the Hamiltonian itself:

$$\frac{dH(x(t),k(t))}{dt} = 0,$$

as can be deduced from (6.1). The second invariant is *angular momentum* and is obtained as follows. Let us first introduce the basis $(\hat{x}, \hat{x}^\perp)$ for two dimensional vectors (this is the usual basis (e_r, e_θ) in polar coordinates). We decompose $k = k_r\hat{x} + k_\theta\hat{x}^\perp$ and $\vartheta = \hat{k}_r\hat{x} + \hat{k}_\theta\hat{x}^\perp$. We verify that

$$\dot{r} = c(r)\hat{k}_r \qquad \text{since} \qquad \dot{x} = \dot{r}\hat{x} + r\dot{\hat{x}} = c(r)\vartheta. \tag{6.6}$$

We also verify that

$$\frac{d(rk_\theta)}{dt} = \frac{dx^\perp \cdot k}{dt} = \dot{x}^\perp \cdot k + x \cdot \dot{k}^\perp = c(r)\vartheta^\perp \cdot k - c'(r)|k|x \cdot \hat{x}^\perp = 0. \tag{6.7}$$

This is conservation of angular momentum, which implies that

$$r(t)k_\theta(t) = k_\theta(0),$$

since $r(0) = 1$.

By symmetry, we observe that the travel time is decomposed into two identical components: the time it takes to go down the Earth until $k_r = 0$, and the time it takes to go back up. On the way up to the surface, k_r is non-negative. Let us denote $p = \hat{k}_\theta(0)$

with $0 < p < 1$. The lowest point is reached when $\hat{k}_\theta = 1$. This means at a point r_p such that

$$\frac{r_p}{c(r_p)} = \frac{p}{c(1)}.$$

To make sure that such a point is uniquely defined, we impose that the function $rc^{-1}(r)$ be *increasing* on $(0,1)$ since it cannot be decreasing. This is equivalent to the constraint:

$$c'(r) < \frac{c(r)}{r}, \qquad 0 < r < 1. \tag{6.8}$$

This assumption ensures the uniqueness of a point r_p such that $pc(r_p) = c(1)r_p$.

Since the Hamiltonian $c(r)|k|$ is conserved, we deduce that

$$\dot{r} = c(r)\sqrt{1-\hat{k}_\theta^2} = c(r)\sqrt{1-\Big(\frac{\hat{k}_\theta(0)c(r)}{rc(1)}\Big)^2},$$

so that

$$dt = \frac{dr}{c(r)\sqrt{1-\Big(\frac{\hat{k}_\theta(0)c(r)}{rc(1)}\Big)^2}}. \tag{6.9}$$

Notice that the right-hand side depends only on r and no longer on functions such as $\hat{k}_r$ that depend on time. The travel time as a function of $p = \hat{k}_\theta(0)$ is now given by twice the time it takes to go back to the surface:

$$T(p) = 2\int_{t(r_p)}^{1} dt = 2\int_{r_p}^{1} \frac{dr}{c(r)\sqrt{1-\Big(\frac{\hat{k}_\theta(0)c(r)}{rc(1)}\Big)^2}}. \tag{6.10}$$

Our measurements are $T(p)$ for $0 < p < 1$ and our objective is to reconstruct $c(r)$ on $(0,1)$. We need a theory to invert this integral transform. Let us define

$$u = \frac{c^2(1)r^2}{c^2(r)} \quad \text{so that} \quad du = \frac{2rc^2(1)}{c^2(r)}\Big(1 - \frac{rc'(r)}{c(r)}\Big)dr.$$

Upon using this change of variables we deduce that

$$T(p) = 2\int_{p^2}^{1} \Big(\frac{dr}{du}\frac{u}{r}\Big)(u)\frac{du}{\sqrt{u-p^2}}. \tag{6.11}$$

It turns out that the function in parenthesis in the above expression can be reconstructed from $T(p)$. This is an *Abel integral*. Before inverting the integral, we need to ensure that the change of variables $r \mapsto u(r)$ is a diffeomorphism (a continuous function with continuous inverse). This implies that du/dr is positive, which in turn is equivalent to (6.8). The constraint (6.8) is therefore useful both to obtain the existence of a minimal point r_p and to ensure that the above change of variables is admissible. The constraint essentially ensures that no rays are trapped in the dynamics so that energy entering the system will eventually exit it. We can certainly consider velocity profiles such that

the energy is attracted at the origin. In such situation the velocity profile cannot be reconstructed.

Let us denote by $f = \frac{dr}{du}\frac{u}{r}$. We will show in the following section that $f(u)$ can be reconstructed from $T(p)$ and is given by

$$f(u) = -\frac{2}{\pi}\frac{d}{du}\int_u^1 \frac{T(\sqrt{p})}{\sqrt{p-u}}dp. \tag{6.12}$$

Now we reconstruct $r(u)$ from the relations

$$\frac{f(u)}{u}du = \frac{dr}{r}, \qquad u(1) = 1, \qquad \text{so that} \qquad r(u) = \exp\Big(\int_u^1 \frac{f(v)dv}{v}\Big).$$

Upon inverting this diffeomorphism, we obtain $u(r)$ and $g(r) = f(u(r))$. Since

$$g(r) = \frac{1}{2}\frac{1}{1 - rc'/c},$$

we now know rc'/c, hence $(\log c)'$. It suffices to integrate $\log c$ from 1 to 0 to obtain $c(r)$ everywhere. This concludes the proof of the reconstruction.

6.1.2 Abel integral and Abel transform

For a smooth function $f(x)$ (continuous will do) defined on the interval $(0, 1)$, we define the Abel transform as

$$g(x) = \int_x^1 \frac{f(y)}{(y-x)^{1/2}}dy. \tag{6.13}$$

This transform can be inverted as follows:

Lemma 6.1.1 *The Abel transform* (6.13) *admits the following inversion*

$$f(y) = -\frac{1}{\pi}\frac{d}{dy}\int_y^1 \frac{g(x)}{(x-y)^{1/2}}dx. \tag{6.14}$$

Proof. Let us calculate

$$\int_z^1 \frac{g(x)}{(x-z)^{1/2}}dx = \int_z^1\int_x^1 \frac{f(y)}{(x-z)^{1/2}(y-x)^{1/2}}dxdy = \int_z^1 dy f(y)k(z,y)dy.$$

The kernel $k(z, y)$ is given by

$$k(z,y) = \int_z^y \frac{dx}{(x-z)^{1/2}(y-x)^{1/2}} = \int_0^1 \frac{dx}{\sqrt{x(1-x)}} = \int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} = \pi.$$

The latter equality comes from differentiating arccos. Thus we have

$$\int_z^1 \frac{g(x)}{(x-z)^{1/2}}dx = \pi\int_z^1 f(y)dy.$$

Upon differentiating both sides with respect to z, we obtain the desired result. □

We can also ask ourselves how well-posed the inversion of the Abel transform is. Since the transforms are defined on bounded intervals, using the Hilbert scale introduced in Chapter 1 would require a few modifications. Instead we will count the number of differentiations. The reconstruction formula shows that the Abel transform is applied once more to g before the result is differentiated. We can thus conclude that the Abel transform regularizes as one half of an integration (since it takes one differentiation to compensate for two Abel transforms). We therefore deduce that the Abel transform is a smoothing operator of order $\alpha = 1/2$ using the terminology introduced in Chapter 1.

6.1.3 Kinematic velocity Inverse Problem

Let us generalize the construction of the Earth velocity from boundary measurements in the setting of a "two-dimensional" Earth.

As in section 4.3.1, we consider a bounded domain $X \subset \mathbb{R}^2$ with smooth surface ∂X parameterized by $0 \leq \tau \leq T$ and points $x = S(\tau)$ with $S(0) = S(T)$ and $|\dot{S}(\tau)| = 1$.

Local travel time and distance are related by

$$ds^2 = \frac{1}{c^2(x)}(dx^2 + dy^2) = n^2(x)(dx^2 + dy^2).$$

which defines a Riemannian metric with tensor proportional to the 2×2 identity matrix.

The geodesics of that metric generate a family of curves $\boldsymbol{F}(t, s, \vartheta)$ as in (4.1) in Chapter 4. We assume that the family of geodesics satisfies the hypotheses stated there, namely the existence of an inverse in (4.2). For a point x in $\bar{X}$ and $0 \leq \tau \leq T$, we recall that $\tilde{\boldsymbol{F}}(x, \tau)$ is the unique curve joining x and $S(\tau)$.

We are interested in reconstructing $n(x)$ from knowledge of

$$G(\tau_1, \tau_2) = \int_{\tilde{F}(S(\tau_1),\tau_2)} d\tau = \int_{\tilde{F}(S(\tau_1),\tau_2)} n(x)\sqrt{dx^2 + dy^2}, \tag{6.15}$$

for every possible boundary points τ_1 and τ_2, where $\gamma(t_1, t_2)$ is an extremal for the above functional, i.e., a geodesic of the Riemannian metric $d\tau^2$. Notice that since the extremals (the geodesics) of the Riemannian metric depend on $n(\mathbf{x})$, the above problem is *non-linear* as in the one-dimensional case.

Let G_k for $k = 1, 2$ correspond to measurements for two slownesses n_k, $k = 1, 2$. We then have the following result

Theorem 6.1.2 *Let n_k be smooth positive functions on X such that the family of extremals are sufficiently regular. Then n_k can uniquely be reconstructed from $G_k(\tau_1, \tau_2)$ and we have the stability estimate*

$$\|n_1 - n_2\|_{L^2(X)} \leq C\left\|\frac{\partial}{\partial\tau_1}(G_1 - G_2)\right\|_{L^2((0,T)\times(0,T))}. \tag{6.16}$$

Proof. Even though the inverse kinematic problem is non-linear, the proof is similar to that of Theorem 4.3.1 for the corresponding linear inverse problem. The reason is that the same energy estimate can be used in both cases. The regular family of curves

F is defined as the geodesics of the Riemannian metric $d\tau^2$. Let us define the travel time

$$\mathrm{t}(\mathrm{x},\tau) = \int_{\tilde{F}(\mathrm{x},\tau)} \mathrm{n}\mathrm{d}\mathrm{s}, \tag{6.17}$$

so that as before $\mathrm{t}(\mathrm{S}(\tau_1),\tau_2) = \mathrm{G}(\tau_1,\tau_2)$. We deduce as in (4.33) that

$$\vartheta \cdot \nabla_x \mathrm{t} = \mathrm{n}(\mathrm{x}). \tag{6.18}$$

We recall $\vartheta(x,\tau) = (\cos\theta(x,\tau), \sin\theta(x,\tau))$ is the unit tangent vector to the curve $\tilde{F}(x,\tau)$ at x and orientated such that (6.18) holds.

Because t is an integration along an extremal curve of the Riemannian metric (this is where we use that the curves are geodesics), we deduce that

$$\vartheta^{\perp} \cdot \nabla \mathrm{t} = 0, \quad \text{so that} \quad \nabla_{\mathrm{x}} \mathrm{t} = \mathrm{n}(\mathrm{x})\vartheta \quad \text{and} \quad |\nabla_{\mathrm{x}} \mathrm{t}|^2(\mathrm{x},\tau) = \mathrm{n}^2(\mathrm{x}).$$

Upon differentiating the latter equality we obtain

$$\frac{\partial}{\partial \tau}|\nabla_x \mathrm{t}|^2 = 0.$$

Let us now define $u = \mathrm{t}_1 - \mathrm{t}_2$, the difference of travel times for the two possible sound speeds c_1 and c_2, so that $\nabla u = n_1\vartheta_1 - n_2\vartheta_2$. We deduce from the above expression that

$$\frac{\partial}{\partial t}(\nabla u \cdot (\vartheta_1 + \frac{n_2}{n_1}\vartheta_2)) = 0.$$

We multiply the above expression by $2\vartheta_1^{\perp} \cdot \nabla u$ and express the product in divergence form. We obtain as in the preceding section that

$$\begin{aligned} &2\vartheta_1^{\perp} \cdot \nabla u \frac{\partial}{\partial t}\vartheta_1 \cdot \nabla u - \frac{\partial}{\partial t}\Big(\vartheta_1^{\perp} \cdot \nabla u \vartheta_1 \cdot \nabla u\Big) \\ &= \dot{\theta}_1 |\nabla u|^2 + \vartheta_1 \cdot \nabla(\vartheta_1^{\perp} \cdot \nabla u u_t) - \vartheta_1^{\perp} \cdot \nabla(\vartheta_1 \cdot \nabla u u_t). \end{aligned}$$

We now show that the second contribution can also be put in divergence form. More precisely, we obtain, since n_1 and n_2 are independent of t, that

$$\begin{aligned} &2\vartheta_1^{\perp} \cdot \nabla u \frac{\partial}{\partial \tau}(\frac{n_2}{n_1}\vartheta_2 \cdot \nabla u) = 2\vartheta_1^{\perp} \cdot (n_1\vartheta_1 - n_2\vartheta_2)\frac{\partial}{\partial \tau}(n_2\vartheta_1 \cdot \vartheta_2 - \frac{n_2^2}{n_1}) \\ &= -2n_2^2\vartheta_1^{\perp} \cdot \vartheta_2(\vartheta_1 \cdot \vartheta_2) = -2n_2^2(\vartheta_1^{\perp} \cdot \vartheta_2)^2\frac{\partial(\theta_1 - \theta_2)}{\partial \tau} = \\ &-2n_2^2 \sin^2(\theta_1 - \theta_2)\frac{\partial(\theta_1 - \theta_2)}{\partial \tau} = \frac{\partial}{\partial \tau}\Big(n_2^2\big[\frac{\sin(2(\theta_1 - \theta_2))}{2} - (\theta_1 - \theta_2)\big]\Big). \end{aligned}$$

The integration of the above term over $X \times (0,T)$ thus yields a vanishing contribution. Following the same derivation as in Chapter 4, we deduce that

$$\int_0^T \int_X \frac{\partial \theta_1}{\partial \tau}|\nabla u|^2 dx d\tau = \int_0^T \int_0^T \frac{\partial \delta G(\tau_1,\tau_2)}{\partial \tau_1}\frac{\partial \delta G(\tau_1,\tau_2)}{\partial \tau_2} d\tau_1 d\tau_2, \tag{6.19}$$

where we have defined $\delta G = G_1 - G_2$. To conclude the proof, notice that

$$\begin{aligned} &\nabla u \cdot \nabla u = |n_1\vartheta_1 - n_2\vartheta_2|^2 = n_1^2 + n_2^2 - 2n_1n_2\vartheta_1 \cdot \vartheta_2 \\ &\geq n_1^2 + n_2^2 - 2n_1n_2 = (n_1 - n_2)^2, \end{aligned}$$

since both n_1 and n_2 are non-negative. With (6.19), this implies that $n_1 = n_2$ when $G_1 = G_2$ and using again the Cauchy-Schwarz inequality yields the stability estimate (6.16). □

6.2 Forward transport problem

So far, the kinetic model accounted for spatial changes in the speed of propagation but not for scattering or absorption mechanisms. We recall that physically, the transport equation models high frequency waves or particles propagating in scattering environments. With the specific form of the Hamiltonian considered in (6.2), and in the time-independent setting to simplify, the transport equation then takes the form

$$c(x)\frac{v}{|v|}\cdot\nabla_x u - |v|\nabla c(x)\cdot\nabla_v u + \sigma(x)u = \int_{\mathbb{R}^d} k(x,v',v)u(x,v')\delta(c(x)|v| - c(x)|v'|)dv', \tag{6.20}$$

where boundary conditions are imposed at the boundary of a domain of interest. Here, $u(x,v)$ still denotes the density of particles at position x with velocity v and $c(x)$ is the speed of propagation. Since we are primarily interested in scattering effects in this chapter, we assume that $c(x) = 1$ is constant and normalized. The equation thus becomes

$$\frac{v}{|v|}\cdot\nabla_x u + \sigma(x)u = \int_{\mathbb{R}^d} k(x,v',v)u(x,v')\delta(\omega - |v'|)dv'. \tag{6.21}$$

Here, $\omega = c|v|$ is the frequency (e.g. the color for light) of the propagating waves. Scattering is assumed to be elastic so that $\omega = c|v|$ is preserved by scattering and hence wave packets with different frequencies satisfy uncoupled equations. We also normalize $\omega = 1$ so that $|v| = \omega = 1$. In other words, v is now the direction of propagation of the wave packets (photons). We thus have an equation of the form

$$v\cdot\nabla_x u + \sigma(x)u = \int_{\mathbb{S}^{d-1}} k(x,v',v)u(x,v')dv', \tag{6.22}$$

where $x \in \mathbb{R}^n$ and $v \in \mathbb{S}^{n-1}$ the unit sphere in $\mathbb{R}^n$. It remains to describe the parameters participating in scattering: $\sigma(x)$ the attenuation (aka total absorption) coefficient, and $k(x,v',v)$ the scattering coefficient. $\sigma(x)$ models the amount of particles that are either absorbed or scattered per unit distance of propagation. This is the same coefficient already encountered in CT and SPECT. Unlike high-energy CT and SPECT, in lower energies such as for visible light, many "absorbed" photons are re-emitted into another direction, i.e., scattered. Then $k(x,v',v)$ gives the density of particles scattered into direction v from a direction v'. The right-hand side in (6.22) corresponds to a re-emission of scattered particles into the direction v.

In an **inverse transport problem**, $\sigma(x)$ and $k(x,v',v)$ are the unknown coefficients. They have to be reconstructed from knowledge of $u(x,v)$ measured, say, at the boundary of the domain of interest. When $k \equiv 0$, this is Computerized Tomography, where the appropriate logarithm of u provides line integrals of σ.

To be more specific, we need to introduce ways to probe the domain of interest X and to model measurements. In the most general setting, photons can enter into X at any point $x \in \partial X$ and with any incoming direction. In the most general setting, photon densities can then be measured at any point $x \in \partial X$ and for any outgoing direction. The sets of incoming conditions Γ_- and outgoing conditions Γ_+ are defined by

$$\Gamma_\pm = \{(x,v) \in \partial X \times V, \text{ s.t. } \pm v\cdot\nu(x) > 0\}, \tag{6.23}$$

where $\nu(x)$ is the outgoing normal vector to X at $x \in \partial X$ and $V = \mathbb{S}^{n-1}$. Denoting by $g(x,v)$ the incoming boundary conditions, we then obtain the following transport equation

$$\begin{aligned} v \cdot \nabla_x u + \sigma(x) u &= \int_{\mathbb{S}^{d-1}} k(x,v',v) u(x,v') dv' \qquad (x,v) \in X \times V \\ u_{|\Gamma_-}(x,v) &= g(x,v) \qquad (x,v) \in \Gamma_-. \end{aligned} \tag{6.24}$$

From the functional analysis point of view, it is natural to consider the L^1 norm of photon densities, which essentially counts numbers of particles (the L^1 norm of the density on a domain is the number of particles inside that domain). Let us introduce the necessary notation.

We say that the optical parameters (σ, k) are admissible when

$$\begin{aligned} &0 \leq \sigma \in L^\infty(X) \\ &0 \leq k(x,v',\cdot) \in L^1(V) \text{ a.e. in } X \times V \\ &\sigma_s(x,v') := \int_V k(x,v',v) dv \in L^\infty(X \times V). \end{aligned} \tag{6.25}$$

Here σ_s is also referred to as the scattering coefficient. In most applications, $\sigma_s(x)$ is independent of v'.

We define the times of escape of free-moving particles from X as

$$\tau_\pm(x,v) = \inf\{s > 0 | x \pm sv \notin X\} \tag{6.26}$$

and $\tau(x,v) = \tau_+(x,v) + \tau_-(x,v)$. On the boundary sets $\Gamma_\pm$, we introduce the measure $d\xi(x,v) = |v \cdot \nu(x)| d\mu(x) dv$, where $d\mu(x)$ is the surface measure on ∂X.

We define the following Banach space

$$W := \big\{ u \in L^1(X \times V) | v \cdot \nabla_x u \in L^1(X \times V), \tau^{-1} u \in L^1(X \times V) \big\}, \tag{6.27}$$

with its natural norm. We recall that τ is defined below (6.26). We have the following trace formula [27]

$$\|f_{|\Gamma_\pm}\|_{L^1(\Gamma_\pm, d\xi)} \leq \|f\|_W, \qquad f \in W. \tag{6.28}$$

This allows us to introduce the following lifting operator

$$\mathcal{I} g(x,v) = \exp\Big(- \int_0^{\tau_-(x,v)} \sigma(x - sv, v) ds \Big) g(x - \tau_-(x,v)v, v). \tag{6.29}$$

It is proved in [27] that $\mathcal{I}$ is a bounded operator from $L^1(\Gamma_-, d\xi)$ to W. Note that $\mathcal{I} g$ is the solution u_0 of

$$v \cdot \nabla_x u_0 + \sigma(x) u_0 = 0 \quad (x,v) \in X \times V, \qquad u_0 = g \quad (x,v) \in \Gamma_-.$$

Exercise 6.2.1 *Prove this. This is the same calculation as for the X-ray transform.*

Let us next define the bounded operators

$$\begin{aligned}\mathcal{K}u(x,v) &= \int_0^{\tau_-(x,v)} \exp\Big(-\int_0^t \sigma(x-sv,v)ds\Big)\int_V k(x-tv,v',v)u(x-tv,v')dv'dt \\ \mathcal{L}S(x,v) &= \int_0^{\tau_-(x,v)} \exp\Big(-\int_0^t \sigma(x-sv,v)ds\Big)S(x-tv,v)dt \end{aligned} \tag{6.30}$$

for $(x,v)\in X\times V$. Note that $\mathcal{L}S$ is the solution u_S of

$$v\cdot\nabla_x u_S+\sigma(x)u_S=S \quad (x,v)\in X\times V, \qquad u_S=0 \quad (x,v)\in\Gamma_-.$$

Exercise 6.2.2 *Prove this.*

Note that

$$\mathcal{K}u(x,v)=\mathcal{L}\Big[\int_V k(x,v',v)u(x,v')dv'\Big](x,v),$$

which allows us to handle the right-hand side in (6.24). Looking for solutions in W, the integro-differential equation (6.24) is thus recast as

$$(I-\mathcal{K})u=\mathcal{I}g. \tag{6.31}$$

Exercise 6.2.3 *Prove this.*

Then we have the following result [12, 27].

Theorem 6.2.1 *Assume that*

$$(I-\mathcal{K}) \text{ admits a bounded inverse in } L^1(X\times V,\tau^{-1}dxdv). \tag{6.32}$$

Then the integral equation (6.31) *admits a unique solution* $u\in W$ *for* $g\in L^1(\Gamma_-,d\xi)$. *Furthermore, the albedo operator*

$$\mathcal{A}: L^1(\Gamma_-,d\xi)\to L^1(\Gamma_+,d\xi), \qquad g\mapsto \mathcal{A}g=u_{|\Gamma_+}, \tag{6.33}$$

is a bounded operator.

The invertibility condition (6.32) *holds under either of the following assumptions*

$$\sigma_a:=\sigma-\sigma_s \geq 0 \tag{6.34}$$

$$\|\tau\sigma_s\|_\infty < 1. \tag{6.35}$$

We shall not prove this theorem here. The salient features are that the transport equation is well-posed provided that (6.32) is satisfied, which is not necessarily true for arbitrary admissible coefficients (σ,k). The conditions (6.34) or (6.35) are sufficient conditions for (6.32) to be satisfied. The first condition is the most natural for us and states that particles that are "created" by scattering into v by $k(x,v',v)$ are particles

that are "lost" for direction v. In other words, the scattering mechanism does not create particles. This is quite natural for photon propagation. In nuclear reactor physics, however, several neutrons may be created by fission for each incoming scattering neutron. There are applications in which (6.34) is therefore not valid. In most medical and geophysical imaging applications, however, (6.34) holds and the transport solution exists. Note that $\sigma_a = \sigma - \sigma_s$ is the *absorption* coefficient, and corresponds to a measure of the particles that are "lost" for direction v' and do not reappear in any direction v (i.e., particles that are absorbed).

Using these operators, we may recast the transport solution as

$$u = \mathcal{I}g + \mathcal{K}\mathcal{I}g + (I - \mathcal{K})^{-1}\mathcal{K}^2\mathcal{I}g, \tag{6.36}$$

where $u_0 := \mathcal{I}g$ is the ballistic component, $u_1 := \mathcal{K}\mathcal{I}g$ the single scattering component and $u_2 := u - u_0 - u_1 = (I - \mathcal{K})^{-1}\mathcal{K}^2\mathcal{I}g$ is the multiple scattering component.

Note that when the problem is subcritical, its solution may be expressed in terms of the following Neumann expansion in $L^1(X \times V)$

$$u = \sum_{m=0}^{\infty} \mathcal{K}^m \mathcal{I}g. \tag{6.37}$$

The contribution $m = 0$ is the ballistic part of u, the contribution $m = 1$ the single scattering part of u, and so on. It is essentially this decomposition of the transport solution into orders of scatterings that allows us to stably reconstruct the optical parameters in the following sections. Note that the above Neumann series expansions has an additional benefit. Since the optical parameters are non-negative, each term in the above series is non-negative provided that g and S are non-negative so that the transport solution itself is non-negative. A little more work allows us to prove the maximum principle, which states that u in $X \times V$ is bounded a.e. by the (essential) supremum of g in Γ_- when $S \equiv 0$.

Finally, the albedo operator $\mathcal{A}$, which maps incoming conditions to outgoing densities models our measurements. We control the fluxes of particles on Γ_- and obtain information about X by measuring the density of particles on Γ_+. This allows us to define the *measurements operator* of *inverse transport*. Let

$$\mathfrak{X} = \{(\sigma, k) \text{ such that } 0 \leq \sigma \in L^\infty(X),\ 0 \leq k \in L^\infty(X \times V \times V),\ \sigma \geq \sigma_s\}, \tag{6.38}$$

and let $\mathfrak{Y} = \mathcal{L}(L^1(\Gamma_-, d\xi), L^1(\Gamma_+, d\xi))$. Then we define the measurement operator $\mathfrak{M}$

$$\mathfrak{M}: \quad \mathfrak{X} \ni (\sigma, k) \mapsto \mathfrak{M}(\sigma, k) = \mathcal{A}[(\sigma, k)] \in \mathfrak{Y}, \tag{6.39}$$

where $\mathcal{A}[(\sigma, k)]$ is the albedo operator constructed in (6.33) with coefficient (σ, k) in (6.24). Note that the measurement operator, as for the Calderón problem in (1.14) is an operator, which to a set of coefficients maps a coefficient-valued operator, the albedo operator.

The main question of inverse transport consists of knowing what can be reconstructed in (σ, k) from knowledge of the full operator $\mathfrak{M}$ or knowledge of only parts of the operator $\mathfrak{M}$. In these notes, we shall mostly be concerned with the full measurement operator.

6.3 Inverse transport problem

One of the main results for inverse transport is the decomposition (6.36). The first term $u_0 := \mathcal{I}g$ is the ballistic component and corresponds to the setting of vanishing scattering. It is therefore the term used in CT and the X-ray transform. It turns out that this term is more *singular*, in a sense that will be made precise below, than the other contributions. It can therefore be extracted from the measurement operator and provide the X-ray transform of the attenuation coefficient σ.

The second term in (6.36) is $u_1 := \mathcal{K}\mathcal{I}g$ and is the the single scattering component of the transport solution. Finally, $u_2 := u - u_0 - u_1 = (I - \mathcal{K})^{-1}\mathcal{K}^2\mathcal{I}g$ is the multiple scattering component, which corresponds to particles that have interacted at least twice with the underlying medium. A fundamental property of the transport equation is that single scattering is also more singular than multiple scattering in dimension three (and higher dimensions), but not in dimension two. We shall describe in more detail below in which sense single scattering is more *singular*. The main conclusion, however, is that the single scattering contribution can also be extracted from the full measurement operator in dimension $n \geq 3$. As we shall see, single scattering provides a linear operator that allows us to invert for the scattering coefficient k once σ is known.

Multiple scattering is then *less singular* than ballistic and single scattering. Intuitively, this means that multiple scattering contributions are smoother functions. In some sense, after multiple scattering, we do not expect the density to depend too much on the exact location of the scattering events. Multiply scattered particles visit a large domain and hence are less specific about the scenery they have visited.

6.3.1 Decomposition of the albedo operator and uniqueness result

Following (6.36), we decompose the albedo operator as

$$\begin{aligned} \mathcal{A}g &= \mathcal{I}g\big|_{\Gamma_+} + \mathcal{K}\mathcal{I}g\big|_{\Gamma_+} + \mathcal{K}^2(I-\mathcal{K})^{-1}\mathcal{I}g\big|_{|\Gamma_+} \\ &:= \mathcal{A}_0 g + \mathcal{A}_1 g + \mathcal{A}_2 g. \end{aligned} \tag{6.40}$$

We denote by α the Schwartz kernel of the albedo operator $\mathcal{A}$:

$$\mathcal{A}g(x,v) = \int_{\Gamma_-} \alpha(x,v,y,w) g(y,w) d\mu(y) dw.$$

Any linear operator, such as the albedo operator, admits such a decomposition. Knowledge of the operator $\mathcal{A}$ is equivalent to knowledge of its kernel α. The decomposition for $\mathcal{A}$ then translates into the decomposition for α:

$$\alpha = \alpha_0 + \alpha_1 + \alpha_2.$$

Here, α_0 corresponds to the ballistic part of the transport solution, α_1 corresponds to the single scattering part of the transport solution, and α_2 corresponds to the rest of the transport solution.

After some algebra, we have the following expression for the first two contributions in the time independent case:

$$\alpha_0(x,v,y,w) = \exp\Big(-\int_0^{\tau_-(x,v)} \sigma(x-sv,v)ds\Big)\delta_v(w)\delta_{\{x-\tau_-(x,v)v\}}(y). \tag{6.41}$$

$$\begin{aligned}\alpha_1(x,v,y,w) \;=\; & \int_0^{\tau_-(x,v)} \exp\Big(-\int_0^t \sigma(x-sv,v)ds - \int_0^{\tau_-(x-tv,w)} \sigma(x-tv-sw,w)ds\Big) \\ & k(x-tv,w,v)\delta_{\{x-tv-\tau_-(x-tv,w)w\}}(y)dt.\end{aligned} \tag{6.42}$$

Here, $\delta_v(w)$ denotes a delta function at v on the unit sphere, i.e., the distribution so that

$$\delta_v(f) = \int_{\mathbb{S}^{n-1}} f(w)\delta_v(w)dw = f(v),$$

for f a continuous function on $\mathbb{S}^{n-1}$. Similarly, $\delta_{\{x\}}(y)$ denotes the delta function at x on the boundary ∂X, i.e., the distribution so that

$$\delta_{\{x\}}(f) = \int_{\partial X} f(y)\delta_{\{x\}}(y)d\mu(y) = f(x),$$

for f a continuous function in the vicinity of $x \in \partial X$.

Exercise 6.3.1 *Prove the above two formulas* (6.41) *and* (6.42). *Note that the first formula is nothing but the expression for the X-ray transform.*

Note that α_0 in any dimension and α_1 in dimension $n \geq 3$ are *distributions*: they involve "delta" functions. In dimension $n = 2$, we verify that α_1 is in fact a *function*. A tedious and lengthy calculation shows that the kernel corresponding to multiple scattering α_2 is also a *function*, and when k is bounded, satisfies that

$$|\nu(y)\cdot w|^{-1}\alpha_2(x,v,y,w) \in L^\infty(\Gamma_-, L^p(\Gamma_+, d\xi)), \quad 1 \leq p < \frac{d+1}{d}. \tag{6.43}$$

We refer the reader to [12] (see also [27]) for the derivation of such a result. The exact regularity of the function α_2 is not very important for us here. For the mathematical derivation of stability estimates, the fact that we can take $p > 1$ above is important. For a picture of the geometry of the singularities, see Fig. 6.1: particles emitted emitted with direction v_0' at $x_0' - Rv_0'$ scatter along the segment $x_0 - (R-t)v_0'$ for $t \geq 0$. Singly scattered photons reach the plane orthogonal to a given direction v on the right-hand side of fig. 6.1 along a segment of the plane. We can verify that photons that are at least doubly scattered may reach any point in that plane and in any other plane. This is consistent with the fact that α_2 is a function.

The strategy to recover σ and k in dimension $n \geq 3$ thus goes as follows: We send beams of particles into the medium that concentrate in the vicinity of a point $(y_0, v_0) \in \Gamma_-$. More precisely, let g_ε be a sequence of normalized L^1 functions on Γ_- converging as $\varepsilon \to 0$ to $\delta_{v_0}(v)\delta_{\{y_0\}}(y)$.

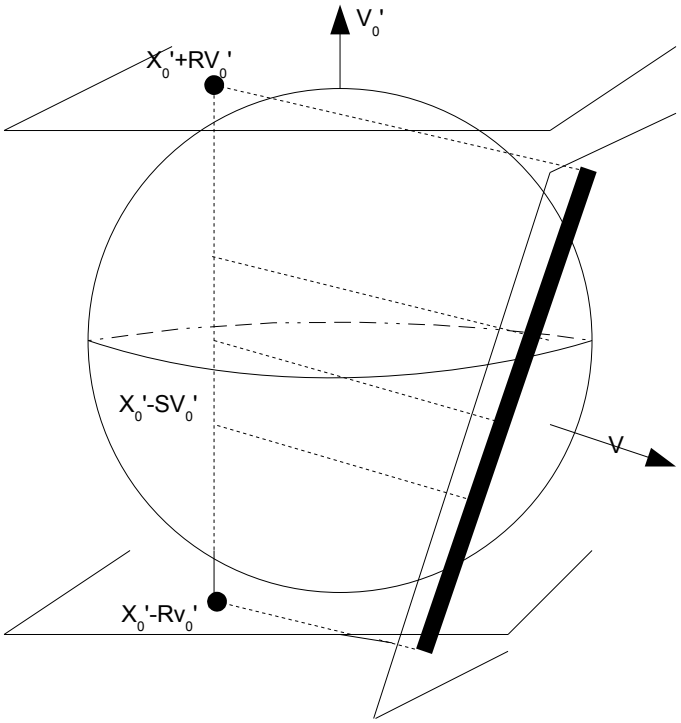

Figure 6.1: Geometry of single scattering for $n = 3$.

Since the ballistic term is more singular than the rest, if we place detectors on the support of the ballistic term, then such detectors will overwhelmingly be measuring ballistic particles and very little scattered particles. Since single scattering is more singular than multiple scattering in dimension $n \geq 3$, we can use the same strategy and place detectors at the location of the support of single scattering. Such detectors will overwhelmingly be measuring singly scattered particles and very little multiply scattered particles. We now briefly explain the rather tedious mathematical construction.

Recovery of the attenuation coefficient $\sigma(x)$

Let $(y_0, v_0) \in \Gamma_-$ be defined as above and $(x_0, v_0) \in \Gamma_+$ such that $y_0 = x_0 - \tau_-(x_0, v_0)v_0$. Let ϕ_ε be a sequence of bounded functions on Γ_+ equal to 1 in the vicinity of (x_0, v_0) and with vanishing support as $\varepsilon \to 0$. Then we verify that

$$\int_{\Gamma_+ \times \Gamma_-} \alpha_m(x, v, y, w)\phi_\varepsilon(x, v)g_\varepsilon(y, w)d\mu(x)dvd\mu(y)dw \xrightarrow{\varepsilon \to 0} 0, \quad m = 1, 2,$$

so that

$$\begin{aligned} \langle \phi_\varepsilon, \mathcal{A}g_\varepsilon \rangle \quad &:= \quad \int_{\Gamma_+ \times \Gamma_-} \alpha(x, v, y, w)\phi_\varepsilon(x, v)g_\varepsilon(y, w)d\mu(x)dvd\mu(y)dw \\ &\xrightarrow{\varepsilon \to 0} \quad \exp\Big(- \int_0^{\tau_-(x_0, v_0)} \sigma(x_0 - sv_0, v_0)ds \Big). \end{aligned} \tag{6.44}$$

Exercise 6.3.2 *Verify* (6.44)*; see [12] for the details.*

In other words, the measurements corresponding to the above choices of functions g_ε and ϕ_ε then converge to the function

$$\langle \phi_\varepsilon, \mathcal{A} g_\varepsilon \rangle \xrightarrow{\varepsilon \to 0} E(x,y) := \exp\Big(- \int_0^{|x-y|} \sigma\big(x - s\frac{x-y}{|x-y|}\big) ds \Big). \tag{6.45}$$

This proves that knowledge of the albedo operator $\mathcal{A}$, which allows one to construct $\langle \phi_\varepsilon, \mathcal{A} g_\varepsilon \rangle$, provides knowledge of $E(x,y)$, the exponential of minus the X-ray transform of σ along the segment (x,y). This can be obtained from any segment (x,y) with x and y on ∂X. As a consequence, knowledge of $\mathcal{A}$ provides knowledge of the X-ray transform of σ. We know that σ is then uniquely and stably reconstructed from such knowledge since the X-ray transform is a smoothing operator by (only) one-half of a derivative.

Recovery of the scattering coefficient $k(x,v,w)$

We assume that $\sigma = \sigma(x)$ is now recovered. Let $z_0 \in X$, $v_0 \in V$, and $v_0 \neq w_0 \in V$. Define $x_0 = z_0 + \tau_+(z_0, v_0)v_0$ so that $(x_0, v_0) \in \Gamma_+$ and $y_0 = z_0 - \tau_-(z_0, w_0)w_0$ so that $(y_0, w_0) \in \Gamma_-$. We formally show how the scattering coefficient may be uniquely reconstructed from full knowledge of $\mathcal{A}$.

Let us define g_{ε_1} as before and ϕ_ε as a sequence of bounded functions on Γ_+ equal to a constant in the vicinity of (x_0, v_0) and with vanishing support as $\varepsilon \to 0$. Since $v_0 \neq w_0$, we find that

$$\int_{\Gamma_+ \times \Gamma_-} \alpha_0(x,v,y,w) \phi_\varepsilon(x,v) g_{\varepsilon_1}(y,w) d\mu(x) dv d\mu(y) dw = 0, \quad 0 \leq \varepsilon, \varepsilon_1 < \varepsilon_0(x_0, v_0, y_0, w_0).$$

i.e., the ballistic contribution vanishes with such measurements. Let us define g_{ε_1} such that $|\nu(y_0) \cdot w_0|^{-1} g_{\varepsilon_1}(y,w)$ converges to a delta function. The factor $|\nu(y_0) \cdot w_0|^{-1}$ is here to ensure that the number of emitted particles is independent of y_0 and w_0. The ballistic part of the transport solution is then approximately concentrated on the line passing through y_0 and with direction w_0. Scattering occurs along this line and particles scattered in direction v_0 are approximately supported on the plane with directions v_0 and w_0 passing through x_0. The intersection of that plane with the boundary ∂X is a one-dimensional *curve* $\mathit{\Gamma}(x_0, v_0, w_0) \subset X$. In two space dimensions, the curve $\mathit{\Gamma}$ has the same dimension as ∂X. As a consequence, α_1 is a function and therefore is not more singular than α_2 in the time independent setting when $n = 2$.

Let $\phi_\varepsilon(x,v)$ be a bounded test function supported in the $\varepsilon-$vicinity of $\mathit{\Gamma}$. Because $\mathit{\Gamma}$ is of measure 0 in ∂X when $d \geq 3$, we find using (6.43) that

$$\int_{\Gamma_+ \times \Gamma_-} \alpha_2(x,v,y,w) \phi_\varepsilon(x,v) g_{\varepsilon_1}(y,w) d\mu(x) dv d\mu(y) dw \xrightarrow{\varepsilon, \varepsilon_1 \to 0} 0,$$

i.e., the multiple scattering contribution is asymptotically negligible with such measurements. Now, choosing $\phi_\varepsilon(x,v)$ properly normalized and supported in the ε_2-vicinity of (x_0, v_0) (for $\varepsilon \ll \varepsilon_2 \ll 1$), we find that

$$\langle \phi_\varepsilon, \mathcal{A} g_{\varepsilon_1} \rangle \xrightarrow{\varepsilon, \varepsilon_1, \varepsilon_2 \to 0} E(y_0, z_0) E(z_0, x_0) k(z_0, w_0, v_0),$$

at each point of continuity of $k(z_0, w_0, v_0)$, where $E(x, y)$ is defined in (6.45). Since $\sigma(x)$ and hence $E(x, y)$ are known from knowledge of $\mathcal{A}$, then so is $k(z_0, w_0, v_0)$ at each point of continuity in $X \times V \times V$ thanks to the above formula.

Exercise 6.3.3 *Verify the above formula; see [12] for the details.*

The above reconstruction of the attenuation coefficient is valid in any dimension $n \geq 2$. The reconstruction of the scattering coefficient, however, is valid only in dimension $n \geq 3$. The reason again is that in dimension $n = 2$, the single scattering contribution is also a function, not a distribution, and thus cannot be separated from the multiple scattering component. What we have obtained so far may be summarized as:

Theorem 6.3.1 ([27]) *Let (σ, k) and $(\tilde{\sigma}, \tilde{k})$ be two admissible pairs of optical parameters associated with the same albedo operator $\mathcal{A}$ and such that σ and $\tilde{\sigma}$ are independent of the velocity variable. Then $\sigma = \tilde{\sigma}$ in dimension $n \geq 2$. Moreover, $k = \tilde{k}$ in dimension $n \geq 3$.*

6.3.2 Stability in inverse transport

Let us assume the existence of two types of measurements $\mathcal{A}$ and $\tilde{\mathcal{A}}$, say, corresponding to the optical parameters (σ, k) and $(\tilde{\sigma}, \tilde{k})$, respectively. The question of the stability of the reconstruction is to bound the errors $\sigma - \tilde{\sigma}$ and $k - \tilde{k}$ as a function of $\mathcal{A} - \tilde{\mathcal{A}}$.

We obtain stability estimates in dimension $n \geq 3$. In dimension $n = 2$, only the estimate on $\sigma(x)$ is valid. The construction of the incoming source $\phi_\varepsilon(x, v)$ is such that $\phi_\varepsilon \in C^1(\Gamma_-)$ is supported in the ε_1 vicinity of (x_0, v_0) and normalized so that $\int_{\Gamma_-} \phi_\varepsilon d\xi = 1$. Let ψ be a compactly support continuous function, which models the array of detectors, on Γ_+ such that $\|\psi\|_\infty \leq 1$. Then

$$\Big| \int_{\Gamma_+} \psi(x, v) \big((\mathcal{A} - \tilde{\mathcal{A}}) \phi_\varepsilon \big)(x, v) d\xi(x, v) \Big| \quad \leq \quad \|(\mathcal{A} - \tilde{\mathcal{A}})\|_{\mathcal{L}(L^1)}, \tag{6.46}$$

where now $\| \cdot \|_{\mathcal{L}(L^1)} = \| \cdot \|_{\mathcal{L}(L^1(\Gamma_-, d\xi), L^1(\Gamma_+, d\xi))}$. We still introduce

$$I_m(\psi, \varepsilon) = \int_{\Gamma_+} \psi(x, v) \big((\mathcal{A}_m - \tilde{\mathcal{A}}_m) \phi_\varepsilon \big)(x, v) d\xi(x, v), \qquad m = 0, 1, 2,$$

and obtain that

$$\begin{aligned} \lim_{\varepsilon \to 0+} I_0(\psi, \varepsilon) &= \psi(y_0, v_0) \Big(E(x_0, y_0) - \tilde{E}(x_0, y_0) \Big) \\ \lim_{\varepsilon \to 0+} I_1(\psi, \varepsilon) &= \int_V \int_0^{\tau_+(x_0, v_0)} \psi(x(s) + \tau_+(x(s), v)v, v) \big(E_+ k - \tilde{E}_+ \tilde{k} \big)(x(s), v_0, v) ds dv \end{aligned} \tag{6.47}$$

where we have introduced $x(s) = x_0 + s v_0$.

The estimate (6.43) allows us to show that

$$\big| I_2(\psi, \varepsilon) \big| \leq C \int_V \Big(\int_{\partial X} |\psi(t, x, v)|^{p'} dx \Big)^{\frac{1}{p'}} dv, \qquad p' > d. \tag{6.48}$$

Multiple scattering is therefore still negligible when the support of $\psi := \psi_\lambda$ tends to 0 when $\lambda \to 0$.

The first sequence of functions ψ_λ is chosen to have a small support concentrated in the vicinity of $(y_0, v_0) \in \Gamma_+$. Then the single scattering contribution $I_1(\psi_\lambda) \to 0$ as $\lambda \to 0$. For w_0 fixed in V, we choose the sequence of functions ψ_λ such that they are concentrated in the vicinity of the curve $\digamma(s)$ on Γ_+ and that they approximately take the value $\text{sign}(E_+k - \tilde{E}_+\tilde{k})(x_0 + sv_0, v_0, w_0)$ along that curve. Since $v_0 \neq w_0$, we verify that $I_0(\psi_\lambda) \to 0$ for such sequence of functions; see [12]. Now, the function ψ_λ has a small support only in dimension $n \geq 3$. Indeed, in dimension $n = 2$, the curve $\digamma$ has the same dimensionality as the boundary ∂X. When $n = 2$, multiple scattering may no longer be separated from single scattering by using the singular structure of the albedo operator $\mathcal{A}$. This allows us to state the result:

Theorem 6.3.2 ([12]) *Assume that $\sigma(x)$ and $k(x, v', v)$ are continuous on $\bar{X}$ and $\bar{X} \times V \times V$, respectively and that $(\tilde{\sigma}, \tilde{k})$ satisfy the same hypotheses. Let $(x_0, v_0) \in \Gamma_-$ and $y_0 = x_0 + \tau_+(x_0, v_0)v_0$. Then we have for $d \geq 2$ that*

$$\left|E(x_0, y_0) - \tilde{E}(x_0, y_0)\right| \leq \|\mathcal{A} - \tilde{\mathcal{A}}\|_{\mathcal{L}(L^1)}, \tag{6.49}$$

while in dimension $n \geq 3$, we have

$$\int_V \int_0^{\tau_+(x_0,v_0)} \left|E_+k - \tilde{E}_+\tilde{k}\right|(x_0 + sv_0, v_0, v)dsdv \leq \|\mathcal{A} - \tilde{\mathcal{A}}\|_{\mathcal{L}(L^1)}. \tag{6.50}$$

The stability obtained above for the X-ray transform of the absorption coefficient is not sufficient to obtain any stability of σ itself without a priori regularity assumptions on σ. This results from the well known fact that the X-ray transform is a smoothing (compact) operator so that the inverse X-ray transform is an unbounded operator. Let us assume that σ belongs to some space $H^s(\mathbb{R}^n)$ for s sufficiently large and that σ_p defined in (6.25) is bounded. More precisely, define

$$\mathcal{M} = \left\{(\sigma, k) \in C^0(\bar{X}) \times C^0(\bar{X} \times V \times V) | \sigma \in H^{\frac{n}{2}+r}(X), \|\sigma\|_{H^{\frac{n}{2}+r}(X)} + \|\sigma_p\|_\infty \leq M\right\}, \tag{6.51}$$

for some $r > 0$ and $M > 0$. Then, we have the following result.

Theorem 6.3.3 ([12, 13]) *Let $n \geq 2$ and assume that $(\sigma, k) \in \mathcal{M}$ and that $(\tilde{\sigma}, \tilde{k}) \in \mathcal{M}$. Then the following is valid:*

$$\|\sigma - \tilde{\sigma}\|_{H^s(X)} \leq C\|\mathcal{A} - \tilde{\mathcal{A}}\|^\kappa_{\mathcal{L}(L^1)}, \tag{6.52}$$

where $-\frac{1}{2} \leq s < \frac{n}{2} + r$ and $\kappa = \frac{n+2(r-s)}{n+1+2r}$.

When $d \geq 3$, we have

$$\|k - \tilde{k}\|_{L^1(X\times V\times V)} \leq \|\mathcal{A} - \tilde{\mathcal{A}}\|^{\kappa'}_{\mathcal{L}(L^1)}\left(1 + \|\mathcal{A} - \tilde{\mathcal{A}}\|^{1-\kappa'}_{\mathcal{L}(L^1)}\right), \tag{6.53}$$

where $\kappa' = \frac{2(r-r')}{n+1+2r}$ and $0 < r' < r$.

Such estimates show that under additional regularization assumptions on σ, we have explicit stability expression of Hölder type on σ and k. The first stability result (6.52) was first established in [63].

The proof of these theorems is fairly technical and will not be presented here in detail; see [12, 13].

Chapter 7

Inverse diffusion and severe ill-posedness

This chapter introduces a classical example of a severely ill-posed problem for which the measurement operator is injective. In the absence of any noise, reconstructions are feasible. However, (in)stability is such that even tiny amounts of noise in moderately high frequency will be so amplified during the reconstruction that the resulting noise in the reconstructed parameters may overwhelm the real parameters. We have seen in earlier chapters that solving the heat equation backward was an example of a severely ill-posed problem. In the preceding chapter, we saw that scattering was responsible for smoothing (in the sense that the multiple scattering contribution of the albedo operator in (6.40) was smoother than the single scattering contribution and that the latter was smoother than the ballistic contribution). In this chapter and in the next chapter, we consider two new examples: the Cauchy problem, which is an inverse source problem, and the Inverse Diffusion problem, also known as the Calderón problem, which is a nonlinear inverse problem. In both problems, the forward modeling, based on a diffusion equation, is highly smoothing because of scattering effects.

This chapter is devoted to the analysis of the Cauchy problem and related inverse problems.

7.1 Cauchy Problem and Electrocardiac potential

Let us consider the imaging of the electrical activity of the heart. A probe is sent inside the heart, where the electrical potential is measured. The inverse problem consists of reconstructing the potential on the endocardial surface (the inside of the cardiac wall) from the measurements on the probe.

Mathematically, this corresponds to solving a Cauchy problem for an elliptic equation. The problem is modeled as follows. Let Γ_0 be a closed smooth surface in $\mathbb{R}^3$ representing the endocardial surface and let Γ_1 be the closed smooth surface inside the volume enclosed by Γ_0 where the measurements are performed. We denote by X the domain with boundary $\partial X = \Gamma_0 \cup \Gamma_1$; see Fig.7.1. The electric potential solves the

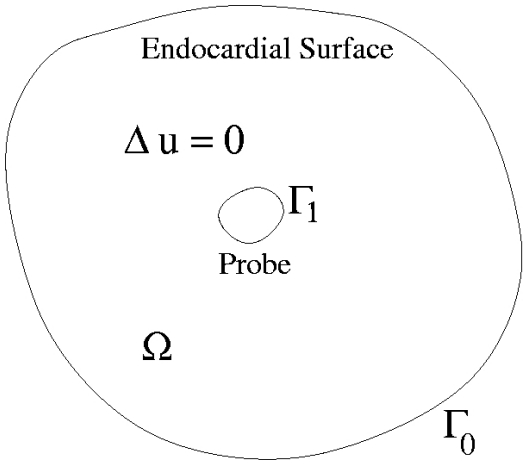

Figure 7.1: Geometry of endocardial measurements

following Laplace equation:

$$\begin{aligned} \Delta u &= 0 && \text{in } X, \\ u &= u_1 && \text{on } \Gamma_1, \\ \frac{\partial u}{\partial n} &= 0 && \text{on } \Gamma_1. \end{aligned} \tag{7.1}$$

The function u_1 models the measurements at the surface of the probe, which is assumed to be insulated so that $n \cdot \nabla u = 0$ on Γ_1. The objective is then to find $u = u_0$ on Γ_0.

7.2 Half Space Problem

Let us consider the simplified geometry where X is the slab $\mathbb{R}^{n-1} \times (0, L)$. We assume that boundary measurements are given at the boundary $x_n = 0$ and wish to obtain the harmonic solution in X. We are thus interested in solving the equation

$$\begin{aligned} \Delta u &= 0, && x = (x', x_n) \in X \\ u(x', 0) &= f(x'), && x' \in \mathbb{R}^{n-1} \\ \frac{\partial u}{\partial x_n}(x', 0) &= g(x'), && x' \in \mathbb{R}^{n-1}. \end{aligned} \tag{7.2}$$

Let us denote by

$$\hat{u}(k', x_n) = (\mathcal{F}_{x' \to k'} u)(k', x_n). \tag{7.3}$$

Upon Fourier transforming (7.2) in the x' variable, we obtain

$$\begin{aligned} -|k'|^2 \hat{u} + \frac{\partial^2 \hat{u}}{\partial {x_n}^2} &= 0, && k' \in \mathbb{R}^{n-1},\ 0 < x_n < L \\ \hat{u}(k', 0) &= \hat{f}(k'), && k' \in \mathbb{R}^{n-1}, \\ \frac{\partial \hat{u}}{\partial y}(k', 0) &= \hat{g}(k'), && k' \in \mathbb{R}^{n-1}. \end{aligned} \tag{7.4}$$

The solution of the above ODE is given by

$$\hat{u}(k', x_n) = \hat{f}(k') \cosh(|k'|x_n) + \frac{\hat{g}(k')}{|k'|} \sinh(|k'|x_n). \tag{7.5}$$

We thus observe that the solution at $x_n > 0$ is composed of an exponentially growing component $e^{|k'|x_n}$ and an exponentially decreasing component $e^{-|k'|x_n}$.

7.2.1 The well posed problem

Let us assume we are in the favorable case where

$$|k'|\hat{f}(k') + \hat{g}(k') = 0, \qquad k' \in \mathbb{R}^{n-1}. \tag{7.6}$$

In the physical domain, this corresponds to satisfying the non-local problem

$$\sqrt{-\Delta} f(x') + g(x') = 0 \qquad x' \in \mathbb{R}^{n-1}, \tag{7.7}$$

with $\sqrt{-\Delta}$ defined as the square root of the Laplacian. It takes the form $\sqrt{-\Delta} = H\frac{d}{dx}$ in two dimensions $n = 2$, where H is the Hilbert transform.

In the Fourier domain, the solution is thus given by

$$\hat{u}(k', x_n) = \hat{f}(k') e^{-|k'|x_n}. \tag{7.8}$$

Exercise 7.2.1 (i) In two dimensions $n = 2$, prove that

$$u(x_1, x_2) = (f * \frac{1}{\pi}\frac{x_2}{x_1^2 + x_2^2})(x) = \frac{1}{\pi}\int_{\mathbb{R}} f(x_1 - z)\frac{x_2}{z^2 + x_2^2} dz. \tag{7.9}$$

Hint: Use (A.9) and show that

$$\frac{1}{2\pi}\int_{\mathbb{R}} e^{-x_2|k|} e^{ix_1k_1} dk_1 = \frac{1}{\pi}\frac{x_2}{x_1^2 + x_2^2}.$$

(ii) Show that $\frac{1}{\pi}\frac{y}{x_1^2+x_2^2}$ is the fundamental solution of (7.2) when $n = 2$ with $f(x_1) = \delta(x_1)$. Calculate the corresponding value of $g(x_1)$.

Provided the above compatibility condition is met, the problem (7.2) admits a unique solution and is well-posed, for instance in the sense that

$$\int_{\mathbb{R}^{n-1}} u^2(x', x_n) dx' \leq \int_{\mathbb{R}^{n-1}} f^2(x') dx', \qquad \text{for all } x_n > 0.$$

This is an immediate consequence of (7.8) and the Parseval relation. In this situation, the construction of u for $x_n > 0$ becomes a well-posed problem.

7.2.2 The electrocardiac application

In general, however, (7.6) is not satisfied. For the electro-cardiac application, $g_0 = 0$ and the harmonic solution is therefore given (for all $x_n > 0$) by

$$\hat{u}(k', x_n) = \hat{f}(k') \cosh |k'| x_n. \tag{7.10}$$

This implies that high frequencies are exponentially amplified. Therefore, the mapping from $f(x')$ to $A_{x_n} f(x') := u(x', x_n)$ for a fixed $x_n > 0$ cannot be bounded from any $H^s(\mathbb{R}^{n-1})$ (even for s very large) to any $H^t(\mathbb{R}^{n-1})$ (even for $-t$ very large). This does not mean that A_{x_n} cannot be inverted. Let us define the space

$$\mathfrak{X}_{x_n}(\mathbb{R}^{n-1}) = \{u \in L^2(\mathbb{R}^{n-1}),\ \cosh(|k'|x_n)\hat{u}(k') \in L^2(\mathbb{R}^{n-1})\}. \tag{7.11}$$

Then, A_{x_n} is indeed continuous in $\mathcal{L}(L^2(\mathbb{R}^{n-1}), \mathfrak{X}_{x_n}(\mathbb{R}^{n-1}))$ with an inverse $A_{x_n}^{-1}$ continuous in $\mathcal{L}(\mathfrak{X}_{x_n}(\mathbb{R}^{n-1}), L^2(\mathbb{R}^{n-1}))$ and given by

$$A_{x_n}^{-1} u = \mathcal{F}^{-1}_{k' \to x'} \cosh(|k'|x_n) \mathcal{F}_{x' \to k'} u. \tag{7.12}$$

This construction is useful when noise is small in $\mathfrak{X}_{x_n}(\mathbb{R}^{n-1})$, which means that it is essentially low-frequency. Such a restrictive assumption on noise is often not verified in practice. When noise is small only in some L^2 sense, for instance, then A_{x_n} above cannot be inverted without assuming prior information about the solution we seek to reconstruct. Chapters 10 and 11 will be devoted to the analysis of such prior information. Here, we look at how stability estimates may still be obtained when simple bounds are available on the harmonic solutions.

7.2.3 Prior bounds and stability estimates

Let us consider the solution (7.10) in the setting where f is small in some sense (modeling the level of noise in the available data since our problem is linear) and where $u(x)$ the harmonic solution is bounded a priori in some sense. We can then obtain a stability result of the form:

Theorem 7.2.1 *Let $u(x)$ be the solution to (7.2) with $g \equiv 0$. Let us assume that*

$$\|f\|_{L^2(\mathbb{R}^{n-1})} \leq \eta \leq E, \qquad \|u(\cdot, L)\|_{L^2(\mathbb{R}^{n-1})} \leq E. \tag{7.13}$$

Then we find that

$$\|u(\cdot, x_n)\|_{L^2(\mathbb{R}^{n-1})} \leq C \eta^{1 - \frac{x_n}{L}} E^{\frac{x_n}{L}}, \tag{7.14}$$

for some universal constant C.

Proof. The proof is based on the following observation: $\hat{u}(k', x_n)$ is close to $\hat{f}(k')$ for small values of $|k'|$, where we wish to use the constraint involving η. For large $|k'|$, the bound in η on $\hat{f}(k')$ is no longer sufficient and we need to use the bound at L, which constrains the growth at high frequencies. We thus calculate

$$\begin{aligned} \int_{\mathbb{R}^{n-1}} |\hat{f}|^2(k') \cosh^2(|k'|x_n) dk' &= \int_{|k'|<k_0} |\hat{f}|^2(k') \cosh^2(|k'|x_n) dk' \\ + \int_{|k'|>k_0} |\hat{f}|^2(k') \cosh^2(|k'|x_n) dk' &\leq \cosh^2(k_0 x_n)\eta^2 + \frac{\cosh^2(k_0 x_n)}{\cosh^2(k_0 L)} E^2. \end{aligned}$$

We now equate both terms. Since k_0 is "large", we verify that $\cosh x \sim \frac{1}{2}e^x$ and choose

$$k_0 = \frac{1}{L} \ln \frac{2E}{\eta}.$$

With this choice, we obtain that

$$\int_{\mathbb{R}^{n-1}} |\hat{f}|^2(k') \cosh^2(|k'|x_n) dk' \leq C\eta^2 \Big(\frac{2E}{\eta}\Big)^{\frac{x_n}{L}} \leq C\Big(\eta^{\frac{x_n}{L}} E^{1-\frac{x_n}{L}}\Big)^2.$$

This proves the result. □

The above result states that even though solving the Cauchy problem is severely ill-posed, if we possess information about the solution at a distance L from where the measurements are collected, then we can reconstruct the elliptic solution for all $0 < x_n < L$ with Hölder-type stability, which is a mildly ill-posed problem. However, at $x_n = L$, which is where we would like to obtain some stable reconstructions in the electro-cardiac problem, then a mere bound at $x_n = L$ in the L^2 norm is not sufficient to obtain any meaningful stability. However, a stronger bound would be sufficient as the following result indicates.

Theorem 7.2.2 *Let $u(x)$ be the solution to (7.2) with $g \equiv 0$. Let us assume that*

$$\|f\|_{L^2(\mathbb{R}^{n-1})} \leq \eta \leq E, \qquad \|u(\cdot, L)\|_{H^s(\mathbb{R}^{n-1})} \leq E, \quad s > 0. \tag{7.15}$$

Then we find that

$$\|u(\cdot, x_n)\|_{L^2(\mathbb{R}^{n-1})} \leq C \Big| \ln \frac{E}{\eta} \Big|^{-s\frac{x_n}{L}} \eta^{1-\frac{x_n}{L}} E^{\frac{x_n}{L}}, \tag{7.16}$$

for some universal constant C.

Proof. The proof is similar to the preceding one. We calculate

$$\int_{\mathbb{R}^{n-1}} |\hat{f}|^2(k') \cosh^2(|k'|x_n) dk' = \int_{|k'|<k_0} |\hat{f}|^2(k') \cosh^2(|k'|x_n) dk'$$
$$+ \int_{|k'|>k_0} |\hat{f}|^2(k') \cosh^2(|k'|x_n) dk' \quad \leq \quad \cosh^2(k_0 x_n)\eta^2 + \frac{\cosh^2(k_0 x_n)}{\cosh^2(k_0 L)} \langle k_0 \rangle^{-2s} E^2.$$

We recall that $\langle k_0 \rangle^2 = 1 + k_0^2$. We then choose

$$k_0 = \frac{1}{L} \ln \frac{2E}{\eta} - \frac{s}{L} \ln \frac{1}{L} \ln \frac{2E}{\eta}.$$

This is a good approximation to the equation $\cosh^2(k_0 L)\eta^2 = \langle k_0 \rangle^{-2s} E^2$. We then find

$$\eta e^{k_0 x_n} = \Big| \frac{1}{L} \ln \frac{2E}{\eta} \Big|^{-s\frac{x_n}{L}} \eta^{1-\frac{x_n}{L}} (2E)^{\frac{x_n}{L}},$$

from which the result follows. □

For $x_n < L$, the latter result is not much more precise than the preceding one. However, when $x_n = L$, we obtain that the error in the $L^2(\mathbb{R}^{n-1})$ norm of $u(x', x_n)$ is of order $|\ln \eta|^{-s}$. This is a *logarithmic* stability result that cannot be improved and indicates the severe ill-posedness of the electro-cardiac problem. Note also that the stability estimate depends on the *strength* of the exponent s in the prior regularity assumption.

Exercise 7.2.2 Let us assume that in Theorem 7.2.2, the prior estimate is replaced by an estimate of the form

$$\|u\|_{H^m(\mathbb{R}^{n-1}\times(0,L))} \leq E, \qquad m \geq 0 \tag{7.17}$$

(i) Obtain a result of the form (7.16) in this setting.
(ii) Show that $\|u(\cdot, L)\|_{L^2(\mathbb{R}^{n-1})}$ satisfies a logarithmic stability estimate when m is sufficiently large (how large?). Relate this result to that of Theorem 7.2.2.

7.2.4 Analytic continuation

The above results in dimension $n = 2$ are intimately connected to the analytic continuation of an analytic function given on the real line.

Let $f(z) = g(z) + ih(z)$ be an analytic function with $g(z)$ and $h(z)$ real valued-functions. Let us assume that $g(z)$ and $h(z)$ are known on the real line $\Im(z) = 0$. The objective is to find them for arbitrary values of z. We identify $z = x + iy$ and assume that $g(x, 0)$ and $h(x, 0)$ are distributions in $\mathcal{S}'(\mathbb{R})$ so that their Fourier transform is defined. Since $f(z)$ is analytic, we have

$$\frac{\partial f}{\partial \bar{z}} = 0,$$

or equivalently that

$$\frac{\partial g}{\partial x} - \frac{\partial h}{\partial y} = 0, \qquad \frac{\partial g}{\partial y} + \frac{\partial h}{\partial x} = 0. \tag{7.18}$$

These Cauchy-Riemann relations imply that g and h are harmonic, i.e., $\Delta g = \Delta h = 0$. They thus solve the following problems

$$\begin{array}{ll} \Delta g = 0, \quad y > 0, & \Delta h = 0, \quad y > 0, \\ \dfrac{\partial g}{\partial y}(x,0) = -\dfrac{\partial h}{\partial x}(x,0), & \dfrac{\partial h}{\partial y}(x,0) = \dfrac{\partial g}{\partial x}(x,0), \\ g(x,0) \text{ known}, & h(x,0) \text{ known}. \end{array} \tag{7.19}$$

Both problems are of the form (7.2). The solutions in the Fourier domain are given by

$$\begin{aligned} \hat{g}(k_x, y) &= \hat{g}(k_x, 0)\cosh(|k_x|y) - i\text{sign}(k_x)\hat{h}(k_x, 0)\sinh(|k_x|y) \\ \hat{h}(k_x, y) &= \hat{h}(k_x, 0)\cosh(|k_x|y) + i\text{sign}(k_x)\hat{g}(k_x, 0)\sinh(|k_x|y) \end{aligned} \tag{7.20}$$

We verify that the problem is well-posed provided that (7.7) is verified, which in this context means

$$\frac{\partial h}{\partial x} = H\frac{\partial g}{\partial x}, \qquad \frac{\partial g}{\partial x} = -H\frac{\partial h}{\partial x}. \tag{7.21}$$

Notice that $H^2 = -I$ so that both equalities above are equivalent. When the above conditions are met, then the analytic continuation is a stable process. When they are not met, we have seen that high frequencies increase exponentially as y increases, which renders the analytic continuation process a severely ill-posed problem.

7.3 General two dimensional case

We now consider a general two-dimensional geometry as described in section 7.1. We use the Riemann mapping theorem to map such a geometry conformally to an annulus (the region lying between two concentric circles) in the plane. We then solve the problem on the annulus. The Riemann mapping gives us a stable way to transform the original domain to an annulus and back. We will see that solving the problem on the annulus is severely ill-posed similarly to what we saw in the preceding section.

7.3.1 Laplace equation on an annulus

Let us begin with a presentation of the problem on the annulus. We assume that the inner circle has radius 1 and the outer circle radius $\rho > 0$. By simple dilation, this is equivalent to the more general case of two circles of arbitrary radius a and b. In polar coordinates the Laplacian takes the form

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial u}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}. \tag{7.22}$$

The general solution to the above equation periodic in θ is decomposed in Fourier modes as

$$u(r,\theta) = a_0 + b_0 \ln r + \sum_{n\in\mathbb{N}^*}\Big(\frac{a_n}{2}r^n + \frac{b_n}{2}r^{-n}\Big)e^{in\theta}. \tag{7.23}$$

Since $\nu\cdot\nabla u = 0$ on at $r = 1$, we deduce that b_0 and $b_n - a_n$ vanish. We then find that the solution to (7.1) on the annulus is given by

$$u(r,\theta) = \sum_{n\in\mathbb{N}}\Big(\frac{1}{2\pi}\int_0^{2\pi} e^{-in\phi}u_1(\phi)d\phi\Big)\frac{r^n + r^{-n}}{2}e^{in\theta}. \tag{7.24}$$

The above solution holds for all $r > 1$.

Exercise 7.3.1 Find the relation that $u_1(\theta) = u(1,\theta)$ and $g_1(\theta) = \nu\cdot\nabla u(1,\theta)$ must satisfy so that the problem

$$\begin{aligned} &\Delta u = 0, \qquad |r| > 1 \\ &u(1,\theta) = u_1(\theta), \quad n\cdot\nabla u(1,\theta) = g_1(\theta), \qquad 0 \le \theta < 2\pi, \end{aligned}$$

is well-posed (in the sense that the energy of $\theta \to u(\rho,\theta)$ is bounded by the energy of $u_1(\theta)$).

We verify that an error of size δ in the measurement of the coefficient a_n is amplified into an error of order

$$A_n(\rho) = \frac{\delta}{2}e^{n\ln\rho} \qquad \text{at} \quad r = \rho.$$

We verify that A_n cannot be bounded by any Cn^α for all $\alpha > 0$, which would correspond to differentiating the noise level α times. This implies that the reconstruction of $u(\rho,\theta)$ from $u_1(\theta)$ using (7.24) is a severely ill-posed problem.

7.3.2 Riemann mapping theorem

Let X be an open smooth two dimensional domain with smooth boundary having two smooth connected components. We denote by Γ_0 the outer component of ∂X and Γ_1 the inner component; see Fig. 7.1.

For $z \in X \subset \mathbb{C}$ we construct a holomorphic function $\Psi(z)$ (i.e., a function such that $\frac{\partial \Psi}{\partial \bar{z}} = 0$) mapping X to an annulus of the form $1 < r < \rho$. The function is constructed as follows. Let first v be the unique solution to the following Dirichlet problem

$$\Delta v = 0 \ \text{ on } X, \quad v_{|\Gamma_0} = 1, \qquad v_{|\Gamma_1} = 0. \tag{7.25}$$

For some c to be fixed later, let $G = cv$. We verify that

$$I = \int_{\Gamma_1} -\frac{\partial G}{\partial y} dx + \frac{\partial G}{\partial x} dy = -c \int_{\Gamma_1} \frac{\partial v}{\partial \nu} ds > 0$$

by the maximum principle (v is positive inside X and its normal derivative must be negative on Γ_1). We fix c such that $I = 2\pi$. In that case we can define a function $H(z)$, a conjugate harmonic of $G(z)$ on X, by

$$H(z) = \int_p^z -\frac{\partial G}{\partial y} dx + \frac{\partial G}{\partial x} dy, \tag{7.26}$$

where p is an arbitrary point in X. Since G is harmonic, we verify that the definition of H is independent of the path chosen between p and z. Moreover we verify that

$$\frac{\partial H}{\partial x} = -\frac{\partial G}{\partial y}, \qquad \frac{\partial H}{\partial y} = \frac{\partial G}{\partial x},$$

so that $G + iH$ is a holomorphic function on X. Then so is

$$\Psi(z) = e^{G(z)+iH(z)}. \tag{7.27}$$

We verify that $\Psi(z)$ maps Γ_0 to the circle $|z| = e^c$ and Γ_1 to the circle $|z| = 1$. Moreover Ψ is a diffeomorphism between X and the annulus $U_c = \{z \in \mathbb{C},\ 1 \le |z| \le e^c\}$. Finally we verify that $\Delta\Psi(z) = 0$ on X since Ψ is holomorphic. Therefore we have replaced the problem (7.1) on X by solving the Laplace equation on U_c with boundary conditions $u_1(\Psi(z))$ on the circle $r = 1$ and vanishing Neumann boundary conditions (we verify that Neumann boundary conditions are preserved by the map Ψ).

Exercise 7.3.2 *Verify the above statements.*

7.4 Backward Heat Equation

Let us now consider the backward heat equation. The forward heat equation is given by

$$\begin{aligned} &\frac{\partial u}{\partial t} - \Delta u = 0 \qquad t > 0, \quad x \in \mathbb{R}^n \\ &u(0, x) = f(x) \qquad\quad x \in \mathbb{R}^n, \end{aligned} \tag{7.28}$$

with $f(x)$ an initial condition. Let us assume that $u(T,x) = \mathfrak{M}f(T,x)$ is measured and that we wish to reconstruct $f(x)$. In this setting, we pass to the Fourier domain and obtain as above that

$$\hat{u}(t,k) = e^{-t|k|^2}\hat{f}(k). \tag{7.29}$$

Therefore, $\hat{f}(k) = e^{T|k|^2}\hat{u}(T,k)$ is the formal inverse to the measurement operator $\mathfrak{M}$. Again, this is a severely ill-posed problem. But with prior information about the solution $u(t,x)$, we can again obtain Hölder and logarithmic type stability results. For instance, we have

Theorem 7.4.1 *Let us assume that*

$$\|u(T,\cdot)\|_{L^2(\mathbb{R}^n)} = \eta \le E, \qquad \|f\|_{H^s(\mathbb{R}^n)} \le E. \tag{7.30}$$

Then we find that

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \le C\Big|\ln\frac{\eta}{E}\Big|^{-\frac{s(T-t)}{2T}}\eta^{\frac{t}{T}}E^{1-\frac{t}{T}}. \tag{7.31}$$

Proof. The proof is the same as above. We use the η bound for $|k| < k_0$ and the other bound for $|k| > k_0$. We then choose k_0 so that

$$k_0^2 = \frac{1}{T}\ln\frac{E}{\eta} - \frac{s}{2T}\ln k_0^2 \approx \frac{1}{T}\ln\frac{E}{\eta} - \frac{s}{2T}\ln\frac{1}{T}\ln\frac{E}{\eta}.$$

The rest of the proof proceeds as above. Note that the main difference with respect to the Cauchy problem is that s is replaced by $\frac{s}{2}$ in the estimate because frequencies are exponentially damped with a strength proportional to $|k|^2$ rather than $|k|$. □

Note that for all $t > 0$, the error on the reconstruction of $u(t,x)$ is Hölder in η, which measures the errors in the measurements. At $t = 0$, which is the problem of interest in the backward heat equation, we obtain a logarithmic stability: the errors in the reconstruction are proportional to $|\ln\eta|^{-\frac{s}{2}}$, which does converge to 0 as $\eta \to 0$, but does so extremely slowly.

The above proof generalizes to a large class of elliptic operators. For instance, consider the problem

$$\begin{aligned} &\frac{\partial u}{\partial t} - Lu = 0, && t > 0,\ x \in X \\ &u(t,x) = 0 && x \in \partial X \\ &u(0,x) = f(x), && x \in X \end{aligned} \tag{7.32}$$

with L a symmetric operator such as for instance $-\nabla\cdot a\nabla + c$ admitting a spectral decomposition

$$-L\phi_n = \lambda_n\phi_n \text{ in } X \quad \phi_n = 0 \text{ on } \partial X, \tag{7.33}$$

with an orthonormal basis of normalized eigenvectors ϕ_n and positive eigenvalues $0 < \lambda_1 < \lambda_2 \le \dots$ with $\lambda_n \to \infty$ as $n \to \infty$. Then we can decompose

$$f(x) = \sum_{n\ge1} f_n\phi_n(x), \qquad u(t,x) = \sum_{n\ge1} u_n(t)\phi_n(x).$$

Let us assume that $u_n(T)$ is measured and that we want to reconstruct $u_n(t)$ for $t \geq 0$ and hence also f_n at $t = 0$. Then we find that

$$u_n(t) = e^{-\lambda_n t} f_n = e^{\lambda_n (T-t)} u_n(T).$$

Let us assume that $f(x)$ admits an a priori bound in $H^s(X)$ of the form

$$\|f\|_{H^s(X)} := \Big(\sum_{n \geq 1} n^{2s} f_n^2 \Big)^{\frac{1}{2}} = E. \tag{7.34}$$

Let us assume that we have an error of the form

$$\|u(T,\cdot)\|_{L^2(X)} := \Big(\sum_{n \geq 1} u_n^2(T) \Big)^{\frac{1}{2}} = \eta \leq E. \tag{7.35}$$

Let us assume that $\lambda_n \approx n^\alpha$ for some $\alpha > 0$ asymptotically. Then we calculate

$$\begin{aligned} \sum_{n \geq 1} u_n^2(t) &= \sum_{n \leq n_0 - 1} (e^{\lambda_n (T-t)} u_n(T))^2 + \sum_{n \geq n_0} (e^{-\lambda_n t} f_n)^2 \\ &\leq e^{2\lambda_{n_0}(T-t)} \eta^2 + e^{-2\lambda_{n_0} t} n_0^{-2s} E^2. \end{aligned}$$

Again, we use the estimate on the noise η for the low frequencies of $u(t)$ and the regularity estimate E for the high frequencies of $u(t)$. It remains to equate these two terms to find that

$$n_0^\alpha \approx \frac{1}{T} \ln \frac{E}{\eta} - \frac{s}{T\alpha} \ln \frac{1}{T} \ln \frac{E}{\eta}.$$

This yields the error estimate

$$\|u(t,\cdot)\|_{L^2(X)} \leq C \Big| \ln \frac{\eta}{E} \Big|^{-\frac{(T-t)s}{T\alpha}} \eta^{\frac{t}{T}} E^{1-\frac{t}{T}}. \tag{7.36}$$

We leave it to the reader to fill in the gaps and state theorems. As usual, we obtain a Hölder estimate for all $0 < t \leq T$. At $t = 0$, the stability estimate becomes again logarithmic and crucially relies on the prior information that f is bounded in some space $H^s(X)$ with $s > 0$.

7.5 Carleman Estimates, UCP and Cauchy problem

We briefly revisit the endocardial problem and show a much more general result for second-order elliptic problems (with isotropic diffusion). The proofs of the (weak) unique continuation principle (UCP) stating that a solution vanishing on an open set vanishes everywhere, of the Cauchy data problem, stating that a solution vanishing on an open set of the boundary with vanishing Neumann conditions on that open set has to vanish everywhere, are based on Carleman estimates, which we consider first.

We want to show that the Laplacian controls lower-order derivatives locally. This is done by means of Carleman estimates.

Let $\psi(x)$ be a function such that $\psi = \exp(\phi)$ with $\phi \geq 0$ on a domain of interest. We are looking at the operator

$$P_{\tau\psi} = e^{\tau\psi}(-\Delta)e^{-\tau\psi} = -e^{\tau\psi}\nabla e^{-\tau\psi} \cdot e^{\tau\psi}\nabla e^{-\tau\psi} = -(\nabla - \tau\nabla\psi)\cdot(\nabla - \tau\nabla\psi)$$

and want to show that for an appropriate function ψ defined on a bounded domain X and for $u \in C_c^\infty(X)$, we have

$$\|P_{\tau\psi}u\| \geq C\Big(\tau^{\frac{1}{2}}\|\nabla u\| + \tau^{\frac{3}{2}}\|u\|\Big)$$

for a positive constant C independent of τ sufficiently large.

The derivation goes as follows. We first observe that

$$P_{\tau\psi} = A + B, \quad A = -\Delta - \tau^2|\nabla\psi|^2, \quad Bu = \tau\big(\nabla\cdot(u\nabla\psi) + \nabla\psi\cdot\nabla u\big).$$

Note that A and iB are Hermitian while B is antisymmetric. We then write

$$\|P_{\tau\psi}u\|^2 = \int_X (A+B)u(A+B)u = \|Au\|^2 + \|Bu\|^2 + ([A,B]u,u)$$

The term $\|Bu\|^2$ is non-negative and will not have any other role. Expanding the commutator $[A,B] = AB - BA$ yields

$$\|P_{\tau\psi}u\|^2 = \|Au\|^2 + \|Bu\|^2 + 4\tau(\nabla^2\psi\nabla u, \nabla u) + 4\tau^3((\nabla^2\psi\nabla\psi,\nabla\psi)u,u) - \tau((\Delta^2\psi)u,u).$$

The control on u will come from a combination of $\|Au\|^2$, the positivity of $(\nabla^2\psi\nabla\psi,\nabla\psi)$, and τ large. Here, $\nabla^2\psi$ is the Hessian of ψ. Using $\psi = \exp\phi$ yields

$$\nabla^2\psi = (\nabla^2\phi + (\nabla\phi)^{\otimes 2})\psi$$

so that

$$4\tau(\nabla^2\psi\nabla u,\nabla u) = 4\tau(\psi\nabla^2\phi\nabla u,\nabla u) + 4\tau(\psi, |\nabla\phi\cdot\nabla u|^2)$$

The last term is non-negative. Define $C_1 = \sup_X |\nabla^2\phi|$ to get

$$\|P_{\tau\psi}u\|^2 \geq \|Au\|^2 + \|Bu\|^2 - C_1\tau(\psi\nabla u,\nabla u) + 4\tau^3((\nabla^2\psi\nabla\psi,\nabla\psi)u,u) - \tau((\Delta^2\psi)u,u).$$

Control over the term involving ∇u has to come from Au and u in some form. Let us write

$$(\psi\nabla u,\nabla u) = \int -(\Delta u)\psi u - (\nabla\psi\cdot\nabla u, u) = (Au,\psi u) + \tau^2(|\nabla\psi|^2 u,\psi u) - (\nabla\psi\cdot\nabla u,u).$$

We find

$$|(\nabla\psi\cdot\nabla u,u)| = (\psi^{\frac{1}{2}}\nabla u, \psi^{-\frac{1}{2}}\nabla\psi u) \leq \frac{1}{2}\|\psi^{\frac{1}{2}}\nabla u\|^2 + \frac{1}{2}\|\psi^{-\frac{1}{2}}\nabla\psi u\|^2$$

so that

$$\frac{1}{2}\|\psi^{\frac{1}{2}}\nabla u\|^2 \leq (Au,\psi u) + 2\tau^2(|\nabla\psi|^2 u,\psi u)$$

for $\tau \geq 1$ and when $\psi|\nabla\psi|^2 \geq \psi^{-\frac{1}{2}}|\nabla\psi|$, which holds when $\psi \geq 1$ and $|\nabla\psi| \geq 1$, say. Under the same constraints, we have

$$\frac{1}{2}\|\psi^{\frac{1}{2}}\nabla u\|^2 \leq \frac{\delta}{2}\|Au\|^2 + (\frac{1}{2\delta} + 2\tau^2)(|\nabla\psi|^2 u, \psi u).$$

Choosing $2C_1\delta\tau = 1$, we get

$$\|P_{\tau\psi}u\|^2 \geq \frac{1}{2}\|Au\|^2 + \Big(\big[4\tau^3(\nabla^2\psi\nabla\psi, \nabla\psi) - C_1\tau(\frac{1}{2\delta} + 2\tau^2)\psi|\nabla\psi|^2 - \tau\Delta^2\psi\big]u, u\Big)$$

We thus need to find a function ψ such that

$$4(\nabla^2\psi\nabla\psi, \nabla\psi) - 2C_1\psi|\nabla\psi|^2 \geq 2\psi^3 c, \quad c > 0. \tag{7.37}$$

uniformly on X. For $\psi \geq 1$ sufficiently smooth and τ sufficiently large, we obtain that

$$\|P_{\tau\psi}u\|^2 \geq \frac{1}{2}\|Au\|^2 + c\tau^3\|u\|^2.$$

The estimate on ∇u

$$(\nabla u, \nabla u) = \int -(\Delta u)u = (Au, u) + \tau^2(|\nabla\psi|^2 u, u) \leq C\big(\|Au\|^2 + \tau^2\|u\|^2\big)$$

provides

$$\tau\|\nabla u\|^2 \leq C\|P_{\tau\psi}u\|^2,$$

which is what we wanted to show.

It remains to find a sufficiently large family of functions ψ such that the above constraints hold. This is so for sufficiently convex functions. Let us construct $\phi = \lambda\varphi$ so that $\psi = e^{\lambda\varphi}$ for $\lambda > 0$ that will be chosen large enough. Then

$$(\nabla^2\psi\nabla\psi, \nabla\psi) = |\nabla\phi|^4\psi^3 + (\nabla^2\phi\nabla\phi, \nabla\phi)\psi^3 = \lambda^4|\nabla\varphi|^4\psi^3 + \lambda^3(\nabla^2\varphi\nabla\varphi, \nabla\varphi)\psi^3.$$

Let φ be a smooth function such that $|\nabla\varphi| \geq c_0 > 0$. Since $C_1 = \lambda C_{10}$ with $C_{10} = \sup_X |\nabla^2\varphi|$ and $|\nabla\psi|^2 = \lambda^2|\nabla\varphi|^2\psi^2$, we observe that for λ sufficiently large, we indeed obtain (7.37).

The result is therefore the following. Consider $\varphi \geq 0$ such that $|\nabla\varphi| \geq c_0 > 0$ on $\bar{X}$. Then for λ sufficiently large and $\psi = \exp(\lambda\varphi)$, there is a positive constant C such that for all τ sufficiently large, we have

$$\|P_{\tau\psi}u\| \geq C\Big(\tau^{\frac{1}{2}}\|\nabla u\| + \tau^{\frac{3}{2}}\|u\|\Big).$$

The derivation obtained for $u \in C_c^\infty(X)$ extends to any function (with the same constants) in $H_0^2(X)$, i.e., function that have two derivatives that are square integrable and that vanish on ∂X, by simple approximation.

Let us relax the positivity constraint on φ and assume that $\varphi + \alpha \geq 0$ for some constant $\alpha \geq 0$. We define $\psi_\alpha = \exp(\lambda(\varphi + \alpha))$ as well as $\psi = \exp(\lambda\varphi)$. For λ sufficiently large and $\tau \geq 1$, we have

$$\|P_{\tau\psi}u\| = \|P_{\tau_\alpha\psi_\alpha}u\| \geq C\Big(\tau_\alpha^{\frac{1}{2}}\|\nabla u\| + \tau_\alpha^{\frac{3}{2}}\|u\|\Big) \geq Ce^{-\frac{3}{2}\lambda\alpha}\Big(\tau^{\frac{1}{2}}\|\nabla u\| + \tau^{\frac{3}{2}}\|u\|\Big), \quad \tau_\alpha e^{\lambda\alpha} = \tau.$$

We thus obtain the following result. Let φ be a smooth function bounded below and such that $|\nabla\varphi| \geq c_0 > 0$ on $\bar{X}$. Then for λ sufficiently large and $\psi = \exp(\lambda\varphi)$, there is a positive constant C independent of τ and $u \in H_0^2(X)$ such that for all τ sufficiently large, we have

$$\|P_{\tau\psi}u\| \geq C\Big(\tau^{\frac{1}{2}}\|\nabla u\| + \tau^{\frac{3}{2}}\|u\|\Big).$$

By replacing u by $e^{\tau\psi}u$, we find, after some simple manipulation, that

$$\|e^{\tau\psi}\Delta u\| \geq C\Big(\tau^{\frac{1}{2}}\|e^{\tau\psi}\nabla u\| + \tau^{\frac{3}{2}}\|e^{\tau\psi}u\|\Big)$$

uniformly for all function $u \in H_0^2(X)$. Note that the results do not change if $\psi = e^{\lambda\varphi} - c$ for any constant c, say $c = 1$.

Now, for $a(x)$ scalar bounded below by a positive constant and a, b, c bounded above (component by component), we deduce that

$$\|e^{\tau\psi}\Delta u\| \leq C\|e^{\tau\psi}(a\Delta u + b\nabla u + cu)\| + C\|e^{\tau\psi}(|\nabla u| + |u|)\|.$$

Therefore, for τ sufficiently large, we obtain more generally that, for a positive constant C,

$$\|e^{\tau\psi}Lu\| \geq C\Big(\tau^{\frac{1}{2}}\|e^{\tau\psi}\nabla u\| + \tau^{\frac{3}{2}}\|e^{\tau\psi}u\|\Big)$$

for any operator of the form $L = a\Delta + b\cdot\nabla + c$ or in divergence form $L = \nabla\cdot a\nabla + b\cdot\nabla + c$ provided a is Lipschitz in the latter case. Note that Lipschitz continuity of a is optimal to obtain unique continuation results as such results are known not to hold for Hölder continuous coefficient $a(x)$.

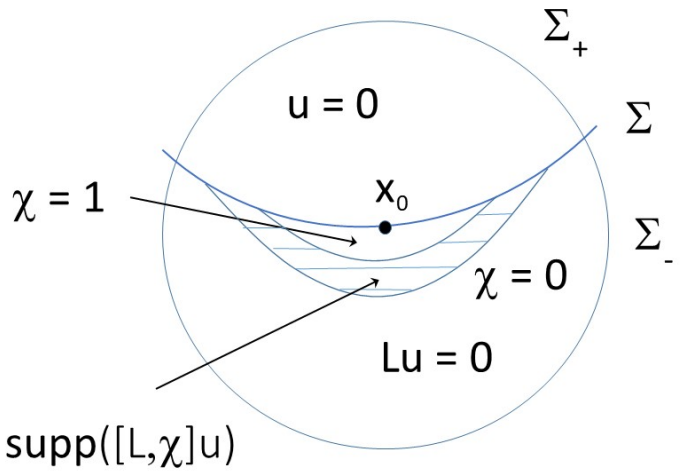

Figure 7.2: Geometry of the Carleman estimate. We obtain from the estimate that $u = 0$ 'below' x_0 where $\chi = 1$.

We now want to use the above Carleman estimate to obtain unique continuation principles.

Let $\phi(x)$ be a smooth real-valued function with non-vanishing $\nabla\phi$ in the vicinity V of a point $x_0 \in \mathbb{R}^n$. Set $\Sigma = \phi^{-1}(0)$ in V and $\Sigma_\pm$ the parts of V where $\pm\phi > 0$. We assume that $u = 0$ on Σ_+ and that $Lu = 0$ on V. We want to show that $u = 0$ in an open neighborhood of x_0. This is done as follows. We define

$$\varphi(x) = \phi(x) + \delta^3 - 3\delta|x - x_0|^2$$

for some $\delta > 0$ sufficiently small. We then verify that $\nabla\varphi \neq 0$ in V and that φ is bounded from below. Note that $\varphi(x_0) = \delta^3 > 0$. The level sets $\varphi = 0$ and $\varphi = -\delta^3$ are hypersurfaces intersecting Σ_- within a distance δ of x_0; see Fig. 7.2.

Let us now define $\chi \in C_c^\infty(\mathbb{R}^n)$ equal to 1 on $\{\varphi > 0\} \cap B(0,\delta)$, equal to 0 on $\{\varphi < -\delta^3\}$ and hence with support of $\nabla\chi$ in Σ_- given by $\{-\delta^3 < \varphi < 0\}$. This is also the domain where $u\nabla\chi$ is supported and hence where $[L,\chi]u$ is supported. Define $\psi = e^{\lambda\varphi} - 1$ as above with λ sufficiently large, so that φ and ψ share the same $0-$level set. Now we have

$$\|u_{|\chi=1}\| \leq \|e^{\tau\psi}u_{|\chi=1}\| \leq \|e^{\tau\psi}u\chi\| \leq \tau^{-\frac{3}{2}}\|e^{\tau\psi}L(\chi u)\| = \tau^{-\frac{3}{2}}\|e^{\tau\psi}[L,\chi]u\| \leq \tau^{-\frac{3}{2}}\|[L,\chi]u\|$$

since $e^{\tau\psi} \geq 1$ where $\chi = 1$ and $u \neq 0$ and $e^{\tau\psi} \leq 1$ where $[L,\chi]u$ is supported.

Finally, for u sufficiently smooth, $\|[L,\chi]u\|$ is bounded so that, sending $\tau \to \infty$, we observe that $u_{|\chi=1} = 0$. This domain includes an open set including the point x_0.

This proves unique continuation across a surface locally. When $u = 0$ on one side of the surface and $Lu = 0$ in the vicinity of the surface, then $u = 0$ in a (possibly smaller) vicinity of the surface simply by displacing x_0 along the surface.

From this, we deduce the weak unique continuation principle for second-order elliptic equations (with scalar coefficients in the above proof although the method extends to the general elliptic (scalar!) case). This goes as follows.

Theorem 7.5.1 *Let X be a bounded connected open domain and $u \in H^2(X)$ with $Lu = 0$ on X and L with sufficiently smooth coefficient that the above Carleman estimates hold. Let X_0 be an open subdomain with $\bar{X}_0 \subset X$ and assume that $u = 0$ on X. Then $u = 0$ on X.*

Proof. Let us first assume that for open balls $B_0 \subset B_1 \subset X$, we have $u = 0$ on B_1 when $u = 0$ on B_0. We define B_t a continuous family of balls such that $B_0 \subset B_s \subset B_t \subset B_1$ for $s < t$. From the above construction, we observe that if $u = 0$ on B_s, then $u = 0$ on a slightly larger ball $B_{s'}$ for $s' > s$. This is obtained by constructing functions ϕ such that the level set of ϕ is the boundary of B_s locally in the vicinity of $x_0 \in \partial B_s$, then φ positive in the vicinity of x_0, and finally ψ such that the Carleman estimate holds. This shows that u vanishes in the vicinity of x_0. Now rotating x_0 along ∂B_s yields the result. We also observe from the construction that the vicinity is independent of s since the coefficients are uniformly sufficiently smooth on X. This shows that $u = 0$ on each ball B_s including B_1.

Now let x be any point in X and V a bounded connected (since X is connected) open domain including x and X_0 such that $\bar{V} \subset X$. By compactness, we can cover $\bar{V}$ with a finite number of open balls supported in X. Let us add to the collection a ball

in X_0 where u vanishes. Assume that u vanishes on a ball B_k and that $B_k \cap B_{k+1} \neq \emptyset$. Then there is a ball in the intersection where u vanishes so that u vanishes on B_{k+1} as well. This shows that u vanishes on all balls and hence at x, which was arbitrary. Therefore, u vanishes on X. □

We also have the local UCP result for Cauchy data

Theorem 7.5.2 *Let X be a bounded connected open domain with smooth boundary and $u \in H^2(X)$ with $Lu = 0$ on X and L with sufficiently smooth coefficient that the above Carleman estimates hold. Let Γ be a non-empty open subset of ∂X and assume that u and ∇u vanish on Γ. Then $u = 0$ in X.*

Proof. Note that the conditions are $u = 0$ and $\nu \cdot \nabla u = 0$ on Γ with $\nu(x)$ the outward unit normal to X at $x \in \partial X$. Let $x_0 \in \Gamma$ and let B be a sufficiently small ball centered at x_0 so that the intersection of ∂X and B_0 is in Γ. Let us extend u by 0 in the open part Y_1 of B that is not in X. We then extend the coefficients in L to smooth coefficients in B that preserve ellipticity. Then $Lu = 0$ in Y_1 and in Y_2, the intersection of B with X. Moreover, the compatibility conditions at $\partial Y_1 \cap \partial Y_2$ ensure that $u \in H^2(B)$. We have therefore constructed an extension u such that $Lu = 0$ on B and $u = 0$ in Y_1. From the preceding theorem, this means that $u = 0$ in the whole of B, and hence in Y_2 and finally the whole of X. □

Chapter 8

Calderón problem

This chapter focuses on the Calderón problem, which is another inverse diffusion problem with a similar behavior to the Cauchy problems considered in the preceding chapter. Because of its central role in inverse problem theory and of the importance of the mathematical tools that are used to solve it, we have devoted a full chapter to its study.

8.1 Introduction

The Calderón problem finds applications in two medical imaging methods, *electrical impedance tomography* (EIT) and *optical tomography* (OT). We begin with the setting of EIT, a non-invasive modality that consists of reconstructing the electrical properties (the conductivity) of tissues from current and voltage boundary measurements. The mathematical framework for the Calderón problem and the definition of the measurement operator were presented in the introduction. For completeness, we repeat this definition here. The Calderón problem is modeled by the following elliptic problem with Dirichlet conditions

$$\begin{aligned} L_\gamma u(x) \equiv \nabla \cdot \gamma(x) \nabla u(x) = 0, \qquad & x \in X \\ u(x) = f(x), \qquad & x \in \partial X, \end{aligned} \tag{8.1}$$

where $X \subset \mathbb{R}^n$ is a bounded domain with smooth boundary ∂X. In what follows, we assume $n \geq 3$. Here, $\gamma(x)$ is a conductivity coefficient, which we assume is a smooth function, and $f(x)$ is a prescribed Dirichlet data for the elliptic problem.

The Dirichlet-to-Neumann or voltage-to-current map is given by

$$\begin{array}{rccl} \Lambda_\gamma : & H^{\frac{1}{2}}(\partial X) & \to & H^{-\frac{1}{2}}(\partial X) \\ & f(x) & \mapsto & \Lambda_\gamma[f](x) = \gamma(x) \dfrac{\partial u}{\partial \nu}(x). \end{array} \tag{8.2}$$

With $\mathfrak{X} = \mathcal{C}^2(\bar{X})$ and $\mathfrak{Y} = \mathcal{L}(H^{\frac{1}{2}}(\partial X), H^{-\frac{1}{2}}(\partial X))$, we define the measurement operator

$$\mathfrak{M} : \quad \mathfrak{X} \ni \gamma \mapsto \mathfrak{M}(\gamma) = \Lambda_\gamma \in \mathfrak{Y}. \tag{8.3}$$

The Calderón problem consists of reconstructing γ from knowledge of the Calderón measurement operator $\mathfrak{M}$. To slightly simplify the derivation of uniqueness, we also

make the (unnecessary) assumption that γ and $\nu \cdot \nabla\gamma$ are *known* on ∂X. The main result of this chapter is the following.

Theorem 8.1.1 *Define the measurement operator $\mathfrak{M}$ as in (8.3). Then $\mathfrak{M}$ is injective in the sense that $\mathfrak{M}(\gamma) = \mathfrak{M}(\tilde{\gamma})$ implies that $\gamma = \tilde{\gamma}$.*

Moreover, we have the following logarithmic stability estimate:

$$\|\gamma(x) - \gamma'(x)\|_{L^\infty(X)} \leq C\Big|\log \|\mathfrak{M}(\gamma) - \mathfrak{M}(\tilde{\gamma})\|_{\mathfrak{Y}}\Big|^{-\delta}, \tag{8.4}$$

for some $\delta > 0$ provided that γ and $\tilde{\gamma}$ are uniformly bounded in $H^s(X)$ for some $s > \frac{n}{2}$.

The proof of the injectivity result was first obtained in [58]. It is based on two main ingredients. The first ingredient consists of recasting the injectivity result as a statement of whether products of functionals of solutions to elliptic equations such as (8.2) are dense in the space of, say, continuous functions. The second step is to construct specific sequences of solutions to (8.2) that positively answer the density question. These specific solutions are Complex Geometric Optics (CGO) solutions. Their construction is detailed in section 8.3. We first start with a section on the uniqueness and stability results.

8.2 Uniqueness and Stability

8.2.1 Reduction to a Schödinger equation

We start with the following lemma

Lemma 8.2.1 *Let Λ_{γ_j} for $j = 1, 2$ be the two operators associated to γ_j and let $f_j \in H^{\frac{1}{2}}(\partial X)$ for $j = 1, 2$ be two Dirichlet conditions. Then we find that*

$$\int_{\partial X} (\Lambda_{\gamma_1} - \Lambda_{\gamma_2}) f_1 f_2 d\mu = \int_X (\gamma_1 - \gamma_2) \nabla u_1 \cdot \nabla u_2 dx, \tag{8.5}$$

where $d\mu$ is the surface measure on ∂X and where u_j is the solution to (8.1) with γ replaced by γ_j and f replaced by f_j.

Here we use the notation on the left-hand side to mean:

$$\int_{\partial X} (\Lambda_{\gamma_1} - \Lambda_{\gamma_2}) f_1(x) f_2(x) d\mu(x) := \langle (\Lambda_{\gamma_1} - \Lambda_{\gamma_2}) f_1, f_2 \rangle_{H^{-\frac{1}{2}}(\partial X), H^{\frac{1}{2}}(\partial X)}.$$

Proof. The proof is a simple integration by parts. Let us consider the equation for u_1, multiply it by u_2 and integrate the product over X. Then we find by application of the divergence theorem that

$$\begin{aligned} 0 = -\int_X u_2 L_{\gamma_1} u_1 dx &= \int_X \gamma_1 \nabla u_1 \cdot \nabla u_2 dx - \int_{\partial X} u_2 \gamma_1 \nu \cdot \nabla u_1 d\mu(x). \\ &= \int_X \gamma_1 \nabla u_1 \cdot \nabla u_2 dx - \int_{\partial X} f_2 \Lambda_{\gamma_1} f_1 d\mu(x). \end{aligned}$$

Exchanging the roles of the indices $j = 1$ and $j = 2$ and subtracting the result to the above equality yields (8.5). ∎

The above lemma shows that when $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, the right-hand-side in (8.1) also vanishes for *any* solutions u_1 and u_2 of (8.1) with γ given by γ_1 and γ_2, respectively. We are now thus faced with the question of whether products of the form $\nabla u_1 \cdot \nabla u_2$ are dense in the space of, say, continuous functions. Unfortunately, answering this question affirmatively seems to be a difficult problem. The main difficulty in the analysis of (8.1) is that the unknown coefficient γ appears in the leading order of the differential operator L_γ. The following Liouville change of variables allows us to treat the unknown coefficient as a *perturbation* to a known operator (with constant coefficients):

$$\gamma^{-\frac{1}{2}} L_\gamma \gamma^{-\frac{1}{2}} = \Delta - q, \qquad q = \frac{\Delta \gamma^{\frac{1}{2}}}{\gamma^{\frac{1}{2}}}. \tag{8.6}$$

Here Δ is the usual Laplacian operator.

Exercise 8.2.1 Prove (8.6).

Therefore if u is a solution of $L_\gamma u = 0$, then $v = \gamma^{\frac{1}{2}} u$ is a solution to the Schrödinger equation $(\Delta - q)v = 0$. We thus wish to replace the problem of the reconstruction of γ by that of q.

Consider the Schrödinger equation (still calling the solution "u" rather than "v")

$$(\Delta - q)u = 0 \quad \text{in } X, \qquad u = f \quad \text{on } \partial X, \tag{8.7}$$

with q given by (8.6). For $f \in H^{\frac{1}{2}}(\partial X)$, we find a unique solution $u \in H^1(X)$ such that $\nu \cdot \nabla u \in H^{-\frac{1}{2}}(\partial X)$. Indeed, the above equation admits a solution since it is equivalent to (8.1) by the change of variables (8.6). We then define the Dirichlet-to-Neumann operator

$$\Lambda_q : \begin{array}{ccc} H^{\frac{1}{2}}(\partial X) & \to & H^{-\frac{1}{2}}(\partial X) \\ f(x) & \mapsto & \Lambda_q[f](x) = \dfrac{\partial u}{\partial \nu}(x), \end{array} \tag{8.8}$$

where u is the solution to (8.7). We then verify that

$$\Lambda_q f = \gamma^{-\frac{1}{2}} \frac{\partial \gamma}{\partial \nu}\Big|_{\partial X} f + \gamma^{-\frac{1}{2}} \Lambda_\gamma (\gamma^{-\frac{1}{2}}\Big|_{\partial X} f), \qquad f \in H^{\frac{1}{2}}(\partial X). \tag{8.9}$$

Exercise 8.2.2 Prove the above result.

We thus observe that knowledge of Λ_γ, $\gamma|_{\partial X}$ and $\nu \cdot \nabla \gamma|_{\partial X}$ implies knowledge of Λ_q. It turns out that knowledge of Λ_γ implies that of $\gamma|_{\partial X}$ and $\nu \cdot \nabla \gamma|_{\partial X}$:

Theorem 8.2.2 *Let us assume that $0 < \gamma_i \in C^m(\bar{X})$ and that $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$. Then we can show that for all $|\alpha| < m$, we have*

$$\partial^\alpha \gamma_1\Big|_{\partial X} = \partial^\alpha \gamma_2\Big|_{\partial X}. \tag{8.10}$$

See [41] for a proof of this result. To simplify the analysis of stability, we have assumed here that $\gamma|_{\partial X}$ and $\nu \cdot \nabla \gamma|_{\partial X}$ were *known*. Thus, Λ_γ uniquely determines Λ_q. Our next step is therefore to reconstruct q from knowledge of Λ_q. We start with the following lemma:

Lemma 8.2.3 *Let Λ_{q_j} for $j = 1, 2$ be the two operators associated to q_j and let $f_j \in H^{\frac{1}{2}}(\partial X)$ for $j = 1, 2$ be two Dirichlet conditions. Then we find that*

$$\int_{\partial X} (\Lambda_{q_1} - \Lambda_{q_2}) f_1 f_2 d\mu = \int_X (q_1 - q_2) u_1 u_2 dx, \tag{8.11}$$

where $d\mu$ is the surface measure on ∂X and where u_j is the solution to (8.7) with q replaced by q_j and f replaced by f_j.

Exercise 8.2.3 *Prove this lemma following the derivation in Lemma 8.2.1.*

The above lemma shows that when $\Lambda_{q_1} = \Lambda_{q_2}$, then the right-hand-side in (8.11) also vanishes for *any* solutions u_1 and u_2 of (8.7) with q replaced by q_1 and q_2, respectively. We are now thus faced with the question of whether products of the form $u_1 u_2$ are *dense* in the space of, say, continuous functions. This is a question that admits an affirmative answer. The main tool in the proof of this density argument is the construction of complex geometric optics solutions. Such solutions are constructed in section 8.3. The main property that we need at the moment is summarized in the following lemma.

Lemma 8.2.4 *Let $\varrho \in \mathbb{C}^n$ be a complex valued vector such that $\varrho \cdot \varrho = 0$. Let $\|q\|_\infty < \infty$ and $|\varrho|$ be sufficiently large. Then there is a solution u of $(\Delta - q)u = 0$ in X of the form*

$$u(x) = e^{\varrho \cdot x}(1 + \varphi(x)), \tag{8.12}$$

such that

$$|\varrho| \|\varphi\|_{L^2(X)} + \|\varphi\|_{H^1(X)} \leq C. \tag{8.13}$$

The proof of this and more general lemmas can be found in section 8.3. The principle of such solutions is this. When $q \equiv 0$, then $e^{\varrho \cdot x}$ is a (complex-valued) harmonic function, i.e., a solution of $\Delta u = 0$. The above result shows that q may be treated as a perturbation of Δ. Solutions of $(\Delta - q)u = 0$ are fundamentally not that different from solutions of $\Delta u = 0$.

Now, coming back to the issue of density of product of elliptic solutions. For u_1 and u_2 solutions of the form (8.12), we find that

$$u_1 u_2 = e^{(\varrho_1 + \varrho_2) \cdot x}(1 + \varphi_1 + \varphi_2 + \varphi_1 \varphi_2). \tag{8.14}$$

If we can choose $\varrho_1 + \varrho_2 = ik$ for a fixed k with $|\varrho_1|$ and $|\varrho_2|$ growing to infinity so that $\varphi_1 + \varphi_2 + \varphi_1 \varphi_2$ becomes negligible in the L^2 sense thanks to (8.13), then we observe that in the limit $u_1 u_2$ equals $e^{ik \cdot x}$. The functions $e^{ik \cdot x}$ for arbitrary $k \in \mathbb{R}^n$ certainly form a dense family of, say, continuous functions.

8.2.2 Proof of injectivity result

Let us make a remark on the nature of the CGO solutions and the measurement operator $\mathfrak{M}$. The CGO solutions are complex valued. Since the equations (8.1) and (8.7) are linear, we can assume that the boundary conditions $f = f_r + if_i$ are complex valued as a superposition of two real-valued boundary conditions f_r and f_i. Moreover, the results (8.5) and (8.11) hold for complex-valued solutions. Our objective is therefore to show

that the product of complex-valued solutions to elliptic equations of the form (8.7) is indeed dense. The construction in dimension $n \geq 3$ goes as follows.

Let $k \in \mathbb{R}^n$ be fixed for $n \geq 3$. We choose $\varrho_{1,2}$ as

$$\varrho_1 = \frac{m}{2} + i\frac{k+l}{2}, \qquad \varrho_2 = -\frac{m}{2} + i\frac{k-l}{2}, \tag{8.15}$$

where the real-valued vectors l, and m are chosen in $\mathbb{R}^n$ such that

$$m \cdot k = m \cdot l = k \cdot l = 0, \qquad |m|^2 = |k|^2 + |l|^2. \tag{8.16}$$

We verify that $\varrho_i \cdot \varrho_i = 0$ and that $|\varrho_i|^2 = \frac{1}{2}(|k|^2 + |l|^2)$. In dimension $n \geq 3$, such vectors can always be found. For instance, changing the system of coordinates so that $k = |k|e_1$, we can choose $l = |l|e_2$ with $|l| > 0$ arbitrary and then $m = \sqrt{|k|^2 + |l|^2}e_3$, where (e_1, e_2, e_3) forms a family of orthonormal vectors in $\mathbb{R}^n$. Note that this construction is possible only when $n \geq 3$. It is important to notice that while k is fixed, $|l|$ can be chosen arbitrarily large so that the norm of ϱ_i can be arbitrarily large while $\varrho_1 + \varrho_2 = k$ is fixed.

Upon combining (8.11) and (8.14), we obtain for the choice (8.15) that $\Lambda_{q_1} = \Lambda_{q_2}$ implies that

$$\Big| \int_X e^{ik\cdot x}(q_1 - q_2)dx \Big| \leq \Big| \int_X e^{ik\cdot x}(q_1 - q_2)(\varphi_1 + \varphi_2 + \varphi_1\varphi_2)dx \Big| \leq \frac{C}{|l|}$$

thanks to (8.13) since $|l|(\varphi_1 + \varphi_2 + \varphi_1\varphi_2)$ is bounded in $L^1(X)$ by an application of the Cauchy-Schwarz inequality and $e^{ik\cdot x}(q_1 - q_2)$ is bounded in $L^\infty(X)$. Since the above inequality holds independent of l, we deduce that the Fourier transform of $(q_1 - q_2)$ (extended by 0 outside of X) vanishes, and hence that $q_1 = q_2$. So far we have thus proved that

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \implies \Lambda_{q_1} = \Lambda_{q_2} \implies q_1 = q_2,$$

where q_j and γ_j are related by (8.6). From (8.6) still, we deduce that

$$0 = \gamma_1^{\frac{1}{2}}\Delta\gamma_2^{\frac{1}{2}} - \gamma_2^{\frac{1}{2}}\Delta\gamma_1^{\frac{1}{2}} = \nabla \cdot (\gamma_1^{\frac{1}{2}}\nabla\gamma_2^{\frac{1}{2}} - \gamma_2^{\frac{1}{2}}\nabla\gamma_1^{\frac{1}{2}}) = \nabla \cdot (\gamma_1 \nabla \Big(\frac{\gamma_2}{\gamma_1}\Big)^{\frac{1}{2}}). \tag{8.17}$$

Since $\gamma_1 = \gamma_2$ on ∂X, this is an elliptic equation for $\big(\frac{\gamma_1}{\gamma_2}\big)^{\frac{1}{2}}$ whose only solution is identically 1. This shows that $\gamma_1 = \gamma_2$. This concludes the proof of the uniqueness result

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \implies \gamma_1 = \gamma_2. \tag{8.18}$$

8.2.3 Proof of the stability result

Let us return to (8.11) and assume that $\Lambda_{q_1} - \Lambda_{q_2}$ no longer vanishes but is (arbitrarily) small. We first want to assess how errors in Λ_q translates into errors in q. For u_j

solutions as stated in Lemma 8.2.3 and of the form (8.12), we find that

$$\begin{aligned}\left|\int_X e^{ik\cdot x}(q_1-q_2)dx\right| &\leq \left|\int_X e^{ik\cdot x}(q_1-q_2)(\varphi_1+\varphi_2+\varphi_1\varphi_2)dx\right| + \left|\int_{\partial X}(\Lambda_{q_1}-\Lambda_{q_2})f_1 f_2 d\mu\right| \\ &\leq \frac{C}{|l|} + \|\Lambda_{q_1}-\Lambda_{q_2}\|_{\mathfrak{Y}}\|f_1\|_{H^{\frac{1}{2}}(\partial X)}\|f_2\|_{H^{\frac{1}{2}}(\partial X)} \\ &\leq \frac{C}{|l|} + C\|\Lambda_{q_1}-\Lambda_{q_2}\|_{\mathfrak{Y}}\|u_1\|_{H^1(X)}\|u_2\|_{H^1(X)} \\ &\leq \frac{C}{|l|} + C|l|\|\Lambda_{q_1}-\Lambda_{q_2}\|_{\mathfrak{Y}}e^{C|l|}.\end{aligned}$$

Indeed, $f_j = u_j|_{\partial X}$ and $\|u\|_{H^{\frac{1}{2}}(\partial X)} \leq C\|u\|_{H^1(X)}$ is a standard estimate. This step is where the ill-posedness of the Calderón problem is best displayed.

Exercise 8.2.4 *Verify that that for some constant C independent of $|l|$ and for u given by (8.12)-(8.13), we have:*

$$\|u\|_{H^1(X)} \leq C|l|e^{C|l|}.$$

Define $\delta q = q_1 - q_2$. So far, we have obtained a control of $\widehat{\delta q}(k)$ uniform in $k \in \mathbb{R}^n$. Upon choosing

$$|l| = \frac{\sigma}{C}\ln\frac{1}{\varepsilon}, \qquad 0<\sigma<1,$$

so that $e^{C|l|} = \varepsilon^{-\sigma}$, we find that for $\varepsilon := \min(1, \|\Lambda_{q_1}-\Lambda_{q_2}\|_{\mathfrak{Y}})$,

$$|\widehat{\delta q}(k)| \leq \eta := C\left|\ln\varepsilon\right|^{-1}. \tag{8.19}$$

Since q is assumed to be bounded and compactly supported, it is square integrable in $\mathbb{R}^n$ so that $\|\delta q\|_{L^2(\mathbb{R}^n)} := E < \infty$. This and the control in (8.19) allows one to obtain a control of δq in $H^{-s}(\mathbb{R}^n)$ for $s>0$. Indeed

$$\begin{aligned}\|\delta q\|^2_{H^{-s}(\mathbb{R}^n)} &= \int \langle k\rangle^{-2s}|\widehat{\delta q}|^2 dk \\ &\leq k_0^n\eta^2 + k_0^{-2s}E^2,\end{aligned}$$

by splitting the integration in k into $|k|<k_0$ and $|k|>k_0$ and choosing $k_0 \geq 1$. We then choose

$$k_0 = \left(\frac{E}{\eta}\right)^{\frac{2}{n+2s}}.$$

This implies

$$\|q_1-q_2\|_{H^{-s}(\mathbb{R}^n)} \leq CE^{\frac{n}{n+2s}}\left|\ln\varepsilon\right|^{-\frac{2s}{n+2s}}. \tag{8.20}$$

Exercise 8.2.5 *Assume that $\|\delta q\|_{H^\varsigma(\mathbb{R}^n)} := E < \infty$. Show that the estimate (8.20) holds with s replaced by $s+\varsigma$ on the right-hand-side.*

It remains to convert the estimate on $q_1 - q_2$ into an estimate for $\gamma_1 - \gamma_2$. We find that (8.17) is replaced by

$$(\gamma_1\gamma_2)^{\frac{1}{2}}(q_1-q_2) = \nabla\cdot(\gamma_1\nabla\Big(\frac{\gamma_2}{\gamma_1}-1\Big)^{\frac{1}{2}}) \quad \text{in} \quad X, \qquad \Big(\frac{\gamma_2}{\gamma_1}-1\Big)^{\frac{1}{2}} = 0 \quad \text{on} \quad \partial X. \tag{8.21}$$

Standard elliptic regularity results and the fact that γ_1 is of class C^2 therefore show that

$$\|\gamma_1 - \gamma_2\|_{H^1(X)} \leq C\|q_1 - q_2\|_{H^{-1}(X)} \leq C|\ln\varepsilon|^{-\delta}, \tag{8.22}$$

with $\delta = \frac{2}{2+n}$ if q is bounded in the L^2 sense and $\delta = \frac{2(1+\varsigma)}{n+2(1+\varsigma)}$ if q is bounded as in Exercise 8.2.5. The final result in (8.4) then follows from interpolating the a priori bound in H^s of $\gamma_1 - \gamma_2$, the above smallness bound in H^1 to obtain a small bound in H^τ for some $\frac{n}{2} < \tau < s$. Then by the Sobolev imbedding of $L^\infty(X)$ into $H^\tau(X)$, we conclude the proof of Theorem 8.1.1.

8.3 Complex Geometric Optics Solutions

The major technical ingredient in the proof of Theorem 8.1.1 is the existence of complex geometrical optics (CGO) solutions of the form (8.12). The proof of Lemma 8.2.4 and more general results that will be useful in subsequent chapters is undertaken in this section.

Let us consider the equation $\Delta u = qu$ in X. When $q = 0$, then a rich family of harmonic solutions is formed by the complex exponentials $e^{\varrho\cdot x}$ for $\varrho \in \mathbb{C}^n$ a complex valued vector such that $\varrho\cdot\varrho = 0$. Indeed, we verify that

$$\Delta e^{x\cdot\varrho} = \varrho\cdot\varrho\, e^{x\cdot\varrho} = 0. \tag{8.23}$$

A vector $\varrho = \varrho_r + i\varrho_i$ is such that $\varrho\cdot\varrho = 0$ if an only if ϱ_r and ϱ_i are orthogonal vectors of the same (Euclidean) length.

When $q \neq 0$, it is tempting to try and write solutions of $\Delta u = qu$ as perturbations of the harmonic solutions $e^{\varrho\cdot x}$, for instance in the form

$$u(x) = e^{\varrho\cdot x}(1 + \varphi(x)).$$

This provides an equation for φ of the form

$$(\Delta + 2\varrho\cdot\nabla)\varphi = q(1+\varphi). \tag{8.24}$$

Exercise 8.3.1 *Check this formula.*

Treating the right-hand side as a source f, the first part of the construction consists of solving the problem

$$(\Delta + 2\varrho\cdot\nabla)\varphi = f, \tag{8.25}$$

for f a source in X and φ defined on X as well. Surprisingly, the analysis of (8.25) is the most challenging technical step in the construction of solutions to (8.24). The construction with $f \in L^2(X)$ is sufficient for the proof of Theorem 8.1.1. In later chapters, we will require more regularity for the solution to (8.25) and thus prove the following result.

Lemma 8.3.1 *Let $f \in H^s(X)$ for $s \geq 0$ and let $|\varrho| \geq c > 0$. Then there exists a solution to (8.25) in $H^{s+1}(X)$ and such that*

$$|\varrho| \|\varphi\|_{H^s(X)} + \|\varphi\|_{H^{s+1}(X)} \leq C \|f\|_{H^s(X)}. \tag{8.26}$$

Proof. We first extend f defined on X to a function still called f defined and compactly supported in $\mathbb{R}^n$ and such that

$$\|f\|_{H^s(\mathbb{R}^n)} \leq C(X) \|f\|_{H^s(X)}.$$

That such an extension exists is proved for instance in [57, Chapter 6, Theorem 5]. We thus wish to solve the problem

$$(\Delta + 2\varrho \cdot \nabla)\varphi = f, \qquad \text{in} \quad \mathbb{R}^n. \tag{8.27}$$

The main difficulty is that the operator $(\Delta + 2\varrho \cdot \nabla)$ has for symbol

$$\mathcal{F}_{x \to \xi}(\Delta + 2\varrho \cdot \nabla)\mathcal{F}^{-1}_{\xi \to x} = -|\xi|^2 + 2i\varrho \cdot \xi.$$

Such a symbol vanishes for $\varrho_r \cdot \xi = 0$ and $2\varrho_i \cdot \xi + |\xi|^2 = 0$. The original proof of the injectivity result of Theorem 8.1.1 in [58] shows the existence and uniqueness of a solution to (8.27) in appropriate functional spaces. Since uniqueness is of no concern to us here, we instead follow a path undertaken by [34, 51] and construct a solution that can be seen of the product of a plane wave with a periodic solution with different period. Let us define

$$\varphi = e^{i\varsigma \cdot x}\mathfrak{p}, \qquad f = e^{i\varsigma \cdot x}\mathfrak{f}$$

for some vector $\varsigma \in \mathbb{R}^n$ to be determined. Then we find

$$\Big(\Delta + 2(\varrho + i\varsigma) \cdot \nabla + (2i\varrho \cdot \varsigma - |\varsigma|^2)\Big)\mathfrak{p} = (\nabla + i\varsigma + 2\varrho) \cdot (\nabla + i\varsigma)\mathfrak{p} = \mathfrak{f}. \tag{8.28}$$

Exercise 8.3.2 *Verify this.*

Let us now assume that $\mathfrak{f}$ is supported in a box Q of size $(-L, L)^n$ for L sufficiently large. Then we decompose as Fourier series:

$$\mathfrak{p} = \sum_{k \in \mathbb{Z}^n} \mathfrak{p}_k e^{i\frac{\pi}{L}k \cdot x}, \qquad \mathfrak{f} = \sum_{k \in \mathbb{Z}^n} \mathfrak{f}_k e^{i\frac{\pi}{L}k \cdot x}. \tag{8.29}$$

We then find that (8.28) is equivalent in the Fourier domain to

$$\mathfrak{p}_k = \frac{1}{-|\frac{\pi}{L}k + \varsigma|^2 + 2i\varrho \cdot (\varsigma + \frac{\pi}{L}k)}\mathfrak{f}_k \tag{8.30}$$

The imaginary part of the denominator is given by $2\varrho_r \cdot (\frac{\pi}{L}k + \varsigma)$. It remains to choose

$$\varsigma = \frac{1}{2}\frac{\pi}{L}\frac{\varrho_r}{|\varrho_r|},$$

to obtain that the above denominator never vanishes since $k \in \mathbb{Z}^n$. Moreover, for such a choice, we deduce that

$$\Big| - \big|\frac{\pi}{L}k + \varsigma\big|^2 + 2i\varrho \cdot \big(\varsigma + \frac{\pi}{L}k\big)\Big| \geq C|\varrho|,$$

for some constant C independent of ϱ. This shows that

$$|\mathfrak{p}_k| \leq C|\varrho|^{-1}|\mathfrak{f}_k|.$$

Since $\mathfrak{f} \in H^s(Q)$, we deduce that $\|\mathfrak{f}\|^2_{H^s(Q)} = \sum_{k\in\mathbb{Z}^n} |k|^{2s}|\mathfrak{f}_k|^2 < \infty$, from which we deduce that

$$\|\mathfrak{p}\|_{H^s(Q)} \leq C|\varrho|^{-1}\|\mathfrak{f}\|_{H^s(Q)}.$$

It remains to restrict the constructed solution to X (and realize that $e^{i\varsigma\cdot x}$ is smooth) to obtain that $|\varrho|\|\varphi\|_{H^s(X)} \leq C\|f\|_{H^s(X)}$ and the first step in (8.26).

The result on $\|\varphi\|_{H^{s+1}(X)}$ requires that we obtain bounds for $|k|\mathfrak{p}_k$. For $|k|$ small, say $|k| \leq \frac{8L}{\pi}|\varrho|$, then we use the same result as above to obtain

$$|k||\mathfrak{p}_k| \leq C|\mathfrak{f}_k|, \qquad |k| \leq \frac{8L}{\pi}|\varrho|.$$

For the other values of $|k|$, we realize that the denominator in (8.30) causes no problem and that

$$|k||\mathfrak{p}_k| \leq C|k|^{-1}|\mathfrak{f}_k|, \qquad |k| > \frac{8L}{\pi}|\varrho|.$$

This shows that $|k||\mathfrak{p}_k| \leq C|\mathfrak{f}_k|$ for some constant C independent of k and $|\varrho|$. The proof that $\|\varphi\|_{H^{s+1}(X)} \leq C\|f\|_{H^s(X)}$ then proceeds as above. This concludes the proof of the fundamental lemma of CGO solutions to Schrödinger equations. □

We now come back to the perturbed problem (8.24). We assume that q is a complex-valued potential in $H^s(X)$ for some $s \geq 0$. We say that $q \in L^\infty(X)$ has regularity s provided that for all $\varphi \in H^s(X)$, we have

$$\|q\varphi\|_{H^s(X)} \leq q_s\|\varphi\|_{H^s(X)}, \tag{8.31}$$

for some constant q_s. For instance, when $s = 0$, when $q_s = \|q\|_{L^\infty(X)}$. Then we have the following result.

Theorem 8.3.2 *Let us assume that $q \in H^s(X)$ is sufficiently smooth so that $q_s < \infty$. Then for $|\varrho|$ sufficiently large, there exists a solution φ to (8.24) that satisfies*

$$|\varrho|\|\varphi\|_{H^s(X)} + \|\varphi\|_{H^{s+1}(X)} \leq C\|q\|_{H^s(X)}. \tag{8.32}$$

Moreover, we have that

$$u(x) = e^{\varrho\cdot x}(1 + \varphi(x)) \tag{8.33}$$

is a Complex Geometrical Optics solution in $H^{s+1}(X)$ to the equation

$$\Delta u = qu \qquad \text{in } X.$$

Proof. Let T be the operator which to $f \in H^s(X)$ associates $\varphi \in H^s(X)$ the solution of (8.27) constructed in the proof of Lemma 8.3.1. Then (8.24) may be recast as

$$(I - Tq)\varphi = Tq.$$

We know that $\|T\|_{\mathcal{L}(H^s(X))} \leq C_s|\varrho|^{-1}$. Choosing $|\varrho|$ sufficiently large so that $|\varrho| > C_s q_s$, we deduce that $(I-Tq)^{-1} = \sum_{m=0}^{\infty}(Tq)^m$ exists and is a bounded operator in $\mathcal{L}(H^s(X))$. We have therefore constructed a solution so that $q(1+\varphi) \in H^s(X)$. The estimate (8.26) yields (8.32) and concludes the proof of the theorem. □

Note that the above theorem with $s = 0$ yields Lemma 8.2.4.

Let us now consider the elliptic equation (8.1). The change of variables in (8.6) shows that $u = \gamma^{-\frac{1}{2}}v$ with v a solution of $\Delta v = qv$, is a solution of (8.1). We therefore have the

Corollary 8.3.3 *Let γ be sufficiently smooth so that $q = \gamma^{-\frac{1}{2}}\Delta\gamma^{\frac{1}{2}}$ verifies the hypotheses of Theorem 8.3.2. Then for $|\varrho|$ sufficiently large, we can find a solution u of $\nabla \cdot \gamma\nabla u = 0$ on X such that*

$$u(x) = \frac{1}{\gamma^{\frac{1}{2}}(x)}e^{\varrho\cdot x}(1 + \varphi(x)), \tag{8.34}$$

and such that (8.32) holds.

For instance, for $s \in \mathbb{N}$, we verify that (8.31) holds provided that γ is of class $C^{s+2}(X)$. The case $s = 0$ with γ of class $C^2(X)$ is the setting of Theorem 8.1.1.

8.4 The Optical Tomography setting

We have seen that optical tomography measurements could be modeled by radiative transfer equations. In the diffusive regime, where photons travel over distances that are much larger than the mean free path, the photon density is accurately modeled by a diffusion equation. Let us assume that the source of photons used to probe the domain of interest is time harmonic with a frequency modulation ω. The density of such time harmonic photons then solves the following elliptic model

$$-\nabla \cdot \gamma\nabla u + (\sigma + i\omega)u = 0 \quad \text{in} \quad X, \qquad u = f \quad \text{on} \quad \partial X. \tag{8.35}$$

We assume here that f is the prescribed distribution of photons at the domain's boundary. This is an approximation as only the density of *incoming* photons can be prescribed. More accurate models would not fundamentally change the main conclusions of this section and we therefore assume Dirichlet conditions to simplify.

The coefficients (γ, σ) are the diffusion and absorption coefficients, respectively. We assume that light speed is normalized to $c = 1$ to simplify (otherwise ω should be replaced by $c^{-1}\omega$).

The Dirichlet-to-Neumann or density-to-current map is given in this setting by

$$\begin{array}{rccl} \Lambda^{\omega}_{\gamma,\sigma}: & H^{\frac{1}{2}}(\partial X) & \to & H^{-\frac{1}{2}}(\partial X) \\ & f(x) & \mapsto & \Lambda^{\omega}_{\gamma,\sigma}[f](x) = \gamma(x)\dfrac{\partial u}{\partial \nu}(x). \end{array} \tag{8.36}$$

With $\mathfrak{X} = \mathcal{C}^2(\bar{X}) \times \mathcal{C}^0(\bar{X})$ and $\mathfrak{Y} = \mathcal{L}(H^{\frac{1}{2}}(\partial X), H^{-\frac{1}{2}}(\partial X))$, we define the measurement operator

$$\mathfrak{M}^\omega : \quad \mathfrak{X} \ni (\gamma, \sigma) \mapsto \mathfrak{M}^\omega(\gamma, \sigma) = \Lambda^\omega_{\gamma,\sigma} \in \mathfrak{Y}. \tag{8.37}$$

The measurement operator is parameterized by the (known) modulation frequency ω and there are now two unknown coefficients γ and σ. Then we have the following result.

Theorem 8.4.1 *Define the measurement operator $\mathfrak{M}^\omega$ as in (8.37). Then for $\omega \neq 0$, we have that $\mathfrak{M}^\omega$ is injective in the sense that $\mathfrak{M}^\omega(\gamma, \sigma) = \mathfrak{M}^\omega(\tilde{\gamma}, \tilde{\sigma})$ implies that $(\gamma, \sigma) = (\tilde{\gamma}, \tilde{\sigma})$.*

Moreover, we have the following logarithmic stability estimate:

$$\|\gamma(x) - \gamma'(x)\|_{L^\infty(X)} + \|\sigma(x) - \sigma'(x)\|_{L^\infty(X)} \leq C \left| \log \|\mathfrak{M}^\omega(\gamma, \sigma) - \mathfrak{M}^\omega(\tilde{\gamma}, \tilde{\sigma})\|_{\mathfrak{Y}} \right|^{-\delta}, \tag{8.38}$$

for some $\delta > 0$ provided that (γ, σ) and $(\tilde{\gamma}, \tilde{\sigma})$ are uniformly bounded in $(H^s(X))^2$ for some $s > \frac{n}{2}$. Here, C depends on ω.

Let now $\omega = 0$. Then $\mathfrak{M}(\gamma, \sigma) = \mathfrak{M}(\tilde{\gamma}, \tilde{\sigma})$ implies that

$$q := \frac{\Delta \gamma^{\frac{1}{2}}}{\gamma^{\frac{1}{2}}} + \frac{\sigma}{\gamma} = \tilde{q} := \frac{\Delta \tilde{\gamma}^{\frac{1}{2}}}{\tilde{\gamma}^{\frac{1}{2}}} + \frac{\tilde{\sigma}}{\tilde{\gamma}}. \tag{8.39}$$

and (γ, σ) are defined up to an arbitrary change of variables that leaves q above invariant. Moreover, q above can reconstructed in $H^{-1}(X)$ as shown in the preceding sections and hence in $L^\infty(X)$ by interpolation.

Proof. The proof is very similar to that of Theorem 8.1.1. We mainly highlight the differences. Let u be a solution of the above equation and $v = \gamma^{\frac{1}{2}} u$. We verify that

$$\Delta v = q_\omega v, \qquad q_\omega = \frac{\Delta \gamma^{\frac{1}{2}}}{\gamma^{\frac{1}{2}}} + \frac{\sigma + i\omega}{\gamma}.$$

Let us assume that $\omega \neq 0$. Then knowledge of $\Lambda_{\gamma,\sigma}$ yields knowledge of Λ_{q_ω} as before. This uniquely determines q_ω and allows to obtain an error bound for $q_\omega - \tilde{q}_\omega$ in $H^{-1}(X)$. The imaginary part of q_ω thus provides a reconstruction of γ in the same space, and hence in L^∞ by interpolation between $H^{-1}(X)$ and $H^s(X)$ for $s > \frac{n}{2}$. Since $n \geq 3$, we thus obtain an error in $\Delta \gamma^{\frac{1}{2}}$ also in H^{-1}, which then yields an error estimate for σ, also in $H^{-1}(X)$. Again, by interpolation, we find a value of δ so that (8.38) holds. Note that the value of δ is a priori worse than the one found for the Calderón problem since we do not take advantage of solving an elliptic equation for $\gamma^{\frac{1}{2}}$.

This concludes the proof when $\omega \neq 0$. When $\omega = 0$, then clearly we can reconstruct q in (8.39) with the corresponding stability estimate. Now let us consider two couples (γ, σ) and $(\tilde{\gamma}, \tilde{\sigma})$ such that (8.39) holds. From (8.9), we deduce that

$$\Lambda_{\gamma,\sigma}(f) = \gamma^{\frac{1}{2}} \Lambda_q \gamma^{\frac{1}{2}} \Big|_{\partial X} f - \frac{\partial \gamma}{\partial \nu}\Big|_{\partial X} \gamma^{\frac{1}{2}} \Big|_{\partial X} f, \qquad f \in H^{\frac{1}{2}}(\partial X). \tag{8.40}$$

Since we assume γ and $\nu \cdot \nabla \gamma$ known on ∂X, then we obtain that $\Lambda_{\gamma,\sigma} = \Lambda_{\tilde{\gamma},\tilde{\sigma}}$. This shows that the reconstruction of (γ, σ) is obtained up to any change of variables that leaves q in (8.39) invariant. □

The above theorem shows that measurements in Optical Tomography with continuous wave (CW) sources corresponding to $\omega = 0$ do not guaranty uniqueness of the reconstruction of (γ, σ). The injectivity of the measurement operator is restored when $\omega \neq 0$. However, as for the Calderón problem, the stability estimates are logarithmic (and cannot fundamentally be improved; only the value of δ may not be optimal in the above derivations).

8.5 Anisotropy and non-uniqueness

Let us return to the equation

$$L_\gamma u(x) \equiv \nabla \cdot \gamma(x) \nabla u(x) = 0 \quad x \in X, \qquad u(x) = f(x) \quad x \in \partial X, \tag{8.41}$$

where now γ is a second-order symmetric tensor rather than a scalar coefficient.

We define the Dirichlet-to-Neumann as

$$\begin{array}{rccl} \Lambda_\gamma : & H^{\frac{1}{2}}(\partial X) & \to & H^{-\frac{1}{2}}(\partial X) \\ & f(x) & \mapsto & \Lambda_\gamma[f](x) = \nu \cdot \gamma(x) \nabla u(x). \end{array} \tag{8.42}$$

We have seen in the preceding section that the Dirichlet to Neumann map Λ_γ uniquely determined a scalar coefficient γ. We now see that the reconstruction of the tensor γ is not unique. In fact, for any γ compatible with a given λ_γ, we can construct a very large family of tensors with the same Dirichlet-to-Neumann map.

Define $F : X \to X$ a diffeomorphism such that $F_{\partial X} = Id$, i.e., an arbitrary change of variables that leaves every point at the boundary ∂X invariant. We then change variables according to $y = F(x)$. We then define the *push-forward*

$$v(y) = F_* u(y) := u \circ F^{-1}(y). \tag{8.43}$$

In other words, $v(y) = u(x)$. We also define $J_F(x)$ the Jacobian of the transformation

$$J_F(x) = \det(DF(x)) \qquad \text{with the differential} \quad (DF)_{ij}(x) = \big(\frac{\partial F_i}{\partial x_j}(x)\big)_{ij}. \tag{8.44}$$

We thus find that

$$u(x) = v \circ F(x), \quad \nabla u(x) = (DF)^t(x) \nabla v(F(x)), \quad dy = J_f(x) dx.$$

Let us consider the Dirichlet form

$$J_\gamma(u) = \int_X \gamma \nabla u \cdot \nabla u dx. \tag{8.45}$$

The unique solution to (8.41) is also the unique minimizer to (8.45) in $H^1(X)$ such that $u = f$ in $H^{\frac{1}{2}}(\partial X)$. Now, using the above change of variables, we find that

$$\begin{array}{rcl} J_\gamma(u) & = & \displaystyle\int_X (DF)(x) \gamma(x) (DF)^t(x) \nabla v(F(x)) \cdot \nabla v(F(x)) dx \\ & = & \displaystyle\int_X \frac{(DF)(x) \gamma(x) (DF)^t(x)}{J_F(x)} \Big|_{x = F^{-1}(y)} \nabla v(y) \cdot \nabla v(y) dy. \end{array}$$

Therefore, we obtain that the Dirichlet form is *invariant* by change of variables if:

$$J_\gamma(u) = J_{F_*\gamma}(F_*u) \qquad \text{if } F_*\gamma(y) := \frac{(DF)(x)\gamma(x)(DF)^t(x)}{J_F(x)}\Big|_{x=F^{-1}(y)}, \tag{8.46}$$

the push-forward of γ by F. Therefore, we find that $v = F_*u$ is the unique minimizer of

$$\nabla \cdot (F_*\gamma)(y)\nabla v(y) = 0 \quad y \in X, \qquad v(y) = f(y) \quad y \in \partial X,$$

since $f(y) = f(F^{-1}(y))$ on ∂X. Now standard integrations by parts done earlier in this chapter show that

$$J_\gamma(u) = \int_{\partial X} \Lambda_\gamma f(x) f(x) d\mu(x) = J_{F_*\gamma}(F_*u) = \int_X \Lambda_{F_*\gamma} f(y) f(y) dy.$$

By polarization, the above expression implies that

$$\Lambda_\gamma = \Lambda_{F_*\gamma}. \tag{8.47}$$

In other words, although the conductivities γ and $F_*\gamma$ are *different*, the corresponding Dirichlet-to-Neumann maps are equal. The Calderón problem with anisotropic conductivities is therefore *not injective*.

In dimension $n = 2$, it is known that the above changes of variables are the only obstruction to injectivity. In other words, if two conductivities share the same Dirichlet-to-Neumann operator, then there exists a diffeomorphism F such that one is the push-forward by F of the other. In dimension $n \geq 3$, this is known only in the case of analytic conductivities. An outstanding open problem related to the Calderón problem is whether such results still hold for (sufficiently smooth) conductivities.

Chapter 9

Coupled-physics IP I: Photo-acoustic Tomography and Transient Elastography

Many inverse diffusion problems, including the Calderón problem, are modeled by measurement operators that are injective but not very stable in the sense that the modulus of continuity is logarithmic. The main reason for such a behavior is that solutions to elliptic equations are very smooth away from where the singularities are generated. As a consequence, singularities in the parameters do not propagate to strong, easily recognizable, signals in the boundary measurements. High frequencies of the parameters are still present in the measurements when the measurement operator is injective. However, they have been exponentially attenuated and the inverse problem is then best described as severely ill-posed and typically displays poor resolution capabilities. Imaging modalities based on inverse diffusion problems are still useful because of the *high contrast* they often display, for instance between healthy and non-healthy tissues in medical imaging. Modalities such as Electrical Impedance Tomography and Optical Tomography may be described as high contrast, low resolution modalities.

In applications where resolution is not paramount, the severe ill-posedness of inverse diffusion problems might not be an issue. In many instances, however, high resolution in the reconstruction is an important objective of the imaging modality. The remedy to such a low resolution is to find methodologies that *combine* the high contrast modality with another well-posed, high-resolution, inversion, such as for instance, those involving inverse wave problems or inverse problems of integral geometry (such as the Radon transform). In order for the combination to exist, we need to be sufficiently fortunate that a measurable physical phenomenon couples the high resolution wave-like mechanism with the high contrast elliptic-like phenomenon. Such examples exist in nature and give rise to *hybrid* imaging techniques that couples the high contrast (but low-resolution) of the diffusion-like equation with the high resolution (but often low contrast) of the wave-like equation.

In this chapter and the next, we consider several such physical couplings. In this chapter, we consider the coupling of optical waves with ultrasound in the so-called photo-acoustic effect. This gives rise to a modality called Photo-acoustic Tomography (PAT). We also consider the coupling of elastic waves with ultrasound in the modality

called Transient Elastography (TE). TE and PAT are relatively similar mathematically and both involve inverse problems with *internal functionals*.

In the next chapter, we consider another coupling of optical waves with ultrasound called ultrasound modulation. Mathematically, this problem also gives rise to an inverse problem with internal functionals, albeit one whose analysis is more complicated than for TE and PAT.

9.1 Introduction to PAT and TE

9.1.1 Modeling of photoacoustic tomography

Photoacoustic tomography (PAT) is a hybrid medical imaging modality that combines the high resolution of acoustic waves with the high contrast of optical waves. When a body is exposed to short pulse radiation, typically emitted in the near infra-red spectrum in PAT, it absorbs energy and expands thermo-elastically by a very small amount; this is the photoacoustic effect. Such an expansion is sufficient to emit acoustic pulses, which travel back to the boundary of the domain of interest where they are measured by arrays of transducers.

Radiation propagation. The propagation of radiation in highly scattering media is modeled by the following diffusion equation

$$\begin{aligned} &\frac{1}{c}\frac{\partial}{\partial t}u - \nabla\cdot\gamma(x)\nabla u + \sigma(x)u = 0, \qquad x \in X \subset \mathbb{R}^n,\ t > 0 \\ &u = f \qquad x\ \in \partial X,\ t > 0 \end{aligned} \tag{9.1}$$

where X is an open, bounded, connected domain in $\mathbb{R}^n$ with C^1 boundary ∂X (embedded in $\mathbb{R}^n$), where n spatial dimension; c is light speed in tissues; $\gamma(x)$ is a (scalar) diffusion coefficient; and $\sigma(x)$ is an absorption coefficient. Throughout the paper, we assume that $\gamma(x)$ and $\sigma(x)$ are bounded from above and below by (strictly) positive constants. The source of incoming radiation is prescribed by $f(t,x)$ on the boundary ∂X and is assumed to be a very short pulse supported on an interval of time $(0,\eta)$ with $c\eta$ of order $O(1)$.

Photoacoustic effect. As radiation propagates, a small fraction is absorbed. This absorbed radiation generates a slight temperature increase, which results in a minute mechanical expansion. The latter expansion is responsible for the emission of ultrasound, which are measured at the domain's boundary. The coupling between the optical and ultrasonic waves is called the photo-acoustic effect. The amount of energy deposited and transformed into acoustic energy is given by:

$$H(t,x) = \Gamma(x)\sigma(x)u(t,x),$$

where $\Gamma(x)$ is the Grüneisen coefficient quantifying the photo-acoustic effect while $\sigma(x)u(t,x)$ is the density of absorbed radiation.

A thermal expansion proportional to H results and acoustic waves are emitted. Such waves are modeled by

$$\frac{1}{c_s^2(x)}\frac{\partial^2 p}{\partial t^2} - \Delta p = \frac{\partial}{\partial t}H(t,x), \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^n, \tag{9.2}$$

with c_s the sound speed. We assume here to simplify that the wave equation is posed in the whole space $\mathbb{R}^n$. This assumption is justified by the fact that waves propagating away from the domain of interest are assumed not to interfere with the measurements and by the fact that the measurement devices themselves do not modify the wave field. In a typical measurement setting, the acoustic pressure $p(t,x)$ is then measured on ∂X as a function of time.

Since light travels much faster than sound with $c_s \ll c$, we may assume for short light pulses that radiation propagation occurs over a very short time at the scale of the acoustic waves. This justifies the simplification:

$$H(t,x) \sim H(x)\delta_0(t), \qquad H(x) = \Gamma(x)\sigma(x)\int_{\mathbb{R}^+} u(t,x)dt.$$

The acoustic signals are therefore modeled by the following wave equation

$$\frac{1}{c_s^2(x)}\frac{\partial^2 p}{\partial t^2} - \Delta p = \frac{\partial \delta_0(t)}{\partial t}H(x). \tag{9.3}$$

We now set the sound speed to $c_s = 1$ to simplify. The acoustic pressure $p(t,x)$ is then measured on ∂X as a function of time. Assuming a system at rest so that $p(t,x) = 0$ for $t < 0$, the wave equation may then be recast as the following equation:

$$\begin{aligned} \frac{\partial^2 p}{\partial t^2} - \Delta p &= 0, && t > 0,\ x \in \mathbb{R}^n \\ p(0,x) &= H(x), && x \in \mathbb{R}^n \\ \partial_t p(0,x) &= 0, && x \in \mathbb{R}^n. \end{aligned} \tag{9.4}$$

Exercise 9.1.1 *Show that under the above simplifying assumptions,* (9.3) *formally leads to the initial value problem* (9.4).

We retrieve the wave equation with unknown source term $H(x)$ given by

$$H(x) = \Gamma(x)\sigma(x)u(x), \qquad u(x) = \int_{\mathbb{R}^+} u(t,x)dt, \tag{9.5}$$

where $u(x)$ is the total intensity of radiation reaching a point x integrated over the time span of the radiation pulse, and thus satisfies the equation

$$\begin{aligned} -\nabla \cdot \gamma(x)\nabla u + \sigma(x)u &= 0, && x \in X \subset \mathbb{R}^n, \\ u &= f && x \in \partial X, \end{aligned} \tag{9.6}$$

where $f(x) = \int_{\mathbb{R}^+} f(t,x)dt$ is the spatial density of photons emitted into the domain.

9.1.2 First step: Inverse wave source problem

The first step in PAT consists of reconstructing $H(x)$ from available ultrasound measurements. Two different settings of ultrasound measurements are considered now.

Inverse wave problem with boundary measurements. The reconstruction of $H(x)$ in (9.4) from a measurement operator given by $\mathfrak{M}H(t,x) = p(t,x)|_{\partial X}$ was considered in Chapter 4. Several explicit reconstruction formulas were presented there, including (5.35) and the reconstruction procedure in the Fourier domain presented in section 5.3.2.

Inverse waves problem with planar measurements. Other measurement settings than point measurements have been considered in practice. One major issue with point measurements such as those considered above is that the acoustic signal is typically rather weak and thus difficult to measure accurately. An alternative to such point measurements is to consider planar detectors, which integrate the pressure field over a larger domain and thus become less sensitive to noise.

Consider the setting of (9.2) with $\mathfrak{f}(x) = H(x)$ an unknown source term. Let $P(s,\omega)$ for $s \in \mathbb{R}$ and $\omega \in \mathbb{S}^2$ be the plane of points $x \in \mathbb{R}^3$ such that $x \cdot \omega = s$. Then we recall that the three-dimensional Radon transform of a function is defined as

$$R\mathfrak{f}(s,\omega) = \int_{\mathbb{R}^3} \mathfrak{f}(x)\delta(s - x\cdot\omega)dx = \int_{P(s,\omega)} \mathfrak{f}(x)d\mu(x),$$

where $d\mu(x)$ is the surface measure on the plane $P(s,\omega)$. We thus obtain that the measurements are given as a function of time t and angle ω by

$$\mathfrak{f} \mapsto \mathfrak{M}_3\mathfrak{f}(t,\omega) = Rp(1,\omega,t), \qquad t > 0, \quad \omega \in \mathbb{S}^2.$$

Again, we have in the Fourier domain the Fourier slice theorem

$$\widehat{R\mathfrak{f}}(\sigma,\omega) = \hat{\mathfrak{f}}(\sigma\omega),$$

from which combined with the following representation of the Laplace operator

$$-\Delta = \mathcal{F}^{-1}_{\xi\to x}|\xi|^2\mathcal{F}_{x\to\xi},$$

we deduce the intertwining property of the Radon transform and the Laplacian:

$$R\Delta\mathfrak{f}(s,\omega) = \frac{\partial^2}{\partial s^2}R\mathfrak{f}(s,\omega).$$

This property holds for smooth functions that decay sufficiently fast at infinity and by extension to a large class of distributions.

Thus (9.2) can be recast for $Rp(s,\omega,t)$ as

$$\frac{\partial^2 Rp}{\partial t^2} - \frac{\partial^2 Rp}{\partial s^2} = 0, \quad t > 0,\ s \in \mathbb{R},\ \omega \in S^2$$

with conditions $\partial_t Rp(s,\omega,0) = 0$ and $Rp(s,\omega,0) = R\mathfrak{f}(s,\omega)$.

This is a one-dimensional wave equation, whose unique solution is given by

$$Rp(s,\omega,t) = \frac{1}{2}\big(Rp(s+t,\omega,0) + Rp(s-t,\omega,0)\big) = \frac{1}{2}\big(R\mathfrak{f}(s+t,\omega) + R\mathfrak{f}(s-t,\omega)\big).$$

For the planes tangent to the unit sphere, $s = 1$, while $t > 0$. Then $R\mathfrak{f}(t+s,\omega) = 0$ since $\mathfrak{f}$ is supported inside the ball of radius 1 so that $R\mathfrak{f}(s,\omega) = 0$ for $|s| \geq 1$. Thus for $0 < t < 2$, we have

$$R\mathfrak{f}(1-t,\omega) = 2Rp(1,\omega,t) = 2\mathfrak{M}_3\mathfrak{f}(t,\omega).$$

Up to a slight smooth change of variables, the data are therefore the three-dimensional Radon transform of $\mathfrak{f}$, i.e., the integral of $\mathfrak{f}$ along all possible hyperplanes (passing through the support of $\mathfrak{f}$).

We recall the explicit inversion of the Radon transform:

$$\mathfrak{f}(x) = \frac{-1}{8\pi^2} R^* \frac{\partial^2}{\partial s^2} R\mathfrak{f}(x) = \frac{-1}{8\pi^2} \int_{\mathbb{S}^2} \big(\frac{\partial^2}{\partial s^2} R\mathfrak{f}\big)(x\cdot\omega,\omega)d\omega.$$

The Radon transform is injective and stable as we saw in Chapter 2. We have the following stability result:

$$\sqrt{2}\|\mathfrak{f}\|_{H^s(\mathbb{R}^3)} \quad \leq \quad \|\mathfrak{M}_3\mathfrak{f}\|_{H^{s+1}(Z)}, \tag{9.7}$$

where $Z = \mathbb{R}\times\mathbb{S}^2$ and $H^s(Z)$ is defined in Chapter 2.

Remarks on the first step of PAT. The inverse wave source problems considered here are therefore well posed problems. When ultrasound is measured with sufficient precision, then the reconstruction of the initial condition $H(x) = \mathfrak{f}(x)$ is stable.

9.1.3 Second step: Inverse problems with internal functionals

Once the first step of PAT is satisfactorily performed, a second step consists of reconstructing the unknown coefficients $(\gamma(x),\sigma(x),\Gamma(x))$ from knowledge of internal functionals of the form

$$H_j(x) = \Gamma(x)\sigma(x)u_j(x), \qquad 1 \leq j \leq J, \tag{9.8}$$

for $J \in \mathbb{N}^*$ illumination maps $f_j(x)$, where u_j is the solution to the steady-state equation

$$\begin{aligned} &-\nabla\cdot\gamma(x)\nabla u_j + \sigma(x)u_j = 0, \qquad x \in X \subset \mathbb{R}^n,\\ &u_j = f_j \qquad x \in \partial X. \end{aligned} \tag{9.9}$$

This second step of PAT is sometimes called quantitative photoacoustic tomography (QPAT). Indeed, the reconstruction of $H_j(x)$ in (9.8) offers important information about the unknown coefficients but depends on the illumination f_j used to probe the domain of interest and cannot be used to quantify intrinsic properties of the parameters. The second step of PAT aims to provide quantitative statements on the unknown parameters $(\gamma(x),\sigma(x),\Gamma(x))$.

Throughout the paper, we assume that the coefficients in (9.9) are known at the domain's boundary ∂X. Our objective is to reconstruct them in X. Formally, the measurement operator in QPAT is therefore

$$(\gamma(x), \sigma(x), \Gamma(x)) \mapsto \mathfrak{M}(\gamma(x), \sigma(x), \Gamma(x)) = (H_j(x))_{1 \le j \le J}.$$

One of the main theoretical results about QPAT is that, unfortunately, no matter how large J and the illuminations f_j may be chosen, we cannot reconstruct all of $(\gamma(x), \sigma(x), \Gamma(x))$ uniquely from QPAT measurements of the form (9.8). In other words, $\mathfrak{M}$ is not injective. However, as we shall see, as soon as one of the coefficients is assumed to be known, then the restriction of $\mathfrak{M}$ in this setting is injective and enjoys good (Lipschitz or Hölder) stability properties.

9.1.4 Reconstruction of one coefficient.

We conclude this introductory section by a simpler problem: the reconstruction of one coefficient in $(\gamma(x), \sigma(x), \Gamma(x))$ when the other two coefficients are known. In practice, it is not uncommon as a first approximation to assume that Γ and γ are known, at least approximatively. Then the important absorption coefficient σ (for instance to reconstruct the oxygenation properties of human tissues) is uniquely and stably determined.

(i) When only Γ is unknown, then we solve (9.9) for u and then construct $\Gamma = \dfrac{H}{\sigma u}$.

(ii) When only σ is unknown, then we solve the following elliptic equation for u

$$\begin{array}{rcll} -\nabla \cdot \gamma \nabla u(x) + \dfrac{H}{\Gamma} & = & 0, & \text{in } X \\ u(x) & = & f(x), & \text{on } \partial X \end{array}, \tag{9.10}$$

and then evaluate $\sigma = \dfrac{H}{\Gamma u}$.

Exercise 9.1.2 *Assume that f is uniformly bounded above and below by positive constants. Show that the reconstruction of σ is Lipschitz-stable in $L^\infty(X)$, i.e., that*

$$\|\sigma - \tilde{\sigma}\|_{L^\infty(X)} \le C \|H - \tilde{H}\|_{L^\infty(X)}, \tag{9.11}$$

where $\tilde{H}$ is acquired as in (9.8)-(9.9) with σ replaced by $\tilde{\sigma}$.

(iii) When only γ is unknown, we obtain $u = \frac{H}{\sigma\Gamma}$ and then the above elliptic equation in (9.10) with $\gamma_{|\partial X}$ known is a transport equation for γ. As soon as $\beta := \nabla u$ is a sufficiently smooth, non-vanishing vector field, then γ is uniquely determined by the linear equation

$$\begin{array}{rcll} \nabla \cdot \gamma\beta = \beta \cdot \nabla\gamma + (\nabla \cdot \beta)\gamma & = & \dfrac{H}{\Gamma}, & \text{in } X \\ \gamma(x) & = & \gamma_{|\partial X}(x), & \text{on } \partial X. \end{array} \tag{9.12}$$

This transport equation will be analyzed in more detail later in the chapter.

What the above results say is that the reconstruction of the optical coefficients is relatively straightforward when two out of the three are already known. The interest of the QPAT theory resides in the fact that the acquisition of more measurements in fact allows us to (uniquely and stably) reconstruct two coefficients but not three.

9.1.5 Introduction to Transient Elastography

Transient elastography images the (slow) propagation of shear waves using ultrasound. For more details, see, e.g., [43] and its extended list of references. As shear waves propagate, the resulting displacements can be imaged by ultra-fast ultrasound. Consider a scalar approximation of the equations of elasticity

$$\begin{aligned} &\nabla \cdot \gamma(x)\nabla u(x,t) = \rho(x)\partial_{tt}u(x,t), && t \in \mathbb{R},\ x \in X \\ &u(x,t) = f(x,t), && t \in \mathbb{R},\ x \in \partial X, \end{aligned} \tag{9.13}$$

where $u(x,t)$ is the (say, downward) displacement, $\gamma(x)$ is one of the Lamé parameters and $\rho(x)$ is density. Using ultra-fast ultrasound measurements, the displacement $u(x,t)$ can be imaged. This results in a very simplified model of transient elastography where we aim to reconstruct (γ,ρ) from knowledge of $u(x,t)$; see [43] for more complex models. We may slightly generalize the model as follows. Upon taking Fourier transforms in the time domain and accounting for possible dispersive effects of the tissues, we obtain

$$\begin{aligned} &\nabla \cdot \gamma(x;\omega)\nabla u(x;\omega) + \omega^2\rho(x;\omega)u(x\omega) = 0, && \omega \in \mathbb{R},\ x \in X \\ &u(x;\omega) = f(x;\omega), && \omega \in \mathbb{R},\ x \in \partial X. \end{aligned} \tag{9.14}$$

The inverse transient elastography problem with dispersion effect would then be the reconstruction of $(\gamma(x;\omega),\rho(x;\omega))$ from knowledge of $u(x;\omega)$ corresponding to one or several boundary conditions $f(x;\omega)$ applied at the boundary ∂X.

This corresponds to the setting of PAT with $\Gamma\sigma = 1$ so that $H = u$. As in PAT, H is an internal functional of the unknown coefficients (γ,ρ). The role of quantitative transient elastography (QTE) is therefore to reconstruct (γ,ρ) from knowledge of one or several internal functionals of the form $H = u$.

9.2 Theory of quantitative PAT and TE

Let us come back to the general theory of photo-acoustic tomography and transient elastic tomography. The two mathematical problems are quite similar. For concreteness, we first focus on the PAT setting and next state how the results should be modified to solve the TE problem.

9.2.1 Uniqueness and stability results in QPAT

We have seen above that one coefficient in (γ,σ,Γ) could be uniquely reconstructed from one internal functional H provided that the other two coefficients were known. It turns out that two coefficients in (γ,σ,Γ) can also be uniquely (and stably) be reconstructed from two (well-chosen) internal functionals H_j, $j = 1,2$ provided that the third coefficient is known. However, these two internal functionals (H_1,H_2) uniquely determine any internal functional H_3 obtained by using an arbitrary illumination f_3 on ∂X. Moreover, the two internal functionals (H_1,H_2) uniquely characterize two explicit functionals of (γ,σ,Γ) that do not allow us to reconstruct all parameters in (γ,σ,Γ) uniquely.

Measurement operator in QPAT. We have mentioned above the notion of *well-chosen* internal functionals H_j, $j = 1, 2$, or equivalently *well-chosen* illuminations f_j, $j = 1, 2$ on ∂X since the different functions H_j are characterized by the probing f_j on ∂X. To make precise statements, we introduce some notation. For $f \in H^{\frac{1}{2}}(\partial X)$, we obtain a solution $u \in H^1(X)$ of (9.9) and we can define the internal functional operator

$$\mathfrak{H}_{(\gamma,\sigma,\Gamma)} : \begin{array}{rcl} H^{\frac{1}{2}}(\partial X) & \to & H^1(X) \\ f & \mapsto & \mathfrak{H}_{(\gamma,\sigma,\Gamma)} f = \Gamma(x)\sigma(x)u(x). \end{array} \tag{9.15}$$

Let $I \in \mathbb{N}^*$ and $f_i \in H^{\frac{1}{2}}(\partial X)$ for $1 \leq i \leq I$ be a given set of I boundary conditions. Define $\mathfrak{f} = (f_1, \ldots, f_I)$. The measurement operator $\mathfrak{M}_\mathfrak{f}$ is then defined as the following collection of internal functionals:

$$\mathfrak{M}_\mathfrak{f} : \begin{array}{rcl} \mathfrak{X} & \to & \mathfrak{Y}^I \\ (\gamma, \sigma, \Gamma) & \mapsto & \mathfrak{M}_\mathfrak{f}(\gamma, \sigma, \Gamma) = (\mathfrak{H}_{(\gamma,\sigma,\Gamma)} f_1, \ldots, \mathfrak{H}_{(\gamma,\sigma,\Gamma)} f_I). \end{array} \tag{9.16}$$

Here, $\mathfrak{X}$ is a subset of a Banach space in which the unknown coefficients are defined; see (**H1**) below). Also $\mathfrak{Y}$ is a subset of $H^1(X)$ where the solutions to (9.9) are defined. We also define $H_j = \mathfrak{H}_{(\gamma,\sigma,\Gamma)} f_j$ for $1 \leq j \leq I$.

Assumptions on the coefficients and the illuminations. Here are now some mathematical assumptions on the coefficients and a definition of illuminations that we call *well-chosen*. Here and below, we denote by $W^{m,p}(X)$ the space of functions with derivatives of order less than or equal to m belonging to $L^p(X)$.

(H1). We denote by $\mathfrak{X}$ the set of coefficients (γ, σ, Γ) that are of class $W^{1,\infty}(X)$, are bounded above and below by fixed positive constants, and such that the traces $(\gamma, \sigma, \Gamma)_{|\partial X}$ on the boundary ∂X are fixed (known) functions.

(H2). The illuminations f_j are positive functions on ∂X that are the restrictions on ∂X of functions of class $C^3(\bar{X})$.

(H3). We say that $\mathfrak{f}_2 = (f_1, f_2)$ is a pair of *well-chosen* illuminations with corresponding functionals $(H_1, H_2) = (\mathfrak{H}_{(\gamma,\sigma,\Gamma)} f_1, \mathfrak{H}_{(\gamma,\sigma,\Gamma)} f_2)$ provided that **(H2)** is satisfied and the vector field

$$\beta := H_1 \nabla H_2 - H_2 \nabla H_1 = H_1^2 \nabla \frac{H_2}{H_1} = H_1^2 \nabla \frac{u_2}{u_1} = -H_2^2 \nabla \frac{H_1}{H_2} \tag{9.17}$$

is a vector field in $W^{1,\infty}(X)$ such that

$$|\beta|(x) \geq \alpha_0 > 0, \qquad \text{a.e. } x \in X. \tag{9.18}$$

(H3'). We say that $\mathfrak{f}_2 = (f_1, f_2)$ is a pair of *weakly well-chosen* illuminations with corresponding functionals $(H_1, H_2) = (\mathfrak{H}_{(\gamma,\sigma,\Gamma)} f_1, \mathfrak{H}_{(\gamma,\sigma,\Gamma)} f_2)$ provided that **(H2)** is satisfied and the vector field β defined in (9.17) is in $W^{1,\infty}(X)$ and $\beta \neq 0$ a.e. in X.

Remark 9.2.1 *Note that* **(H3')** *is satisfied as soon as* $\frac{f_1}{f_2} \neq C$ *is not a constant. Indeed, if* $\beta = 0$ *on a set of positive measure, then* $\nabla \frac{u_2}{u_1} = 0$ *on that same set. Yet,* $\frac{u_2}{u_1}$ *solves the elliptic equation* (9.20) *below. It is known that under sufficient smoothness conditions on the coefficients, the critical points of solutions to such elliptic equations are of measure zero unless these solutions are constant [35, 49].*

Uniqueness/Non-uniqueness result. Hypothesis **(H3)** will be useful in the analysis of the stability of the reconstructions. For the uniqueness result, the weaker hypothesis **(H3')** is sufficient. Note that almost all illumination pairs $\mathfrak{f}_2$ satisfy **(H3')**, which is a mere regularity statement. Beyond the regularity assumptions on (γ, σ, Γ), the domain X, and the boundary conditions f_j, the only real assumption we impose is thus (9.18). In general, there is no guaranty that the gradient of $\frac{u_2}{u_1}$ does not vanish. Not all pairs of illuminations $\mathfrak{f}_2 = (f_1, f_2)$ are *well-chosen* although most are *weakly well-chosen*. That the vector field β does not vanish is a *sufficient* condition for the stability estimates presented below to be satisfied. It is not *necessary*. As we shall see, guaranteeing (9.18) is relatively straightforward in dimension $n = 2$. It is much complicated in dimension $n \geq 3$. The only available methodology to ensure that (9.18) holds for a large class of conductivities is based on the same method of complex geometric optics (CGO) solutions already used to solve the Calderón problem in Chapter 7.

Under these hypotheses, we obtain the following result:

Theorem 9.2.2 *Let* $\mathfrak{X}$ *be defined as in* **(H1)** *and let* $\mathfrak{f}_2$ *be* well chosen illuminations *as indicated in* **(H2)** *and* **(H3')**. *Let* $I \in \mathbb{N}^*$ *and* $\mathfrak{f} = (f_1, \ldots, f_I)$ *be a set of (arbitrary) illuminations satisfying* **(H2)**. *Then we have the following:*

(i). The measurement operator $\mathfrak{M}_{\mathfrak{f}_2}$ *uniquely determines* $\mathfrak{M}_{\mathfrak{f}}$ *(meant in the sense that* $\mathfrak{M}_{\mathfrak{f}_2}(\gamma, \sigma, \Gamma) = \mathfrak{M}_{\mathfrak{f}_2}(\tilde{\gamma}, \tilde{\sigma}, \tilde{\Gamma})$ *implies that* $\mathfrak{M}_{\mathfrak{f}}(\gamma, \sigma, \Gamma) = \mathfrak{M}_{\mathfrak{f}}(\tilde{\gamma}, \tilde{\sigma}, \tilde{\Gamma})$*).*

(ii). The measurement operator $\mathfrak{M}_{\mathfrak{f}_2}$ *uniquely determines the two following functionals of* (γ, σ, Γ) *(meant in the same sense as above):*

$$\chi(x) := \frac{\sqrt{\gamma}}{\Gamma\sigma}(x), \qquad q(x) := \Big(\frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}} + \frac{\sigma}{\gamma}\Big)(x). \tag{9.19}$$

(iii). Knowledge of the two functionals χ *and* q *uniquely determines (in the same sense as above)* $\mathfrak{M}_{\mathfrak{f}_2} = (H_1, H_2)$. *In other words, the reconstruction of* (γ, σ, Γ) *is unique up to any transformation that leaves* (χ, q) *invariant.*

Proof. Let us start with (i). Upon multiplying the equation for u_1 by u_2, the equation for u_2 by u_1, and subtracting both relations, we obtain

$$\begin{aligned} -\nabla \cdot (\gamma u_1^2) \nabla \frac{H_2}{H_1} &= 0, && \text{in } X \\ \gamma u_1^2 &= \gamma_{|\partial X} f_1^2, && \text{on } \partial X. \end{aligned} \tag{9.20}$$

This is a transport equation in conservative form for γu_1^2. More precisely, this is a transport equation $\nabla \cdot \rho\tilde{\beta} = 0$ for ρ with $\rho_{|\partial X} = 1$ and

$$\tilde{\beta} = \chi^2 \beta = (\gamma u_1^2) \nabla \frac{H_2}{H_1}.$$

Since $\tilde{\beta} \in W^{1,\infty}(X)$ and is divergence free, the above equation for ρ admits the unique solution $\rho \equiv 1$ since (9.18) holds. Indeed, we find that $\nabla \cdot (\rho-1)^2\tilde{\beta} = 0$ by application of the chain rule with $\rho_{|\partial X} - 1 = 0$ on ∂X. Upon multiplying the equation by $\frac{H_2}{H_1}$ and integrating by parts, we find

$$\int_X (\rho-1)^2 \chi^2 H_1^2 \Big| \nabla \frac{H_2}{H_1} \Big|^2 dx = 0.$$

Using **(H3')** and the result of remark 9.2.1, we deduce that $\rho = 1$ on X by continuity. This proves that γu_1^2 is uniquely determined. Dividing by $H_1^2 = (\Gamma\sigma)^2 u_1^2$, this implies that $\chi > 0$ defined in (9.19) is uniquely determined. Note that we do not need the full $W^{1,\infty}(X)$ regularity of β in order to obtain the above result. However, we still need to be able to apply the chain rule to obtain an equation for $(\rho-1)^2$ and conclude that the solution to the transport equation is unique.

Let now f be an arbitrary boundary condition and let u be the solution to (9.9) and $H = \mathfrak{H}_{(\gamma,\sigma,\Gamma)} f$ defined by (9.8). Replacing H_2 above by H yields

$$\begin{aligned} -\nabla \cdot \chi^2 H_1^2 \nabla \frac{H}{H_1} &= 0, && \text{in } X \\ H &= \Gamma_{|\partial X} \sigma_{|\partial X} f, && \text{on } \partial X. \end{aligned} \tag{9.21}$$

This is a well-defined elliptic equation with a unique solution $H \in H^1(X)$ for $f \in H^{\frac{1}{2}}(\partial X)$. This proves that $H = \mathfrak{H}_{(\gamma,\sigma,\Gamma)} f$ is uniquely determined by (H_1, H_2) and concludes the proof of (i).

Let us next prove (ii). We have already seen that χ was determined by $\mathfrak{M}_{\mathfrak{f}_2} = (H_1, H_2)$. Define now $v = \sqrt{\gamma} u_1$, which is also uniquely determined based on the results in (i). Define

$$q = \frac{\Delta v}{v} = \frac{\Delta(\sqrt{\gamma} u_1)}{\sqrt{D} u_1},$$

which is the Liouville change of variables used in Chapter 7 to solve the Calderón problem. Since u_1 is bounded from below, is sufficiently smooth, and solves (9.9), the following calculations show that q is given by (9.19). Indeed, we find that

$$\nabla \cdot \gamma \nabla u_1 = \nabla \cdot (\sqrt{\gamma} \nabla v) - \nabla \cdot ((\nabla \sqrt{\gamma}) v) = \sqrt{\gamma} \Delta v - (\Delta \sqrt{\gamma}) v = \sigma u_1 = \frac{\sigma}{\sqrt{\gamma}} v. \tag{9.22}$$

Finally, we prove (iii). Since q is now known, we can solve

$$(\Delta - q) v_j = 0, \quad X, \qquad v_j = \sqrt{\gamma_{|\partial X}} g_j \quad \partial X, \qquad j = 1, 2.$$

Because q is of the specific form (9.19) as a prescribed functional of (γ, σ, Γ), it is known that $(\Delta - q)$ does not admit 0 as a (Dirichlet) eigenvalue, for otherwise, 0 would also be a (Dirichlet) eigenvalue of the elliptic operator

$$(-\nabla \cdot \gamma \nabla + \sigma) \cdot = (-\sqrt{\gamma}(\Delta - q)\sqrt{\gamma}) \cdot . \tag{9.23}$$

The latter calculation follows from (9.22). Thus v_j is uniquely determined for $j = 1, 2$. Now,

$$H_j = \Gamma \sigma u_j = \frac{\Gamma\sigma}{\sqrt{\gamma}} v_j = \frac{v_j}{\chi}, \qquad j = 1, 2,$$

and is therefore uniquely determined by (χ, q). This concludes the proof that (χ, q) uniquely determines $\mathfrak{M}_{\mathfrak{f}_2}$. □

Reconstruction of two coefficients. The above result shows that the unique reconstruction of (γ, σ, Γ) is not possible even from knowledge of a measurement operator $\mathfrak{M}_{\mathfrak{f}}$ corresponding to an arbitrary (in fact possibly infinite) number of internal functionals I. We therefore face this peculiar situation that two well-chosen illuminations uniquely determine the functionals (χ, q) but that acquiring additional measurements does not provide any new information.

However, we can prove the following positive result that if one of the coefficients in (γ, σ, Γ) is known, then the other two coefficients are uniquely determined.

Corollary 9.2.3 *Under the hypotheses of the previous theorem, let* (χ, q) *in* (9.19) *be known. Then*

(a) If Γ *is known, then* (γ, σ) *are uniquely determined.*

(b) If γ *is known, then* (σ, Γ) *are uniquely determined.*

(c) If σ *is known, then* (γ, Γ) *are uniquely determined.*

Proof. (a) is probably the most practical case as Γ is often assumed to be constant or known. Since Γ is known, then so is $\Gamma\chi = \sqrt{\gamma}/\sigma$ so that we have the elliptic equation for $\sqrt{\gamma}$:

$$(\Delta - q)\sqrt{\gamma} + \frac{1}{\Gamma\chi} = 0, \quad X, \qquad \sqrt{\gamma}_{|\partial X} = \sqrt{\gamma_{|\partial X}}, \quad \partial X. \tag{9.24}$$

Again, because of the specific form of q, $(\Delta - q)$ is invertible and the above equation admits a unique solution. Once $\sqrt{\gamma}$, hence γ, is known, then so is $\sigma = \frac{\sqrt{\gamma}}{\Gamma\chi}$.

If γ is known in (b), then σ is known from q and Γ is known from χ.

Finally in (c), we obtain that from the expression for q that

$$\sqrt{\gamma}(\Delta - q)\sqrt{\gamma} + \sigma = 0 \quad X, \qquad \sqrt{\gamma}_{|\partial X} = \sqrt{\gamma_{|\partial X}}, \quad \partial X. \tag{9.25}$$

We need to prove a uniqueness result for the above nonlinear equation for $\sqrt{\gamma}$. Let us assume that $\sqrt{\gamma}$ and another solution $\tau\sqrt{\gamma}$ for $0 < \tau(x)$ satisfy the above equation for σ fixed. We have

$$-\sqrt{\gamma}(\Delta - q)\sqrt{\gamma}\tau - \frac{\sigma}{\tau} = 0 \quad X.$$

Thanks to (9.23), this implies the following equation for τ:

$$-\nabla \cdot \gamma\nabla\tau + \sigma(\tau - \frac{1}{\tau}) = 0, \quad X, \qquad \tau = 1, \quad \partial X.$$

Upon multiplying by $\tau - 1$ and integrating by parts, we find that

$$\int_X \gamma|\nabla(\tau - 1)|^2 dx + \int_X \sigma|\tau - 1|^2 \frac{\tau + 1}{\tau} dx = 0.$$

Since $\tau > 0$, we deduce from the above that $\tau \equiv 1$ and that γ is uniquely determined by q. We then retrieve Γ from knowledge of χ. □

Reconstruction formulas. Note that the above uniqueness results provide a *constructive* reconstruction procedure. In all cases, we first need to solve a transport equation for the functional χ:

$$-\nabla \cdot (\chi^2 \beta) = 0 \qquad \text{in } X, \qquad \chi_{|\partial X} \text{ known on } \partial X, \tag{9.26}$$

with β the vector field defined in (9.17). This uniquely defines $\chi > 0$. Then we find that

$$q(x) = \frac{\Delta(H_1\chi)}{H_1\chi} = \frac{\Delta(H_2\chi)}{H_2\chi}. \tag{9.27}$$

This provides explicit reconstructions for (χ, q). In case (b), no further equation needs to be solved. In cases (a) and (c), we need to solve an elliptic equation for $\sqrt{\gamma}$, which is the linear equation (9.24) in (a) and the nonlinear equation (9.25) in (c). These steps have been implemented numerically with very satisfactory results in [17].

Stability of the solution of the transport equation for $\chi(x)$. We now derive a stability result for the reconstruction of χ obtained from analyzing the transport equation (9.20). Similar stability results can be obtained for q and then for (γ, σ, Γ) depending on the reconstruction considered.

Theorem 9.2.4 *Let $\mathfrak{M}_{\mathfrak{f}_2}(\gamma, \sigma, \Gamma) = (H_1, H_2)$ be the measurements corresponding to the coefficients (γ, σ, Γ) such that* **(H1), (H2), (H3)** *hold. Let $\mathfrak{M}_{\mathfrak{f}_2}(\tilde\gamma, \tilde\sigma, \tilde\Gamma) = (\tilde H_1, \tilde H_2)$ be the measurements corresponding to the same illuminations $\mathfrak{f}_2 = (f_1, f_2)$ with another set of coefficients $(\tilde\gamma, \tilde\sigma, \tilde\Gamma)$ such that* **(H1), (H2)** *hold. Define $\delta\mathfrak{M}_{\mathfrak{f}_2} = \mathfrak{M}_{\mathfrak{f}_2}(\tilde\gamma, \tilde\sigma, \tilde\Gamma) - \mathfrak{M}_{\mathfrak{f}_2}(\gamma, \sigma, \Gamma)$. Then we find that*

$$\|\chi - \tilde\chi\|_{L^p(X)} \le C \|\delta\mathfrak{M}_{\mathfrak{f}_2}\|^{\frac{1}{2}}_{(W^{1,\frac{p}{2}}(X))^2}, \qquad \text{for all } 2 \le p < \infty. \tag{9.28}$$

Let us assume, moreover, that $\gamma(x)$ is of class $C^3(\bar X)$. Then we have the estimate

$$\|\chi - \tilde\chi\|_{L^p(X)} \le C \|\delta\mathfrak{M}_{\mathfrak{f}_2}\|^{\frac{1}{3}}_{(L^{\frac{p}{2}}(X))^2}, \qquad \text{for all } 2 \le p < \infty. \tag{9.29}$$

By interpolation, the latter result implies that

$$\|\chi - \tilde\chi\|_{L^\infty(X)} \le C \|\delta\mathfrak{M}_{\mathfrak{f}_2}\|^{\frac{p}{3(d+p)}}_{(L^{\frac{p}{2}}(X))^2}, \qquad \text{for all } 2 \le p < \infty. \tag{9.30}$$

We may for instance choose $p = 4$ above to measure the noise level in the measurement $\mathfrak{M}_{\mathfrak{f}_2}$ in the square integrable norm when noise is described by its power spectrum in the Fourier domain.

Proof. Define $\nu = \chi^2$ and $\tilde\nu = \tilde\chi^2$ with χ defined in (9.19) and β and $\tilde\beta$ as in (9.17). Then we find that

$$\nabla \cdot \frac{\nu - \tilde\nu}{\nu}(\nu\beta) + \nabla \cdot \tilde\nu(\beta - \tilde\beta) = 0.$$

Note that $\nu\beta = \chi^2 H_1^2 \nabla \frac{H_2}{H_1}$ is a divergence-free field. Let φ be a twice differentiable, non-negative, function from $\mathbb{R}$ to $\mathbb{R}$ with $\varphi(0) = \varphi'(0) = 0$. Then we find that

$$\nabla \cdot \varphi\Big(\frac{\nu - \tilde\nu}{\nu}\Big)(\nu\beta) + \varphi'\Big(\frac{\nu - \tilde\nu}{\nu}\Big)\nabla \cdot \tilde\nu(\beta - \tilde\beta) = 0.$$

Let us multiply this equation by a test function $\zeta \in H^1(X)$ and integrate by parts. Since $\nu = \nu'$ on ∂X, we find

$$\int_X \varphi\Big(\frac{\nu-\tilde\nu}{\nu}\Big)\nu\beta\cdot\nabla\zeta dx + \int_X \tilde\nu(\beta-\tilde\beta)\nabla\cdot\Big[\zeta\varphi'\Big(\frac{\nu-\tilde\nu}{\nu}\Big)\Big]dx = 0.$$

Upon choosing $\zeta = \frac{H_2}{H_1}$, we find

$$\int_X \varphi\nu H_1^2\Big|\nabla\frac{H_2}{H_1}\Big|^2 dx + \int_X \tilde\nu(\beta-\tilde\beta)\cdot\nabla\frac{H_2}{H_1}\varphi' dx + \int_X \tilde\nu(\beta-\tilde\beta)\cdot\nabla\frac{\nu-\tilde\nu}{\nu}\frac{H_2}{H_1}\varphi'' dx = 0.$$

Above, φ stands for $\varphi(\frac{\nu-\tilde\nu}{\nu})$ in all integrals. By assumption on the coefficients, $\nabla\frac{\nu-\tilde\nu}{\nu}$ is bounded a.e.. This is one of our main motivations for assuming that the optical coefficients are Lipschitz. The middle term is seen to be smaller than the third term and so we focus on the latter one. Upon taking $\varphi(x) = |x|^p$ for $p \geq 2$ and using assumption **(H3)**, we find that

$$\|\nu-\tilde\nu\|^p_{L^p(X)} \leq C\int_X |\beta-\tilde\beta||\nu-\tilde\nu|^{p-2}dx.$$

By an application of the Hölder inequality, we deduce that

$$\|\nu-\tilde\nu\|_{L^p(X)} \leq C\|\beta-\tilde\beta\|^{\frac12}_{L^{\frac{p}{2}}(X)}.$$

We next write $\beta-\tilde\beta = (H_1-\tilde H_1)\nabla H_2 + \tilde H_1(\nabla(H_2-\tilde H_2) - \dots$ and use the fact that the solutions to (9.9) and the coefficients are in $W^{1,\infty}(X)$ to conclude that (9.28) holds.

The other results are obtained by regularity theory and interpolation. Indeed from standard regularity results with coefficients in $W^{1,\infty}(X)$, we find that the solutions to (9.9) are of class $W^{3,q}(X)$ for all $1 \leq q < \infty$. Since the coefficient γ is of class $C^3(\bar X)$, then the measurements H_j are of class $W^{3,q}(X)$ for all $1 \leq q < \infty$. Standard Sobolev estimates show that

$$\|H_j-\tilde H_j\|_{W^{1,q}(X)} \leq C\|H_j-\tilde H_j\|^{\frac23}_{L^q(X)}\|H_j-\tilde H_j\|^{\frac13}_{W^{3,q}(X)}.$$

The last term is bounded by a constant, which gives (9.29) for $q = \frac{p}{2}$. Another interpolation result states that

$$\|\varphi\|_\infty \leq \|\nabla\varphi\|^\theta_\infty\|\varphi\|^{1-\theta}_p, \qquad \theta = \frac{d}{d+p}.$$

This provides the stability result in the uniform norm (9.30). □

Exercise 9.2.1 *Find similar stability estimates for q and (γ,σ,Γ) in the different settings considered in Corollary 9.2.3 and in section 9.1.4.*

9.2.2 Application to Quantitative Transient Elastography

We can apply the above results to the time-harmonic reconstruction in a simplified model of transient elastography. Let us assume that γ and ρ are unknown functions of

$x \in X$ and $\omega \in \mathbb{R}$. Recall that the displacement solves (9.14). Assuming that $u(x;\omega)$ is known after step 1 of the reconstruction using the ultrasound measurements, then we are in the setting of Theorem 9.2.2 with $\Gamma\sigma = 1$. Let us then assume that the two illuminations $f_1(x;\omega)$ and $f_2(x;\omega)$ are chosen such that for u_1 and u_2 the corresponding solutions of (9.14), we have that **(H3')** holds. Then, (9.19) shows that the reconstructed function χ uniquely determines the Lamé parameter $\gamma(x;\omega)$ and that the reconstructed function q then uniquely determines $\omega^2\rho$ and hence the density parameter $\rho(x;\omega)$. The reconstructions are performed for each frequency ω independently. We may summarize this as follows:

Corollary 9.2.5 *Under the hypotheses Theorem 9.2.2 and the hypotheses described above, let (χ, q) in (9.19) be known. Then $(\gamma(x;\omega), \rho(x;\omega))$ are uniquely determined by two well-chosen measurements. Moreover, if* **(H3)** *holds, the stability results in Theorem 9.2.4 hold.*

Alternatively, we may assume that in a given range of frequencies, $\gamma(x)$ and $\rho(x)$ are independent of ω. In such a setting, we expect that one measurement $u(x;\omega)$ for two different frequencies will provide sufficient information to reconstruct $(\gamma(x), \rho(x))$. Assume that $u(x;\omega)$ is known for $\omega = \omega_j$, $j = 1, 2$ and define $0 < \alpha = \omega_2^2\omega_1^{-2} \neq 1$. Then straightforward calculations show that

$$\nabla \cdot \gamma\beta_\alpha = 0, \quad \beta_\alpha = \big(u_1\nabla u_2 - \alpha u_2 \nabla u_1\big). \tag{9.31}$$

This provides a transport equation for γ that can be solved stably provided that $|\beta_\alpha| \geq c_0 > 0$, i.e., a hypothesis of the form **(H3)** applies and β_α does not vanish on X. Then, Theorem 9.2.2 and Theorem 9.2.4 apply in this setting.

9.3 *Well-chosen* illuminations in PAT and TE

The stability results presented for QPAT and QTE involve *well-chosen* illuminations so that **(H3)** and (9.18) hold. Such a constraint is clearly not satisfied for all possible pairs (f_1, f_2). In two dimensions of space, i.e., when $n = 2$, a very large class of illuminations can be proved to be *well-chosen*. In dimensions $n \geq 3$, the proofs are much more complicated and involve the CGO solutions constructed in Chapter 7.

9.3.1 The two dimensional case

In dimension $n = 2$, we have:

Lemma 9.3.1 *Assume that $h = \frac{f_2}{f_1}$ on ∂X is an almost two-to-one function, i.e., a function that is a two-to-one map except possibly at its minimum and at its maximum. Then (9.18) is satisfied.*

Proof. Upon multiplying the equation for u_1 by u_2, the equation for u_2 by u_1, and subtracting both relations, we obtain

$$-\nabla \cdot (\gamma u_1^2)\nabla\frac{u_2}{u_1} = 0, \quad \text{in } X \qquad \frac{u_2}{u_1} = \frac{f_2}{f_1}, \quad \text{on} \quad \partial X. \tag{9.32}$$

This implies that $v := \frac{u_2}{u_1}$ satisfies an elliptic equation with a diffusion coefficient $\tilde{\gamma} = \gamma u_1^2$ bounded from above and below by positive constants. Note that $\beta = H_1^2 \nabla v$. Results in, e.g., [2, Theorem 1.2] show that ∇v cannot vanish inside X. In other words, v does not admit any *critical point*. By the maximum principle and the assumption on h, no critical point of v can occur on ∂X either. This implies that $|\nabla v| > 0$ and that we can find a constant such that (9.18) holds since H_1^2 is bounded from below by a positive constant and by continuity $|\nabla v|$ attains its (strictly positive) minimum in $\bar{X}$. □

9.3.2 The n dimensional case

In dimension $n \geq 3$, the above result on the (absence of) critical points of elliptic solutions no longer holds. However, by continuity, we may verify that (9.18) is satisfied for a large class of illuminations when γ is close to a constant and σ is sufficiently small. For arbitrary coefficients (γ, σ) in dimension $n \geq 3$, the only available proof that (9.18) is satisfied for an open set of illuminations is the one obtained by means of complex geometrical optics solutions; see [21]. The main result is:

Theorem 9.3.2 *Let* **(H1)** *hold. Then there is an open set in* $C^3(\partial X)$ *of illuminations* $\mathfrak{f}_2 = (f_1, f_2)$ *such that* **(H3)** *holds.*

The open set, unlike the result obtained in two dimensions in Lemma 9.3.1, is not explicitly constructed.

Proof. Let us consider the elliptic equation (9.6). In chapter 7, we have constructed solutions of the form

$$u_\varrho(x) = \frac{1}{\sqrt{\gamma}} e^{\varrho \cdot x} \big(1 + \psi_\varrho(x)\big), \tag{9.33}$$

with $|\varrho| \psi_\varrho(x)$ bounded uniformly in $H^s(X)$ for arbitrary $s \geq 0$ provided that γ and σ are sufficiently smooth coefficients. Using the construction of Chapter 7, we can prove the following lemma:

Lemma 9.3.3 *Let* u_{ϱ_j} *for* $j = 1, 2$ *be CGO solutions with* (γ, σ) *sufficiently smooth for both* ϱ_j *and* $k \geq 1$ *and with* $c_0^{-1}|\varrho_1| \leq |\varrho_2| \leq c_0|\varrho_1|$ *for some* $c_0 > 0$. *Then we have*

$$\hat{\beta} := \frac{1}{2|\varrho_1|} e^{-(\varrho_1 + \varrho_2)\cdot x} \Big(u_{\varrho_1} \nabla u_{\varrho_2} - u_{\varrho_2} \nabla u_{\varrho_1} \Big) = \frac{\varrho_1 - \varrho_2}{2|\varrho_1|} + \hat{h}, \tag{9.34}$$

where the vector field $\hat{h}$ *satisfies the constraint*

$$\|\hat{h}\|_{C^k(\bar{X})} \leq \frac{C_0}{|\varrho_1|}, \tag{9.35}$$

for some constant C_0 *independent of* ϱ_j, $j = 1, 2$.

Exercise 9.3.1 *Prove the above lemma using the results obtained in Chapter 7.*

With $\varrho_2 = \overline{\varrho_1}$ so that $u_{\varrho_2} = \overline{u_{\varrho_1}}$, the imaginary part of (9.34) is a vector field that does not vanish on X for $|\varrho_1|$ sufficiently large. Moreover, let $u_{\varrho_1} = v + iw$ and $u_{\varrho_2} = v - iw$

for v and w real-valued functions. Then the imaginary and real parts of (9.34) are given by

$$\Im\hat{\beta} = \frac{1}{|\varrho_1|}e^{-2\Re\varrho_1\cdot x}(w\nabla v - v\nabla w) = \frac{\Im\varrho_1}{|\varrho_1|} + \Im\hat{h}, \qquad \Re\hat{\beta} = 0.$$

Let u_1 and u_2 be solutions of the elliptic problem (9.6) on X such that $u_1 + iu_2$ on ∂X is close (in the $C^3(\partial X)$ (strong) topology) to the trace of u_{ϱ_1}. The above result shows that

$$|u_1\nabla u_2 - u_2\nabla u_1| \geq c_0 > 0 \quad \text{in } X.$$

This yields (9.18) and the proof of the theorem. □

The set of illuminations $\mathfrak{f}_2 = (f_1, f_2)$ for which (9.18) is guaranteed is not known explicitly. All we know is that if $\mathfrak{f}_2$ is chosen sufficiently close to the traces of CGO solutions constructed above, then indeed the vector field β will satisfy (9.18). One major drawback with such a result is that the CGO solutions depend on the unknown coefficient (γ, σ). That said, there does exist an open set of illuminations $\mathfrak{f}_2$ such that (9.18) holds.

This result should be contrasted with the case in two dimensions, where we were able to prove that (9.18) held for a very large class of (non-oscillatory) illuminations $\mathfrak{f}_2$.

9.4 The case of anisotropic coefficients

9.4.1 Linear algebra problem

We saw in the analysis of the Calderón problem that anisotropic coefficients could not be uniquely reconstructed from boundary measurements. The setting of PAT or TE is different since the first step of the reconstruction provides internal information, including where 'each point x really is'. It turns out that much more information is available on the coefficients in PAT and TE than in elliptic boundary inverse problems.

We consider the most general second-order scalar elliptic equation (with real-valued coefficients to simplify the presentation)

$$Lu := \nabla \cdot a\nabla u + b \cdot \nabla u + cu = 0 \ \text{ in } \ X, \qquad u = f \ \text{ on } \ \partial X.$$

Let us assume that we have made I_n measurements H_j of the PAT or TE type, with I_n large enough. Our measurements are linear in u so that ratios of measurements gives for known $v_j = u_{j+1}/u_1$ the following equations:

$$\nabla \cdot u_1^2 a\nabla v_j + u_1^2 b \cdot \nabla v_j = 0, \qquad \alpha : \nabla^{\otimes 2} v_j + \beta \cdot \nabla v_j = 0$$

where we have defined $\alpha = u_1^2 a$ and $\beta = u_1^2 b + \nabla \cdot (u_1^2 a)$. It is clear that we can multiply the last equation by τ so that α and β can be reconstructed only up to multiplication by a scalar coefficient. This is what we now do.

We assume the construction of v_j such that $(\nabla v_1, \ldots, \nabla v_n)$ form a local frame (locally a basis at each point x). Define $H_{jk} = \nabla v_j \cdot \nabla v_k$ the symmetric invertible matrix with inverse given by coefficients denoted by H^{jk}. For $m \geq 1$, we can decompose any other gradient as

$$-\nabla v_{n+m} = \tilde{\theta}_{mj}\nabla v_j, \qquad \tilde{\theta}_{mj} = -H^{jk}\nabla v_{n+m} \cdot \nabla v_k.$$

Define now θ_{jm} equal to $\tilde{\theta}_{jm}$ for $1 \leq j \leq n$, equal to 1 for $j = m + n$, and equal to 0 otherwise. The construction is made so that

$$\sum_{j=1}^{I_n - 1} \theta_{jm} \nabla v_j = 0.$$

Define next the matrices

$$M^m = \sum_j \theta_{jm} \nabla^{\otimes 2} v_j, \qquad 1 \leq m \leq \frac{1}{2} n(n+1) - 1.$$

We assume that the matrices so constructed as linearly independent and hence form a span of dimension $\frac{1}{2}n(n+1) - 1$ in the space of symmetric matrices. This is the dimension of the space of such matrices minus 1. We therefore deduce the existence of a matrix M^0 such that

$$M^m : M^0 = \mathrm{tr}(M^m M^0) = \delta_{m0}, \qquad 0 \leq m \leq \frac{1}{2} n(n+1) - 1.$$

We define $\alpha = M^0$ since we know that the latter tensor can only be defined up to multiplication by a scalar constant.

Once α is known, we can reconstruct β from the equation using the first n measurements:

$$\beta = -H^{jk}(M^0 : \nabla^{\otimes 2} v_j) \nabla v_k,$$

as may easily be verified.

Let us assume that we can find sets of such functions v_j for each $x \in X$. We then observe that the reconstruction of α and β involves two derivatives of v_j. The above expressions therefore allow us to state that

$$\|\delta\alpha\|_{C^{m,\alpha}(\bar{X})} + \|\delta\beta\|_{C^{m,\alpha}(\bar{X})} \lesssim \|\{\delta v_j\}\|_{C^{m+2,\alpha}(\bar{X})} \sim \|\{\delta H_j\}\|_{C^{m+2,\alpha}(\bar{X})},$$

where δH_j models measurement error and $\delta\alpha$ and $\delta\beta$ quantify reconstruction errors.

9.4.2 Explicit reconstructions

Let us assume that (α, β) above are reconstructed. Then a and b are such that $u_1^2 a = \tau\alpha$ and $u_1^2 b + \nabla \cdot u_1^2 a = \tau\beta$ for some undetermined scalar τ. Let us assume that $b = 0$, i.e., the drift term vanishes. This assumption holds in both PAT and TE. We also define $\gamma = -a$ and $\sigma = c$ for consistency with previous sections. We then find that

$$\nabla \cdot (\alpha\tau) = \tau\beta, \qquad \text{or} \qquad \alpha\nabla\tau = (\beta - \nabla \cdot \alpha)\tau,$$

which is an overdetermined elliptic problem for τ. We can take the divergence of such an equation to obtain a well-posed problem for τ provided that τ is known on ∂X. This is an elliptic equation for τ, which we know to be invertible by an application of the maximum principle. Note that the latter principle holds only because τ is scalar-valued here. In such a setting, τ, and hence $u_1^2 a$ is uniquely reconstructed. Moreover, from elliptic regularity, we observe that τ is one derivative smoother than α.

When u_1 is known, then so is the full anisotropic tensor a. When $H_1 = \Gamma\sigma u_1$ is known, then what we reconstruct is the tensor $(\Gamma\sigma)^{-2}\gamma$, which is similar to χ^2 introduced earlier. Note that only the determinant of a remains unknown in QPAT. The anisotropic structure of a is fully reconstructed at this stage.

Let us denote $a = \hat{a}B^2$ with $\hat{a}$ the anisotropic part with determinant set to 1. We know $u_1^2 a = u_1^2 B^2 \hat{a}$. As in the scalar case, let us introduce $v = Bu_1$ and deduce from the equation for u_1 that

$$-\nabla \cdot \hat{a}B\nabla v + \nabla \cdot \hat{a}v\nabla B + \frac{\sigma v}{B} = 0.$$

This is, with $-\hat{a}\nabla v \cdot \nabla B + \hat{a}\nabla B \cdot \nabla v = 0$,

$$-B\nabla \cdot \hat{a}\nabla v + v\nabla\hat{a}\nabla B + \frac{\sigma v}{B} = 0,$$

or

$$\nabla \cdot \hat{a}\nabla v = qv, \qquad q = \frac{\nabla \cdot \hat{a}\nabla B}{B} + \frac{\sigma}{B^2}.$$

This shows that q is known since v and $\hat{a}$ are known. As in the scalar case, if we assume that the Grüneisen coefficient Γ is known, then q and B/σ are known and we deduce from the equation

$$-(\nabla \cdot \hat{a}\nabla - q)B = \frac{\sigma}{B}$$

that B can be reconstructed when it is known on ∂X. Note that the above equation is also elliptic. This comes from the above derivation, which we recast more explicitly as

$$B^{-1}\nabla \cdot \hat{a}B^2\nabla B^{-1} = \nabla\hat{a}\nabla - B^{-1}(\nabla \cdot \hat{a}\nabla B) = \nabla\hat{a}\nabla - (q - \frac{\sigma}{B^2}).$$

This concludes the derivation of QPAT in the anisotropic setting. In general, all we can reconstruct is $(\Gamma\sigma)^{-2}\gamma$ and q above. When Γ is known, then (γ, σ) are uniquely characterized from the available measurements.

In terms of stability, we observed that the reconstruction of $\hat{a}$ and that of $\beta = \nabla\cdot(\hat{a}v^2)$ involved a loss of two derivatives compared to $\{H_j\}$. The reconstruction of τ, $B^2u_1^2$, v, and σ/B involves the loss of one derivative. The loss for the reconstruction of q is 3 and then by elliptic regularity 1 for that of B. We now verify that

$$-\nabla\hat{a}v^2\nabla\frac{1}{u_1} = -\hat{a}v^2 : \nabla^{\otimes 2}\frac{1}{u_1} - \nabla \cdot (\hat{a}v^2)\nabla\frac{1}{u_1} = H_1.$$

This shows that u_1^{-1} is reconstructed with no loss of regularity compared to H by Schauder elliptic estimates since the error in the reconstruction of u_1 is two derivatives smoother than the errors in the coefficients $\hat{a}v^2$ and its divergence. This implies the same result for σ.

This is summarized, with any $m \geq 0$ and $0 < \alpha < 1$.

$$\|\delta\hat{a}\|_{C^{m,\alpha}} + \|\delta B\|_{C^{m+1,\alpha}} + \|\delta\sigma\|_{C^{m+2,\alpha}} \lesssim \|\{\delta H_j\}\|_{C^{m+2,\alpha}}. \tag{9.36}$$

When $\hat{a}$ is known a priori, then we may choose any $m \geq -1$ above. Note that the errors on the reconstructions of σ and the isotropic part B are optimal. The reconstruction of the anisotropic structure involves the loss of one additional derivative compared to that of the determinant, a fact that is shared by many other hybrid inverse problems.

9.4.3 Control problem

The above estimates hinge on appropriate linear independence of derivatives of the solutions u_j and v_j. This is obtained in two steps.

The first step is to consider a point $x_0 \in X$, to freeze the coefficients at their values at x_0, and then to construct solutions of the elliptic equation with constant coefficients. We let the reader verify (as a lengthy exercise) that we can construct locally, by generalizing the notion of harmonic polynomial, local solutions that satisfy all the constraints of linear independence (for gradients and Hessians) we mentioned in the preceding section. Now choosing a small ball B of radius r centered at x_0, we can impose the boundary conditions obtained from the solutions of the constant coefficient equations to the equation with non-constant coefficients restricted to B. When r is sufficiently small, the linear independence is clearly still satisfied and so we have constructed linearly independent solutions of the original equation on B. The question is whether we can find boundary conditions f_j such that the solutions obtained on X are well approximated by the local solutions when restricted to B. When this is the case, then the problem is solved and estimates such as, e.g., (9.36) hold.

The problem at hand is therefore a control problem. Can one control solutions in B from solutions in X? It should be clear that such a control cannot be exact. Solutions in B need not be smooth on ∂B. When the coefficients in the vicinity of ∂B are smooth, then solutions on X have to be smooth there. So control has to be approximate at best. What makes the approximate control possible is the unique continuation principle (UCP) we already used in the analysis of the Cauchy problem for elliptic equations.

We denote $L = \nabla \cdot a\nabla + b \cdot \nabla + c$ and recall that UCP holds for such an operator when a is elliptic and of class C^1 while all other coefficients are bounded. We proved such a result in the case where a is scalar. Such results extend to the anisotropic setting as well. We assume L invertible on X when augmented with Dirichlet conditions.

Theorem 9.4.1 (Runge approximation) *Let X_0 be an open subset with closure in X and let u be a solution in $H^1(X_0)$ of $Lu = 0$ on X_0. Then there is a sequence of function $u_\varepsilon \in H^1(X)$ solutions of $Lu_\varepsilon = 0$ in X such that $v_\varepsilon \to u$ in $L^2(X_0)$, where $v_\varepsilon = u_{\varepsilon|X_0}$ is the restriction of u_ε to X_0.*

Proof. The proof goes by contradiction as an application of a geometric version of the Hahn-Banach theorem. Let $F = \{v_{|X_0};\ v \in H^1(X),\ Lv = 0 \text{ on } X\}$ and $G = \{u \in H^1(X_0),\ Lu = 0 \text{ on } X_0\}$. Both are subspaces of $L^2(X_0)$. Let $\bar{F}$ and $\bar{G}$ be the closures of F and G for the $L^2(X_0)$ topology. The Runge approximation states that $\bar{F} = \bar{G}$. Assume otherwise. We verify that $\bar{F}$ is a closed convex subset of $L^2(X_0)$ since F is a linear space. Assume the existence of $u \in \bar{G}$ with $u \notin \bar{F}$. Then $\{u\}$ is a compact subset of $L^2(X_0)$ and Hahn-Banach states that $\bar{F}$ and $\{u\}$ are separated, in other words, there is an element $f \in L^2(X_0)$ identified with its dual such that $(f, v) < \alpha < (f, u)$ for some $\alpha > 0$, say. Since v may be replaced by λv for any $\lambda \in \mathbb{R}$, this implies $(f, v) = 0$ for each $v \in \bar{F}$ while $(f, u) > 0$. Let us now prove that this contradicts the unique continuation principle (UCP).

Let us extend f by 0 in X outside X_0 and still call f the extension. Let us solve

$$L^*w = f, \quad \text{in} \quad X, \quad w = 0, \quad \text{on } \partial X.$$

This equation admits a unique solution by assumption on L (and the Fredhiolm alternative). Then, for $v \in F$, we have

$$0 = (Lv, w) - (v, L^* w) = -\int_{\partial X} (an \cdot \nabla w) v d\sigma.$$

This holds for arbitrary trace $v_{|\partial X}$ in $H^{\frac{1}{2}}(\partial X)$, from which we deduce that $an \cdot \nabla w = 0$ on ∂X. However, we also have $w = 0$ on ∂X so that all Cauchy data associated to w vanish. From the UCP, we deduce that $w \equiv 0$, and hence that $f \equiv 0$, which is incompatible with the existence of $u \notin \bar{F}$. This proves the result. □

We are not entirely done. The approximation we obtain is in the $L^2(X_0)$ sense. This is not sufficient to obtain point-wise linear independence of Hessians of v_j. However, such an estimate clearly comes from elliptic interior regularity. Indeed, we have $u_\varepsilon - u$ small in X_0 and $L(u_\varepsilon - u_0) = 0$ in X_0. Elliptic regularity with coefficients in $C^{p,\alpha}$ shows that $u_\varepsilon - u$ is also small in $C^{p,\alpha}(X_1)$ for any X_1 open with closure in X_0. This concludes our control of internal derivatives of elliptic solutions from the boundary. The boundary controls are obviously the sequence of traces $f_\varepsilon = u_{\varepsilon|\partial X}$. We choose ε small enough so that the Hessian of u_ε and that of u are sufficiently small. This proves the linear independent of Hessians and gradients of the functions $u_{j,\varepsilon}$ approximating u_j. By continuity of elliptic solutions with respect to perturbations in the boundary conditions, we therefore obtain the existence of an open set of boundary conditions f_j such that the resulting solutions u_j satisfy the independence conditions on the domain X_1. This may be repeated for a covering of X by domains of the form X_1 (including by domains that cover the boundary ∂X using a slightly different regularity theory leading to the same results).

Chapter 10

Coupled-physics IP II: Ultrasound Modulation Tomography

10.1 Ultrasound Modulation Tomography

The preceding chapter analyzed inverse problems related to the photo-acoustic effect. In this chapter, we consider another physical mechanism that couples optical waves with ultrasound, namely the ultrasound modulation effect. To simplify the presentation, and because most results are known in this simplified setting, we assume that the elliptic equations involve an unknown diffusion coefficient but that the absorption coefficient is assumed to vanish. We refer the reader to [20] to generalizations to the practically physical setting of an elliptic equation with unknown diffusion and absorption coefficients.

Consider thus the following elliptic equation

$$-\nabla \cdot \gamma(x)\nabla u = 0 \quad \text{in } X, \qquad u = f \quad \text{on } \partial X. \tag{10.1}$$

Here, γ is the unknown diffusion coefficient, which we assume for the moment is a real-valued, scalar, function defined on a domain $X \subset \mathbb{R}^n$ for $n = 2$ or $n = 3$. We assume that γ is bounded above and below by positive constants so that the above equation admits a unique solution. We also assume that γ is sufficiently smooth so that the solution to the above equation is continuously differentiable on $\bar{X}$, the closure of X [30]. As before, we denote by $f(x)$ the imposed (sufficiently smooth) Dirichlet boundary conditions.

As we have seen already, the coefficient $\gamma(x)$ may model the electrical conductivity in the setting of electrical impedance tomography (EIT) or a diffusion coefficient of particles (photons) in the setting of optical tomography (OT). Both EIT and OT are modalities with high contrast, in the sense that $\gamma(x)$ takes different values in different tissues and allows one to discriminate between healthy and non-healthy tissues. In OT, high contrasts are mostly observed in the absorption coefficient, which we recall is not modeled here; see [20].

A methodology to couple high contrast with high resolution consists of perturbing the diffusion coefficient acoustically. Let an acoustic signal propagate through the domain. We assume here that the sound speed is constant and that the acoustic signal is a plane wave of the form $p\cos(k \cdot x + \varphi)$ where p is the amplitude of the acoustic signal, k its wave-number and φ an additional phase. The acoustic signal modifies the properties of

the diffusion equation. We assume that such an effect is small and that the coefficient in (10.1) is modified as

$$\gamma_\varepsilon(x) = \gamma(x)(1 + \zeta\varepsilon\mathfrak{c}), \tag{10.2}$$

where we have defined $\mathfrak{c} = \mathfrak{c}(x) = \cos(k \cdot x + \varphi)$ and where $\varepsilon = p\Gamma$ is the product of the acoustic amplitude $p \in \mathbb{R}$ and a measure $\Gamma > 0$ of the coupling between the acoustic signal and the modulations of the constitutive parameter in (10.1). We assume that $\varepsilon \ll 1$ so that the influence of the acoustic signal on γ_ε admits an asymptotic expansion that we truncated at the second order as displayed in (10.2). The size of the terms in the expansion are physically characterized by ζ and depend on the specific application.

Let u and v be solutions of (10.1) with fixed boundary conditions g and h, respectively. When the acoustic field is turned on, the coefficients are modified as described in (10.2) and we denote by u_ε and v_ε the corresponding solutions. Note that $u_{-\varepsilon}$ is the solution obtained by changing the sign of p or equivalently by replacing φ by $\varphi + \pi$.

By the standard continuity property of the solution to (10.1) with respect to changes in the coefficients and regular perturbation arguments, we find that $u_\varepsilon = u_0 + \varepsilon u_1 + O(\varepsilon^2)$. Let us multiply the equation for u_ε by $v_{-\varepsilon}$ and the equation for $v_{-\varepsilon}$ by u_ε, subtract the resulting equalities, and use standard integrations by parts. We obtain that

$$\int_X (\gamma_\varepsilon - \gamma_{-\varepsilon})\nabla u_\varepsilon \cdot \nabla v_{-\varepsilon} dx = \int_{\partial X} \gamma_{-\varepsilon}\frac{\partial v_{-\varepsilon}}{\partial \nu} u_\varepsilon - \gamma_\varepsilon \frac{\partial u_\varepsilon}{\partial \nu} v_{-\varepsilon} d\mu(x). \tag{10.3}$$

Exercise 10.1.1 *Verify the above result.*

Here, $d\mu(x)$ is the standard surface measure on ∂X. We assume that $\gamma_\varepsilon \partial_\nu u_\varepsilon$ and $\gamma_\varepsilon \partial_\nu v_\varepsilon$ are measured on ∂X, at least on the support of $v_\varepsilon = h$ and $u_\varepsilon = g$, respectively, for all values ε of interest. Note that the above equation holds if the Dirichlet boundary conditions are replaced by Neumann boundary conditions. Let us define

$$J_\varepsilon := \frac{1}{2}\int_{\partial X} \gamma_{-\varepsilon}\frac{\partial v_{-\varepsilon}}{\partial \nu} u_\varepsilon - \gamma_\varepsilon \frac{\partial u_\varepsilon}{\partial \nu} v_{-\varepsilon} d\mu(x) \;=\; \varepsilon J_1 + \varepsilon^2 J_2 + O(\varepsilon^3). \tag{10.4}$$

We assume that the real valued functions $J_m = J_m(k, \varphi)$ are known (measured functions). Notice that such knowledge is based on the physical boundary measurement of the Cauchy data of the form $(u_\varepsilon, \gamma_\varepsilon \partial_\nu u_\varepsilon)$ and $(v_\varepsilon, \gamma_\varepsilon \partial_\nu v_\varepsilon)$ on ∂X.

Equating like powers of ε, we find at the leading order that

$$\int_X \left[\zeta\gamma(x)\nabla u_0 \cdot \nabla v_0(x)\right] \cos(k \cdot x + \varphi) dx = J_1(k, \varphi). \tag{10.5}$$

This may be acquired for all $k \in \mathbb{R}^n$ and $\varphi = 0, \frac{\pi}{2}$, and hence provides the Fourier transform of

$$H[u_0, v_0](x) = \zeta\gamma(x)\nabla u_0 \cdot \nabla v_0(x). \tag{10.6}$$

Note that when $v_\varepsilon = u_\varepsilon$, then we find from the expression in (10.3) that $J_2 = 0$ in (10.4) so that the expression for J_1 may be obtained from available measurements in (10.4) with an accuracy of order $O(\varepsilon^2)$. Note also that

$$H[u_0, v_0](x) = \frac{1}{4}\big(H[u_0 + v_0, u_0 + v_0] - H[u_0 - v_0, u_0 - v_0]\big)$$

by polarization. In other words, the limiting measurements (for small ε) in (10.6) may also be obtained by considering expressions of the form (10.3) with $u_\varepsilon = v_\varepsilon$.

In the setting of optical tomography, the coefficient γ_ε in (10.2) takes the form

$$\gamma_\varepsilon(x) = \frac{\tilde{\gamma}_\varepsilon}{c_\varepsilon^{n-1}}(x),$$

where $\tilde{\gamma}_\varepsilon$ is the diffusion coefficient, c_ε is the light speed, and n is spatial dimension. When the pressure field is turned on, the location of the scatterers is modified by compression and dilation. Since the diffusion coefficient is inversely proportional to the scattering coefficient, we find that

$$\frac{1}{\gamma_\varepsilon(x)} = \frac{1}{\gamma(x)}\big(1 + \varepsilon \mathfrak{c}(x)\big).$$

Moreover, the pressure field changes the index of refraction (the speed) of light as follows

$$c_\varepsilon(x) = c(x)(1 + \mathit{\Gamma}\,\varepsilon \mathfrak{c}(x)),$$

where $\mathit{\Gamma}$ is a constant (roughly equal to $\frac{1}{3}$ for water). This shows that

$$\zeta = -(1 + (d-1)\mathit{\Gamma}). \tag{10.7}$$

In the setting of electrical impedance tomography, we simply assume that ζ models the coupling between the acoustic signal and the change in the electrical conductivity of the underlying material. The value of ζ thus depends on the application.

10.2 Inverse problems in ultrasound modulation.

Assuming the validity of the above derivation, the objective of ultrasound modulated tomography is to reconstruct the coefficient $\gamma(x)$ from knowledge of the interior functionals

$$H_{ij}(x) = \gamma(x)\nabla u_i(x) \cdot \nabla u_j(x), \qquad 1 \le i, j \le m, \tag{10.8}$$

where u_j is the solution to the equation

$$\nabla \cdot (\gamma \nabla u_j) = 0 \quad \text{in} \quad X, \qquad u_j = f_j \quad \text{on} \quad \partial X, \qquad 1 \le j \le m, \tag{10.9}$$

for appropriate choices of the boundary conditions f_j on ∂X. In practice, the ultrasound modulation effect is extremely weak because the coupling coefficient Γ is small. However, its mathematical analysis shows that UMOT and UMEIT are well posed problems unlike the (non ultrasound-modulated) OT and EIT problems.

We need some notation to introduce the main result of this chapter. We first define the UMT measurement operator. We again define $\mathfrak{f} = (f_1, \ldots, f_m)$ the vector of illuminations at the domain's boundary. The corresponding solutions to (10.9) are denoted by u_j. The measurement operator $\mathfrak{M}_\mathfrak{f}$ is then defined as the following collection of internal functionals:

$$\begin{array}{rccl} \mathfrak{M}_\mathfrak{f}: & \mathfrak{X} & \to & \mathcal{M}^\sigma(\mathbb{R}^m; \mathfrak{Y}) \\ & \gamma & \mapsto & \mathfrak{M}_\mathfrak{f}\gamma = (\gamma \nabla u_i \cdot \nabla u_j)_{1 \le i,j \le m}. \end{array} \tag{10.10}$$

Here, $\mathfrak{X}$ is a subset of a Banach space in which the unknown coefficients are defined, $\mathfrak{Y}$ is a subset of $H^1(X)$ where the solutions to (10.9) are defined, and $\mathcal{M}^\sigma(\mathbb{R}^m;\mathfrak{Y})$ is the space of symmetric second-order tensors of order m with values in $\mathfrak{Y}$.

Again, we observe that $\mathfrak{M}_\mathfrak{f}$ is parameterized by the illuminations $\mathfrak{f}$.

Remark 10.2.1 *Note that for $m = 1$, the internal functional is of the form $H = \gamma|\nabla u|^2$. The unknown coefficient γ can then be eliminated and formally, u solves the following non-linear equation*

$$\nabla \cdot \frac{H}{|\nabla u|^2}\nabla u = 0 \quad in \quad X, \qquad u = f \quad on \quad \partial X. \tag{10.11}$$

This nonlinear problem is in fact hyperbolic in nature rather than elliptic and its solution is therefore far from guaranteed.

In some sense, the UM inverse problem has two unknowns (γ, u). In the one-functional setting, eliminationg of γ to get an equation for u is rather trivial. However, the resulting equation is difficult to analyze and may not have unique solutions. The multi-functional setting when $m \geq 2$ aims to simplify the solution of the resulting (redundant system of) equation(s). However, the elimination *of unknowns becomes significantly more challenging.*

We first perform the change of unknown functions $S_i = \gamma^{\frac{1}{2}}\nabla u_i$ for every i and define

$$F(x) := \nabla \log \gamma(x). \tag{10.12}$$

Let $(e_1, \ldots, e_n)$ be the canonical basis in $\mathbb{R}^n$. For a given vector field $V = V^i e_i$ defined on X, we define the corresponding one-form $V^\flat := V^i dx^i$, where dx^i is the dual basis (of 1-forms) to e_i in the sense that $dx^i(e_j) = \delta_{ij}$, i.e., 0 if $i \neq j$ and 1 if $i = j$. With this notation, we obtain that the vector fields S_j satisfy the system of equations

$$\nabla \cdot S_j = -\frac{1}{2}F \cdot S_j, \tag{10.13}$$

$$dS_j^\flat = \frac{1}{2}F^\flat \wedge S_j^\flat, \qquad 1 \leq j \leq m, \tag{10.14}$$

where $\wedge$ and d denote the usual exterior product and exterior derivative, respectively. The first equation stems directly from (10.9) whereas the second one states that the one-form $\gamma^{-\frac{1}{2}}S_j^\flat = du_j$ is exact, therefore closed, and hence $d(\gamma^{-\frac{1}{2}}S_j^\flat) = 0$. In dimension $n = 3$, this means that $\nabla u = \gamma^{-\frac{1}{2}}S$ is a gradient so that its curl vanishes. The above equations are generalizations to arbitrary dimensions.

When $n = 2, 3$, equation (10.14) is recast as:

$$n = 2: \quad [\nabla, S_j] - \alpha[F, S_j] = 0, \qquad n = 3: \quad \text{curl } S_j - \alpha F \times S_j = 0,$$

where for $n = 2$, we define $[A, B] := A_x B_y - A_y B_x$ and $[\nabla, A] := \partial_x A_y - \partial_y A_x$ and for $n = 3$, $\times$ denotes the standard cross-product.

Exercise 10.2.1 *Check the above formulas.*

The strategy now is the following: we wish to *eliminate* F from the above equations. This will result in an overdetermined system of equations for the vector fields S that we will prove admits a unique solution. Once all vector fields S_j are known, we obtain an explicit expression for F, from which the reconstruction of γ is relatively straightforward. The elimination of F requires that a certain system be invertible. As for condition **(H3)** of **(H3')** in chapter 8, we therefore need to find *well-chosen* illuminations $\mathfrak{f}$ for which such an invertibility condition holds. We will again see that a large class of illuminations are *well-chosen* in two dimensions $n = 2$. In dimensions $n \geq 3$, the construction of *well-chosen* illuminations will again be based on CGO solutions.

The invertibility condition is that the m gradients ∇u_j have maximal rank in $\mathbb{R}^n$ at every point $x \in X$. This hypothesis can be formalized by the somewhat stronger statement: there exists a finite open covering $\mathcal{O} = \{\Omega_k\}_{1\leq k\leq N}$ of X (i.e. $X \subset \cup_{k=1}^N \Omega_k$), an indexing function $\tau : [1, N] \ni i \mapsto \tau(i) = (\tau(i)_1, \ldots, \tau(i)_n) \in [1, m]^n$ and a positive constant c_0 such that

$$\min_{1\leq i\leq N} \inf_{x\in\Omega_i} \det(S_{\tau(i)_1}(x), \ldots, S_{\tau(i)_n}(x)) \geq c_0 > 0. \tag{10.15}$$

This assumption is equivalent to imposing the following condition on the data

$$\min_{1\leq i\leq N} \inf_{x\in\Omega_i} \det H^{\tau(i)}(x) \geq c_0^2 > 0, \tag{10.16}$$

where $H^{\tau(i)}$ stands for the $n \times n$ matrix of elements $H_{kl}^{\tau(i)} = S_{\tau(i)_k} \cdot S_{\tau(i)_l}$. We state that $\mathfrak{f}$ is a well-posed set of illuminations when (10.15) or equivalently (10.16) holds.

This complicated expression simply states that at each point $x \in X$, we can find n linearly independent vectors $\nabla u_{j(x)}$ with determinant bounded from below uniformly in $x \in X$. The elimination of F is then guaranteed and can be done in a stable fashion as the following lemma indicates.

Lemma 10.2.2 *Let $\Omega \subset X$ open where, up to renumbering of solutions, we have*

$$\inf_{x\in\Omega} \det(S(x)) \geq c_0 > 0, \qquad S(x) := (S_1(x)|\ldots|S_n(x)).$$

Let us define $H(x) := \{S_i(x) \cdot S_j(x)\}_{1\leq i,j\leq n}$ and $D(x) = \sqrt{\det H(x)}$. Then we have:

$$\begin{aligned} F &= \frac{c_F}{D} \sum_{i,j=1}^n (\nabla(DH^{ij}) \cdot S_i)S_j = c_F\Big(\nabla \log D + \sum_{i,j=1}^n (\nabla H^{ij} \cdot S_i)S_j\Big), \\ c_F &:= (\frac{1}{2}(n-2)+1)^{-1}. \end{aligned} \tag{10.17}$$

Here H^{ij} denotes the element (i, j) of the matrix H^{-1}.

Once F is eliminated, we can write a system of equations for the vectors S_j that admits a unique solution provided that γ and all S_j's are known at one point $x_0 \in \bar{X}$, for instance at the domain's boundary ∂X. This leads to well-posed reconstructions as stated in the following:

Theorem 10.2.3 *Let $X \subset \mathbb{R}^n, n \geq 2$ be an open convex bounded domain, and let two sets of $m \geq n$ solutions of (10.9) generate measurements $(H = \mathfrak{M}_{\mathfrak{f}}(\gamma), \tilde{H} = \mathfrak{M}_{\mathfrak{f}}(\tilde{\gamma}))$ whose components belong to $W^{1,\infty}(X)$, and who jointly satisfy condition (10.16) with the same triple $(\mathcal{O}, \tau, c_0)$. In other words, we assume that $\mathfrak{f}$ is* well-chosen *for both γ and $\tilde{\gamma}$.*

Let $x_0 \in \overline{\Omega_{i_0}} \subset \overline{X}$ and $\gamma(x_0), \tilde{\gamma}(x_0)$ and $\{S_{\tau(i_0)_i}(x_0), \tilde{S}_{\tau(i_0)_i}(x_0)\}_{1 \leq i \leq n}$ be given. Then we have the stability estimate:

$$\| \log \gamma - \log \tilde{\gamma} \|_{W^{1,\infty}(X)} \leq C \left(\varepsilon_0 + \| H - \tilde{H} \|_{W^{1,\infty}(X)} \right), \tag{10.18}$$

where ε_0 is the error at x_0:

$$\varepsilon_0 := | \log \gamma(x_0) - \log \tilde{\gamma}(x_0) | + \sum_{i=1}^{n} \| S_{\tau(i_0)_i}(x_0) - \tilde{S}_{\tau(i_0)_i}(x_0) \|.$$

10.3 Eliminations and redundant systems of ordinary differential equations

The proof of Theorem 10.2.3, which we complete in this section, involves first proving Lemma 10.2.2 and next solving a redundant system of equations for the vectors S_j.

10.3.1 Elimination of F

We now prove Lemma 10.2.2 and first recall some notation from tensor calculus. For $0 \leq k \leq n$, Λ^k denotes the space of $k-$ forms. We recall the definition of the *Hodge star operator* $\star : \Lambda^k \mapsto \Lambda^{n-k}$ for $0 \leq k \leq n$, such that for any elementary k-form $dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$, we have

$$\star dx^I = \sigma dx^J, \text{ where } \quad \sigma = \text{sign}((1 \ldots n) \mapsto (I, J)). \tag{10.19}$$

Here, J is implicitly defined by the fact that $(1 \ldots n) \mapsto (I, J)$ is a permutation. Note that σdx^J is independent of the ordering of the $n-k$ indices in J. We recall the following useful identities:

$$\star\star = (-1)^{k(n-k)} \quad \text{on } \Lambda^k, \qquad \star(u^\flat \wedge \star v^\flat) = u \cdot v, \qquad \star d \star u^\flat = \nabla \cdot u, \qquad u, v \in \Lambda^1.$$

Because $S_1(x), \ldots, S_n(x)$ forms a basis of $\mathbb{R}^n$, a vector V can be represented in this basis by the following representation

$$V = H^{ij}(V \cdot S_i) S_j. \tag{10.20}$$

For $j = 1, \ldots, n$, let us introduce the following 1-forms:

$$X_j^\flat := (-1)^{n-1} \sigma_j * (S_{i_1}^\flat \wedge \cdots \wedge S_{i_{n-1}}^\flat), \quad (i_1, \ldots, i_{n-1}) = (1, \ldots, \hat{j}, \ldots, n), \tag{10.21}$$

where the hat indicates an omission and $\sigma_j = (-1)^{j-1}$. We now show that the vector fields X_j satisfy a simple divergence equation. We compute

$$\begin{aligned}\nabla \cdot X_j = \star d \star X_j^\flat = \sigma_j \star d(S_{i_1}^\flat \wedge \cdots \wedge S_{i_{n-1}}^\flat) = \sigma_j \star \sum_{k=1}^{n-1} (-1)^k S_{i_1}^\flat \wedge \cdots \wedge dS_{i_k}^\flat \wedge \cdots \wedge S_{i_{n-1}}^\flat \\ = \sigma_j \star \sum_{k=1}^{n-1} (-1)^k S_{i_1}^\flat \wedge \cdots \wedge \frac{1}{2}(F^\flat \wedge S_{i_k}^\flat) \wedge \cdots \wedge S_{i_{n-1}}^\flat \\ = \frac{1}{2}(n-1) \star (F^\flat \wedge \star X_j^\flat).\end{aligned}$$

Using the identity $\star(u^\flat \wedge \star v^\flat) = u \cdot v$, we deduce that

$$\nabla \cdot X_j = \frac{1}{2}(n-1) F \cdot X_j, \quad j = 1 \dots n. \tag{10.22}$$

We now decompose X_j in the basis $(S_1, \dots, S_n)$. For $k \neq j$, there is an l such that $i_l = k$ and we have

$$X_j \cdot S_k = \det(S_1, \dots, S_{j-1}, S_k, S_{j+1}, \dots, S_n) = 0,$$

by repetition of the term S_k in the determinant. Now for $k = j$, we have

$$X_j \cdot S_j = \det(S_1, \dots, S_n) = \det S = D.$$

Using formula (10.20), we deduce that X_j admits the expression

$$X_j = D H^{ij} S_i.$$

Plugging this expression into equation (10.22), and using $\nabla \cdot (\varphi V) = \nabla \varphi \cdot V + \varphi \nabla \cdot V$, we obtain

$$\begin{aligned} & \nabla(DH^{ij}) \cdot S_i + DH^{ij} \nabla \cdot S_i &&= \tfrac{1}{2}(n-1) F \cdot (DH^{ij} S_i) \\ \Leftrightarrow \quad & \nabla(DH^{ij}) \cdot S_i - DH^{ij} \tfrac{1}{2} F \cdot S_i &&= \tfrac{1}{2}(n-1) DH^{ij} F \cdot S_i \\ \Leftrightarrow \quad & \nabla(DH^{ij}) \cdot S_i &&= c_F^{-1} DH^{ij} F \cdot S_i. \end{aligned}$$

Finally, using the representation (10.20) for F itself yields

$$F = (H^{ij} F \cdot S_i) S_j = \frac{c_F}{D} (\nabla(DH^{ij}) \cdot S_i) S_j. \tag{10.23}$$

We can also recast the previous expression as

$$F = c_F \left[\nabla \log D + ((\nabla H^{ij}) \cdot S_i) S_j \right], \tag{10.24}$$

and the proof of the lemma is complete.

10.3.2 System of ODEs for S_j

In this section, we obtain a redundant system of ordinary differential equations for the matrix S. Let S be the matrix formed with the column vectors S_j. Then $H = S^T S$ is known from the measurement operator. Moreover, we have just found an equation of the form $dS_j^\flat = F^\flat(S) \wedge S_j^\flat$, with $F = F(S)$ given above. We thus possess information about the symmetric part of S and the skew-symmetric part of S. It remains to know whether this is enough to write an equation of the form $\nabla \otimes S_j = \mathcal{F}_j(S)$. The answer is affirmative as we now show.

We first need to introduce other standard geometric notation, without which the derivations become quickly intractable. Let us denote the Euclidean orthonormal frame $e_i = \partial_{x^i}$ and $e^i = dx^i$. We work on a convex set $\Omega \subset \mathbb{R}^n$ with the Euclidean metric $g(X,Y) \equiv X \cdot Y = \delta_{ij} X^i Y^j$ on $\mathbb{R}^n$. Following [42], we denote by $\overline{\nabla}$ the Euclidean connection, which here has the expression we can take as a definition:

$$\overline{\nabla}_X f = X \cdot \nabla f = X^i \partial_i f, \quad \text{and} \quad \overline{\nabla}_X Y = (X \cdot \nabla Y^j) e_j = X^i (\partial_i Y^j) e_j,$$

for given vector fields $X = X^i e_i$ and $Y = Y^i e_i$. An important identity for the sequel is the following characterization of the exterior derivative of a one-form ω:

$$d\omega(X,Y) = \overline{\nabla}_X(\omega(Y)) - \overline{\nabla}_Y(\omega(X)) - \omega([X,Y]), \tag{10.25}$$

or equivalently in the Euclidean metric, writing $\omega = S_i^\flat$, $X = S_j$ and $Y = S_k$,

$$S_i \cdot [S_j, S_k] = S_j \cdot \nabla(S_i \cdot S_k) - S_k \cdot \nabla(S_i \cdot S_j) - dS_i^\flat(S_j, S_k), \tag{10.26}$$

where the Lie bracket (commutator) of X and Y may be "defined" here as $[X,Y] = \overline{\nabla}_X Y - \overline{\nabla}_Y X = X \cdot \nabla Y - Y \cdot \nabla X$ seen as a vector field.

Exercise 10.3.1 *Verify* (10.26) *directly.*

Note that the right-hand side in (10.26) involves no derivatives in the unknown S since $S_i \cdot S_k = H_{ik}$ is known and $dS_i^\flat$ is a known functional of S by (10.14) and (10.17).

The following relation between inner products and Lie brackets of a given frame (see e.g. [42, Eq. 5.1 p. 69]) is very useful

$$\begin{aligned} 2(X \cdot \nabla Y) \cdot Z \; &= X \cdot \nabla(Y \cdot Z) + Y \cdot \nabla(Z \cdot X) - Z \cdot \nabla(X \cdot Y) \\ &\quad - Y \cdot [X,Z] - Z \cdot [Y,X] + X \cdot [Z,Y]. \end{aligned} \tag{10.27}$$

Exercise 10.3.2 *Check the above expression directly.*

We thus find using (10.14) and (10.26) that

$$2(S_i \cdot \nabla S_j) \cdot S_k = S_i \cdot \nabla H_{jk} - S_j \cdot \nabla H_{ik} + S_k \cdot \nabla H_{ij} - 2F \cdot S_k H_{ij} + 2F \cdot S_j H_{ik}. \tag{10.28}$$

Note that the right-hand-side no longer involves any derivative of S_j. Moreover, S_j forms a frame. We can therefore extract the full gradient of S_j from the terms $2(S_i \cdot \nabla S_j) \cdot S_k$. Geometrically, gradients generalize to tensors via the total covariant derivative, which maps a vector field X to a tensor of type $(1,1)$ defined by

$$\overline{\nabla} X(\omega, Y) = \omega(\overline{\nabla}_Y X). \tag{10.29}$$

S is a frame provided that the determinant condition $\inf_{x\in\Omega} \det S \geq c_0 > 0$ holds. In the frame S, we may express $\overline{\nabla} S_i$ in the basis $\{S_j \otimes S_k^\flat\}_{j,k=1}^n$ of such tensors by writing $\overline{\nabla} S_i = a_{ijk} S_j \otimes S_k^\flat$ and identifying the coefficients a_{ijk} by writing

$$\overline{\nabla} S_i(S_p^\flat, S_q) = S_p^\flat(\overline{\nabla}_{S_q} S_i) = S_q \cdot \nabla S_i \cdot S_p,$$

and also

$$\overline{\nabla} S_i(S_p^\flat, S_q) = a_{ijk} S_j \otimes S_k^\flat(S_p^\flat, S_q) = a_{ijk} H_{jp} H_{kq}.$$

Equating the two, we obtain the representation

$$\overline{\nabla} S_i = H^{qk} H^{jp} [(S_q \cdot \nabla S_i) \cdot S_p] S_j \otimes S_k^\flat, \tag{10.30}$$

where H^{ij} denote the coefficients of the matrix H^{-1}.

Exercise 10.3.3 *Seeing $\nabla \otimes S_j$ as the matrix with components (i,k) given by $\partial_{x_i}(S_j)_k$ and $S_i \otimes S_m$ as the matrix with components (j,k) given by $(S_i)_k (S_m)_k$, show that*

$$\nabla \otimes S_j = \sum_{i,k,l,m} H^{ik} (S_k \cdot \nabla S_j) \cdot S_l H^{lm} S_i \otimes S_m = \mathcal{F}_j(S).$$

Now plugging (10.28) into (10.30), and using $H_{ij} H^{jk} = \delta_{ik}$, we obtain

$$\begin{aligned} 2\overline{\nabla} S_i &= 2H^{qk} H^{jp} (\overline{\nabla}_{S_q} S_i \cdot S_p) S_j \otimes S_k^\flat \\ &= H^{qk} H^{jp} \left(\nabla H_{iq} \cdot S_p + \nabla H_{ip} \cdot S_q - \nabla H_{pq} \cdot S_i + (H_{pq}(F \cdot S_i) - H_{qi}(F \cdot S_p)) \right) S_j \otimes S_k^\flat \\ &= (H^{jp} U_{ik} \cdot S_p + H^{qk} U_{ij} \cdot S_q + \nabla H^{jk} \cdot S_i + (H^{jk}(F \cdot S_i) - H^{jp} \delta_{ik} (F \cdot S_p))) S_j \otimes S_k^\flat, \end{aligned}$$

where we have used $\nabla H^{jk} = -H^{jp} (\nabla H_{pq}) H^{qk}$ and have defined

$$U_{jk} := (\nabla H_{jp}) H^{pk} = -H_{jp} \nabla H^{pk}, \quad 1 \leq j,k \leq n. \tag{10.31}$$

Using formulas $H^{jk} S_j \otimes S_k^\flat = e_i \otimes e^i$ and $H^{kl}(V \cdot S_k) S_l = V$ for any smooth vector field V, we obtain for $1 \leq i \leq n$

$$\overline{\nabla} S_i = \frac{1}{2} \left(U_{ik} \otimes S_k^\flat + S_k \otimes U_{ik}^\flat + (\nabla H^{jk} \cdot S_i) S_j \otimes S_k^\flat \right) + \frac{1}{2} (F \cdot S_i) I_n - \frac{1}{2} F \otimes S_i^\flat. \tag{10.32}$$

Using (10.23), we observe that $\overline{\nabla} S_i$ is equal to a polynomial of degree at most three in the frame S with coefficients involving the known inner products H_{ij}. For each $1 \leq i,k \leq n$, $\partial_k S_i$ is nothing but $\overline{\nabla}_{e_k} S_i = \overline{\nabla} S_i(\cdot, e_k)$, which can be obtained from (10.32). Denoting $\mathbf{S} := (S_1^T, \ldots, S_n^T)^T$, we are then able to construct the system of equations

$$\partial_k \mathbf{S} = \sum_{|\beta| \leq 3} Q_\beta^k \mathbf{S}^\beta, \qquad \mathbf{S}^\beta = \prod_{i=1}^{n^2} \mathbf{S}_i^{\beta_i}, \quad 1 \leq k \leq n, \tag{10.33}$$

where Q_β^k depends only on the data and β is an n^2-index. This redundant system can then be integrated along any curve (where it becomes a system of ordinary differential equations with Lipschitz right-hand sides ensuring uniqueness of the solution) in order to solve for the matrix-valued function $\mathbf{S}$.

10.3.3 ODE solution and stability estimates

Once (10.33) has been obtained, the derivation of Theorem 10.2.3 follows rapidly. We leave this step as an exercise.

Exercise 10.3.4 *Prove the stability estimates in Theorem 10.2.3 from* (10.33), (10.24), *and the definition* (10.12).

10.4 Well-chosen illuminations

It remains to find boundary conditions such that (10.15) holds. As in the preceding chapter, we need to distinguish dimension $n = 2$ from dimensions $n \geq 3$.

10.4.1 The case $n = 2$

In dimension $n = 2$, the critical points of u (points x where $\nabla u(x) = 0$) are necessarily isolated as is shown in, e.g., [2]. From this and techniques of quasiconformal mappings that are also restricted to two dimensions of space, we can show the following results.

Lemma 10.4.1 ([3]) *Let u_1 and u_2 be the solutions of* (10.1) *on X simply connected with boundary conditions $f_1 = x_1$ and $f_2 = x_2$ on ∂X, respectively, where $x = (x_1, x_2)$ are Cartesian coordinates on X. Assume that γ is sufficiently smooth. Then $(x_1, x_2) \mapsto (u_1, u_2)$ from X to its image is a diffeomorphism. In other words, $\det(\nabla u_1, \nabla u_2) > 0$ uniformly on $\bar{X}$.*

In other words, in two dimensions of space, there are explicit boundary conditions, such as those above with f_j the trace of x_j on ∂X for $j = 1, 2$ that guarantees that (10.15) holds uniformly on the whole domain X. It is shown in [24] that the appropriate extension of this result is false in dimension $n \geq 3$.

10.4.2 The case $n \geq 3$

In dimension $n \geq 3$, we have the following result:

Lemma 10.4.2 *Let $n \geq 3$ and $\gamma \in H^{\frac{n}{2}+3+\varepsilon}(X)$ for some $\varepsilon > 0$ be bounded from below by a positive constant. Then for n even, there exists a open set G of illuminations $\{f_1, \ldots, f_n\}$ such that for any $\mathfrak{f} \in G$, the condition* (10.15) *holds with $\mathcal{O} = \{X\}$ for some constant $c_0 > 0$.*
For n odd, there exists an open set G of illuminations $\{f_1, .., f_{n+1}\}$ such that for any $\mathfrak{f} \in G$ there exists an open cover of X of the form $\{\Omega_{2i-1}, \Omega_{2i}\}_{1 \leq i \leq N}$ and a constant $c_0 > 0$ such that

$$\inf_{x \in \Omega_{2i-1}} \det(S_1, \ldots, S_{n-1}, \epsilon_i S_n) \geq c_0 \quad \text{and} \quad \inf_{x \in \Omega_{2i}} \det(S_1, \ldots, S_{n-1}, \tilde{\epsilon}_i S_{n+1}) \geq c_0, \tag{10.34}$$

for $1 \leq i \leq N$ and with $\epsilon_i, \tilde{\epsilon}_i = \pm 1$.

In other words, this lemma indicates that for appropriate boundary conditions f_j, we can always find n corresponding solutions whose gradients form a basis of $\mathbb{R}^n$.

Proof. Consider the problem $\nabla \cdot \gamma(x)\nabla u = 0$ on $\mathbb{R}^n$ with $\gamma(x)$ extended in a continuous manner outside of X and such that γ equals 1 outside of a large ball. Let $q(x) = -\frac{\Delta\sqrt{\gamma}}{\gamma}$ on $\mathbb{R}^n$. Then $q \in H^{\frac{n}{2}+1+\varepsilon}(\mathbb{R}^n)$ since $\gamma - 1 \in H^{\frac{n}{2}+3+\varepsilon}(\mathbb{R}^n)$ for some $\varepsilon > 0$. By Sobolev imbedding, γ is of class $C^3(\overline{X})$ while q is of class $C^1(\overline{X})$. With the above hypotheses, we can apply Corollary 8.3.3.

Let $v = \sqrt{\gamma}u$ so that $\Delta v + qv = 0$ on $\mathbb{R}^n$. Let $\varrho \in \mathbb{C}^n$ be of the form $\varrho = \rho(\vartheta + i\vartheta^\perp)$ with $\vartheta, \vartheta^\perp \in \mathbb{S}^{n-1}$, $\vartheta \cdot \vartheta^\perp = 0$, and $\rho = |\varrho|/\sqrt{2} > 0$. Now, as we showed in Corollary 8.3.3, we have

$$v_\varrho = \sqrt{\gamma}u_\varrho = e^{\varrho\cdot x}(1 + \psi_\varrho), \qquad \rho\psi_{\varrho|X} = O(1) \text{ in } C^1(\overline{X}),$$

with $(\Delta + q)v_\varrho = 0$ and hence $\nabla \cdot \gamma\nabla u_\varrho = 0$ in $\mathbb{R}^n$. We have used again the Sobolev imbedding stating that functions in $H^{\frac{n}{2}+k+\varepsilon}(Y)$ are of class $C^k(Y)$ for a bounded domain Y. Taking gradients of the previous equation and rearranging terms, we obtain that

$$\sqrt{\gamma}\nabla u_\varrho = e^{\varrho\cdot x}(\varrho + \varphi_\varrho), \quad \text{with} \quad \varphi_\varrho := \nabla\psi_\varrho + \psi_\varrho\varrho - (1 + \psi_\varrho)\nabla\sqrt{\gamma}.$$

Because $\nabla\sqrt{\gamma}$ is bounded and $\rho\psi_{\varrho|X} = O(1)$ in $C^1(\overline{X})$, the $\mathbb{C}^n$-valued function φ_ϱ satisfies $\sup_{\overline{X}} |\varphi_\varrho| \le C$ independent of ϱ. Moreover, the constant C is in fact independent of γ provided that the norm of the latter is bounded by a uniform constant in $H^{\frac{n}{2}+3+\varepsilon}(X)$ according to Corollary 8.3.3.

Both the real and imaginary parts of u_ϱ, denoted $u_\varrho^\Re$ and $u_\varrho^\Im$, are solutions of the free-space conductivity equation. Thus, $\sqrt{\gamma}\nabla u_\varrho^\Re$ and $\sqrt{\gamma}\nabla u_\varrho^\Im$ can serve as vectors S_i. More precisely, we have

$$\begin{aligned}
\sqrt{\gamma}\nabla u_\varrho^\Re &= \rho e^{\rho\vartheta\cdot x}\left((\vartheta + \rho^{-1}\varphi_\varrho^\Re)\cos(\rho\vartheta^\perp\cdot x) - (\vartheta^\perp + \rho^{-1}\varphi_\varrho^\Im)\sin(\rho\vartheta^\perp\cdot x)\right),\\
\sqrt{\gamma}\nabla u_\varrho^\Im &= \rho e^{\rho\vartheta\cdot x}\left((\vartheta^\perp + \rho^{-1}\varphi_\varrho^\Im)\cos(\rho\vartheta^\perp\cdot x) + (\vartheta + \rho^{-1}\varphi_\varrho^\Re)\sin(\rho\vartheta^\perp\cdot x)\right).
\end{aligned} \tag{10.35}$$

Case n even: Set $n = 2p$, define $\varrho_l = \rho(e_{2l} + ie_{2l-1})$ for $1 \le l \le p$, and construct

$$S_{2l-1} = \sqrt{\gamma}\nabla u_{\varrho_l}^\Re \quad \text{and} \quad S_{2l} = \sqrt{\gamma}\nabla u_{\varrho_l}^\Im, \quad 1 \le l \le p.$$

Using (10.35), we obtain that

$$\det(S_1, \ldots, S_n) = \rho^n e^{2\rho\sum_{l=1}^p x_{2l}}(1 + f(x)),$$

where $\lim_{\rho\to\infty}\sup_{\overline{X}}|f| = 0$. Letting ρ so large that $\sup_{\overline{X}}|f| \le \frac{1}{2}$ and denoting $\gamma_0 := \min_{x\in\overline{X}}(\rho^n e^{2\rho\sum_{l=1}^p x_{2l}}) > 0$, we have $\inf_{x\in\overline{X}}\det(S_1, \ldots, S_n) \ge \frac{\gamma_0}{2} > 0$.

Case n odd: Set $n = 2p - 1$, define $\varrho_l = \rho(e_{2l} + ie_{2l-1})$ for $1 \le l \le p-1$, and $\varrho_p = \rho(e_n + ie_1)$ and construct

$$S_{2l-1} = \sqrt{\gamma}\nabla u_{\varrho_l}^\Re \quad \text{and} \quad S_{2l} = \sqrt{\gamma}\nabla u_{\varrho_l}^\Im, \quad 1 \le l \le p.$$

Using (10.35), we obtain that

$$\det(S_1, \ldots, S_{n-1}, S_n) = \rho^n e^{\rho(x_n + 2\sum_{l=1}^{p-1} x_{2l})} \left(-\cos(\rho x_1) + f_1(x)\right),$$
$$\det(S_1, \ldots, S_{n-1}, S_{n+1}) = \rho^n e^{\rho(x_n + 2\sum_{l=1}^{p-1} x_{2l})} \left(-\sin(\rho x_1) + f_2(x)\right),$$

where $\lim_{\rho\to\infty} \sup_{\overline{X}} |f_1| = \lim_{\rho\to\infty} \sup_{\overline{X}} |f_2| = 0$. Letting ρ so large that $\sup_{\overline{X}}(|f_1|, |f_2|) \leq \frac{1}{4}$ and denoting $\gamma_1 := \min_{x\in\overline{X}}(\rho^n e^{\rho(x_n+2\sum_{l=1}^{p-1} x_{2l})}) > 0$, we have that $|\det(S_1, \ldots, S_{n-1}, S_n)| \geq \frac{\gamma_1}{4}$ on sets of the form $X \cap \{\rho x_1 \in]\frac{-\pi}{3}, \frac{\pi}{3}[+m\pi\}$ and $|\det(S_1, \ldots, S_{n-1}, S_{n+1})| \geq \frac{\gamma_1}{4}$ on sets of the form $X \cap \{\rho x_1 \in]\frac{\pi}{6}, \frac{5\pi}{6}[+m\pi\}$, where m is a signed integer. Since the previous sets are open and a finite number of them covers X (because X is bounded and ρ is finite), we therefore have fulfilled the desired requirements of the construction. Upon changing the sign of S_n or S_{n+1} on each of these sets if necessary, we can assume that the determinants are all positive. □

Note that as in the preceding paragraph, we have obtained the existence of an open set of illuminations $\mathfrak{f}$ such that appropriate determinants remain strictly positive throughout the domain X. However, these illuminations $\mathfrak{f}$ are not characterized explicitly.

10.5 Remarks on hybrid inverse problems

In the past two chapters, we have briefly presented two hybrid inverse problems based on the photo-acoustic effect (with a similar theory for the imaging modality Transient Elastography) and the ultrasound modulation effect. What characterizes these hybrid inverse problems is that after a preliminary step (involving and inverse wave problem in photoacoustics and an inverse Fourier transform in ultrasound modulation) we obtain an inverse problem with *internal functionals* of the unknown parameters.

These internal functionals have an immediate advantage: singularities of the unknown coefficients no longer need to be propagated to the boundary of the domain by an elliptic operator that severely damps high frequencies. The main reason for ill-posedness of invertible operators, namely the smoothing property of such operators, is therefore no longer an issue. However, injectivity of the measurement operator is not guaranteed. We have seen in QPAT that only two out of three coefficients could be reconstructed in QPAT. QPAT data acquired at one frequency are thus not sufficient to reconstruct three coefficients. With appropriate prior information about the dependency of coefficients with respect to a frequency parameter, then injectivity of the measurement operator can be restored [18]. But again, this requires prior information that one may not wish to make. The alternative is then to come up with *other* prior models that restore injectivity or to combine QPAT measurements with additional measurements. Hybrid inverse problems face the same shortcomings as any other inverse problem.

Once injectivity is guaranteed, then stability of the reconstructions are guaranteed in principle by the fact that singularities no longer need to propagate. We have seen a few such stability results. Note that these results typically require a certain degree of smoothness of the unknown coefficients. This is a shortcoming of the theories presented above. The reason why we have recourse to hybrid inverse problems is to obtain high resolution. The reason we typically need high resolution is because coefficients may

vary rapidly and we wish to quantify such variations. It would therefore be useful to understand how stability estimates degrade when the coefficients are not smooth.

That said, numerical experiments conducted in e.g. [18, 17, 19] show that reconstructions based on algorithms similar to those presented above do allow us to obtain very accurate reconstructions even for highly discontinuous coefficients, and this even in the presence of relatively significant noise.

For additional general references to hybrid inverse problems, we refer the reader to [4, 52].

Chapter 11

Priors and Regularization

As we have mentioned several times in these notes, the influence of "*noise*" is largely subjective: we either find it *acceptable* or *unacceptable*. Mathematically, this influence is typically acceptable in a given norm while it may be unacceptable in another (more constraining) norm. In which norms that influence is controlled for a given problem is characterized by stability estimates, which we have presented for the problems considered in these notes.

Once we have decided that "*noise*" had too large an effect in a setting of interest, something must be done. That something inescapably requires that we add *prior information*. Several techniques have been developed to do so. The simplest and most developed is the *regularization* methodology. Typically, such a regularization assumes that the object we are interested in reconstructing has a prior smoothness. We may for instance assume that the object belongs to $H^s(X)$ for some $s > 0$. This assumption indicates that the Fourier transform object of interest decreases rapidly as frequency increases. High frequencies, which are not present, thus do not need to be reconstructed with high accuracy. This allows us to mitigate the effect of high frequency noise in the reconstructions.

The main drawback of regularization theory is that objects of interest may not necessarily be smooth. Smooth means that the first coefficients in a Fourier series expansion are big while the other coefficients are small. In several settings of interest, the objects may be represented by a few big coefficients and a large number of small coefficients, but not in the basis of Fourier coefficients. In other words, the object may be *sparse* in a different, known basis. The objective of sparse regularization is to devise methods to find these coefficients.

In some settings, the problems are so ill-posed that looking even for the first coefficients in a given basis may not provide sufficient accuracy. Other sorts of prior information may be necessary, for instance assuming that the objects of interest are small inclusions with specific structures, or that next to a blue pixel in an image, red pixels are extremely unlikely. In such situations, reconstructions are typically not very accurate and it is often important to characterize this inaccuracy. A very versatile setting to do so is the statistical Bayesian framework. In such a setting, objects of interest are modeled by a set of possible outcomes with prior probabilities of happening. This is the *prior probability distribution*. Then data are acquired with a given noise model. The probability of such data happening conditioned on given parameters is called the *like-*

lihood probability distribution. Using the Bayes rule, a *posterior probability distribution* gives the probability density of the parameters based on availability of the data.

We now briefly consider these three settings, the smoothness regularization methodology, the sparsity regularization methodology, and the Bayesian framework and show their relations.

11.1 Smoothness Regularization

We have seen that many of the inverse problems we have considered were either mildly ill-posed (with $\alpha > 0$ in the sense of Chapter 1) or severely ill-posed (as for instance the Calderón problem or the Cauchy problems for elliptic equations). We present here some techniques to regularize the inversion. Such techniques typically work for mildly ill-posed problems but are often not sufficient for severely ill-posed problems. But this the influence of "*noise*" is subjective anyway, these regularization techniques should be applied first as they are the simplest both theoretically and computationally. We refer the reader to [29, 39, 61] for additional information on these regularization techniques.

11.1.1 Ill-posed problems and compact operators

Let A be an injective and compact operator defined on an infinite dimensional (separable) Hilbert space H with range $\mathrm{Range}(A)$ in H:

$$A : H \to \mathrm{Range}(A) \subset H. \tag{11.1}$$

We recall that compact operators map the unit ball in H to a subset of H whose closure (with respect to the usual norm in H) is compact, i.e., verifies that every bounded (with respect to the usual norm in H) family of points admits a converging (with respect to the usual norm in H) subsequence in the compact set.

Since A is injective (i.e., $A\mathrm{x} = 0$ implies $\mathrm{x} = 0$), we can define the inverse operator A^{-1} with domain of definition $\mathrm{Range}(A)$ and Range H:

$$A^{-1} : D(A^{-1}) = \mathrm{Range}(A) \to H. \tag{11.2}$$

The problem is that A^{-1} is *never* a continuous operator from $\mathrm{Range}(A)$ to H when both spaces are equipped with the usual norm in H:

Lemma 11.1.1 *For A as above, there exists a sequence x_n such that*

$$\|\mathrm{x}_n\|_H = 1, \qquad \|A\mathrm{x}_n\|_H \to 0. \tag{11.3}$$

The same holds true with $\|\mathrm{x}_n\|_H \to \infty$.

Proof. The proof holds in more complicated settings than Hilbert spaces. The Hilbert structure gives us a very simple proof and is based on the existence of an orthonormal basis in H, i.e., vectors x_n such that $\|\mathrm{x}_n\|_H = 1$ and $(\mathrm{x}_n, \mathrm{x}_m)_H = 0$ for $n \neq m$. Since these vectors belong to the unit ball, we deduce that $\mathrm{y}_n = A\mathrm{x}_n$ is a converging sequence (up to taking subsequences), say to $\mathrm{y} \in H$. Take now $\tilde{\mathrm{x}}_n = 2^{-1/2}(\mathrm{x}_n - \mathrm{x}_{n+1})$. We verify that $\tilde{\mathrm{x}}_n$ satisfies (11.3). Now define $\mathrm{z}_n = \tilde{\mathrm{x}}_n / \|A\tilde{\mathrm{x}}_n\|_H^{\frac{1}{2}}$ when the latter denominator does

not vanish and $z_n = n\tilde{x}_n$ otherwise. Then Az_n still converges to 0 while $\|z_n\|_H$ converges to ∞. □

This simple lemma shows that inverting a compact operator can never be a well-posed problem in the sense that A^{-1} is not continuous from $D(A^{-1})$ to H with the H norm. Indeed take the sequence $y_n = Ax_n/\|Ax_n\|$ in $D(A^{-1})$, where x_n is the sequence in (11.3). Then $\|y_n\|_H = 1$ while $\|A^{-1}y_n\|_H$ tends to ∞.

The implication for inverse problems is the following. If δy_n is the measurement noise for n large, then $\delta A^{-1}y_n$ models the error in the reconstruction, which may thus be arbitrarily larger than the norm of the true object we aim to reconstruct. More precisely, if $Ax = b$ is the real problem and $A\tilde{x} = \tilde{b}$ is the exact reconstruction from noisy data, then arbitrarily small errors $\|b - \tilde{b}\|$ in the measurements is still compatible with arbitrary large errors $\|x - \tilde{x}\|$ in the space of parameters. This shows that the problem needs to be regularized before any inversion is carried out.

11.1.2 Regularity assumptions and error bound

The calculations we have carried out in the preceding section show that an ill-posed inverse problem cannot satisfactorily be solved if no other *assumptions* on the problem are added. A sometimes reasonable and practically useful assumption is to impose, before we start the reconstruction process, that the object we want to reconstruct is *sufficiently smooth*. This allows us to filter out high frequencies that may appear in the reconstruction because we know they are part of the noise and not of the object we want to reconstruct. We present two mathematical frameworks for such a regularization.

In the first framework, we introduce the adjoint operator A^* to A, defined from H to Range(A^*) by the relation

$$(Ax, y)_H = (x, A^*y)_H, \qquad \text{for all } x, y \in H.$$

Since A is compact and injective, then so is A^*. We can also define the inverse operator $A^{-*} = (A^*)^{-1}$ from Range(A^*) to H.

We may now assume that x, the object we aim at reconstructing, is sufficiently smooth that is belongs to the range of A^*, i.e., there exists y such that $x = A^*y$. Since A and A^* are compact operators, hence smoothing operators, the above hypothesis means that we assume *a priori* that x is smoother than being merely an element in H. We then define the *stronger* norm

$$\|x\|_1 = \|A^{-*}x\|_H. \tag{11.4}$$

We may also assume that the object x is even smoother than being in the range of A^*. For instance let us assume that x belongs to the range of A^*A, i.e., there exists y such that $x = A^*Ay$. Note that since both A and A^* are smoothing operators (because they are compact), the assumption on x is stronger than simply being in the range of A^*. We define the even stronger norm

$$\|x\|_2 = \|(A^*A)^{-1}x\|_H. \tag{11.5}$$

We want to use these definitions to show that if the solution x is *a priori* bounded for the $\|\cdot\|_1$ or the $\|\cdot\|_2$ norm and "*noise*" is small, then the error in the reconstruction

is small. For instance, assume that $\mathrm{y}_j = A\mathrm{x}_j$ for $j = 1, 2$ so that $\mathrm{y} = A\mathrm{x}$ for $\mathrm{y} = \mathrm{y}_1 - \mathrm{y}_2$ and $\mathrm{x} = \mathrm{x}_1 - \mathrm{x}_2$. If both x_j, $j = 1, 2$ are bounded and y is small, then how small is x? For such questions, we have the following result:

Theorem 11.1.2 *Let $\mathrm{x} \in H$ such that $\|\mathrm{x}\|_1 \leq E$ and $\|A\mathrm{x}\|_H \leq \delta$. Then we have:*

$$\|\mathrm{x}\|_H \leq \sqrt{E\delta}. \tag{11.6}$$

If we now assume that $\|\mathrm{x}\|_2 \leq E$ instead, we obtain the better bound

$$\|\mathrm{x}\|_H \leq E^{\frac{1}{3}}\delta^{\frac{2}{3}}. \tag{11.7}$$

Proof. Let $\mathrm{y} = A^{-*}\mathrm{x}$ so that $\|\mathrm{y}\|_H \leq E$. We have then

$$\|\mathrm{x}\|_H^2 = (\mathrm{x}, A^*\mathrm{y}) = (A\mathrm{x}, \mathrm{y}) \leq \|A\mathrm{x}\|_H\|\mathrm{y}\|_H \leq \delta E.$$

This proves (11.6). For the second bound let $\mathrm{z} = (A^*A)^{-1}\mathrm{x}$ so that $\|\mathrm{z}\|_H \leq E$ and compute:

$$\|\mathrm{x}\|_H^2 = (\mathrm{x}, A^*A\mathrm{z}) = (A\mathrm{x}, A\mathrm{z}) \leq \delta\|A\mathrm{z}\| = \delta(A\mathrm{z}, A\mathrm{z})^{\frac{1}{2}} \leq \delta(\mathrm{z}, \mathrm{x})^{\frac{1}{2}} \leq \delta E^{\frac{1}{2}}\|\mathrm{x}\|_H^{\frac{1}{2}}.$$

This proves the second bound (11.7). □

The theorem should be interpreted as follows. Consider that $A\mathrm{x}$ is the noise level in the measured data and that $\|\mathrm{x}\|_1 < E$ or $\|\mathrm{x}\|_2 < E$ is *a priori* smoothness information we have on the object we want to reconstruct. Then the worst error we can make on the reconstruction (provided we find an appropriate inversion method; see below) is given by the bounds (11.6) and (11.7). Note that the latter bound is better (since $\delta^{\frac{2}{3}} \ll \delta^{\frac{1}{2}}$). This results from a more stringent assumption on the image x.

Let us now consider smoothing operators in the (second) framework of the Hilbert scale $H^s(\mathbb{R})$ we have introduced in Chapter 1. Then we have the following result:

Theorem 11.1.3 *Let us assume that the operator A is mildly ill-posed of order $\alpha > 0$ so that*

$$\|Af\|_{L^2(\mathbb{R})} \geq m\|f\|_{H^{-\alpha}(\mathbb{R})}. \tag{11.8}$$

Suppose now that the measurement error is small and that the function we want to reconstruct is regular in the sense that

$$\|Af\|_{L^2(\mathbb{R})} \leq \delta m, \qquad \textit{and} \qquad \|f\|_{H^\beta(\mathbb{R})} \leq E, \tag{11.9}$$

for some $\delta > 0$, $\beta > 0$, and $E > 0$. Then we have

$$\|f\|_{L^2(\mathbb{R})} \leq \delta^{\frac{\beta}{\alpha+\beta}} E^{\frac{\alpha}{\alpha+\beta}}. \tag{11.10}$$

Proof. The proof is a simple but interesting exercise in interpolation theory. Notice that the hypotheses are

$$\|f\|_{H^\beta(\mathbb{R})} \leq E, \qquad \text{and} \qquad \|f\|_{H^{-\alpha}(\mathbb{R})} \leq \delta,$$

and our objective is to find a bound for $\|f\|_{L^2(\mathbb{R})}$. Let us denote $\langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}}$. We have

$$\begin{aligned}(2\pi)^n\|f\|^2_{L^2(\mathbb{R})} &= \int_{\mathbb{R}^n} |\hat{f}(\xi)|^{2\theta}\langle\xi\rangle^{2\gamma}|\hat{f}(\xi)|^{2(1-\theta)}\langle\xi\rangle^{-2\gamma}d\xi \\ &\leq \Big(\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2\langle\xi\rangle^{2\gamma/\theta}d\xi\Big)^{\theta}\Big(\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2\langle\xi\rangle^{-2\gamma/(1-\theta)}d\xi\Big)^{1-\theta},\end{aligned}$$

thanks to Hölder's inequality

$$\Big|\int_{\mathbb{R}} f(x)g(x)dx\Big| \leq \|f\|_{L^p(\mathbb{R})}\|g\|_{L^q(\mathbb{R})},$$

which holds for all $p \geq 1$ and $q \geq 1$ such that $p^{-1}+q^{-1}=1$, where we have defined for all $p \geq 1$,

$$\|f\|_{L^p(\mathbb{R})} = \Big(\int_{\mathbb{R}} |f(x)|^p dx\Big)^{1/p}. \tag{11.11}$$

Choosing $\theta = \frac{\alpha}{\alpha+\beta}$ and $\gamma = \frac{\alpha\beta}{\alpha+\beta}$ gives (11.10).

Let us briefly recall the proof of the Hölder's inequality [60]. We first verify that

$$x^{\frac{1}{p}} \leq \frac{x}{p} + \frac{1}{q}, \qquad x > 0,$$

for $p^{-1}+q^{-1}=1$ and $p \geq 1$, since $x^{\frac{1}{p}} - \frac{x}{p}$ attains its maximum at $x=1$ where it is equal to q^{-1}. For $y > 0$ we use the above inequality for x/y and multiply by y to obtain

$$x^{\frac{1}{p}}y^{\frac{1}{q}} \leq \frac{x}{p} + \frac{y}{q}, \qquad x > 0,\ y > 0. \tag{11.12}$$

Choosing $x = |tf(x)|^p$ and $y = |t^{-1}g(x)|^q$, we deduce that

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leq \frac{1}{p}\|tf\|^p_{L^p(\mathbb{R})} + \frac{1}{q}\|t^{-1}g\|^q_{L^q(\mathbb{R})} = \frac{t^p}{p}\|f\|^p_{L^p(\mathbb{R})} + \frac{t^q}{q}\|g\|^q_{L^q(\mathbb{R})},$$

for all $t > 0$. Maximizing over t gives the Hölder inequality. □

The last theorem applies to a less general class of operators than compact operators (although it applies to operators that are not necessarily compact) but it gives us an accurate result. We should still consider δ as the noise level and E as an a priori bound we have on the object we want to reconstruct. Then depending on the a priori smoothness of the object, we obtain different possible accuracies in the reconstructions. What is important is the relative regularity of the object compared to the smoothing effect of the operator A. When $\beta = \alpha$, this corresponds to assuming the same regularity as $\|x\|_1 \leq E$ in Theorem 11.1.2. We thus obtain an accuracy of order $\delta^{\frac{1}{2}}$ in the reconstruction. When $\beta = 2\alpha$, this corresponds to $\|x\|_2 \leq E$ since $f = (A^*A)^{-1}g$ for some $g \in L^2(\mathbb{R})$ means that f is twice as regular as A is smoothing. We thus recover the accuracy of order $\delta^{\frac{2}{3}}$ as in Theorem 11.1.2. Theorem 11.1.3 allows us to deal with arbitrary values of β. Notice that as $\beta \to \infty$, we recover that the problem is almost well-posed since the error in the reconstruction is asymptotically of the same order δ as the noise level.

11.1.3 Regularization methods

Now that we know how noise can optimally be controlled in the reconstruction based on the regularity of the object we want to reconstruct, we need to devise algorithms that indeed control noise amplification in the reconstruction.

Since A^{-1} is an unbounded operator with domain of definition Range(A), a proper subset of H, we first need to introduce approximations of the inverse operator. We denote by R_γ defined from H to H for $\gamma > 0$ a sequence of regularizations of A^{-1} such that

$$\lim_{\gamma \to 0} R_\gamma A \mathrm{x} = \mathrm{x} \qquad \text{for all } \mathrm{x} \in H. \tag{11.13}$$

Under the hypotheses of Lemma 11.1.1, we can show that the sequence of operators R_γ is not uniformly bounded. A uniform bound would indeed imply that A^{-1} is bounded. Thus, $R_\gamma A$ converges to identity *strongly* (since (11.13) is the definition of strong convergence of operators) but not uniformly in the sense that $\|R_\gamma A - I\|$ does not converge to 0.

Exercise 11.1.1 Prove this.

One of the main objectives of the regularization technique is to handle noise in an optimal fashion. Let us denote by y^δ our measurements and assume that $\|\mathrm{y}^\delta - A\mathrm{x}\|_H \leq \delta$. We then define

$$\mathrm{x}^{\gamma,\delta} = R_\gamma \mathrm{y}^\delta. \tag{11.14}$$

We want to find sequences R_γ that deal with noise in an optimal fashion. For instance assuming that $\|\mathrm{x}\|_1 \leq E$ and that $\|\mathrm{y}^\delta - A\mathrm{x}\|_H \leq \delta$, we want to be able to show that

$$\|\mathrm{x} - \mathrm{x}^{\gamma,\delta}\| \leq C\sqrt{E\delta},$$

at least for some values of γ. We know from Theorem 11.1.3 that such a bound is optimal. We will consider three regularization techniques: singular value decomposition, Tikhonov regularization, and Landweber iterations.

The choice of a parameter γ is then obviously of crucial importance as the above bound will not hold independently of γ. More precisely, the reconstruction error can be decomposed as

$$\|\mathrm{x}^{\gamma,\delta} - \mathrm{x}\|_H \leq \delta \|R_\gamma\|_H + \|R_\gamma A\mathrm{x} - \mathrm{x}\|_H. \tag{11.15}$$

Exercise 11.1.2 Prove this. The operator norm $\|R_\gamma\|_H$ is defined as the supremum of $\|R_\gamma \mathrm{x}\|_H$ under the constraint $\|\mathrm{x}\|_H \leq 1$.

We thus observe that two competing effects enter (11.15). The first effect comes from the ill-posedness: as $\gamma \to 0$, the norm $\|R_\gamma\|_H$ tends to ∞ so γ should not be chosen too small. The second effect comes from the regularization: as γ increases, $R_\gamma A$ becomes a less accurate approximation of identity so γ should not be chosen too large. Only intermediate values of γ will provide an optimal reconstruction.

Singular Value Decomposition

For a compact and injective operator A defined on an infinite dimensional Hilbert space H, let us assume that we know it singular value decomposition defined as follows. Let A^* be the adjoint operator to A and $\lambda_j > 0$, $j \in \mathbb{N}$ the eigenvalues of the symmetric operator A^*A. Then, the sequence $\mu_j = \sqrt{\lambda_j}$ for $j \in \mathbb{N}$ are called the *singular values* of A. Since $\mu_j \leq \|A\|_H$, we order the singular values such that

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq \cdots > 0.$$

Multiple eigenvalues are repeated as many times as their multiplicity (which is necessarily finite since the associated eigenspace for A^*A needs to be compact).

Then there exist two orthonormal systems $(\mathrm{x}_j)_{j\in\mathbb{N}}$ and $(\mathrm{y}_j)_{j\in\mathbb{N}}$ in H such that

$$A\mathrm{x}_j = \mu_j \mathrm{y}_j \qquad \text{and} \qquad A^*\mathrm{y}_j = \mu_j \mathrm{x}_j, \qquad \text{for all} \quad j \in J. \tag{11.16}$$

We call $(\mu_j, \mathrm{x}_j, \mathrm{y}_j)$ the singular system for A. Notice that

$$A\mathrm{x} = \sum_{j=1}^{\infty} \mu_j (\mathrm{x}, \mathrm{x}_j)\mathrm{y}_j, \qquad A^*\mathrm{y} = \sum_{j=1}^{\infty} \mu_j (\mathrm{y}, \mathrm{y}_j)\mathrm{x}_j.$$

Here $(\mathrm{x}, \mathrm{x}_j)$ is the inner product in H, $(\mathrm{x}, \mathrm{x}_j)_H$. We then have the very useful characterization of the Range of the compact and injective operator A:

Lemma 11.1.4 (Picard) *The equation $A\mathrm{x} = \mathrm{y}$ is solvable in H if and only if*

$$\sum_{j\in\mathbb{N}} \frac{1}{\mu_j^2} |(\mathrm{y}, \mathrm{y}_j)|^2 < \infty, \tag{11.17}$$

in which case the solution is given by

$$\mathrm{x} = A^{-1}\mathrm{y} = \sum_{j\in\mathbb{N}} \frac{1}{\mu_j} (\mathrm{y}, \mathrm{y}_j)\mathrm{x}_j. \tag{11.18}$$

The ill-posedness of the inverse problem appears very clearly in the singular value decomposition. As $j \to \infty$, the singular values μ_j tend to 0. And they do so all the faster that the inverse problem is ill-posed. We can extend the definition of ill-posed problems in the sense that a compact operator generates a mildly ill-posed inverse problem of order $\alpha > 0$ when the singular values decay like $j^{-\alpha}$ and generates a severely ill-posed problem when the singular values decay faster than any j^{-m} for $m \in \mathbb{N}$.

So in order to regularize the problem, all we have to do is to replace too small singular values by larger values. Let us define $q(\gamma, \mu)$ for $\gamma > 0$ and $\mu \in [0, \|A\|]$ such that

$$|q(\gamma, \mu)| < 1, \quad |q(\gamma, \mu)| \leq c(\gamma)\mu, \quad \text{and} \quad q(\gamma, \mu) - 1 \to 0 \quad \text{as} \quad \gamma \to 0, \tag{11.19}$$

(not uniformly in μ obviously). Then we define the regularizing sequence

$$R_\gamma \mathrm{y} = \sum_{j\in\mathbb{N}} \frac{q(\gamma, \mu_j)}{\mu_j} (\mathrm{y}, \mathrm{y}_j)\mathrm{x}_j. \tag{11.20}$$

Compare to (11.18). As $\gamma \to 0$, R_γ converges to A^{-1} pointwise. We are interested in estimating (11.15) and showing that the error is optimal based on the assumed regularity of x. The total error is estimated by using

$$\|R_\gamma\|_H \leq c(\gamma), \qquad \|R_\gamma A\mathrm{x} - \mathrm{x}\|_H = \sum_{j=1}^{\infty} \big(q(\gamma, \mu_j) - 1\big)^2 |(\mathrm{x}, \mathrm{x}_j)|^2. \tag{11.21}$$

Exercise 11.1.3 Prove these relations.

We can now prove the following results:

Theorem 11.1.5 *(i) Let us assume that* $\mathrm{x} = A^*\mathrm{z}$ *with* $\|\mathrm{z}\|_H \leq E$ *and that* $\|\mathrm{y}^\delta - A\mathrm{x}\| \leq \delta$*, where* y^δ *is the measurements. Choose* $q(\gamma, \mu)$ *and* γ *such that*

$$|q(\gamma, \mu) - 1| \leq C_1 \frac{\sqrt{\gamma}}{\mu}, \qquad c(\gamma) \leq \frac{C_2}{\sqrt{\gamma}}, \qquad \gamma = \frac{C_3 \delta}{E}. \tag{11.22}$$

Then we have that

$$\|\mathrm{x}^{\gamma,\delta} - \mathrm{x}\|_H \leq \Big(\frac{C_2}{\sqrt{C_3}} + C_1\sqrt{C_3}\Big)\sqrt{\delta E}. \tag{11.23}$$

(ii) Let us assume that $\mathrm{x} = A^*A\mathrm{z}$ *with* $\|\mathrm{z}\|_H \leq E$ *and that* $\|\mathrm{y}^\delta - A\mathrm{x}\| \leq \delta$*, where* y^δ *is the measurements. Choose* $q(\gamma, \mu)$ *and* γ *such that*

$$|q(\gamma, \mu) - 1| \leq C_4 \frac{\gamma}{\mu^2}, \qquad c(\gamma) \leq \frac{C_5}{\sqrt{\gamma}}, \qquad \gamma = C_6\Big(\frac{\delta}{E}\Big)^{\frac{2}{3}}. \tag{11.24}$$

Then we have that

$$\|\mathrm{x}^{\gamma,\delta} - \mathrm{x}\|_H \leq \Big(\frac{C_5}{\sqrt{C_6}} + C_4 C_6\Big)\delta^{\frac{2}{3}} E^{\frac{1}{3}}. \tag{11.25}$$

Proof. Since $\mathrm{x} = A^*\mathrm{z}$, we verify that $(\mathrm{x}, \mathrm{x}_j) = \mu_j(\mathrm{y}, \mathrm{y}_j)$ so that

$$\|R_\gamma A\mathrm{x} - \mathrm{x}\|_H^2 = \sum_{j=1}^{\infty} \big(q(\gamma, \mu_j) - 1\big)^2 |(\mathrm{z}, \mathrm{y}_j)|^2 \leq C_1^2 \gamma \|\mathrm{z}\|_H^2.$$

This implies that

$$\delta\|R_\gamma\|_H + \|R_\gamma A\mathrm{x} - \mathrm{x}\|_H \leq \frac{C_2\delta}{\sqrt{\gamma}} + C_1\sqrt{\gamma}E.$$

Using (11.15) and the expression for γ yields (11.23).

Exercise 11.1.4 Using similar arguments, prove (11.25).

This concludes the proof. □

We have thus defined an optimal regularization scheme for the inversion of $A\mathrm{x} = \mathrm{y}$. Indeed from the theory in Theorem 11.1.2 we know that up to some multiplicative constants, the above estimates are optimal.

It remains to find filters $q(\gamma, \mu)$ satisfying the above hypotheses. We propose two:

$$q(\gamma, \mu) = \frac{\mu^2}{\gamma + \mu^2}, \tag{11.26}$$

$$q(\gamma, \mu) = \begin{cases} 1, & \mu^2 \geq \gamma, \\ 0, & \mu^2 < \gamma. \end{cases} \tag{11.27}$$

Exercise 11.1.5 Show that the above choices verify the hypotheses of Theorem 11.1.5.

Tikhonov Regularization

One of the main drawbacks of the theory presented in the preceding section is that in most cases, the singular value decomposition of the operator is not analytically available (although it is for the Radon transform; see [45, 46]), and is quite expensive to compute numerically once the continuous problem has been discretized. It is therefore useful to consider regularization techniques that do not depend on the SVD. One of the most popular regularization techniques is the Tikhonov-Phillips regularization technique.

Solving $A\mathrm{x} = \mathrm{y}$ corresponds to minimizing $\|A\mathrm{x} - \mathrm{y}\|_H$. Instead one may want to minimize the regularized Tikhonov functional

$$J_\gamma(x) = \|A\mathrm{x} - \mathrm{y}\|_H^2 + \gamma \|\mathrm{x}\|_H^2, \qquad \mathrm{x} \in H. \tag{11.28}$$

For $\gamma > 0$ and A a linear bounded operator on H, we can show that the above functional admits a unique minimizer x^γ solving the following normal equations

$$A^* A\mathrm{x}^\gamma + \gamma \mathrm{x}^\gamma = A^* \mathrm{y}. \tag{11.29}$$

Exercise 11.1.6 *Prove the above statement.*

We can thus define the regularizing sequence

$$R_\gamma = (\gamma + A^* A)^{-1} A^*. \tag{11.30}$$

The operator is bounded in H by $\|R_\gamma\|_H \leq C\gamma^{-1/2}$ for all $\gamma > 0$. Notice that for a compact operator A with singular system (μ_i, x_i, y_i), we verify that the singular value decomposition of R_γ is

$$R_\gamma \mathrm{y} = \sum_{j=1}^{\infty} \frac{\mu_j}{\gamma + \mu_j^2} (\mathrm{y}, \mathrm{y}_j) \mathrm{x}_j. \tag{11.31}$$

This means that the Tikhonov regularization corresponds to the SVD regularization with filter given by (11.26) and implies that the Tikhonov regularization is optimal to inverse problem with a priori regularity $\|\mathrm{x}\|_1 \leq E$ or $\|\mathrm{x}\|_2 \leq E$. It is interesting to observe that the Tikhonov regularization is no longer optimal when the a priori regularity of x is better than $\|\mathrm{x}\|_2 \leq E$ (see [39]).

Let us make this observation more explicit. Let us consider the operator A given in the Fourier domain by multiplication by $\langle\xi\rangle^{-\alpha}$ for some $\alpha > 0$. We verify that $A^* = A$ and that R_γ is given in the Fourier domain by

$$R_\gamma = \mathcal{F}^{-1}_{\xi\to x}\frac{\langle\xi\rangle^{-\alpha}}{\langle\xi\rangle^{-2\alpha}+\gamma}\mathcal{F}_{x\to\xi}, \qquad \text{so that} \qquad \|R_\gamma\| \leq \frac{1}{2\sqrt{\gamma}}.$$

Indeed, we check that $x/(x^2+\gamma) \leq 1/(2\sqrt{\gamma})$ and attains its maximum at $x = \sqrt{\gamma}$. We now verify that

$$I - R_\gamma A = \mathcal{F}^{-1}_{\xi\to x}\frac{\gamma}{\langle\xi\rangle^{-2\alpha}+\gamma}\mathcal{F}_{x\to\xi},$$

so that for a function $f \in H^\beta(\mathbb{R}^n)$, we have

$$\|f - R_\gamma A f\| \leq \sup_{\langle\xi\rangle\geq 1}\frac{\gamma\langle\xi\rangle^{-\beta}}{\langle\xi\rangle^{-2\alpha}+\gamma}\|f\|_{H^\beta(\mathbb{R}^n)}.$$

Moreover the inequality is sharp in the sense that there exists functions f such that the reverse inequality holds (up to a multiplicative constant independent of γ; Check this). For $\beta > 2\alpha$, the best estimate we can have for the above multiplier is that it is of order $O(\gamma)$ (choose for instance $\langle\xi\rangle = 1$).

Exercise 11.1.7 Using (11.12) show that

$$\frac{\gamma\langle\xi\rangle^{-2\alpha\theta}}{\langle\xi\rangle^{-2\alpha}+\gamma} \leq \gamma^\theta, \qquad 0 \leq \theta \leq 1.$$

Show that the above inequality is sharp.

Let $Af = g$ be the problem we want to solve and g^δ the measurements so that $\|Af - g^\delta\|_{L^2(\mathbb{R}^n)} \leq \delta$. Let us assume that f belongs to $H^\beta(\mathbb{R}^n)$. We verify using (11.15) that the error of the regularized problem is given by

$$\|f - R_\gamma g^\delta\| \leq \frac{\delta}{2\sqrt{\gamma}} + \gamma^{\frac{\beta}{2\alpha}\wedge 1}\|f\|_{H^\beta(\mathbb{R}^n)}. \tag{11.32}$$

Here, $a \wedge b = \min(a, b)$. This implies that

$$\|f - R_\gamma g^\delta\| \leq C\delta^{\frac{\beta}{\alpha+\beta}\wedge\frac{2}{3}}\|f\|^{\frac{\alpha}{\alpha+\beta}\wedge\frac{1}{3}}_{H^\beta(\mathbb{R}^n)}, \tag{11.33}$$

for a universal constant C. We therefore obtain that the Tikhonov regularization is optimal according to Theorem 11.1.3 when $0 < \beta \leq 2\alpha$. However, for all $\beta > 2\alpha$, the error between the Tikhonov regularization and the exact solution will be of order $\delta^{\frac{2}{3}}$ instead of $\delta^{\frac{\beta}{\beta+\alpha}}$.

Exercise 11.1.8 More generally, consider an operator A with symbol $a(\xi)$, i.e.,

$$A = \mathcal{F}^{-1}_{\xi\to x}a(\xi)\mathcal{F}_{x\to\xi},$$

such that $0 < a(\xi) \in C^\infty(\mathbb{R}^n)$ and for some $\alpha > 0$ and $a_\infty \neq 0$,

$$\frac{a(\xi)}{\langle\xi\rangle^\alpha} \to a_\infty, \qquad \text{as} \qquad |\xi| \to \infty. \tag{11.34}$$

(i) Show that A^*, the adjoint of A for the $L^2(\mathbb{R}^n)$ inner product, satisfies the same hypothesis (11.34).
(ii) Show that R_γ and $S_\gamma = R_\gamma A - I$ are bounded operator with symbols given by

$$r_\gamma(\xi) = (|a(\xi)|^2 + \gamma)^{-1}\bar{a}(\xi), \qquad s_\gamma(\xi) = \gamma(|a(\xi)|^2 + \gamma)^{-1},$$

respectively.
(iii) Assuming that $f \in H^\beta(\mathbb{R})$, show that (11.33) holds.

These results show that for the Radon transform, an a priori regularity of the function $f(x)$ in $H^1(\mathbb{R}^2)$ is sufficient to obtain an error of order $\delta^{\frac{2}{3}}$. When the function is smoother, a different technique from Tikhonov regularization is necessary to get a more accurate reconstruction.

Landweber iterations

The drawback of the Tikhonov regularization is that it requires to invert the regularization of the normal operator $\gamma + A^*A$. This inversion may be computationally expensive in practice. The Landweber iteration method is an iterative technique in which no inversion is necessary. It is defined to solve the equation $A\mathrm{x} = \mathrm{y}$ as follows

$$\mathrm{x}_0 = 0, \qquad \mathrm{x}_{n+1} = (I - rA^*A)\mathrm{x}_n + rA^*\mathrm{y}, \qquad n \geq 0, \tag{11.35}$$

for some $r > 0$. By induction, we verify that $\mathrm{x}_n = R_n\mathrm{y}$, where

$$R_n = r\sum_{k=0}^{n-1}(I - rA^*A)^kA^*, \qquad n \geq 1. \tag{11.36}$$

Consider a compact operator A with singular system $(\mu_j, \mathrm{x}_j, \mathrm{y}_j)$. We thus verify that

$$R_n\mathrm{y} = \sum_{j=1}^{\infty}\frac{1}{\mu_j}\big(1 - (1 - r\mu_j^2)^n\big)(\mathrm{y}, \mathrm{y}_j)\mathrm{x}_j. \tag{11.37}$$

Exercise 11.1.9 Check (11.37).

This implies that R_n is of the form R_γ in (11.20) with $\gamma = n^{-1}$ and

$$q(\gamma, \mu) = 1 - (1 - r\mu^2)^{1/\gamma}.$$

Exercise 11.1.10 *Show that the above filter verifies the hypotheses (11.19) and those of Theorem 11.1.5.*

This implies that the Landweber iteration method is an optimal inversion method by Theorem 11.1.5.

Exercise 11.1.11 Show that the hypotheses of Theorem 11.1.5 are met provided that the number of iterations n is chosen as

$$n = c\frac{E}{\delta}, \qquad n = c\Big(\frac{E}{\delta}\Big)^{\frac{2}{3}},$$

when $\|\mathrm{x}\|_1 \leq E$ and $\|\mathrm{x}\|_2 \leq E$, respectively.

The above result shows that the number of iterations should be chosen carefully: when n is too small, then $R_\gamma A$ is a poor approximation of I, and when n is too large, then $\|R_\gamma\|_H$ is too large. Unlike the Tikhonov regularization, we can show that the Landweber iteration method is also optimal for stronger regularity assumptions on x than those given in Theorem 11.1.5 (see [39] for instance).

Let us come back to the operator A with symbol $a(\xi) = \langle\xi\rangle^{-\alpha}$. We verify that R_n and $S_n = R_n A - I$ have respective symbols

$$r_n(\xi) = \frac{1-(1-r\langle\xi\rangle^{-2\alpha})^n}{\langle\xi\rangle^{-\alpha}}, \qquad s_n(\xi) = -(1-r\langle\xi\rangle^{-2\alpha})^n.$$

Exercise 11.1.12 (i) Show that $s_n(\xi)\langle\xi\rangle^{-\beta}$ is bounded by $Cn^{-\beta/(2\alpha)}$ for $\langle\xi\rangle$ of order $n^{1/(2\alpha)}$. Deduce that for $f \in H^\beta(\mathbb{R}^n)$, we have

$$\|S_n f\| \leq Cn^{-\frac{\beta}{2\alpha}}\|f\|_{H^\beta(\mathbb{R}^n)}.$$

(ii) Show that provided that n is chosen as

$$n = C\delta^{\frac{-2\alpha}{\alpha+\beta}}\|f\|^{\frac{2\alpha}{\alpha+\beta}}_{H^\beta(\mathbb{R}^n)},$$

we have the estimate

$$\delta\|R_n\| + \|S_n f\| \leq C\delta^{\frac{\beta}{\alpha+\beta}}\|f\|^{\frac{\alpha}{\alpha+\beta}}_{H^\beta(\mathbb{R}^n)}. \tag{11.38}$$

(iii) Deduce that the Landweber iteration method is an optimal regularization technique for all $\beta > 0$.
(iv) Generalize the above results for the operators described in Exercise 11.1.8.

We have thus obtained the interesting result that unlike the Tikhonov regularization method described in (11.33), the Landweber iteration regularization can be made optimal (by choosing the appropriate number of iterations n) for all choices on the regularity in $H^\beta(\mathbb{R}^n)$ of the object f.

Let us conclude this section by the following summarizing remark. The reason why regularization was necessary was because *the user* decided that noise was too amplified during the not-regularized inversions. Smoothness priors were then considered to restore well-posedness. This corresponds to a choice of the factor β and a bound E. *In addition*, we need to choose a *regularization* parameter, γ in the Tikhonov regularization algorithm and the stopping iteration n in the Landweber iteration. How these choices are made depends on the bound E but also on the *estimated error* δ. Various techniques have been developed to choose γ or n a posteriori (Morozov principle, L-curve). All these techniques require that δ be known. There is no free lunch. Regularization does require prior information about the solution to mitigate the perceived lack of information in the available data.

11.2 Sparsity and other Regularization Priors

The regularization methods considered in the preceding section have a huge advantage: the replace ill-posed linear systems of equations by well-posed, at least better-posed (better-conditioned), *linear* systems as well. For instance, the inversion of A has been replaced by that of $A^*A + \gamma I$ in the simplest version of Tikhonov regularization. Their main disadvantage is that they render the regularized solution typically smoother than the "exact" solution. Such a smoothing is unavoidable with such regularizations.

The reason why smooth objects are well reconstructed by the smoothing regularization method is that such objects can be represented by a small number of large coefficients (e.g., the first Fourier modes in a Fourier series expansion). It turns out that some objects are better represented in other bases. For instance, an image tends to have sharp discontinuities, for instance between a bright area and a dark area. Some bases, such as for instance those based on wavelets, will be much better than Fourier bases to represent this type of information.

A general framework to account for such prior information is to recast the inverse problem as seeking the minimum of an appropriate functional. Solving the inverse problem then amounts to solving a *optimization problem*. Let us assume that we have a problem of the form

$$\mathfrak{M}(u) = v,$$

and assume that v_d are given data. Let us assume that a functional $u \mapsto \mathcal{R}(u)$ incorporates the prior information about u in the sense that $\mathcal{R}(u)$ is small when u satisfies the constraints. Let us assume also that $\rho(u, v)$ is a function that we wish to use to quantify the error between the available data v_d and the forward model $\mathfrak{M}(u)$. Then we want to minimize $\rho(v, v_d)$ and at the same time minimize $\mathcal{R}(u)$. Both constraints can be achieved by introducing a *regularization parameter* α and minimizing the sum

$$\mathcal{F}_\alpha(u) = \rho\big(\mathfrak{M}(u), v_d\big) + \alpha\mathcal{R}(u). \tag{11.39}$$

Solving the inverse problem consists of minimizing the above functional to get

$$\hat{u}_\alpha = \operatorname{argmin} \mathcal{F}_\alpha(u). \tag{11.40}$$

These minimization problems often go by the name of Tikhonov regularization and may be seen as generalizations of (11.28).

The main objective of regularization (or sparsity) theory is then to devise functions ρ, $\mathcal{R}$ and a regularization parameter α, that best fits our prior information about the problem of interest. Once such a problem has been formulated, it remains to devise a method to solve such an optimization problem numerically. There is a vast literature on the subject; for recent references see the two books [52, 53].

11.2.1 Smoothness Prior and Minimizations

The simplest class of regularizations consists of choosing $\rho(u, v) = \frac{1}{2}\|D(u - v)\|_H^2$ in the $H = L^2$ sense for an operator D that may be identity or an operator of differentiation if small errors on derivatives of the solution matter in practice, and choosing $\mathcal{R}(u)$ also as a quadratic functional, for instance $\mathcal{R}(u) = \frac{1}{2}\|Ru\|_H^2$, again for an operator R that may

be identity or a differential operator. Then associated to the linear problem $Au = v$, we have the minimization problem:

$$\mathcal{F}_\alpha(u) = \frac{1}{2}\|D(Au - v_d)\|_H^2 + \frac{\alpha}{2}\|Ru\|_H^2. \tag{11.41}$$

The main advantage of the above quadratic expression is that the Euler-Lagrange equations associated to the above minimization problem is the following linear system of equations

$$\big((DA)^*(DA) + \alpha R^* R\big)u = (DA)^* D v_d. \tag{11.42}$$

When $D = R = I$, this is nothing but (11.29). Provided that R is an invertible matrix, then the above problem can be solved for all $\alpha > 0$. We have seen in the preceding section how the method converged (at least when $D = R = I$) as the noise in the data and the regularization parameter α tend to 0.

11.2.2 Sparsity Prior and Minimizations

Choosing $\rho(u, v) = \frac{1}{2}\|D(u - v)\|_H^2$ for the misfit to the data may appear as relatively "natural" as it corresponds to measuring noise in the $H = L^2$ sense. The quadratic functional $\mathcal{R}(u) = \frac{1}{2}\|Ru\|_H^2$ is, however, much less justified in many settings.

Sometimes, prior knowledge about the object we wish to reconstruct shows that the latter is sparse in a given representation (a given basis, say). Sparse means here that the object is represented by a small number of large coefficients. For instance, an audio signal may be represented by a finite number of frequencies. Images typically display sharp edges that can be represented in a more economical fashion than pixel by pixel values.

Let us assume that u is discrete and A a matrix to simplify the presentation. Let us also assume that Bu is sparse, where B is a known matrix. Sparsity will be encoded by the fact that the l_1 norm of Bu is small. Penalizing the residual and the l_1 norm yields the minimization of

$$\mathcal{F}_\mu(u) = \|Bu\|_{l_1} + \frac{\mu}{2}\|Au - v_d\|_{l_2}^2. \tag{11.43}$$

This and similar minimization problems have been applied very successfully for a large class of imaging problems.

However, minimizing $\mathcal{F}_\mu$ above is computationally more intensive than solving (11.42). Several algorithms have been developed to solve such minimization problems. We present one strategy, called the split Bregman iteration, that is both efficient and relatively easy to explain. The main idea is that when B and A are the identity operators, then the above minimization can be performed for each component of u separately. In the general case, we introduce

$$d = Bu,$$

and replace the above minimization by

$$\min_{u,d} \ \|d\|_{l_1} + \frac{\mu}{2}\|Au - v_d\|_{l_2}^2 + \frac{\lambda}{2}\|d - Bu\|_{l_2}^2. \tag{11.44}$$

Choosing λ sufficiently large provides a good approximation of the problem we wish to solve. Alternatively, we can solve a series of problems of the above form and show that

we minimize (11.43) in the limit; we do not present the details here and refer the reader to [32].

Now the minimization of (11.44) can be performed iteratively by successively minimizing for u and for d. The minimization for u becomes a linear problem while the minimization for d can be performed for each coordinate independently (this is called soft shrinkage). The iterative algorithm then converges [32]. More precisely, the solution of

$$\min_u \ \frac{\mu}{2}\|Au - v_d\|_{l_2}^2 + \frac{\lambda}{2}\|d - Bu\|_{l_2}^2,$$

is given by

$$(\mu A^* A + \lambda B^* B)u = \mu A^* v_d + \lambda d. \tag{11.45}$$

This is a linear problem that admits a unique solution. Now the solution of

$$\min_d \ \|d\|_{l_1} + \frac{\lambda}{2}\|d - Bu\|_{l_2}^2 = \min_d \sum_{j=1}^{J} |d_j| + \frac{\lambda}{2}|d_j - (Bu)_j|^2 = \sum_{j=1}^{J} \min_{d_j} |d_j| + \frac{\lambda}{2}|d_j - (Bu)_j|^2.$$

Each element in the sum can then be minimized separately. We find that the solution of

$$\min_d |d| + \frac{\lambda}{2}|d - a|^2, \tag{11.46}$$

is given by the soft thresholding

$$d = \operatorname{sgn}(a)\ \max\left(|a| - \frac{1}{\lambda}, 0\right).$$

We have presented this algorithm to show that replacing a smoothness regularization as in (11.41) by a sparsity regularization as in (11.43) increased the computational complexity of the reconstruction algorithm: instead of solving one linear system, we have to iteratively solve linear systems of the form (11.45) and soft thresholdings given by (11.46). When the sparsity assumptions are valid, however, these methods have shown to be greatly beneficial in many practical settings of medical imaging; see [52, 53].

11.3 Bayesian framework and regularization

The penalty regularization framework seen in the preceding two sections is very efficient when the data are sufficiently informative. When the data are very informative and noise relatively small, then no real regularization is necessary. When data are less informative but still quite informative, prior information becomes necessary and smoothness and sparsity type priors allow us to still obtain very accurate reconstructions. When data are even less informative, for instance because noise is very large, or because the problem is severely ill-posed, then sparsity priors are typically no longer sufficient. What one typically obtains as a result is a function that resembles the minimum of the penalization term. In some cases, that may not be desirable. Also, additional prior information may occasionally be known, for instance that next to a black pixel, there is never a blue pixel. Such information is difficult to include in a penalization method.

A fairly versatile methodology to include various prior informations is the Bayesian framework. To simplify the presentation slightly, let us assume that the problem of interest is

$$y = \mathfrak{M}(x) + n, \tag{11.47}$$

where $\mathfrak{M}$ is the measurement operator, x the unknown set of coefficients, y the measurements and n models additive noise in the data.

The main assumption of Bayesian inversions is to assume that x belongs to a class of possible models $\mathfrak{X}$ and that each $x \in \mathfrak{X}$ is given an *a priori* probability of being the "true" coefficient. The associated probability (density) $\pi(x)$ is called the *prior* distribution. A second ingredient in the Bayesian framework is the model for the noise n. We denote by $\pi_n(n)$ the probability distribution of n.

Let us now define

$$\pi(y|x) = \frac{\pi(x,y)}{\pi(x)}$$

the conditional probability density of y *knowing* x with $\pi(x,y)$ the probability density of x *and* y. Note that $\pi(x) = \int \pi(x,y)dy$ as a marginal density so that the above conditional probability density is indeed a probability density (integrating to 1 in y). Note that knowledge of $\pi(y|x)$ is equivalent to knowledge of π_n since

$$\pi(y|x) = \pi_n(y - \mathfrak{M}(x)) \quad \text{for each fixed } x.$$

Bayes' rule then essentially states that

$$\pi(x|y)\pi(y) = \pi(y|x)\pi(x) = \pi(x,y). \tag{11.48}$$

In our inverse problem where y is the measured data and x the unknown coefficients, this means

$$\pi(x|y) = \frac{1}{\pi(y)}\pi(y|x)\pi(x) \propto \pi(y|x)\pi(x), \tag{11.49}$$

where $\propto$ means proportional to, i.e., up to a normalizing constant (here $1/\pi(y)$). In other words, if we know the prior density $\pi(x)$ and the likelihood function $\pi(y|x)$, then by Bayes' rule, we know $\pi(x|y)$, which is the *posterior* probability (density).

Let us recapitulate the main ingredients of the Bayesian formalism. We assume the *prior* distribution $\pi(x)$ known as an indication of our prior beliefs about the coefficients before data are acquired. We assume knowledge of the likelihood function $\pi(y|x)$, which as we have seen is a statement about the noise model in the experiment. From these two prior assumptions, we use Bayes' rule to infer the *posterior* distribution $\pi(x|y)$ for x knowing the data.

11.3.1 Penalization methods and Bayesian framework

Before going into the advantages and drawbacks of the method, we first show that penalization methods can be seen as an application of the Bayesian framework. Let $\mathcal{R}(x)$ be a given function and assume that the prior is given by the Gibbs distribution:

$$\pi(x) \propto e^{-\mathcal{R}(x)}.$$

Now assume that the likelihood function is of the form

$$\pi(y|x) \propto e^{-\rho(y-\mathfrak{M}(x))},$$

where ρ is a distance function. Then by Bayes' rule, we find that

$$\pi(x|y) \propto e^{-\left(\rho(y-\mathfrak{M}(x))+\mathcal{R}(x)\right)}.$$

The Maximum A Posteriori (MAP) x_{MAP} is the parameter that maximizes the posterior distribution, or equivalently the minimum of the functional

$$\mathcal{F}(x) = \rho(y - \mathfrak{M}(x)) + \mathcal{R}(x).$$

Therefore, for appropriate choices of the prior and likelihood function, we retrieve the penalization methods seen in the preceding section.

Note that the minimization problem is solved by linear algebra when both ρ and $\mathcal{R}$ are quadratic functionals. For instance if $\pi_n(n) \sim \mathcal{N}(0, \Sigma)$ a multivariate Gaussian with correlation matrix Σ, then we have $\rho(n) \propto e^{-\frac{1}{2}n^t\Sigma^{-1}n}$. Similarly, for $\pi(x) \sim \mathcal{N}(0, \Gamma)$, then $\mathcal{R}(x) \propto e^{-\frac{1}{2}x^t\Gamma^{-1}x}$ so that we need to minimize

$$\mathcal{F}(x) = \frac{1}{2}(y - \mathfrak{M}x)^t\Sigma^{-1}(y - \mathfrak{M}x) + \frac{1}{2}x^t\Gamma^{-1}x. \tag{11.50}$$

If $\mathfrak{M}$ is a linear operator, then the solution to the minimization problem is, as we already saw, solution of

$$(\mathfrak{M}^*\Sigma^{-1}\mathfrak{M} + \Gamma^{-1})x = \mathfrak{M}^*\Sigma^{-1}y. \tag{11.51}$$

The Bayesian framework can then be used to recover the Tikhonov regularization of linear equations. Moreover, it gives an explicit characterization of the correlation matrices Σ and Γ as the co-variance functions of the measurement noise and of the prior assumptions on the coefficients, respectively.

Note that the l_1 minimization corresponds to a choice $\mathcal{R}(x) = \sum_i |x_i|$. This corresponds to assuming that each pixel value satisfies independent and identically distributed random variables with a Laplace distribution. We thus also recover the sparsity regularizations using the Bayesian framework. If we expect nearby pixels to be correlated, then more complex prior models or functionals $\mathcal{R}(x)$ need to be constructed. This is a very active area of research. Although the derivation of the "best" functional is often more an art than grounded in first principles, the Bayesian framework sometimes allows for very "pleasing" reconstructions (we recall that the ill-posedness of an inverse problem is a subjective notion).

11.3.2 Computational and psychological costs of the Bayesian framework

We have seen that the Bayesian framework reduced to an optimization problem when the Maximum A Posteriori (MAP) x_{MAP} is what we are looking for. The Bayesian framework allows one to obtain much more information, at least in theory, since the output of the procedure is the full posterior distribution $\pi(x|y)$ and not only its argmax.

In practice, however, we are faced with daunting tasks: first of all, how do we sample what is often a very high dimensional distribution $\pi(x|y)$? And second of all, even if sampling is possible, how does one represent such a huge object practically? These two questions severely limit the applicability of Bayesian frameworks in practice.

A third, and in some sense more fundamental and structural, question pertains to the choice of the prior $\pi(x)$. Where should that prior information come from? There is no "good" answer to this fundamental yet ill-formulated question. In some sense, we have already partially answered it: since *the user* decided that adding no prior information was not working in the sense that "*noise*" had too large an effect on the reconstruction, then *the user* has to come up with *another* model. There is no such a thing as a "non-informative" prior, since *the user* decided that a prior was necessary. (If data alone were sufficient to obtain "good" reconstructions, then the Bayesian framework would not be necessary.) If data are not sufficient, then the Bayesian framework provides a very versatile framework for *the user* to provide information about the problem that helps to compensate for what is not present in the data. Some researchers will not be satisfied with this way of addressing the inverse problem and the notion of "compensating" for the lack of data. This is a perfectly reasonable position. However, the Bayesian framework at least has this very appealing feature: it provides a logical mapping from the prior information, namely the prior distribution and the likelihood function, to the outcome of the procedure, namely the posterior distribution. If nothing else, it can therefore serve as a very valuable tool to guide intuition and to search what types of prior informations are necessary for a given set of constraints on the posterior distribution. Moreover, nothing prevents us from estimating how the posterior distribution is sensitive to variations in the prior information. This strategy, sometimes referred to as Robust Bayesian analysis, allows one understand which features of the reconstructed parameters strongly or weakly depend on the choices of the prior density and likelihood functions.

Now that the psychological cost of the Bayesian framework has been taken into account, let us come back to its computational cost, which still poses enormous challenges. Let us first address the representation of the posterior distribution. Typically, moments of the posterior distribution are what we are interested in. For instance, one may be interested in the first moment (a vector) and the variance (a matrix)

$$x_m = \int x\pi(x|y)dx, \qquad \Gamma_x = \int x \otimes x\, \pi(x|y)dx - x_m \otimes x_m. \tag{11.52}$$

Of interest are also various quantiles of the posterior distribution, for instance the probability that x_j be larger than a number γ: $\int \pi(x|y)\chi(x_j > \gamma)dx$, where $\chi(X)$ is the indicatrix function of the set X equal to 1 on X and to 0 otherwise.

For each of these moments of the posterior distribution, we need to be able to sample $\pi(x|y)dx$. In a few cases, the sampling of $\pi(x|y)dx$ may be straightforward, for instance when $\pi(y|x)$ has a Gaussian structure. In most cases, however, sampling is a difficult exercise. The most versatile method to perform such a sampling is arguably the Markov Chain Monte Carlo (MCMC) method. The objective of MCMC samplers is to generate a Markov chain X^i for $i \in \mathbb{N}$ whose invariant distribution (distribution at convergence when the algorithm converges) is the posterior distribution. There are two main MCMC samplers, the Gibbs sampler and the Metropolis-Hastings sampler. The latter is defined as follows. We assume that we want to sample a distribution $\pi(x|y)$.

Let $q(x, x')$ be a given, positive, transition density from the vector x to the vector x' (it thus sums to 1 integrated in all possible vectors x' for each x). Let us then define

$$\alpha(x, x') := \min\Big(\frac{q(x, x')\pi(x'|y)}{q(x', x)\pi(x|y)}, 1\Big). \tag{11.53}$$

Note that the above quantity, which is all we need about $\pi(x|y)$ in the Metropolis-Hastings sampler, depends only on $\frac{\pi(x'|y)}{\pi(x|y)}$ and thus is independent of the normalizing constant of $\pi(x|y)$, which is typically not known in the Bayesian framework, and whose estimation is typically expensive computationally.

Let X^i the current state of the Markov chain. Let $\tilde{X}^{i+1}$ be drawn from the transition kernel $q(X^i, x')$. Then with probability $\alpha(X^i, \tilde{X}^{i+1})$, we accept the transition and set $X^{i+1} = \tilde{X}^{i+1}$ while with probability $1 - \alpha(X^i, \tilde{X}^{i+1})$, we reject the transition and set $X^{i+1} = X^i$.

The transition probability of the chain from x to x' is thus $p(x, x') = \alpha(x, x')q(x, x')$ while the probability to stay put at x is $1 - \int p(x, x')dx'$. The construction is such that $\pi(x|y)p(x, x') = p(x', x)\pi(x'|y)$, which means that $\pi(x|y)dy$ is indeed the invariant distribution of the Markov chain. In practice, we want independent samples of $\pi(x|y)$ so that the following Monte Carlo integration follows from an application of the law of large numbers (ergodicity), for instance:

$$\int f(x)\pi(x|y)dx \sim \frac{1}{|I|}\sum_{i\in I} f(X^i), \tag{11.54}$$

for any reasonable (continuous) functional f. Such a rule is accurate if the X^i are sampled according to $\pi(x|y)dy$ and are sufficiently independent. This is for instance achieved by choosing $I = \{1 \leq i \leq i_{\max}, \quad i = Nj, \quad j \in \mathbb{N}\}$. For instance, we can take $i_{\max} = 10^7$ and $N = 1000$ so that I is composed of $|I| = 10^4$ points chosen every 1000 points in the Metropolis-Hastings Markov chain. For an accuracy equal to $\sqrt{|I|} = 0.01$ (as an application of the central limit theorem to estimate the error in (11.54)), we thus need 10^7 evaluations of $\pi(x|y)$. Using Bayes' rule, this is proportional to $\pi(x)\pi(y|x)$, where the latter likelihood function requires that we solve a forward problem (for a given x drawn from the prior $\pi(x)$) to estimate the law of the "data" y. In other words, the construction of the above statistical moment with an accuracy of order 10^{-2} requires that we solve 10^7 forward problems. In many practical situations, this is an unsurmountable computational cost. Moreover, this assumes that the transition $q(x, x')$ has been chosen in such a way that every 1000 samples X^i are indeed sufficiently independent. This is very difficult to achieve in practice and is typically obtained by experienced users rather than from sound, physics- or mathematics- based principles.

Note that in practice, it has been observed that I in (11.54) should be the set of all runs $1 \leq i \leq i_{\max}$. In other words, there is no gain in throwing away points X^i in the evaluation of the integrals. However, the above heuristics are correct: the error in the approximation (11.54) is indeed proportional to the square root of the number of independent components in $\{X^i\}$ and not $\sqrt{i_{\max}}$.

Many methodologies have been developed to improve the efficiency of MCMC algorithms. It is however fair to say that even with nowadays computational capabilities, many problems of interest are totally out of reach using the standard Bayesian framework. That said, it is still a very versatile methodology that goes a long way to address

the main problem of this chapter, which we recall is: *the user* decided that adding no prior information was not working and thus something had to be done.

General references on the use of the Bayesian framework in inverse problems include [37, 59].

Chapter 12

Geometric Priors and Parameterizations

In this chapter, we consider a few other methods to include prior information one might possess about the unknown objects of interest. The Bayesian framework introduced in the preceding chapter is quite versatile, but it is computationally extremely expensive because deterministic reconstructions are replaced by a probability measure on possible reconstructions, which is often daunting to sample and visualize from a numerical point of view.

Other physically motivated methods have been developed to incorporate prior information. In one such method, we aim to reconstruct the support of an inclusion rather than its full description and assume that such an inclusion is embedded in a *known* medium. Several techniques have been developed to do this and we focus here on a method called the *factorization* method. The factorization method is a functional analytical tool that allows us to separate the influence of the unknown inclusion from that of the known background. It is considered in a simple setting in section 12.1.

In another method, we give up the hope to perform global reconstructions and replace the unknown object by a low-dimensional parameterization. Any finite dimensional problem that is injective is necessarily well-posed essentially as an application of the Fredholm alternative. One such quite useful parameterization consists of assuming that the inclusion has *small* volume. We then perform asymptotic expansions in its volume to understand the leading influence of such an inclusion on available measurements. The reconstruction then focuses on the first coefficients appearing in the asymptotic expansion assuming the surrounding background to be known. The method is presented in a very simple setting in section 12.2.

12.1 Reconstructing the domain of inclusions

The reconstruction of physical parameters in an elliptic equation from boundary measurements, such as the Neumann-to-Dirichlet map, is a severely ill-posed problem. One should therefore not expect to stably reconstruct more than a few coefficients modeling the physical parameters, such as for instance the first Fourier modes in a Fourier series expansion as we saw in the smoothness penalization considered in Chapter 11.

In certain applications, knowing the first few coefficients in a Fourier series expansion

is not what one is interested in. In this chapter, we assume that the physical parameters are given by a background, which is known, and an inclusion, from which we only know that it differs from the background. Moreover, we are not so much interested in the detailed structure of the inclusion as in its location. We thus wish to reconstruct an interface separating the background from the inclusion.

To reconstruct this interface, we use the method of *factorization*. The method provides a constructive method to obtain the support of the inclusion from the Neumann-to-Dirichlet (NtD) boundary measurements. Notice that the NtD measurements allow us a priori to reconstruct much more than the support of the inclusion. However, because we restrict ourselves to this specific reconstruction, we can expect to obtain more accurate results on location of the inclusion than by directly reconstructing the physical parameters on the whole domain.

12.1.1 Forward Problem

We consider here the problem in impedance tomography. The theory generalizes to a certain extent to problems in optical tomography.

Let $\gamma(x)$ be a conductivity tensor in an open bounded domain $X \in \mathbb{R}^n$ with Lipschitz boundary ∂X. We define Σ, a smooth surface in X, as the boundary of the inclusion. We denote by D the simply connected bounded open domain such that $\Sigma = \partial D$. This means that D is the domain "inside" the surface Σ. We also define $D^c = X \backslash \overline{D}$, of boundary $\partial D^c = \partial X \cup \Sigma$. We assume that $\gamma(x)$ is a smooth known background $\gamma(x) = \gamma_0(x)$ on D^c, and that γ and γ_0 are smooth but different on D. For γ_0 a smooth known tensor on the full domain X, this means that γ jumps across Σ so that Σ is the surface of discontinuity of γ. More precisely, we assume that the $n \times n$ symmetric tensor $\gamma_0(x)$ is of class $C^2(\overline{X})$ and positive definite such that $\xi_i \xi_j \gamma_{0ij}(x) \geq \alpha_0 > 0$ uniformly in $x \in X$ and in $\{\xi_i\}_{i=1}^n = \xi \in \mathbb{S}^{n-1}$, the unit sphere in $\mathbb{R}^n$. Similarly, the $n \times n$ symmetric tensor $\gamma(x)$ is of class $C^2(\overline{D}) \otimes C^2(\overline{D^c})$ (in the sense that $\gamma(x)|_D$ can be extended as a function of class $C^2(D)$ and similarly for $\gamma(x)|_{D^c}$) and positive definite such that $\xi_i \xi_j \gamma_{ij}(x) \geq \alpha_0 > 0$ uniformly in $x \in X$ and in $\{\xi_i\}_{i=1}^n = \xi \in \mathbb{S}^{n-1}$.

The equation for the electric potential $u(x)$ is given by

$$\begin{aligned} \nabla \cdot \gamma \nabla u &= 0, && \text{in } X \\ \nu \cdot \gamma \nabla u &= g && \text{on } \partial X \\ \int_{\partial X} u \, d\sigma &= 0. \end{aligned} \tag{12.1}$$

Here, $\nu(x)$ is the outward unit normal to X at $x \in \partial X$. We also denote by $\nu(x)$ the outward unit normal to D at $x \in \Sigma$. Finally $g(x)$ is a mean-zero current, i.e., $\int_{\partial X} g d\sigma = 0$, imposed at the boundary of the domain.

The above problem admits a unique solution $H_0^1(X)$, the space of functions in $u \in H^1(X)$ such that $\int_{\partial X} u d\sigma = 0$. This results from the variational formulation of the above equation

$$b(u, \phi) \equiv \int_X \gamma \nabla u \cdot \nabla \phi dx = \int_{\partial X} g \phi d\sigma(x) \equiv l(\phi), \tag{12.2}$$

holding for any test function $\phi \in H_0^1(X)$. Indeed from a Poincaré-like inequality, we deduce that $b(u,v)$ is a coercive and bounded bilinear form on $H_0^1(X)$ and the existence result follows from the Lax-Milgram theorem. Classical trace estimates show that $u|_{\partial X} \in H_0^{\frac{1}{2}}(\partial X)$, the space of functions $v \in H^{\frac{1}{2}}(\partial X)$ such that $\int_{\partial X} v d\sigma = 0$.

We define the Neumann-to-Dirichlet operator Λ_Σ, depending on the location of the discontinuity Σ, as

$$\Lambda_\Sigma \; : \; H_0^{-\frac{1}{2}}(\partial X) \; \longrightarrow \; H_0^{\frac{1}{2}}(\partial X), \qquad g \; \longmapsto \; u_{|\partial X}, \tag{12.3}$$

where $u(x)$ is the solution to (12.1) with boundary normal current $g(x)$. Similarly, we introduce the "background" Neumann-to-Dirichlet operator Λ_0 defined as above with γ replaced by the known background γ_0. To model that the inclusion has a different conductivity from the background, we assume that γ satisfies either one of the following hypotheses

$$\gamma(x) - \gamma_0(x) \geq \alpha_1 > 0 \quad \text{on } D, \qquad \gamma_0(x) = \gamma(x), \quad \text{on } D^c, \tag{12.4}$$

$$\gamma_0(x) - \gamma(x) \geq \alpha_1 > 0 \quad \text{on } D, \qquad \gamma_0(x) = \gamma(x), \quad \text{on } D^c, \tag{12.5}$$

for some constant positive definite tensor α_1. The tensor inequality $\gamma_1 \geq \gamma_2$ is meant in the sense that $\xi_i \xi_j (\gamma_{1,ij} - \gamma_{2,ij}) \geq 0$ for all $\xi \in \mathbb{R}^n$.

12.1.2 Factorization method

The purpose of the factorization method is to show that

$$\Lambda_0 - \Lambda_\Sigma = L^* F L, \tag{12.6}$$

where L and L^* are operators in duality that depend only on $\gamma|_D = (\gamma_0)|_D$ and F or $-F$ is an operator that generates a coercive form on $H_0^{\frac{1}{2}}(\Sigma)$ when (12.4) or (12.5) are satisfied, respectively. The operators are constructed as follows. Let v and w be the solutions of

$$\begin{array}{rcll} \nabla \cdot \gamma \nabla v &=& 0, & \text{in } D^c \\ \nu \cdot \gamma \nabla v &=& \phi & \text{on } \partial X \\ \nu \cdot \gamma \nabla v &=& 0 & \text{on } \Sigma \\ \displaystyle\int_\Sigma v d\sigma &=& 0, & \end{array} \qquad \begin{array}{rcll} \nabla \cdot \gamma \nabla w &=& 0, & \text{in } D^c \\ \nu \cdot \gamma \nabla w &=& 0 & \text{on } \partial X \\ \nu \cdot \gamma \nabla w &=& -\phi & \text{on } \Sigma \\ \displaystyle\int_\Sigma w d\sigma &=& 0. & \end{array} \tag{12.7}$$

These equations are well-posed in the sense that they admit solutions in $H^1(D^c)$ with traces in $H^{\frac{1}{2}}(\Sigma)$ and in $H^{\frac{1}{2}}(\partial X)$ at the boundary of D^c. We then define the operator L, which maps $\phi \in H_0^{-\frac{1}{2}}(\partial X)$ to $v|_\Sigma \in H_0^{\frac{1}{2}}(\Sigma)$, where v is the unique solution to the left equation in (12.7), and the operator L^*, which maps $\phi \in H_0^{-\frac{1}{2}}(\Sigma)$ to $w|_{\partial X}$, where w is the unique solution to the right equation in (12.7). We verify that both operators are in duality in the sense that

$$(L\phi, \psi)_\Sigma \equiv \int_\Sigma \psi L\phi \; d\sigma = \int_{\partial X} \phi L^* \psi \; d\sigma \equiv (\phi, L^*\psi)_{\partial X}.$$

Let us now define two operators G_Σ and G_Σ^* as follows. For any quantity f defined on $D \cup D^c$, we denote by $f^+(x)$ for $x \in \Sigma$ the limit of $f(y)$ as $y \to x$ and $y \in D^c$, and by $f^-(x)$ the limit of $f(y)$ as $y \to x$ and $y \in D$. Let v and w be the unique solutions to the following problems

$$\begin{array}{rcll} \nabla \cdot \gamma \nabla v &=& 0, & \text{in } X \backslash \Sigma \\ [v] &=& 0, & \text{on } \Sigma \\ [\nu \cdot \gamma \nabla v] &=& 0 & \text{on } \Sigma \\ \nu \cdot \gamma \nabla v &=& g & \text{on } \partial X \\ \displaystyle\int_\Sigma v \, d\sigma &=& 0 & \end{array} \qquad \begin{array}{rcll} \nabla \cdot \gamma \nabla w &=& 0, & \text{in } X \backslash \Sigma \\ [w] &=& \phi, & \text{on } \Sigma \\ [\nu \cdot \gamma \nabla w] &=& 0 & \text{on } \Sigma \\ \nu \cdot \gamma \nabla w &=& 0 & \text{on } \partial X \\ \displaystyle\int_{\partial X} w \, d\sigma &=& 0. & \end{array} \tag{12.8}$$

We define G_Σ as the operator mapping $g \in H_0^{-\frac{1}{2}}(\partial X)$ to $G_\Sigma g = \nu \cdot \gamma \nabla v^+_{|\Sigma} \in H_0^{-\frac{1}{2}}(\Sigma)$ and the G_Σ^* as the operator mapping $\phi \in H_0^{\frac{1}{2}}(\Sigma)$ to $G_\Sigma^* \phi = w_{|\partial X} \in H_0^{\frac{1}{2}}(\partial X)$, where v and w are the unique solutions to the above equations (12.8).

Except for the normalization $\int_\Sigma v \, d\sigma = 0$, the equation for v is the same as (12.1) and thus admits a unique solution in $H^1(X)$, say. Moreover integrations by parts on D^c imply that

$$\int_\Sigma \nu \cdot \gamma \nabla v^+ \, d\sigma = \int_{\partial X} g \, d\sigma = 0.$$

This justifies the well-posedness of the operator G_Σ as it is described above. The operator G_Σ^* is more delicate. We first obtain that for any smooth test function ψ,

$$\begin{array}{rcl} \displaystyle\int_{D^c} \gamma \nabla w \cdot \nabla \psi \, dx + \int_\Sigma \nu \cdot \gamma \nabla w \psi^+ \, d\sigma &=& 0 \\ \displaystyle\int_D \gamma \nabla w \cdot \nabla \psi \, dx - \int_\Sigma \nu \cdot \gamma \nabla w \psi^- \, d\sigma &=& 0, \end{array}$$

so that

$$\int_X \gamma \nabla w \cdot \nabla \psi \, dx = \int_\Sigma (-\nu \cdot \gamma \nabla w) \, [\psi] \, d\sigma. \tag{12.9}$$

It turns out that $\|\nu \cdot \gamma \nabla w\|_{H_0^{-\frac{1}{2}}(\Sigma)}$ is bounded by the norm of $\gamma \nabla w$ in $H(\mathrm{div}, \mathrm{X})$ (see [31]). This and a Poincaré-type inequality shows that the above right-hand side with $\psi = w$ is bounded by $C\|\phi\|^2_{H_0^{\frac{1}{2}}(\Sigma)}$. Existence and uniqueness of the solution $w \in H^1(D) \otimes H^1(D^c)$ to (12.8) is then ensured by an application of the Lax-Milgram theorem. This also shows that the operator G_Σ^* as defined above is well-posed.

Integrations by parts in the equation for v in (12.8) by a test function φ yields

$$\begin{array}{rcl} \displaystyle\int_D \gamma \nabla u \cdot \nabla \varphi \, dx - \int_\Sigma \nu \cdot \gamma \nabla v \varphi^- d\sigma &=& 0 \\ \displaystyle\int_{D^c} \gamma \nabla v \cdot \nabla \varphi \, dx + \int_\Sigma \nu \cdot \gamma \nabla v \varphi^+ d\sigma &=& \displaystyle\int_{\partial X} g\varphi \, d\sigma, \end{array} \tag{12.10}$$

from which we deduce that

$$\int_X \gamma \nabla v \cdot \nabla \varphi = \int_{\partial X} g\varphi - \int_\Sigma (G_\Sigma g)[\varphi]. \tag{12.11}$$

That G_Σ and G_Σ^* are in duality in the sense that

$$\int_\Sigma G_\Sigma g\ \phi\ d\sigma = \int_{\partial X} g\ G_\Sigma^* \phi\ d\sigma, \tag{12.12}$$

follows from (12.11) with $\varphi = w$ and (12.9) with $\psi = v$ since $[v] = 0$.

We finally define F_Σ as the operator mapping $\phi \in H_0^{\frac{1}{2}}(\Sigma)$ to $F_\Sigma \phi = -\nu \cdot \gamma \nabla w \in H_0^{-\frac{1}{2}}(\Sigma)$, where w is the solution to (12.8). Based on the above results, this is a well-posed operator. Moreover, we deduce from (12.9) that

$$(F_\Sigma[w], [\psi])_\Sigma = \int_X \gamma \nabla w \cdot \nabla \psi\ dx = ([w], F_\Sigma[\psi])_\Sigma, \tag{12.13}$$

so that $F_\Sigma = F_\Sigma^*$. Upon choosing $[w] = [\psi]$, we find that F_Σ is coercive on $H_0^{\frac{1}{2}}(\Sigma)$. This implies among other things that F_Σ is injective.

We now notice that

$$G_\Sigma^* = L^* F_\Sigma.$$

This follows from the uniqueness of the solution to the elliptic problem on D^c with conditions defined on $\partial D^c = \Sigma \cup \partial X$. By duality, this also implies that $G_\Sigma = F_\Sigma L$. The operators G_0 and F_0 are defined similarly except that γ is replaced by γ_0 in (12.8). Let us finally define the operator M, which maps $g \in H_0^{-\frac{1}{2}}(\partial X)$ to $u_{|\partial X} \in H_0^{\frac{1}{2}}(\partial X)$, where u is the solution to

$$\begin{aligned} \nabla \cdot \gamma \nabla u &= 0, && \text{in } D^c \\ \nu \cdot \gamma \nabla u &= 0, && \text{on } \Sigma \\ \nu \cdot \gamma \nabla u &= g, && \text{on } \partial X \\ \int_{\partial X} u\ d\sigma &= 0. \end{aligned} \tag{12.14}$$

Except for the normalization, the operator M is the same as the operator L (so that $L - M$ is proportional to identity) and is thus well-posed. We now verify from the linearity of the elliptic problems that

$$\Lambda_\Sigma = M - L^* G_\Sigma = M - L^* F_\Sigma L, \qquad \Lambda_0 = M - L^* G_0 = M - L^* F_0 L. \tag{12.15}$$

We thus deduce the main factorization result of this section, namely that

$$\Lambda_0 - \Lambda_\Sigma = L^* F L, \qquad F = F_\Sigma - F_0. \tag{12.16}$$

The above result would not be very useful if F did not have specific properties. We now show that F or $-F$ generates a coercive form on $H_0^{\frac{1}{2}}(\Sigma)$ and may be written as B^*B for some *surjective* operator B^*. Note that $F^* = F$ since both F_Σ and F_0 are self-adjoint.

We denote by w_Σ the solution w to (12.8) and by w_0 the solution to the same equation with γ replaced by γ_0. Upon multiplying the equation for w_Σ by w_0 and subtracting the equation for w_0 multiplied by w_Σ, we obtain since $\gamma = \gamma_0$ on D^c that

$$\begin{aligned} \int_D (\gamma - \gamma_0) \nabla w_0 \cdot \nabla w_\Sigma\, dx &= \int_\Sigma (\nu \cdot \gamma \nabla w_\Sigma w_0^- - \nu \cdot \gamma \nabla w_0 w_\Sigma^-) d\sigma \\ 0 &= \int_\Sigma (\nu \cdot \gamma \nabla w_\Sigma w_0^+ - \nu \cdot \gamma \nabla w_0 w_\Sigma^+) d\sigma. \end{aligned}$$

Notice that both γ and ∇w_Σ jump across Σ but that $\nu \cdot \gamma \nabla w_\Sigma$ does not. This yields that

$$\int_D (\gamma - \gamma_0) \nabla w_0 \cdot \nabla w_\Sigma \, dx = \int_\Sigma (F_\Sigma - F_0) \phi \, \phi \, d\sigma = \int_\Sigma F\phi \, \phi \, d\sigma. \tag{12.17}$$

Let us now introduce $\delta w = w_0 - w_\Sigma$. Upon multiplying $\nabla \cdot \gamma_0 \nabla \delta w + \nabla \cdot (\gamma_0 - \gamma) \nabla w_\Sigma = 0$ by δw and integrating by parts on D^c and D we deduce that

$$\int_X \gamma_0 \nabla \delta w \cdot \nabla \delta w \, dx + \int_D (\gamma - \gamma_0) \nabla w_\Sigma \cdot \nabla w_\Sigma \, dx = \int_D (\gamma - \gamma_0) \nabla w_0 \cdot \nabla w_\Sigma \, dx.$$

By exchanging the roles of the indices Σ and 0 we also obtain

$$\int_X \gamma \nabla \delta w \cdot \nabla \delta w \, dx + \int_D (\gamma_0 - \gamma) \nabla w_0 \cdot \nabla w_0 \, dx = \int_D (\gamma_0 - \gamma) \nabla w_0 \cdot \nabla w_\Sigma \, dx.$$

Combining these results with (12.17) we deduce that

$$\begin{aligned} \int_\Sigma F\phi \, \phi \, d\sigma &= \int_X \gamma_0 \nabla \delta w \cdot \nabla \delta w \, dx + \int_D (\gamma - \gamma_0) \nabla w_\Sigma \cdot \nabla w_\Sigma \, dx \\ \int_\Sigma -F\phi \, \phi \, d\sigma &= \int_X \gamma \nabla \delta w \cdot \nabla \delta w \, dx + \int_D (\gamma_0 - \gamma) \nabla w_0 \cdot \nabla w_0 \, dx. \end{aligned} \tag{12.18}$$

Let us assume that (12.4) holds. Then F generates a coercive form on $H_0^{\frac{1}{2}}(\Sigma)$. Indeed, let us assume that the right-hand side of the first equality above is bounded. Then by a Poincaré-type inequality, we have $\delta w \in H^1(X)$ and $w_\Sigma|_D \in H^1(D)$ thanks to (12.4). This implies that $(\nu \cdot \gamma \nabla w_\Sigma)_{|\Sigma} \in H^{-\frac{1}{2}}(\Sigma)$ and thus based on (12.8) that $w_\Sigma|_{D^c} \in H^1(D^c)$. This in turn implies that both w_Σ^+ and w_Σ^- belong to $H^{\frac{1}{2}}(\Sigma)$ so that their difference $\phi \in H_0^{\frac{1}{2}}(\Sigma)$. Thus, we have shown the existence of a positive constant C such that

$$\|\phi\|_{H_0^{\frac{1}{2}}(\Sigma)} \leq C(F\phi, \phi)_\Sigma^{\frac{1}{2}}. \tag{12.19}$$

Exchanging the indices Σ and 0 also yields the existence of a constant C under hypothesis (12.5) such that

$$\|\phi\|_{H_0^{\frac{1}{2}}(\Sigma)} \leq C(-F\phi, \phi)_\Sigma^{\frac{1}{2}}. \tag{12.20}$$

In what follows, we assume that (12.4) and (12.19) hold to fix notation. The final results are not modified when (12.5) and (12.20) hold instead.

The operator F is defined from $H_0^{\frac{1}{2}}(\Sigma)$ to $H_0^{-\frac{1}{2}}(\Sigma)$, which have not been identified yet. So writing $F = B^*B$ requires a little bit of work. Let $\mathcal{I}$ be the canonical isomorphism between $H_0^{-\frac{1}{2}}(\Sigma)$ and $H_0^{\frac{1}{2}}(\Sigma)$. Since it is positive definite, we can decompose it as

$$\mathcal{I} = \mathcal{J}^* \mathcal{J}, \qquad \mathcal{J} : \; H_0^{-\frac{1}{2}}(\Sigma) \; \to \; L_0^2(\Sigma), \quad \mathcal{J}^* : \; L_0^2(\Sigma) \; \to \; H_0^{\frac{1}{2}}(\Sigma).$$

Both $\mathcal{J}$ and $\mathcal{J}^*$ are isometries as defined above. We can thus recast the coercivity of F as

$$(F\phi, \phi) = (F\mathcal{J}^* u, \mathcal{J}^* u) = (\mathcal{J} F \mathcal{J}^* u, u) \geq \alpha \|\phi\|^2_{H_0^{\frac{1}{2}}(\Sigma)} = \alpha \|u\|^2_{L_0^2(\Sigma)}.$$

So $\mathcal{J}F\mathcal{J}^*$ as a self-adjoint positive definite operator on $L^2(\Sigma)$ can be written as C^*C, where C and C^* are bounded operators from $L^2_0(\Sigma)$ to $L^2_0(\Sigma)$. Since

$$\|Cu\|^2_{L^2_0(\Sigma)} \geq \alpha \|u\|^2_{L^2_0(\Sigma)},$$

we deduce that C^* is surjective. We thus obtain that $F = B^*B$ where $B = C(\mathcal{J}^*)^{-1}$ maps $H^{\frac{1}{2}}_0(\Sigma)$ to $L^2_0(\Sigma)$ and its adjoint operator $B^* = \mathcal{J}^{-1}C^*$ maps $L^2_0(\Sigma)$ to $H^{\frac{1}{2}}_0(\Sigma)$. Since $\mathcal{J}$ is an isomorphism, we deduce that B^* is surjective.

From the above calculations we obtain that

$$\Lambda_0 - \Lambda_\Sigma = L^*FL = L^*B^*(L^*B^*)^* = A^*A, \qquad A = BL.$$

Since the Range of $(A^*A)^{\frac{1}{2}}$ for A acting on Hilbert spaces is equal to the Range of A^*, we deduce that

$$\mathcal{R}((\Lambda_0 - \Lambda_\Sigma)^{\frac{1}{2}}) = \mathcal{R}(L^*B^*) = \mathcal{R}(L^*) \tag{12.21}$$

since B^* is surjective. The latter is shown as follows. We always have that $\mathcal{R}(L^*B^*) \subset \mathcal{R}(L^*)$. Now for $y \in \mathcal{R}(L^*)$ there is x such that $y = L^*x$ and since B^* is surjective u such that $y = L^*B^*x$ so that $y \in \mathcal{R}(L^*B^*)$; whence $\mathcal{R}(L^*) \subset \mathcal{R}(L^*B^*)$.

When (12.5) and (12.20) hold instead of (12.4) and (12.19), we deduce that

$$\mathcal{R}((\Lambda_\Sigma - \Lambda_0)^{\frac{1}{2}}) = \mathcal{R}(L^*), \tag{12.22}$$

instead of (12.21). In both cases, we have shown the following result.

Theorem 12.1.1 *Provided that* (12.4) *or* (12.5) *holds, the range of the operator* L^* *defined in* (12.7) *is determined by the Neumann-to-Dirichlet operator* Λ_Σ *and* (12.22) *holds.*

12.1.3 Reconstruction of Σ

The above theorem gives us a method to reconstruct Σ: we need to find a family of functions that belong to the Range of L^* when some probe covers D and do not belong to it when the probe covers D^c. Notice that the operator L^* does not depend on the domain D and thus depends only on the tensor γ_0 and the surface Σ. Consequently, the reconstruction of Σ is independent of γ on D, except for the existence of a positive definite tensor α_0 such that (12.4) or (12.5) holds.

Let us now introduce the family of functions $N(\cdot; y)$ indexed by the parameter $y \in X$ solution of

$$\begin{aligned} \nabla \cdot \gamma_0 \nabla N(\cdot; y) &= \delta(\cdot - y), && \text{in } X \\ \nu \cdot \gamma_0 \nabla N(\cdot; y) &= 0 && \text{on } \partial X \\ \int_{\partial X} N(\cdot; y)\, d\sigma &= 0. \end{aligned} \tag{12.23}$$

We define the family of functions $g_y(x) = N(x; y)_{|\partial X}$ on ∂X. Then we have the following result:

Theorem 12.1.2 *The function* $g_y(x)$ *belongs to* $\mathcal{R}(L^*)$ *when* $y \in D$ *and does not belong to* $\mathcal{R}(L^*)$ *when* $y \in D^c$.

This theorem provides us with a constructive method to image $\Sigma = \partial D$. For each $y \in X$, all we have to do is to solve (12.23) and verify whether the trace on ∂X belongs to the Range of $(\pm(\Lambda_0 - \Lambda_\Sigma))^{\frac{1}{2}}$, which can be evaluated from the known boundary measurements. Only when the verification is positive do we deduce that $y \in D$.

Proof. The proof of the theorem is as follows. When $y \in D$, we have that $\nu \cdot \gamma \nabla N(x;y)_{|\Sigma} \in H_0^{-\frac{1}{2}}(\Sigma)$ and $\nabla \cdot \gamma_0 \nabla N(\cdot;y) = 0$ on D^c so that $g_y \in \mathcal{R}(L^*)$. Let us now assume that $y \in D^c$ and $g_y(x) \in \mathcal{R}(L^*)$. Then there exists $\phi \in H_0^{-\frac{1}{2}}(\Sigma)$ such that $g_y = L^*\phi = w_{|\partial X}$, where w is the solution to (12.7).Let $B(y;\varepsilon)$ be the ball of radius ε centered at y for ε sufficiently small. On $D^c \backslash \overline{B_\varepsilon}$, both w and g_y satisfy the same equation. By uniqueness of the solution to the Cauchy problem imposing Dirichlet data and vanishing Neumann data on ∂X, we deduce that $w = g_y$ on $D^c \backslash \overline{B_\varepsilon}$. On $\omega_\varepsilon = B_{\varepsilon_0} \backslash \overline{B_\varepsilon}$ for some fixed $\varepsilon_0 > 0$, we verify that the $H^1(\omega_\varepsilon)$ norm of w remains bounded independently of ε, which is not the case for the fundamental solution g_y; whence the contradiction implying that g_y is not in the Range of L^* when $y \in D^c$. □

The factorization method is one example in the class of so-called qualitative reconstruction methodologies. The interested reader is referred to [25, 40, 48] for additional information about such methods.

12.2 Reconstructing small inclusions

This second section concerns the reconstruction of small inclusions. We have seen that the reconstruction of diffusion or absorption coefficients in an elliptic equation resulted in a severely ill-posed problem. The previous section dealt with the issue by reconstructing the support of an inclusion instead of its detailed structure. Because the support of the inclusion remains an infinite dimensional object, the stability of the reconstruction is still a severely ill-posed problem. Here we further simplify the problem by assuming that the inclusions have small support. This introduces a small parameter allowing us to perform asymptotic expansions. We can then characterize the influence of the inclusion on the boundary measurements by successive terms in the expansion. The interest of such a procedure is the following. Since high-order terms in the expansion become asymptotically negligible, the procedure tells us which parameters can be reconstruction from a given noise level in the measurements and which parameters cannot possibly be estimated. Moreover, each term in the asymptotic expansion is characterized by a finite number of parameters. This implies that by truncating the expansion, we are parameterizing the reconstruction with a finite number of parameters. Unlike the previous reconstructions, this becomes a well-posed problem since the ill-posedness comes from the infinite dimensionality of the parameters we want to reconstruct, at least as long as the mapping from the object to be reconstructed to the noise-free measurements is one-to one (injective).

We consider in this chapter a mathematically very simple problem, namely the reconstruction of inclusions characterized by a variation in the absorption coefficient. We also restrict ourselves to the reconstruction from the leading term in the aforementioned asymptotic expansion. The interesting problem of variations of the diffusion coefficient is mathematically more difficult, although the main conclusions are in the end very similar. The presentation follows that in [7].

12.2.1 First-order effects

Let us consider the problem of optical tomography modeled by a diffusion equation on a domain X with current density $g(x)$ prescribed at the boundary ∂X. We assume here that the diffusion coefficient is known and to simplify, is set to $\gamma \equiv 1$. Our main hypothesis on the absorption coefficient is that is is the superposition of a background absorption, to simplify the constant σ_0, and a finite number of fluctuations of arbitrary size $\sigma_m - \sigma_0$, with σ_m constant to simplify, but of small volume. Smallness of the volume of the inclusions compared to the volume of the whole domain is characterized by the small parameter $\varepsilon \ll 1$. The diffusion equation with small absorption inclusions then takes the form

$$\begin{aligned} -\Delta u_\varepsilon(x) + \sigma_\varepsilon(x) u_\varepsilon(x) &= 0, \qquad X \\ \frac{\partial u_\varepsilon}{\partial \nu} &= g, \qquad \partial X, \end{aligned} \tag{12.24}$$

where absorption is given by

$$\sigma_\varepsilon(x) = \sigma_0 + \sum_{m=1}^{M} \sigma_m \chi_{z_m + \varepsilon B_m}(x). \tag{12.25}$$

We have introduced here εB_m as the shape of the mth inclusion centered at z_m, and $\chi_{z_m+\varepsilon B_m}(x) = 1$ if $x - z_m \in \varepsilon B_m$ and 0 otherwise. The inclusions are centered at *zero* in the sense that

$$\int_{B_m} x dx = 0 \qquad \text{for all } m, \tag{12.26}$$

and are assumed to be at a distance greater than $d > 0$, independent of ε, of each-other and of the boundary ∂X. The parameter ε is a measure of the diameter of the inclusions. In the three-dimensional setting, which we assume from now on, this implies that the volume of the inclusions is of order ε^3.

We want to derive an asymptotic expansion for u_ε in powers of ε and observe which information about the inclusions we can deduce from the first terms in the expansion. Let us first define the Green function of the corresponding homogeneous problem

$$\begin{aligned} -\Delta G(x;z) + \sigma_0 G(x;z) &= \delta(x-z), \qquad X \\ \frac{\partial G}{\partial \nu}(x;z) &= 0, \qquad \partial X, \end{aligned} \tag{12.27}$$

and the homogeneous-domain solution $U(x)$ of

$$\begin{aligned} -\Delta U(x) + \sigma_0 U(x) &= 0, \qquad X \\ \frac{\partial U}{\partial \nu}(x) &= g(x), \qquad \partial X. \end{aligned} \tag{12.28}$$

As $\varepsilon \to 0$, the volume of the inclusions tends to 0 and u_ε converges to U. To show this, we multiply (12.27) by u_ε and integrate by parts to obtain

$$u_\varepsilon(z) = \int_{\partial X} g(x) G(x;z) d\sigma(x) - \sum_{m=1}^{M} \int_{z_m + \varepsilon B_m} \sigma_m G(x;z) u_\varepsilon(x) dx.$$

Using the same procedure for $U(x)$, we obtain

$$u_\varepsilon(z) = U(z) - \sum_{m=1}^{M} \int_{z_m+\varepsilon B_m} \sigma_m G(x;z) u_\varepsilon(x) dx. \tag{12.29}$$

In three space dimensions, the Green function is given by

$$G(x;z) = \frac{e^{-\sqrt{\sigma_0}|z-x|}}{4\pi|z-x|} + w(x;z), \tag{12.30}$$

where $w(x;z)$ is a smooth function (because it solves (12.28) with smooth boundary conditions) provided that ∂X is smooth. For z at a distance greater than $d > 0$ away from the inclusions $x_m + \varepsilon B_m$, we then deduce from the L^∞ bound on u_ε (because g and ∂X are assumed to be sufficiently regular) that

$$u_\varepsilon(z) = U(z) + O(\varepsilon^3).$$

In the vicinity of the inclusions, we deduce from the relation

$$\int_{z_m+\varepsilon B_m} G(x;z) dx = O(\varepsilon^2), \qquad z - z_m \in \varepsilon B_m,$$

that $u_\varepsilon(z) - U(z)$ is of order ε^2 when z is sufficiently close to an inclusion. This also shows that the operator

$$K_\varepsilon u_\varepsilon(z) = -\sum_{m=1}^{M} \int_{z_m+\varepsilon B_m} \sigma_m G(x;z) u_\varepsilon(x) dx \tag{12.31}$$

is a bounded linear operator in $\mathcal{L}(L^\infty(X))$ with a norm of order ε^2. This implies that for sufficiently small values of ε, we can write

$$u_\varepsilon(z) = \sum_{k=0}^{\infty} K_\varepsilon^k U(z). \tag{12.32}$$

The above series converges fast when ε is small. Notice however that the series does not converge as fast as ε^3, the volume of the inclusions, because of the singular behavior of the Green function $G(x;z)$ when x is close to z.

Let us now use that

$$\begin{aligned} u_\varepsilon(z) = U(z) &- \sum_{m=1}^{M} \int_{z_m+\varepsilon B_m} \sigma_m G(x;z) U(x) dx \\ &+ \sum_{m=1}^{M} \sum_{n=1}^{M} \int_{z_m+\varepsilon B_m} \int_{z_n+\varepsilon B_n} \sigma_m \sigma_n G(x;z) G(y;x) u_\varepsilon(y) dy dx. \end{aligned} \tag{12.33}$$

For the same reasons as above, the last term is of order ε^5, and expanding smooth solutions $U(x)$ and $G(x;z)$ inside inclusions of diameter ε, we obtain that

$$u_\varepsilon(x) = U(x) - \sum_{m=1}^{M} G(z;z_m) C_m U(z_m) + O(\varepsilon^5), \tag{12.34}$$

where C_m is given by

$$C_m = \varepsilon^3 |B_m| \sigma_m. \tag{12.35}$$

The reason why we obtain a correction term of order ε^5 in (12.34) comes from the fact that (12.26) holds so that the terms of order ε^4, proportional to $x \cdot \nabla U$ or $x \cdot \nabla G$, vanish.

12.2.2 Stability of the reconstruction

The above analysis tells us the following. Provided that our measurement errors are of order $O(\varepsilon^5)$, the only information that can possibly be retrieved on the inclusions is its location z_m and the product $C_m = \varepsilon^3 \sigma_m B_m$ of the absorption fluctuation with the volume of the inclusion. More refined information requires data with less noise. Assuming that the inclusions are sufficiently small so that the above asymptotic expansion makes sense, no other information can be obtained in a stable fashion from the data.

Note that the problem we now wish to solve is finite-dimensional. Indeed, each inclusion is represented by four real numbers, namely the three components of the position z_m and the product C_m. Assuming that only M inclusions are present, this leaves us with $4M$ parameters to reconstruct. The main advantage of reconstructing a finite number of parameters is that it is natural to expect stability of the reconstruction. We can even show stability of the reconstruction from boundary measurements corresponding to one current density $g(x)$ provided that the homogeneous solution $U(x)$ is uniformly positive inside the domain. Here is how it can be proved.

Let us assume that the boundary measurements have an accuracy of order $O(\varepsilon^5)$, which is consistent with

$$u_\varepsilon(z) = U(z) - \sum_{m=1}^{M} C_m(G(z_m; z)U(z_m)) + O(\varepsilon^5). \tag{12.36}$$

We denote by u_ε and u'_ε the solution of two problems with absorption coefficients σ_ε and σ'_ε of the form (12.25). Using (12.36), we obtain that

$$u_\varepsilon(z) - u'_\varepsilon(z) = F(z) + O(\varepsilon^5),$$

with

$$F(z) = -\sum_{m=1}^{M} \Big(C_m(G(z_m; z)U(z_m)) - C'_m(G(z'_m; z)U(z'_m)) \Big). \tag{12.37}$$

Here we use $M = \max(M, M')$ with a small abuse of notation; we will see shortly that $M = M'$. The function $F(z)$ satisfies the homogeneous equation $-\Delta F + \sigma_0 F = 0$ on X except at the points z_m and z'_m. Moreover, we have that $\frac{\partial F}{\partial \nu} = 0$ at ∂X. If $F = 0$ on ∂X, we deduce from the uniqueness of the Cauchy problem for the operator $-\Delta + \sigma_0$ that $F \equiv 0$ in X. As $\varepsilon \to 0$ and $u_\varepsilon - u'_\varepsilon \to 0$, we deduce that $F(z)$ becomes small not only at ∂X but also inside X (the continuation of F from ∂X to $X \backslash \{z_m \cup z'_m\}$ is independent of ε). However, the functions $G(z_m; z)U(z_m)$ form an independent family. Each term must therefore be compensated by a term from the sum over the *prime* coefficients. We thus obtain that $M = M'$ and that

$$\Big| C_m(G(z_m; z)U(z_m)) - C'_m(G(z'_m; z)U(z'_m)) \Big| \quad \leq \quad C\|u_\varepsilon - u'_\varepsilon\|_{L^\infty(\partial X)} + O(\varepsilon^5).$$

The left-hand side can be recast as

$$(C_m - C'_m)G(z_m; z)U(z_m) + C'_m(z_m - z'_m)\partial_{z_m}(G(\bar{z}_m; z)U(\bar{z}_m))$$

where $\bar{z}_m = \theta z_m + (1-\theta)z'_m$ for some $\theta \in (0, 1)$. Again these two functions are linearly independent and so we deduce that

$$|C_m - C'_m| + |C'_m||z_m - z'_m| \leq C\|u_\varepsilon - u'_\varepsilon\|_{L^\infty(\partial X)} + O(\varepsilon^5).$$

Using (12.34) and (12.35), we then obtain assuming that $\|u_\varepsilon - u'_\varepsilon\|_{L^\infty(\partial X)} \approx \varepsilon^5$, that

$$\left|B_m\sigma_m - B'_m\sigma'_m\right| + |z_m - z'_m| \leq C\varepsilon^{-3}\|u_\varepsilon - u'_\varepsilon\|_{L^\infty(\partial X)} \approx \varepsilon^2. \tag{12.38}$$

Assuming that the accuracy of the measured data is compatible with the expansion (12.34), i.e. that the u_ε is known on ∂X up to an error term of order ε^5, we can then reconstruct the location z_m of the heterogeneities up to an error of order ε^2. The product of the volume of the inclusion and the absorption fluctuation is also known with the same accuracy.

The analysis of small volume expansions has been analyzed in much more general settings than the one presented above. We refer the reader to [5] for additional details. In the context of the reconstruction of absorbing inclusions as was considered above, the expression of order $O(\varepsilon^5)$ and the additional information it provides about inclusions may be found in [7].

Appendix A

Notation and results

A.1 Fourier transform

Let $f(x)$ be a complex-valued function in $L^2(\mathbb{R}^n)$ for some $n \in \mathbb{N}^*$, which means a (measurable) function on $\mathbb{R}^n$ that is square integrable in the sense that

$$\|f\|^2 = \int_{\mathbb{R}^n} |f(x)|^2 dx < \infty. \tag{A.1}$$

Here $\|f\|$ is the $L^2(\mathbb{R}^n)$-norm of f and dx the Lebesgue (volume) measure on $\mathbb{R}^n$. We define the Fourier transform of f as

$$\hat{f}(k) = [\mathcal{F}_{x\to k} f](k) = \int_{\mathbb{R}^n} e^{-ik\cdot x} f(x) dx. \tag{A.2}$$

It is a well-known results about Fourier transforms that $\hat{f}(k) \in L^2(\mathbb{R}^n)$ and the Fourier transform admits an inverse on $L^2(\mathbb{R}^n)$ given by

$$f(x) = [\mathcal{F}^{-1}_{k\to x} \hat{f}](x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ik\cdot x} \hat{f}(k) dk. \tag{A.3}$$

More precisely we have the Parseval relation

$$(\hat{f}, \hat{g}) = (2\pi)^n (f, g) \quad \text{and} \quad \|\hat{f}\| = (2\pi)^{\frac{n}{2}} \|f\|. \tag{A.4}$$

where the Hermitian product is given by

$$(f, g) = \int_{\mathbb{R}^n} f(x) \overline{g}(x) dx. \tag{A.5}$$

Here $\overline{g}$ is the complex conjugate to g. So up to the factor $(2\pi)^{\frac{n}{2}}$, the Fourier transform and its inverse are isometries.

Important properties of the Fourier transform for us here are how they interact with differentiation and convolutions. Let $\alpha = (\alpha_1, \cdots, \alpha_n)$ be a multi-index of non-negative components $\alpha_j \geq 0$, $1 \leq j \leq n$ and let $|\alpha| = \sum_{i=1}^n \alpha_j$ be the length of the multi-index. We then define the differentiation D^α of degree $|\alpha|$ as

$$D^\alpha = \prod_{i=1}^n \frac{\partial^{\alpha_i}}{\partial x_i^{\alpha_i}}. \tag{A.6}$$

We then deduce from the definition (A.2) that

$$\mathcal{F}_{x\to k}[D^\alpha f](k) \;=\; \Big(\prod_{j=1}^{n}(ik_j)^{\alpha_j}\Big)\;[\mathcal{F}_{x\to k}f](k). \tag{A.7}$$

Let us now define the convolution as

$$f\star g(x)=\int_{\mathbb{R}^n} f(x-y)g(y)dy. \tag{A.8}$$

We then verify that

$$\begin{array}{rclcrcl} \mathcal{F}_{x\to k}(f\star g) &=& \mathcal{F}_{x\to k}f\mathcal{F}_{x\to k}g, & \text{i.e.} & \widehat{f\star g} &=& \hat f\hat g,\\ \mathcal{F}^{-1}_{k\to x}(\hat f\star \hat g) &=& (2\pi)^n f\,g & \text{i.e.} & \hat f\star\hat g &=& (2\pi)^d\,\widehat{fg}. \end{array} \tag{A.9}$$

So the Fourier transform *diagonalizes* differential operators (replaces them by multiplication in the Fourier domain). However Fourier transforms replace products by non-local convolutions.

The Fourier transform is a well-posed operator from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ since the inverse Fourier transform is also defined from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ and is bounded as shown in (A.4). Let us assume that we measure

$$\hat d(k)=\hat f(k)+\hat N(k),$$

where we believe that $\delta=\|\hat N\|$ is relatively small. Then the error in the reconstruction will also be of order δ in the $L^2(\mathbb{R}^n)$ norm. Indeed let $d(x)$ be the reconstructed function from the data $d(k)$ and $f(x)$ be the real function we are after. Then we have

$$\|d-f\|=\|\mathcal{F}^{-1}_{k\to x}\hat d-\mathcal{F}^{-1}_{k\to x}\hat f\|=\|\mathcal{F}^{-1}_{k\to x}(\hat d-\hat f)\|=(2\pi)^{-\frac{n}{2}}\delta. \tag{A.10}$$

In other words, measurements errors can still be seen in the reconstruction. The resulting image is not perfect. However the error due to the noise has not been amplified too drastically.

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Index

Λ tomography, 52

Bayesian framework, 4, 5

Calderón problem, 8
canonical graph, 76
computerized tomography, 8
coupled-physics modality, 19

elliptic ΨDO , 65

Fourier integral operator, 47

hybrid inverse problem, 19

ill-posed, 3, 13
injectivity, 2

left-parametrix, 65
likelihood function, 5
Local tomography, 52

measurement operator, 2
$\mathfrak{M}$, 2
modeling, 3
multi-wave modality, 19

noise model, 3
non-degenerate phase function, 78

parametrix, 61, 65
penalization, 4
penalization theory, 5
posterior distribution, 5
prior distribution, 5
prior model, 4
propagation of singularities, 18
proper set, 62
properly supported operators, 62
Pseudo-differential operator, 47

qualitative methods, 4

Radon transform, 8
regularization, 4
regularization theory, 5

Schwartz kernel, 46
singular support, 48
sparsity, 4
stability estimate, 2