Inverse Transport Theory of Photoacoustics

Guillaume Bal

Department of Applied Physics & Applied Mathematics, Columbia University, New York, NY 10027

E-mail: gb2030@columbia.edu

Alexandre Jollivet

Department of Applied Physics & Applied Mathematics, Columbia University, New York, NY 10027

E-mail: aj2315@columbia.edu

Vincent Jugnon

Centre de mathématiques appliquées, Ecole Polytechnique, 91128 Palaiseau, France and Department of Applied Physics & Applied Mathematics, Columbia University, New York, NY 10027

E-mail: jugnon@cmapx.polytechnique.fr

Abstract. We consider the reconstruction of optical parameters in a domain of interest from photoacoustic data. Photoacoustic tomography (PAT) radiates high frequency electromagnetic waves into the domain and measures acoustic signals emitted by the resulting thermal expansion. Acoustic signals are then used to construct the deposited thermal energy map. The latter depends on the constitutive optical parameters in a nontrivial manner. In this paper, we develop and use an inverse transport theory with internal measurements to extract information on the optical coefficients from knowledge of the deposited thermal energy map. We consider the multi-measurement setting in which many electromagnetic radiation patterns are used to probe the domain of interest. By developing an expansion of the measurement operator into singular components, we show that the spatial variations of the intrinsic attenuation and the scattering coefficients may be reconstructed. We also reconstruct coefficients describing anisotropic scattering of photons, such as the anisotropy coefficient $g(x)$ in a Henyey-Greenstein phase function model. Finally, we derive stability estimates for the reconstructions.

Keywords. Photoacoustics, Optoacoustics, transport equation, inverse problems, stability estimates, internal measurements.

1. Introduction

Photoacoustic imaging is a recent medical imaging technique combining the large contrast between healthy and unhealthy tissues of their optical parameters with the high spatial resolution of acoustic (ultrasonic) waves. Electromagnetic radiation, sent through a domain of interest, generates some heating and a resulting thermal expansion of the underlying tissues. The mechanical displacement of the tissues generates acoustic waves, which then propagate through the medium and are recorded by an array of detectors (ultrasound transducers). The photoacoustic effect is now being actively investigated for its promising applications in medical imaging. We refer the reader to e.g. [12, 15, 17] for recent reviews on the practical and theoretical aspects of the method.

In an idealized setting revisited below, the electromagnetic source is a very short pulse that propagates through the domain at a scale faster than that of the acoustic waves. The measured acoustic signals may then be seen as being emitted by unknown initial conditions. A first step in the inversion thus consists in reconstructing this initial condition by solving an inverse source problem for a wave equation. This inversion is relatively simple when the sound speed is constant and full measurements are available. It becomes much more challenging when only partial measurements are available and the sound speed is not constant; see e.g. [1, 8, 12, 15, 16].

A second step consists of analyzing the initial condition reconstructed in the first step and extracting information about the optical coefficients of the domain of interest. The second step is much less studied. The energy deposited by the radiation is given by the product of $\sigma_a(x)$, the attenuation in the tissue and of $I(x)$, the radiation intensity. The question is therefore what information on the medium may be extracted from $\sigma_a I$. The product can be plotted as a proxy for σ_a when I is more or less uniform. This, however, generates image distortions as has been reported e.g., in [13].

Two different regimes of radiation propagation should then be considered. In thermoacoustic tomography (TAT), low frequency (radio-frequency) waves with wavelengths much larger than the domain of interest, are being used. We do not consider this modality here. Rather, we assume that high frequency radiation is generated in the near infra red (NIR) spectrum. NIR photons have the advantage that they may propagate over fairly large distances (with an absorption mean free path of order one centimeter) before being absorbed. Moreover, their absorption properties have a very large contrast between healthy and cancerous tissues. In this regime, $I(x)$ may be interpreted as a spatial density of photons propagating in the domain of interest. The density of photons is then modeled by a transport equation that accounts for photon propagation, absorption, and scattering; see (2.12) below.

This paper concerns the reconstruction of the optical parameters in a steady-state transport equation from knowledge of $H(x) := \sigma_a(x)I(x)$. The derivation of the transport equation given in (2.12) below is addressed in section 2. We are concerned here with the setting of measurements of $H(x)$ for different radiation patterns. Our most general measurement operator A is then the operator which to arbitrary radiation patterns at the domain's boundary maps the deposited energy $H(x)$.

We analyze the reconstruction of the absorption and scattering properties of the photons from knowledge of A. The main tool used in the analysis is the decomposition of

A into singular components, in a spirit very similar to what was done e.g. in [4, 3, 7] (see also [2]) in the presence of boundary measurements rather than internal measurements. The most singular component is related to the ballistic photons, as in the application of X-ray Computerized Tomography. We show that the analysis of that component allows us to reconstruct the attenuation coefficient σ_a and the spatial component σ_s of the scattering coefficient. The anisotropic behavior of scattering is partially determined by the second most singular term in A, which accounts for photons having scattered only once in the domain. Although the full phase function of the scattering coefficient cannot be reconstructed with the techniques described in this paper, we show that the anisotropy coefficient $g(x)$ that appears in the classical Henyey-Greenstein phase function is uniquely determined by the measurements in spatial dimensions $n = 2$ and $n = 3$. Moreover, all the parameters that can be reconstructed are obtained with Hölder-type stability. We present the stability results in detail.

When scattering is large so that mean free path l^* is small, then the amount of ballistic and single scattering photons is exponentially small and proportional to $\exp\left(-\frac{\mathrm{diam}X}{l^*}\right)$, where $\mathrm{diam}X$ is the diameter of the domain of interest. In such a regime, measurements may no longer be sufficiently accurate to separate the ballistic and single scattering contributions from multiply scattered photons. Radiation is then best modeled by a diffusion equation characterized by two unknown coefficients, the diffusion coefficient $D(x)$ and the attenuation coefficient $\sigma_a(x)$. This regime is briefly mentioned in section 2.8.

The rest of the paper is structured as follows. Section 2 is devoted to the derivation of the stationary inverse transport problem starting from the transient equation for the short electromagnetic pulse. The main uniqueness and stability results are also presented in detail in this section. The derivation of the uniqueness and stability results is postponed to the technical sections 3 and 4. The former section is devoted to the decomposition of the albedo operator into singular components. Useful results on the transport equation are also recalled. The latter section presents in detail the proofs of the stability results given in section 2.

2. Derivation and main results

2.1. Transport and inverse wave problem

The propagation of radiation is modeled by the following radiative transfer equation

$$\frac{1}{c}\frac{\partial}{\partial t}u(t,x,v) + Tu(t,x,v) = S(t,x,v), \quad t \in \mathbb{R},\, x \in \mathbb{R}^n,\, v \in \mathbb{S}^{n-1}. \tag{2.1}$$

We assume here that $S(t,x,v)$ is compactly supported in $t \geq 0$ and in x outside of a bounded domain of interest X we wish to probe. The domain X is assumed to be an open subset of $\mathbb{R}^n$ with C^1 boundary. For the sake of simplicity, X is also assumed to be convex although all uniqueness and stability results in this paper remain valid when X is not convex. Here, c is light speed, $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$ and T is the transport operator defined as

$$Tu = v \cdot \nabla_x u + \sigma(x,v)u - \int_{\mathbb{S}^{n-1}} k(x,v',v)u(t,x,v')dv', \tag{2.2}$$

where $\sigma(x,v)$ is the total attenuation coefficient and $k(x,v',v)$ is the scattering coefficient. Both coefficients are assumed to be non-negative and bounded by a constant $M < \infty$ throughout the paper. We define

$$\sigma_s(x,v) = \int_{\mathbb{S}^{n-1}} k(x,v,v')dv', \qquad \sigma_a(x,v) = \sigma(x,v) - \sigma_s(x,v). \tag{2.3}$$

We also refer to σ_s as a scattering coefficient and define σ_a as the intrinsic attenuation coefficient. We assume that for all $(x,v) \in \mathbb{R}^n \times \mathbb{S}^{n-1}$, we have

$$0 < \sigma_0 \leq \sigma_a(x,v) \leq \sigma_1. \tag{2.4}$$

We also assume that $k(x,v,v') = 0$ for all $x \notin X$.

The optical coefficients $k(x,v',v)$ and $\sigma(x,v)$ are the unknown coefficients inside X that we would like to reconstruct by probing the domain X by radiation modeled by $S(t,x,v)$. In photo-acoustics, the emitted radiation generates some heating inside the domain X. Heating then causes some dilation, which mechanically induces acoustic waves. Such acoustic waves are measured at the boundary of the domain X. After time reversion, the latter measurements allow us to infer the intensity of the source of heating. This gives us internal measurements of the solution u of the transport equation (2.1). The objective of this paper is to understand which parts of the optical parameters may be reconstructed from such information and with which stability.

Before doing so, we need an accurate description of the propagation of the acoustic waves generated by the radiative heating. The proper model for the acoustic pressure is given by the following wave equation (see e.g. [11])

$$\Box p(t,x) = \beta \frac{\partial}{\partial t} H(t,x), \tag{2.5}$$

where $\Box$ is the d'Alembertian defined as

$$\Box p = \frac{1}{c_s^2(x)} \frac{\partial^2 p}{\partial t^2} - \Delta p, \tag{2.6}$$

with c_s the sound speed, where β is a coupling coefficient assumed to be constant and known, and where $H(t,x)$ is the thermal energy deposited by the radiation given by

$$H(t,x) = \int_{\mathbb{S}^{n-1}} \sigma_a(x,v')u(t,x,v')dv'. \tag{2.7}$$

Not surprisingly, the amount of heating generated by radiation is proportional to the amount of radiation u and to the rate of (intrinsic) absorption σ_a.

As it stands, the problem of the reconstruction of the source term $H(t,x)$ inside X from measurements of $p(t,x)$ on the boundary ∂X is ill-posed, because H is

$(n+1)-$dimensional whereas information on $(t,x) \in \mathbb{R}_+ \times \partial X$ is $n-$dimensional. What allows us to simplify the inverse acoustic problem is the difference of time scales between the sound speed c_s and the light speed c.

To simplify the analysis, we assume that c_s is constant and rescale time so that $c_s = 1$. Then c in (2.1) is replaced by $\frac{c}{c_s}$, which in water is of order $2.3\,10^8/1.5\,10^3 \approx 1.5\,10^5 := \frac{1}{\varepsilon} \gg 1$. The transport scale is therefore considerably faster than the acoustic scale. As a consequence, when the radiation source term $S(t,x,v)$ is supported on a scale much faster than the acoustic scale, then u is also supported on a scale much faster than the acoustic scale and as a result $H(t,x)$ can be approximated by a source term supported at $t=0$.

More specifically, let us assume that the source of radiation is defined at the scale $\varepsilon = \frac{c_s}{c}$ so that S is replaced by

$$S_\varepsilon(t,x,v) = \frac{1}{\varepsilon}\rho\big(\frac{t}{\varepsilon}\big)S_0(x,v), \tag{2.8}$$

where $\rho \geq 0$ is a function compactly supported in $t \in (0,\infty)$ such that $\int_{\mathbb{R}_+} \rho(t)dt = 1$. The transport solution then solves

$$\frac{1}{c}\frac{\partial}{\partial t}u_\varepsilon(t,x,v) + Tu_\varepsilon(t,x,v) = S_\varepsilon(t,x,v).$$

With $c_s = 1$, we verify that u_ε is given by

$$u_\varepsilon(t,x,v) = \frac{1}{\varepsilon}u\Big(\frac{t}{\varepsilon},x,v\Big), \tag{2.9}$$

where u solves (2.1) with $S(t,x,v) = \rho(t)S_0(x,v)$. Because $\sigma_a \geq \sigma_0 > 0$, we verify that u decays exponentially in time. This shows that u_ε lives at the time scale ε so that $H_\varepsilon(t,x) = \int_{\mathbb{S}^{n-1}} \sigma_a(x,v)u_\varepsilon(t,x,v)dv$ is also primarily supported in the vicinity of $t=0$.

Let us formally derive the equation satisfied by p_ε when $\varepsilon \to 0$. Let $\varphi(t,x)$ be a test function and define $(\cdot,\cdot)$ as the standard inner product on $\mathbb{R}\times\mathbb{R}^n$. Then we find that

$$(\Box p_\varepsilon, \varphi) = \beta(\frac{\partial H_\varepsilon}{\partial t},\varphi) = -\beta(H_\varepsilon,\frac{\partial\varphi}{\partial t}) = -\beta(H,\frac{\partial\varphi}{\partial t}(\varepsilon t))),$$

where $H(t,x) = \int_{\mathbb{S}^{n-1}} \sigma_a(x,v)u(t,x,v)dv$ with u defined in (2.9) and where we have used the change of variables $t \to \varepsilon t$. The latter term is therefore equal to

$$-\beta(H,\frac{\partial\varphi}{\partial t}(0)) = -\beta\frac{\partial\varphi}{\partial t}(0)\int_{\mathbb{R}\times\mathbb{S}^{n-1}} \sigma_a(x,v)u(t,x,v)dtdv,$$

up to a small term for φ sufficiently smooth. We thus find that p_ε converges weakly as $\varepsilon \to 0$ to the solution p of the following wave equation

$$\begin{aligned}
&\Box p = 0 && t>0, \quad x\in\mathbb{R}^n \\
&p(0,x) = H_0(x) := \int_{\mathbb{R}\times\mathbb{S}^{n-1}} \sigma_a(x,v)u(t,x,v)dtdv, && x\in\mathbb{R}^n \\
&\frac{\partial p}{\partial t}(0,x) = 0 && x\in\mathbb{R}^n.
\end{aligned} \tag{2.10}$$

The inverse problem for the wave equation is now well-posed. The objective is to reconstruct $H_0(x)$ for $x \in X$ from measurements of $p(t,x)$ for $t \geq 0$ and $x \in \partial X$. Such an inverse problem has been extensively studied in the literature; see e.g. [1, 8, 12, 15, 16]. In typical experiments with 10ns pulses and measurements of acoustic signals bandlimited to about $f = 50$MHz, we obtain in ideal situations a spatial resolution $\delta x = c_s/f \sim 30\mu m$.

In this paper, we assume that $H_0(x)$ has been reconstructed accurately as a functional of the radiation source $S_0(x,v)$. Our objective is to understand which parts of the optical parameters $\sigma(x,v)$ and $k(x,v',v)$ can be reconstructed from knowledge of $H_0(x)$ for a given set of radiations $S_0(x,v)$. Note that we will allow ourselves to generate many such $S_0(x,v)$ and thus consider a multi-measurement setting.

The time average $u(x,v) := \int u(t,x,v)dt$ satisfies a closed-form steady state transport equation given by

$$Tu = S_0(x,v), \qquad (x,v) \in \mathbb{R}^n \times \mathbb{S}^{n-1}, \tag{2.11}$$

as can be seen by averaging (2.1) in time since $\int \rho(t)dt = 1$.

2.2. Inverse transport with internal measurements

Since $S_0(x,v)$ is assumed to be supported outside of the domain X and scattering $k(x,v',v) = 0$ outside of X, the above transport equation may be replaced by a boundary value problem of the form

$$\begin{aligned} & v \cdot \nabla_x u + \sigma(x,v)u - \int_{\mathbb{S}^{n-1}} k(x,v',v)u(x,v')dv' = 0, && (x,v) \in X \times \mathbb{S}^{n-1} \\ & u(x,v) = \phi(x,v) && (x,v) \in \Gamma_-, \end{aligned} \tag{2.12}$$

where the sets of outgoing and incoming boundary radiations are given by

$$\Gamma_\pm = \{(x,v) \in \partial X \times \mathbb{S}^{n-1}, \quad \pm v \cdot \nu(x) > 0\}, \tag{2.13}$$

where $\nu(x)$ is the outward normal to X at $x \in \partial X$ and $\phi(x,v)$ is an appropriate set of incoming radiation conditions obtained by solving $v \cdot \nabla_x u + \sigma_a(x,v)u = S_0$ outside of X assuming that $\sigma_a(x,v)$ is known outside of X.

We are now ready to state the inverse transport problem of interest in this paper. It is well known that (2.12) admits a unique solution in $L^1(X \times \mathbb{S}^{n-1})$ when $\phi(x,v) \in L^1(\Gamma_-, d\xi)$, where $d\xi = |v \cdot \nu(x)| d\mu(x) dv$ with $d\mu$ the surface measure on ∂X. We thus define the albedo operator as

$$\begin{aligned} A: \quad L^1(\Gamma_-, d\xi) &\to L^1(X) \\ \phi(x,v) &\mapsto A\phi(x) = H(x) := \int_{\mathbb{S}^{n-1}} \sigma_a(x,v)u(x,v)dv. \end{aligned} \tag{2.14}$$

The inverse transport problem with angularly averaged internal measurements thus consists of understanding what can be reconstructed of the optical parameters $\sigma(x,v)$ and $k(x,v',v)$ from complete or partial knowledge of the albedo operator A. We also wish to understand the stability of such reconstructions.

2.3. Albedo operator and decomposition

The inverse transport problem and its stability properties are solved by looking at a decomposition of the albedo operator into singular components. Let $\alpha(x, x', v')$ be the Schwartz kernel of the albedo operator A, i.e., the distribution such that

$$A\phi(x) = \int_{\Gamma_-} \alpha(x, x', v')\phi(x', v')d\mu(x')dv'. \tag{2.15}$$

The kernel $\alpha(x, x', v')$ corresponds to measurements of $H(x)$ at $x \in X$ for a radiation condition concentrated at $x' \in \partial X$ and propagating with direction $v' \in \mathbb{S}^{n-1}$. Such a kernel can thus be obtained as a limit of physical experiments with sources concentrated in the vicinity of (x', v') and detectors concentrated in the vicinity of x.

The kernel $\alpha(x, x', v')$ accounts for radiation propagation inside X, including all orders of scattering of the radiation with the underlying structure. It turns out that we can extract from $\alpha(x, x', v')$ singular components that are not affected by multiple scattering. Such singular components provide useful information on the optical coefficients. Let us define the ballistic part of transport as the solution of

$$v \cdot \nabla_x u_0 + \sigma(x, v)u_0 = 0, \quad \text{in } X \times \mathbb{S}^{n-1}, \qquad u_0 = \phi, \quad \text{on } \Gamma_-. \tag{2.16}$$

Then for $m \geq 1$, we define iteratively

$$v \cdot \nabla_x u_m + \sigma(x, v)u_m = \int k(x, v', v)u_{m-1}(x, v')dv', \text{ in } X \times \mathbb{S}^{n-1}, \quad u_m = 0, \text{ on } \Gamma_-. \tag{2.17}$$

This allows us to decompose the albedo operator as

$$A = A_0 + A_1 + \mathcal{G}_2, \tag{2.18}$$

where A_k for $k = 0, 1$ are defined as A in (2.14) with u replaced by u_k and where $\mathcal{G}_2$ is defined as $A - A_0 - A_1$. Thus, A_0 is the contribution in A of particles that have not scattered at all with the underlying structure while A_1 is the contribution of particles that have scattered exactly once and $\mathcal{G}_2$ the contribution of particles that have scattered at least twice.

Let α_k for $k = 0, 1$ be the Schwartz kernel of A_k and Γ_2 the Schwartz kernel of $\mathcal{G}_2$ using the same convention as in (2.15). We define $\tau_\pm(x, v)$, $\tau(x, v)$ for $x \in X$ and $v \in \mathbb{S}^{n-1}$ as $\tau_\pm(x, v) = \inf\{s \in \mathbb{R}_+ | x \pm sv \notin X\}$ and $\tau(x, v) = (\tau_- + \tau_+)(x, v)$. Thus, $\tau_\pm(x, v)$ indicates the time of escape from X of a particle at x moving in direction $\pm v$ while $\tau(x, v)$ measures the total time (or distance since speed is normalized to 1) inside X of a particle with direction v passing through x. On ∂X, we also define $\delta_{\{x\}}(y)$ as the distribution such that $\int_{\partial X} \delta_{\{x\}}(y)\phi(y)d\mu(y) = \phi(x)$ for any continuous function ϕ on ∂X. Finally, we define the following terms that quantify attenuation. We define the function $E(x_0, x_1)$ on $X \times \partial X$ as

$$E(x_0, x_1) = \exp\Big(- \int_0^{|x_0 - x_1|} \sigma\big(x_0 - s\frac{x_0 - x_1}{|x_0 - x_1|}, \frac{x_0 - x_1}{|x_0 - x_1|}\big)ds\Big). \tag{2.19}$$

We still denote by E the function defined above for x_1 in X. Then by induction on m, we define

$$E(x_1,\ldots,x_m) = E(x_1,\ldots,x_{m-1})E(x_{m-1},x_m). \tag{2.20}$$

The latter term measures the attenuation along the broken path $[x_1,\ldots,x_m]$.

Then we have the following result.

Theorem 2.1 *Let α_0, α_1, and Γ_2 be the Schwartz kernels defined as above. Then we have:*

$$\begin{aligned}
\alpha_0(x,x',v') &= \sigma_a(x,v')\exp\Big(-\int_0^{\tau_-(x,v')}\sigma(x-sv',v')ds\Big)\delta_{\{x-\tau_-(x,v')v'\}}(x') \\
\alpha_1(x,x',v') &= |\nu(x')\cdot v'| \\
&\quad \int_0^{\tau_+(x',v')}\sigma_a(x,v)\frac{E(x,x'+t'v',x')}{|x-x'-t'v'|^{n-1}}k(x'+t'v',v',v)|_{v=\frac{x-x'-t'v'}{|x-x'-t'v'|}}dt'.
\end{aligned} \tag{2.21}$$

Moreover, we have the bound

$$\begin{aligned}
\frac{\Gamma_2(x,x',v')}{|\nu(x')\cdot v'|} &\in L^\infty(X\times\Gamma_-) \quad \textit{when} \quad n=2 \\
\frac{\Gamma_2(x,x',v')}{|\nu(x')\cdot v'|\ln(|x-x'-((x-x')\cdot v')v'|)} &\in L^\infty(X\times\Gamma_-) \quad \textit{when} \quad n=3 \\
\frac{|x-x'-((x-x')\cdot v')v'|^{n-3}\Gamma_2(x,x',v')}{|\nu(x')\cdot v'|} &\in L^\infty(X\times\Gamma_-) \quad \textit{when} \quad n\geq 4.
\end{aligned} \tag{2.22}$$

This theorem will be proved in section 3. The results show that the ballistic contribution α_0 is more singular than the scattering contributions α_1 and Γ_2. It turns out that the single scattering contribution α_1 is also more singular than the multiple scattering contribution Γ_2 in the sense that it is asymptotically much larger than Γ_2 in the vicinity of the support of the ballistic part α_0. That this is the case is the object of the following result (see also Lemma 3.2 (3.9) and (3.11) for "$m=1$"). For any topological space Y, we denote by $\mathcal{C}_b(Y)$ the set of the bounded continuous functions from Y to $\mathbb{R}$.

Theorem 2.2 *Let us assume that $\sigma\in\mathcal{C}_b(X\times\mathbb{S}^{n-1})$ and that $k\in\mathcal{C}_b(X\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})$. For $(x',v')\in\Gamma_-$, let $x=x'+t_0'v'$ for some $t_0'\in(0,\tau_+(x',v'))$. Let $v'^\perp\in\mathbb{S}^{n-1}$ be such that $v'\cdot v'^\perp=0$. Then we have the following asymptotic expansion:*

$$\begin{aligned}
\frac{\alpha_1(x+\varepsilon v'^\perp,x',v')}{E(x,x')|\nu(x')\cdot v'|} &= \Big(\ln\frac{1}{\varepsilon}\Big)\big(\chi(x,v',v')+\chi(x,v',-v')\big)+o\Big(\ln\frac{1}{\varepsilon}\Big) \\
\frac{\alpha_1(x+\varepsilon v'^\perp,x',v')}{E(x,x')|\nu(x')\cdot v'|} &= \frac{1}{\varepsilon^{n-2}}\int_0^\pi \sin^{n-3}\theta\chi(x,v',v(\theta))d\theta+o\Big(\frac{1}{\varepsilon^{n-2}}\Big),
\end{aligned} \tag{2.23}$$

for $n=2$ and $n\geq 3$, respectively, where we have defined the functions $\chi(x,v',v)=\sigma_a(x,v)k(x,v',v)$ and $v(\theta)=\cos\theta v'+\sin\theta v'^\perp$.

Theorem 2.2 will be proved in section 3.

Thus (2.22) and (2.23) show that α_1 is asymptotically larger than Γ_2 as $\varepsilon\to 0$, i.e., as the observation point x becomes closer to the segment where the ballistic term α_0 is supported. This singularity allows us to obtain information on the optical coefficients that is not contained in the ballistic part α_0. Moreover, because of the singular behavior of α_1, such information can be reconstructed in a stable manner.

2.4. Stability estimates

As mentioned above, the singular behaviors of α_0 and α_1 allow us to extract them from the full measurement α. Moreover, such an extraction can be carried out in a stable fashion, in the sense that small errors in the measurement of the albedo operator translate into small errors in the extraction of the terms characterizing α_0 and α_1.

More precisely, let A be the albedo operator corresponding to the optical parameters (σ, k) and $\tilde{A}$ the operator corresponding to the optical parameters $(\tilde{\sigma}, \tilde{k})$. From now on, a term superimposed with the ˜ sign means a term calculated using the optical parameters $(\tilde{\sigma}, \tilde{k})$ instead of (σ, k). For instance $\tilde{E}(x, y)$ is the equivalent of $E(x, y)$ defined in (2.19) with (σ, k) replaced by $(\tilde{\sigma}, \tilde{k})$.

We first derive the stability of useful functionals of the optical parameters in terms of errors made on the measurements. Let us assume that A is the "real" albedo operator and that $\tilde{A}$ is the "measured" operator. We want to obtain error estimates on the useful functionals of the optical parameters in terms of appropriate metrics for $A - \tilde{A}$. We obtain the following two results. The first result pertains to the stability of the ballistic term in the albedo operator:

Theorem 2.3 *Let A and $\tilde{A}$ be two albedo operators and $(x', v') \in \Gamma_-$. Then we obtain that*

$$\begin{aligned} &\int_0^{\tau_+(x',v')} \left| \sigma_a(x'+tv',v') e^{-\int_0^t \sigma(x'+sv',v')ds} - \tilde{\sigma}_a(x'+tv',v') e^{-\int_0^t \tilde{\sigma}(x'+sv',v')ds} \right| dt \\ \leq\ & \|A - \tilde{A}\|_{\mathcal{L}(L^1(\Gamma_-,d\xi);L^1(X))}. \end{aligned} \tag{2.24}$$

Theorem 2.3 is proved in section 4.1.

The $\mathcal{L}(L^1(\Gamma_-, d\xi); L^1(X))$ norm on A may physically be interpreted as follows. For any illumination $\psi \in L^1(\Gamma_-, d\xi)$ and for any modeling of the detector array by a function $\phi \in L^\infty(X)$, then the "measurement" $\int_X \phi(x) A\psi(x) dx$ is well-defined. This allows us to define measurement errors as in (4.1) below.

Other metrics may also be introduced to quantify the quality of measurements. We consider a stronger metric, which allows one to obtain better reconstructions for the optical coefficients provided that more accurate measurements are available. Results for the stronger metric will be stated without proof and are motivated as follows. When ψ approximates a delta source term $\delta(x - x_0)\delta(v - v_0)$, then the theory recalled in section 3 shows that $A\psi(x)$ is bounded along the line $t \to x_0 + tv_0$. This shows that detectors can be chosen as functions in L^1 along this line rather than in L^∞ and thus can detect point-wise values rather than values integrated along segments. Provided that such point-wise measurements are accurate, then $A - \tilde{A}$ is small in a more constraining norm than $\mathcal{L}(L^1(\Gamma_-, d\xi); L^1(X))$. Such a norm, written in terms of the Schwartz kernels α, takes the form

$$\|A\|_* := \sup_{\substack{(\phi,\psi)\in L^1(\mathbb{R})\times L^1(\Gamma_-,d\xi) \\ \|\phi\|_{L^1(\mathbb{R})}\le 1,\ \|\psi\|_{L^1(\Gamma_-,d\xi)}\le 1}} \left\| \int_{\Gamma_-} \alpha(x,x',v')\phi(\tau_-(x,v'))\psi(x',v')d\mu(x')dv' \right\|_{L^1(X_x)}. \tag{2.25}$$

The theory of section 3 shows that the above norm on the kernel α is indeed well-defined. For such a stronger constraint on the measurements, we obtain an improved version of (2.24) given by:

Theorem 2.4 *Let A and $\tilde{A}$ be two albedo operators and $(x',v') \in \Gamma_-$. Then we obtain that*

$$\begin{aligned} &\sup_{(x,v)\in X\times\mathbb{S}^{n-1}} \left| \sigma_a(x,v)e^{-\int_0^{\tau_-(x,v)}\sigma(x-sv,v)ds} - \tilde{\sigma}_a(x,v)e^{-\int_0^{\tau_-(x,v)}\tilde{\sigma}(x-sv,v)ds} \right| \\ \le\ & \|A-\tilde{A}\|_*. \end{aligned} \tag{2.26}$$

Theorem 2.4 is given here without proof although its derivation requires minor modifications from that of Theorem 2.3.

We now turn to a stability result obtained from the single scattering component of the measurement operator. The result is based on the singular behavior of the transport solution in the vicinity of the support of the ballistic component. Such a behavior cannot be captured by the L^1 norm (or the $\|\cdot\|_*$ norm) used above. Instead, we define $\Gamma_1 = \alpha - \alpha_0$ as the Schwartz kernel of the albedo operator where the ballistic part has been removed, i.e., for measurements that are performed away from the support of the ballistic part. Our stability results are obtained in terms of errors on Γ_1 rather than on A. We can then show the following stability result.

Theorem 2.5 *Let us assume that $(\sigma,\tilde{\sigma}) \in \mathcal{C}_b(X\times\mathbb{S}^{n-1})^2$ and that $(k,\tilde{k}) \in \mathcal{C}_b(X\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})^2$. Let $(x,x') \in X\times\partial X$ and define $v' = \frac{x-x'}{|x-x'|}$. In dimension $n=2$, we have*

$$\begin{aligned} &\left| E(x,x')(\chi(x,v',v')+\chi(x,v',-v')) - \tilde{E}(x,x')(\tilde{\chi}(x,v',v')+\tilde{\chi}(x,v',-v')) \right| \\ \le\ & \left\| \frac{(\Gamma_1-\tilde{\Gamma}_1)(x,x',v')}{|\nu(x')\cdot v'|w_2(x,x',v')} \right\|_{L^\infty(X\times\Gamma_-)}, \end{aligned} \tag{2.27}$$

where $w_2(x,x',v') = 1 + \ln\left(\frac{|x-x'-\tau_+(x',v')v'|-(x-x'-\tau_+(x',v')v')\cdot v'}{|x-x'|-(x-x')\cdot v'}\right)$. When $n\ge 3$, we have

$$\begin{aligned} &\left| \int_0^\pi \sin^{n-3}(\theta)\Big(E(x,x')\chi(x,v',v(\theta)) - \tilde{E}(x,x')\tilde{\chi}(x,v',v(\theta))\Big)d\theta \right| \\ &\le \left\| \frac{(\Gamma_1-\tilde{\Gamma}_1)(x,x',v')}{|\nu(x')\cdot v'|w_n(x,x',v')} \right\|_{L^\infty(X\times\Gamma_-)}, \end{aligned} \tag{2.28}$$

where $w_n(x,x',v') = |x-x'-((x-x')\cdot v')v'|^{2-n}$ and where we use the same notation as in Theorem 2.2.

Theorem 2.5 is proved in section 4.2.

Such results do not grant uniqueness of the reconstruction of the optical parameters in the most general setting. However, they do provide stable, unique, reconstructions in several settings of interest.

2.5. Scattering-free setting

Let us first assume that $k \equiv 0$ so that $\sigma \equiv \sigma_a$. Then knowledge of the albedo operator uniquely determines $\sigma_a(x, v)$ for all $x \in X$ and $v \in \mathbb{S}^{n-1}$.

Indeed, we deduce from Theorem 2.3 that

$$\sigma_a(x' + tv', v')e^{-\int_0^t \sigma_a(x'+sv',v')ds} = -\frac{d}{dt}\Big(e^{-\int_0^t \sigma_a(x'+sv',v')ds}\Big)$$

is uniquely determined and hence $e^{-\int_0^t \sigma_a(x'+sv',v')ds}$ since the latter equals 1 when $t = 0$. Taking the derivative of the negative of the logarithm of the latter expression gives us $\sigma_a(x'+tv', v')$ for all $(x', v') \in \Gamma_-$ and $t > 0$ and hence $\sigma_a(x, v)$ for all $(x, v) \in X \times \mathbb{S}^{n-1}$.

The stability of the reconstruction is optimally written in terms of anisotropic norms on X. For $s \in \{-1, 0\}$ and $p \in [1, \infty]$ and for any measurable function f from $X \times \mathbb{S}^{n-1}$ to $\mathbb{C}$, we define

$$\|f\|_{L^\infty(\Gamma_{-x',v'};W^{s,p}(0,\tau_+(x',v')))} := \sup_{(x',v')\in\Gamma_-} \|f(x' + tv', v')\|_{W_t^{s,p}(0,\tau_+(x',v'))}. \tag{2.29}$$

Loosely speaking, we thus consider functions that are in $W^{s,p}$ in the “spatial” variable in the direction v' and in L^∞ in all other variables. In the following theorem, we use the above norm for $W^{s,p} = W^{0,1} \equiv L^1$.

Theorem 2.6 *Recalling that $\sigma_a(x, v)$ is bounded from above by the constant M, we find that when $k \equiv 0$,*

$$\Big\|\frac{e^{-Mt}}{1 + M\tau_+(x', v')}(\sigma_a - \tilde\sigma_a)(x' + tv', v')\Big\|_{L^\infty(\Gamma_{-x',v'};L^1_t(0,\tau_+(x',v')))} \le \|A-\tilde A\|_{\mathcal{L}(L^1(\Gamma_-,d\xi);L^1(X))}. \tag{2.30}$$

The above theorem is proved in section 4.1.

Remark 2.7 *(i) The above result is local in x' and v'. In other words, $\sigma_a(x, v')$ is uniquely determined by $\{\alpha(y, x - \tau_-(x, v')v', v')|y = x + tv'$ for $-\tau_-(x, v') < t < \tau_+(x, v')\}$, that is by the experiment that consists of sending a beam of radiation in direction v' passing through the point x (at least asymptotically since such a transport solution is not an element in $L^1(X \times \mathbb{S}^{n-1})$).*

(ii) The estimate degrades exponentially as e^{-Mt} as t increases, which means that the error on $\sigma_a(x)$ grows exponentially as the distance $\tau_-(x, v)$ between x and the boundary ∂X increases.

(iii) As a consequence of (2.30), we have the following estimate

$$\|\sigma_a - \tilde\sigma_a\|_{L^\infty(\mathbb{S}^{n-1};L^1(X))} \le C\|A - \tilde A\|_{\mathcal{L}(L^1(\Gamma_-,d\xi);L^1(X))}. \tag{2.31}$$

Here, C is a constant that depends on the uniform bound M and that is of order $e^{M\mathrm{diam}(X)}(1 + M\mathrm{diam}(X))$.

(iv) The analog of (2.30) for the norm $\|.\|_*$ defined by (2.25) is

$$\Big\|\frac{e^{-M\tau_-(x,v)}}{1 + M\tau_-(x, v)}(\sigma_a - \tilde\sigma_a)(x, v)\Big\|_{L^\infty(X\times\mathbb{S}^{n-1})} \le \|A - \tilde A\|_*. \tag{2.32}$$

2.6. Reconstruction of the spatial structure of the optical parameters

We now assume that $k \neq 0$. Then the ballistic component of A and the estimate in Theorem 2.3 allow us to uniquely reconstruct both $\sigma_a(x,v)$ and $\sigma(x,v)$ under the assumption that

$$\sigma_a(x,v) = \sigma_a(x,-v), \qquad \sigma(x,v) = \sigma(x,-v). \tag{2.33}$$

Indeed, we deduce from Theorem 2.3 that

$$\mathcal{D}(x',v',t) := \sigma_a(x'+tv',v')e^{-\int_0^t \sigma(x'+sv',v')ds},\ (x',v') \in \Gamma_-, \tag{2.34}$$

is uniquely determined and hence $e^{-\int_0^t \sigma(x'+sv',v') + \int_t^{\tau_+(x',v')} \sigma(x'+s'v')ds}$ since the latter equals $\mathcal{D}(x',v',t)/\mathcal{D}(x'+\tau_+(x',v')v', -v', \tau_+(x',v')-t)$. Taking the derivative of the negative of the logarithm of the latter expression gives us $2\sigma(x'+tv',v')$ for a.e. $(x',v') \in \Gamma_-$ and $t > 0$ and hence $2\sigma(x,v)$ for a.e. $(x,v) \in X \times \mathbb{S}^{n-1}$. We conclude that σ is uniquely determined and hence so is σ_a (see (2.34)).

We recall that $\sigma_a(x,v)$ satisfies (2.4). For technical reasons for the study of stability, we also assume that $\sigma(x,v)$ is known in the δ_0-vicinity of ∂X, i.e., for all $(x,v) \in \mathbb{R}^n \times \mathbb{S}^{n-1}$ such that $\mathrm{dist}(x,\partial X) < \delta_0$ for some $\delta_0 > 0$. Such a hypothesis is not very restrictive from a practical viewpoint. We denote by $W^{-1,1}(0,t)$, $t > 0$, the Banach space of the continuous linear functionals on the Banach space $W_0^{1,\infty}(0,t) := \{\phi \in L^\infty((0,t)) \mid \mathrm{supp}\phi \subset (0,t),\ \frac{d\phi}{dt} \in L^\infty((0,t),\mathbb{C}^n)\}$ (where the derivative d/dt is understood in the distributional sense). The spaces $W^{s,p}(0,t)$, $t > 0$ for $-1 \le s \le 0$ and $1 \le p \le \infty$ are defined similarly. The norm $\|f\|_{L^\infty(\Gamma_{-x',v'};W^{-1,1}(0,\tau_+(x',v')))}$ in then defined in (2.29).

Under the above assumptions, we have the following stability result.

Theorem 2.8 *Let A and $\tilde{A}$ be two albedo operators and $(x',v') \in \Gamma_-$. Then we obtain that*

$$\left\| \sigma_0 e^{-M\tau_+(x',v')}(\sigma - \tilde{\sigma})(x'+tv',v') \right\|_{L^\infty(\Gamma_{-x',v'};W_t^{-1,1}(0,\tau_+(x',v')))} \tag{2.35}$$
$$+ \left\| \frac{\sigma_0\delta_0 e^{-M\tau_+(x',v')}}{2\sigma_1(\delta_0 + \tau_+(x',v'))}(\sigma_a - \tilde{\sigma}_a)(x'+tv',v') \right\|_{L^\infty(\Gamma_{-x',v'};L_t^1(0,\tau_+(x',v')))} \le 3\|A - \tilde{A}\|_{\mathcal{L}(L^1(\Gamma_-,d\xi);L^1(X))}.$$

This theorem is proved in section 4.1.

Stability estimates may be obtained in stronger norms for σ provided that a priori regularity assumptions are imposed. We show the

Corollary 2.9 *Let us assume that σ and $\tilde{\sigma}$ are bounded in $L^\infty(\Gamma_{-x',v'}, W^{r,p}(0,\tau_+(x',v')))$ by C_0 for $p > 1$ and $r > -1$. Then for all $-1 \le s \le r$, we have*

$$\|\sigma - \tilde{\sigma}\|_{L^\infty(\Gamma_{-x',v'};W^{s,p}(0,\tau_+(x',v')))} \le C_0^{\frac{s+1}{r+1}} C^{\frac{r-s}{r+1}} \|A - \tilde{A}\|_{\mathcal{L}(L^1(\Gamma_-,d\xi);L^1(X))}^{\frac{1}{p}\frac{r-s}{1+r}}, \tag{2.36}$$

where the constant $C := (M\tau_+(x',v'))^{\frac{p-1}{p}} e^{\frac{M\tau_+(x',v')}{p}} 2^{\frac{1}{p}-1}\sigma_0^{-\frac{1}{p}}$.

The corollary is proved in section 4.1.

Remark 2.10 *(i) The result in* (2.35) *is also local in* x' *and* v'*. In other words,* $\sigma_a(x,v')$ *and* $\sigma(x,v')$ *are uniquely determined by* $\{\alpha(y,x-\tau_-(x,v')v',v')|y = x+tv' \text{ for } -\tau_-(x,v') < t < \tau_+(x,v')\}$*, i.e., by the experiment that consists of sending a beam of radiation in direction* v' *passing through the point* x*.*

(ii) Unlike the case $k=0$*, the reconstruction of* $\sigma_a(x,v)$ *and* $\sigma(x,v)$ *is based on using* (2.34) *for both* v' *and* $-v'$ *and thus degrades when* $\tau_+(x,v)+\tau_-(x,v)$ *increases (rather than* $\tau_-(x,v)$ *as in the case* $k=0$*). As a consequence, the reconstruction is a priori not better in the vicinity of* ∂X *than it is away from it. It is easy to verify that when* k *is known, then the estimate* (2.30) *for* σ_a *is valid again with now* M *an upper bound for* $\sigma(x,v)$ *rather than for* $\sigma_a(x,v)$*.*

(iii) The stability estimates (2.35) *and* (2.36) *yield*

$$\|\sigma-\tilde\sigma\|_{L^\infty(\mathbb{S}^{n-1};W^{-1,1}(X))} + \|\sigma_a-\tilde\sigma_a\|_{L^\infty(\mathbb{S}^{n-1};L^1(X))} \le C_1\|A-\tilde A\|_{\mathcal{L}(L^1(\Gamma_-,d\xi);L^1(X))}, \quad (2.37)$$

$$\|\sigma-\tilde\sigma\|_{L^\infty(\mathbb{S}^{n-1};W^{s,p}(X))} \le C_2\|A-\tilde A\|^{\frac{1}{p}\frac{r-s}{1+r}}_{\mathcal{L}(L^1(\Gamma_-,d\xi);L^1(X))}, \quad (2.38)$$

where C_1 *(resp.* C_2*) depends on* σ_0*,* M*,* $\mathrm{diam}(X)$*,* δ_0 *(resp.* σ_0*,* M*,* $\mathrm{diam}(X)$*,* δ_0*,* r*,* s*,* p*,* C_0*). The constant* C_2 *is of order* $(M\mathrm{diam}(X))^{\frac{p-1}{p}}e^{\frac{M\mathrm{diam}(X)}{p}}$*.*

(iv) The analog of estimate (2.35) *for the norm* $\|.\|_*$ *defined by* (2.25) *is*

$$\begin{aligned}&\left\|\frac{\sigma_0}{\sigma_1}e^{-M\tau(x,v)}(\sigma_a-\tilde\sigma_a)(x,v)\right\|_{L^\infty(X\times\mathbb{S}^{n-1})}\\ &+\left\|\sigma_0e^{-M\tau(x,v)}(\sigma-\tilde\sigma)(x,v)\right\|_{L^\infty(\Gamma_{-x',v'};W^{-1,\infty}(0,\tau_+(x',v')))} \le 6\|A-\tilde A\|_*.\end{aligned} \quad (2.39)$$

For the above estimate, "σ *is known in a* δ_0*-vicinity of* ∂X*" is not required.*

2.7. Anisotropic structure of scattering for Henyey-Greenstein kernels

Let us assume that $\sigma \in \mathcal{C}_b(X\times\mathbb{S}^{n-1})$ and that $k\in\mathcal{C}_b(X\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})$ and let us assume again that $\sigma(x)$ is known in the δ_0-vicinity of ∂X, i.e., for all $x\in\mathbb{R}^n$ such that $\mathrm{dist}(x,\partial X)<\delta_0$ for some $\delta_0>0$.

The stability estimate (2.24) allows one to uniquely reconstruct σ_a and σ under the symmetry hypothesis (2.33), which is quite general physically. Indeed, even when attenuation is anisotropic, there is no reason to observe different attenuations in direction v and direction $-v$. The stability estimate in Theorem 2.5 provides additional information on the optical coefficients, but not enough to fully reconstruct the scattering kernel $k(x,v',v)$.

In dimension $n=2$, we gain information only on $k(x,v',v')+k(x,v',-v')$. In dimension $n\ge 3$, we garner information about $\int_0^\pi \chi(x,v',\cos\theta v'+\sin\theta v'^{\perp})d\theta$ for all $v'^{\perp}$ orthogonal to v'. The integration in θ means that one dimension of information is lost in the measurements. Thus, $3n-3$ dimensions of information are available on the $(3n-2)-$dimensional object $k(x,v',v)$.

Let us consider the case of an isotropic absorption $\sigma_a=\sigma_a(x)$ and isotropic scattering in the sense that $k(x,v',v)=k(x,v'\cdot v)$. In such a setting, $k(x,v'\cdot v)$ becomes

$(n+1)-$dimensional. Yet, available data in dimension $n \geq 2$ give us information on

$$\sigma_g(x) := k(x,-1) + k(x,1), \text{ when } n = 2, \tag{2.40}$$

$$\sigma_g(x) := \int_0^\pi k(x,\cos\theta)\sin^{n-3}\theta d\theta, \text{ when } n \geq 3. \tag{2.41}$$

This is different information from the normalization in (2.3)

$$\sigma_s(x) = \int_{\mathbb{S}^{n-1}} k(x, v'\cdot v)dv' = |\mathbb{S}^{n-2}| \int_0^\pi k(x,\cos\theta)\sin^{n-2}\theta d\theta.$$

As a consequence, if $k(x,\cos\theta)$ is of the form $\sigma_s(x)f(x,\cos\theta)$, where $f(x,\cos\theta)$ is parameterized by one function $g(x)$, then we have a chance of reconstructing $g(x)$ from knowledge of $\sigma_g(x)$ and $\sigma_s(x)$ provided $\sigma_s(x) > 0$ (where $\sigma_s = \sigma - \sigma_a$).

This occurs for the classical Henyey-Greenstein (HG) phase function in dimensions $n = 2$ and $n = 3$, where

$$k(x,\lambda) := \sigma_s(x)\frac{1-g^2(x)}{2\pi(1+g(x)^2 - 2g(x)\lambda)}, \text{ when } n = 2, \tag{2.42}$$

$$k(x,\lambda) := \sigma_s(x)\frac{1-g^2(x)}{4\pi(1+g(x)^2 - 2g(x)\lambda)^{\frac{3}{2}}}, \text{ when } n = 3, \tag{2.43}$$

where $g \in C_b(X)$ and $0 \leq g(x) < 1$ for a.e. $x \in X$. Note that

$$\sigma_g(x) = \sigma_s(x)h(g(x)), \tag{2.44}$$

for $x \in X$ where the function $h : [0,1) \to \mathbb{R}$ is given by

$$h(\kappa) := \frac{1+\kappa^2}{\pi(1-\kappa^2)}, \text{ when } n = 2, \tag{2.45}$$

$$h(\kappa) := \int_0^\pi \frac{1-\kappa^2}{4\pi(1+\kappa^2 - 2\kappa\cos(\theta))^{\frac{3}{2}}}d\theta, \text{ when } n = 3. \tag{2.46}$$

Theorem 2.11 *In the HG phase function in dimension $n = 2, 3$, the parameter $g(x)$ is uniquely determined by the data provided $\sigma_s(x) > 0$ for a.e. $x \in X$.*

Theorem 2.11 follows from Theorems 2.8, 2.5, and from (2.40), (2.41), (2.44) and the following Lemma.

Lemma 2.12 *The function h is strictly increasing on $[0,1)$, $\dot{h}(0) = 0$, $\ddot{h}(0) > 0$, $\lim_{g\to 1^-}(1-g)h(g) = c(n)$, where $\dot{h}(g) = \frac{dh}{dg}(g)$ and $c(2) = \pi^{-1}$ and $c(3) = (2\pi)^{-1}$.*

Considering (2.45), Lemma 2.12 in dimension $n = 2$ is trivial. We give a proof of Lemma 2.12 in dimension $n = 3$ in Appendix A.

Moreover we have the following stability estimates.

Theorem 2.13 *Let*

$$\varepsilon := \left\| \frac{(\Gamma_1 - \tilde{\Gamma}_1)(x,x',v')}{|\nu(x')\cdot v'|w_n(x,x',v')} \right\|_\infty + \|A - \tilde{A}\|_{\mathcal{L}(L^1(\Gamma_-,d\xi);L^1(X))},$$

where $w_n(x,x',v')$ *is defined in Theorem 2.5. In dimension* $n \geq 2$*, we have*

$$\|\sigma_g(x) - \tilde{\sigma}_g(x)\|_{L^1(X)} \leq C\varepsilon, \tag{2.47}$$

where the constant C *depends on* δ_0*,* σ_0 *and* M*.*

In addition, for the HG phase function in dimension $n = 2, 3$*, we have the following stability estimates. Assume that* $\min(\sigma_s(x), \tilde{\sigma}_s(x)) \geq \sigma_{s,0} > 0$ *and* $(g, \tilde{g}) \in W^{1,\infty}(X)^2$*,* $\max\big(\|\nabla g\|_{L^\infty(X)}, \|\nabla \tilde{g}\|_{L^\infty(X)}\big) \leq c$ *for some nonnegative constant* c*. Then we have that*

$$\|g^2 - \tilde{g}^2\|_{L^1(X)} \leq C\varepsilon, \tag{2.48}$$

and

$$\|g - \tilde{g}\|_{L^1(X)} \leq C(g,\tilde{g})\varepsilon, \ \textit{for } C(g,\tilde{g}) := C\frac{1 - \max\left(\inf_X g, \inf_X \tilde{g}\right)}{\inf_X(g + \tilde{g})}, \ \inf_X(g+\tilde{g}) > 0, \tag{2.49}$$

where the constant C *depends on* c*,* δ_0*,* σ_0*,* $\sigma_{s,0}$ *and* M*.*

Theorem 2.13 is proved in section 4.2.

The sensitivity of the reconstruction of $g(x)$ degrades as g converges to 0, in the sense that the constant $C(g,\tilde{g}) \to +\infty$ in (2.49) as g, $\tilde{g} \to 0$ in $L^\infty(X)$ with $\inf_X(g+\tilde{g}) > 0$. However g^2 is stably reconstructed as can be seen in (2.48). Similarly, the reconstruction of $g(x)$ is very accurate for $g(x)$ close to 1, i.e., in the case of very anisotropic media, in the sense that $C(g,\tilde{g}) \to 0$ in (2.49) as g, $\tilde{g} \to 1^-$ in $L^\infty(X)$.

2.8. Reconstructions in the diffusive regime.

When scattering is large so that the mean free path $l^* = \frac{1}{\sigma}$ is small and intrinsic attenuation $\sigma_a(x)$ is small, then the ballistic and single scattering photons are of order $\exp\big(-\frac{\mathrm{diam} X}{l^*}\big)$. This explains the presence of the exponential weights in the estimates of Theorems 2.6, 2.8 and 2.13. Although theoretically still feasible, the separation of the singular components from multiply scattered photons becomes increasingly difficult in practice when scattering increases.

In such regimes, radiation inside the domain X is best modeled by a diffusion equation

$$\begin{aligned} &-\nabla \cdot D(x)\nabla I(x) + \sigma_a(x) I(x) = 0 \qquad && x \in X \\ &I(x) = \phi(x) && x \in \partial X, \end{aligned} \tag{2.50}$$

where $I(x) = \int_{S^{n-1}} u(x,v)dv$ is the spatial density of photons and $D(x)$ is the diffusion coefficient. We refer the reader to e.g. [2, 10] for references on the diffusion approximation. When scattering is e.g. isotropic, i.e., when $k(x,\theta',\theta) = k(x)$, then we find that $D(x) = \frac{1}{n\sigma_s(x)}$, where σ_s is introduced in (2.3) and n is the spatial dimension.

When $D(x)$ is known, then the reconstruction of $\sigma_a(x)$ may be easily obtained by using only one measurement. Indeed, the measurement $H(x) = \sigma_a(x)I(x)$ so that $I(x)$ may be obtained by solving (2.50). Once $I(x)$ is known, it will be positive in X provided that $\phi(x)$ is non trivial and non-negative. Then $\sigma_a(x)$ is obtained by dividing

H by I. When $\phi(x)$ is bounded from below by a positive constant, then we see that the reconstruction of σ_a is unique and clearly stable.

When $(D(x), \sigma_a(x))$ are both unknown, then reconstructions are possible (i) either when multiple (at least two) measurements corresponding to multiple illuminations ϕ are available (see e.g. [5] for recent results on this problem); (ii) or multiple frequency measurements are available; see e.g. [9].

This concludes the section on the derivation and the display of the main results. The mathematical proofs are presented in the following two sections.

3. Transport equation and estimates

In this section, we introduce several results on the decomposition of the albedo operator (Lemmas 3.1, 3.2 and 3.3) and prove Theorems 2.1 and 2.2.

We first recall the well-posedness of the boundary value problem (2.12) (here $\sigma \in L^\infty(X \times \mathbb{S}^{n-1})$, $k \in L^\infty(X \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$ and $\sigma_a \geq \sigma_0 > 0$) and give a decomposition of the albedo operator. The boundary value problem (2.12) is equivalent to the integral equation

$$(I - K)u = J\phi \tag{3.1}$$

for $u \in L^1(X \times \mathbb{S}^{n-1})$ and $\phi \in L^1(\Gamma_-, d\xi)$, where I denotes the identity operator and K is the bounded operator in $L^1(X \times \mathbb{S}^{n-1})$ defined by

$$Ku = \int_0^{\tau_-(x,v)} E(x, x - tv) \int_{\mathbb{S}^{n-1}} k(x - tv, v', v)u(x - tv, v')dv'dt, \tag{3.2}$$

for a.e. $(x, v) \in X \times \mathbb{S}^{n-1}$ and for $u \in L^1(X \times \mathbb{S}^{n-1})$, and J is the bounded operator from $L^1(\Gamma_-, d\xi)$ to $L^1(X \times \mathbb{S}^{n-1})$ defined by

$$J\phi(x, v) = e^{-\int_0^{\tau_-(x,v)} \sigma(x-pv,v)dp}\phi(x - \tau_-(x, v)v, v), \tag{3.3}$$

for a.e. $(x, v) \in X \times \mathbb{S}^{n-1}$.

Since $\sigma_a = \sigma - \sigma_s \geq 0$, it turns out that $I - K$ is invertible in $L^1(X \times S^{n-1})$ [3, Lemma 2.4], so that the albedo operator $A : L^1(\Gamma_-, d\xi) \to L^1(X)$ is well-defined by (2.14) and so that the solution u of (2.12) with boundary condition $\phi \in L^1(\Gamma_-, d\xi)$ satisfies

$$u = \sum_{m=0}^{n} K^m J\phi + K^{n+1}(I - K)^{-1}J\phi. \tag{3.4}$$

It follows that

$$\begin{aligned} A\phi = &\int_{\Gamma_-} \left(\sum_{m=0}^{n} \alpha_m(x, x', v') \right) \phi(x', v')d\mu(x')dv' \\ &+ \int_{X \times \mathbb{S}^{n-1}} \gamma_{n+1}(x, y, w)((I - K)^{-1}J\phi)(y, w)dydw, \end{aligned} \tag{3.5}$$

where γ_m, $m \geq 0$, is the distributional kernel of $\bar{K^m} : L^1(X \times \mathbb{S}^{n-1}) \to L^1(X)$ defined by

$$\bar{K^m}u(x) = \int_{\mathbb{S}^{n-1}} \sigma_a(x,v) K^m u(x,v) dv, \tag{3.6}$$

for a.e. $x \in X$ and $u \in L^1(X \times \mathbb{S}^{n-1})$, and where α_m, $m \geq 0$, is the distributional kernel of $\bar{K^m}J : L^1(\Gamma_-, d\xi) \to L^1(X)$.

We give the explicit expression of the distributional kernels α_m, $m \geq 2$, and γ_m, $m \geq 3$ in Lemma 3.1 and study the boundedness of α_m in Lemma 3.2 and of γ_{n+1} in Lemma 3.3. We then prove Theorems 2.1 and 2.2 and Lemma 3.3. The proof of Lemmas 3.1 and 3.2 are given in Appendix A. For $w \in \mathbb{R}^n$, $w \neq 0$, we set $\hat{w} := \frac{w}{|w|}$.

Lemma 3.1 *For $m \geq 2$ and a.e. $(z_0, z_m, v_m) \in X \times \Gamma_-$, we have*

$$\begin{aligned} \alpha_m(z_0, z_m, v_m) = |\nu(z_m) \cdot v_m| \int_{X^{m-1}} \int_0^{\tau_+(z_m,v_m)} &[\sigma_a(z_0, v_0) \\ \times \frac{E(z_0, \ldots, z_{m-1}, z_m + t'v_m, z_m) k(z_m + t'v_m, v_m, v_{m-1})}{|z_m + t'v_m - z_{m-1}|^{n-1} \Pi_{i=0}^{m-2} |z_i - z_{i+1}|^{n+1}} \\ \times \Pi_{i=1}^{m-1} k(z_i, v_i, v_{i-1})\big]_{\substack{v_i = \widehat{z_i - z_{i+1}},\ i=0\ldots m-2, \\ v_{m-1} = \widehat{z_{m-1} - z_m - t'z_m}}} dt' dz_1 \ldots dz_{m-1}. \end{aligned} \tag{3.7}$$

For $m \geq 3$ and a.e. $(z_0, z_m, v_m) \in X \times X \times \mathbb{S}^{n-1}$, we have

$$\begin{aligned} \gamma_m(z_0, z_m, v_m) = \int_{X^{m-1}} \frac{E(z_0, \ldots, z_m)}{\Pi_{i=1}^m |z_i - z_{i-1}|^{n-1}} \\ \times [\sigma_a(z_0, v_0) \Pi_{i=1}^m k(z_i, v_i, v_{i-1})]_{v_i = \widehat{z_i - z_{i+1}},\ i=0\ldots m-1} dz_1 \ldots dz_{m-1}. \end{aligned} \tag{3.8}$$

Lemma 3.2 *For $n = 2$,*

$$\frac{\alpha_1(x, x', v')}{|\nu(x') \cdot v'| \ln\left(\frac{|x - x' - \tau_+(x',v')v'| - (x - x' - \tau_+(x',v')v') \cdot v'}{|x-x'| - (x-x') \cdot v'}\right)} \in L^\infty(X \times \Gamma_-), \tag{3.9}$$

$$\frac{\alpha_2(x, x', v')}{|\nu(x') \cdot v'|} \in L^\infty(X \times \Gamma_-). \tag{3.10}$$

For $n \geq 3$, we have

$$\frac{|x - x' - ((x - x') \cdot v')v'|^{n-1-m} \alpha_m(x, x', v')}{|\nu(x') \cdot v'|} \in L^\infty(X \times \Gamma_-),\ 1 \leq m \leq n-2, \tag{3.11}$$

$$\frac{\alpha_{n-1}(x, x', v')}{|\nu(x') \cdot v'| \ln(|x - x' - ((x - x') \cdot v')v'|)} \in L^\infty(X \times \Gamma_-), \tag{3.12}$$

$$\frac{\alpha_n(x, x', v')}{|\nu(x') \cdot v'|} \in L^\infty(X \times \Gamma_-). \tag{3.13}$$

We do not use (3.9) and (3.11) for $m = 1$ in order to prove (2.22). However we will use them in the proof of the stability estimates given in Theorem 2.5.

Lemma 3.3 *For $n \geq 2$, we have*

$$\gamma_{n+1} \in L^\infty(X \times X \times \mathbb{S}^{n-1}). \tag{3.14}$$

In addition, we have

$$\bar{K}^{n+1}(I-K)^{-1}J\phi(x) = \int_{\Gamma_-} \Gamma_{n+1}(x,x',v')\phi(x',v')d\mu(x')dv', \tag{3.15}$$

for a.e. $x \in X$ and for $\phi \in L^1(\Gamma_-, d\xi)$, where

$$\frac{\Gamma_{n+1}(x,x',v')}{|\nu(x')\cdot v'|} \in L^\infty(X\times\Gamma_-). \tag{3.16}$$

Proof of Theorem 2.1. The equality (2.21) for α_0 follows from the definition of the operator J (3.3). From (3.6), (3.2) and (3.3) it follows that

$$\bar{K}J\phi(x) = \int_{\mathbb{S}^{n-1}} \sigma_a(x,v)\int_0^{\tau_-(x,v)}\int_{\mathbb{S}^{n-1}} E(x, x-tv, x-tv-\tau_-(x-tv,v')v') \times k(x-tv,v',v)\phi(x-tv-\tau_-(x-tv,v')v',v')dv'dtdv, \tag{3.17}$$

for a.e. $x\in X$ and $\phi \in L^1(\Gamma_-, d\xi)$. Performing the change of variables $z = x - tv$ ($dz = t^{n-1}dtdv$, $t = |z-x|$) on the right-hand side of (3.17), we obtain

$$\bar{K}J\phi(x) = \int_{X\times\mathbb{S}^{n-1}} \frac{\sigma_a(x,\widehat{x-z})}{|x-z|^{n-1}} E(x,z,z-\tau_-(z,v')v') \times k(z,v',\widehat{x-z})\phi(z-\tau_-(z,v')v',v')dv'dz, \tag{3.18}$$

for a.e. $x\in X$ and $\phi \in L^1(\Gamma_-, d\xi)$. Performing the change of variables $z = x' + tv'$ ($x'\in\partial X$, $t>0$, $dz = |\nu(x')\cdot v'|dtd\mu(x')$) on the right hand side of (3.18), we obtain

$$\bar{K}J\phi(x) = \int_{\Gamma_-}\int_0^{\tau_+(x',v')} \frac{\sigma_a(x,\widehat{x-x'-t'v'})}{|x-x'-t'v'|^{n-1}} E(x,x'+t'v',x') \times k(x'+t'v',v',\widehat{x-x'-t'v'})\phi(x',v')dtd\xi(x',v'), \tag{3.19}$$

for a.e. $x\in X$ and $\phi\in L^1(\Gamma_-, d\xi)$, which yields (2.21) for α_1.

Now set $\Gamma_2 := \sum_{m=2}^n \alpha_m + \Gamma_{n+1}$ when $n\geq 2$. Taking account of Lemma 3.2 (3.10)–(3.13) and Lemma 3.3 (3.16), we obtain (2.22). □

Proof of Theorem 2.2. We assume that $\sigma\in\mathcal{C}_b(X\times\mathbb{S}^{n-1})$ and $k\in\mathcal{C}_b(X\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})$. Let $(x',v')\in\Gamma_-$ and $t'_0\in(0,\tau_+(x',v'))$ and let $v'^\perp$ be such that $v'\cdot v'^\perp = 0$. Set $x = x'+t'_0v'$. From (2.21), it follows that

$$\frac{\alpha_1(x+\varepsilon v'^\perp, x', v')}{|\nu(x')\cdot v'|} = (L_+ + L_-)(\varepsilon), \qquad \text{where} \tag{3.20}$$

$$L_+(\varepsilon) := \int_0^{t'_0} \frac{E(x+\varepsilon v'^\perp, x'+t'v', x')}{((t'-t'_0)^2+\varepsilon^2)^{\frac{n-1}{2}}}\sigma_a(x+\varepsilon v'^\perp, v_{t',\varepsilon})k(x'+t'v',v',v_{t',\varepsilon})dt', \tag{3.21}$$

$$L_-(\varepsilon) := \int_{t'_0}^{\tau_+(x',v')} \frac{E(x+\varepsilon v'^\perp, x'+t'v', x')}{((t'-t'_0)^2+\varepsilon^2)^{\frac{n-1}{2}}}\sigma_a(x+\varepsilon v'^\perp, v_{t',\varepsilon})k(x'+t'v',v',v_{t',\varepsilon})dt', \tag{3.22}$$

for $\varepsilon\in(0,\tau_+(x,v'^{\perp}))$, where $v_{t',\varepsilon}=\widehat{\frac{(t'_0-t')}{\varepsilon}v'+v'^{\perp}}$ for $t'\in\mathbb{R}$.

We prove (2.23) for $n=2$. Consider the function $\mathrm{arcsinh}:\mathbb{R}\to\mathbb{R}$ defined by $\mathrm{arcsinh}(y):=\ln(y+\sqrt{1+y^2})$, for $y\in\mathbb{R}$. Then performing the change of variables $\eta:=\frac{\mathrm{arcsinh}(\frac{t'_0-t'}{\varepsilon})}{\mathrm{arcsinh}(\frac{t'_0}{\varepsilon})}$ on the right hand side of (3.21) and performing the change of variables $\eta:=\frac{\mathrm{arcsinh}(\frac{t'-t'_0}{\varepsilon})}{\mathrm{arcsinh}(\frac{\tau_+(x',v')-t'_0}{\varepsilon})}$ on the right hand side of (3.22), we obtain

$$L_\pm(\varepsilon):=\mathrm{arcsinh}(\frac{s_\pm}{\varepsilon})L'_\pm(\varepsilon), \tag{3.23}$$

for $\varepsilon\in(0,\tau_+(x,v'^{\perp}))$, where

$$s_+:=t'_0,\ s_-:=\tau_+(x',v')-t'_0, \tag{3.24}$$

$$\begin{aligned} L'_\pm(\varepsilon) &:= \int_0^1 E(x+\varepsilon v'^{\perp},x'+t'_\pm(\eta,\varepsilon)v',x') \\ &\qquad \sigma_a(x+\varepsilon v'^{\perp},v_\pm(\eta,\varepsilon))k(x'+t'_\pm(\eta,\varepsilon)v',v',v_\pm(\eta,\varepsilon))d\eta \qquad (3.25)\\ t'_\pm(\eta,\varepsilon) &:= t'_0\mp\varepsilon\sinh\Big(\eta\mathrm{arcsinh}(\frac{s_\pm}{\varepsilon})\Big), \qquad (3.26)\\ v_\pm(\eta,\varepsilon) &:= \frac{\pm\sinh\big(\eta\mathrm{arcsinh}(\frac{s_\pm}{\varepsilon})\big)v'+v'^{\perp}}{\sqrt{\sinh\big(\eta\mathrm{arcsinh}(\frac{s_\pm}{\varepsilon})\big)^2+1}}, \qquad (3.27)\end{aligned}$$

for $\eta\in(0,1)$ (we recall that $\sinh(y)=\frac{e^y-e^{-y}}{2}$, $y\in\mathbb{R}$). Note that using the definition of sinh and arcsinh, we obtain

$$|\varepsilon\sinh\Big(\eta\mathrm{arcsinh}(\frac{s_\pm}{\varepsilon})\Big)|\le\frac{e^{\eta\ln(s_\pm+\sqrt{s_\pm^2+1})}\varepsilon^{1-\eta}}{2}, \tag{3.28}$$

for $\eta\in(0,1)$. Therefore using (3.26) we obtain

$$t'_\pm(\eta,\varepsilon)\to t'_0,\ \text{as }\varepsilon\to0^+ \tag{3.29}$$

for $\eta\in(0,1)$ and $i=1,2$. Note also that from (3.27) it follows that

$$v_\pm(\eta,\varepsilon)=\frac{\pm v'+\sinh\big(\eta\mathrm{arcsinh}(\frac{s_\pm}{\varepsilon})\big)^{-1}v'^{\perp}}{\sqrt{1+\sinh\big(\eta\mathrm{arcsinh}(\frac{s_\pm}{\varepsilon})\big)^{-2}}}\underset{\varepsilon\to0^+}{\longrightarrow}\pm v', \tag{3.30}$$

for $\eta\in(0,1)$ (we used the limit $\sinh\big(\eta\mathrm{arcsinh}(\frac{s}{\varepsilon})\big)\to+\infty$ as $\varepsilon\to0^+$ which holds for any positive real numbers s and η).

Using (3.25), (3.29), (3.30) and continuity and boundedness of σ and k and σ_a, and using Lebesgue dominated convergence theorem, we obtain

$$L'_\pm(\varepsilon)\to\sigma_a(x,\pm v')E(x,x')k(x'+t'_0v',v',\pm v')\text{ as }\varepsilon\to0^+. \tag{3.31}$$

Finally note that

$$\mathrm{arcsinh}(\frac{s_\pm}{\varepsilon})=\ln\big(\frac{1}{\varepsilon}\big)+o\Big(\frac{1}{\varepsilon}\Big),\ \text{as }\varepsilon\to0^+. \tag{3.32}$$

Combining (3.20), (3.23), (3.31) and (3.32), we obtain (2.23) for $n=2$.

We prove (2.23) for $n\geq 3$. Performing the change of variables $\eta=\frac{t'-t'_0}{\varepsilon}$ on the right-hand side of (3.21), we obtain

$$L_+(\varepsilon)=\varepsilon^{2-n}\int_{-\frac{t'_0}{\varepsilon}}^{0}E(x+\varepsilon v'^{\perp},x'+t'_3(\varepsilon,\eta)v',x')\,\frac{\sigma_a(x+\varepsilon v'^{\perp},v)}{\sqrt{\eta^2+1}^{\,n-1}}k(x'+t'_3(\varepsilon,\eta)v',v',v)_{|v=\widehat{-\eta v'+v'^{\perp}}}d\eta, \tag{3.33}$$

for $\varepsilon\in(0,\tau_+(x,v'^{\perp}))$, where $t'_3(\varepsilon,\eta)=t'_0+\varepsilon\eta$ for $\eta\in\mathbb{R}$. Note that

$$t'_3(\varepsilon,\eta)\to t'_0,\ \text{as}\ \varepsilon\to 0^+. \tag{3.34}$$

Therefore, using the Lebesgue dominated convergence theorem and continuity and boundedness of (σ,k,σ_a), we obtain

$$L_+(\varepsilon)=\varepsilon^{2-n}E(x,x')\int_{-\infty}^{0}\frac{\sigma_a(x,v)k(x,v',v)_{|v=\widehat{-\eta v'+v'^{\perp}}}}{\sqrt{\eta^2+1}^{\,n-1}}d\eta+o(\varepsilon^{2-n}), \tag{3.35}$$

as $\varepsilon\to 0^+$. Similarly performing the change of variables $\eta=\frac{t'-t'_0}{\varepsilon}$ on the right hand side of (3.22), and using Lebesgue dominated convergence theorem and continuity and boundedness of (σ,k,σ_a), we obtain

$$L_-(\varepsilon)=\varepsilon^{2-n}E(x,x')\int_{0}^{+\infty}\frac{\sigma_a(x,v)k(x,v',v)_{|v=\widehat{-\eta v'+v'^{\perp}}}}{\sqrt{\eta^2+1}^{\,n-1}}d\eta+o(\varepsilon^{2-n}), \tag{3.36}$$

as $\varepsilon\to 0^+$. Note that performing the change of variables $\cos(\theta)=\frac{-\eta}{\sqrt{1+\eta^2}}$, $\theta\in(0,\pi)$, ($\eta=-\frac{\cos(\theta)}{\sin(\theta)}$, $d\eta=\frac{1}{\sin(\theta)^2}d\theta$) we have

$$\int_{-\infty}^{+\infty}\frac{\sigma_a(x,v)k(x,v',v)_{|v=\widehat{-\eta v'+v'^{\perp}}}}{\sqrt{\eta^2+1}^{\,n-1}}d\eta=\int_0^{\pi}\sin(\theta)^{n-3}\sigma_a(x,v(\theta))k(x,v',v(\theta))d\theta. \tag{3.37}$$

Adding (3.35) and (3.36) and using (3.37) and (3.20), we obtain (2.23) for $n\geq 3$. □

Proof of Lemma 3.3. We first prove the estimates (3.38) and (3.39) given below

$$\int_X\frac{dz_1}{|z_0-z_1||z_1-z|^{n-1}}\leq C_2-C'_2\ln(|z_0-z|),\ \text{when}\ n=2 \tag{3.38}$$

$$\int_X\frac{dz_1}{|z_0-z_1|^m|z_1-z|^{n-1}}\leq\frac{C_n}{|z_0-z|^{m-1}},\ \text{when}\ n\geq 3, \tag{3.39}$$

for $(z_0,z)\in X^2$ and for $m\in\mathbb{N}$ such that $z_0\neq z$ and $2\leq m\leq n-1$, where the positive constants C_n, C'_2 do not depend on (z_0,z).

Let $(z_0,z)\in X^2$ and let $m\in\mathbb{N}$ be such that $z_0\neq z$ and $1\leq m\leq n-1$. Performing the change of variables $z_1=z+r_1\Omega_1$, $(r_1,\Omega_1)\in(0,D)\times\mathbb{S}^{n-1}$ (where D denotes the diameter of X), we obtain

$$\int_X\frac{dz_1}{|z_0-z_1|^m|z_1-z|^{n-1}}\leq\int_0^D\int_{\mathbb{S}^{n-1}}\frac{d\Omega_1}{|r\Omega-r_1\Omega_1|^m}dr_1=\int_0^D\int_{\mathbb{S}^{n-1}}\frac{d\Omega_1}{|re_1-r_1\Omega_1|^m}dr_1 \tag{3.40}$$

where $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$ (we also used a rotation that maps Ω to the vector e_1) and

$$r = |z - z_0|,\ \Omega = \frac{z_0 - z}{|z_0 - z|}. \tag{3.41}$$

Performing the change of variables $\Omega_1 = (\sin(\theta_1), \cos(\theta_1)\Theta_1)$, $(\theta_1, \Theta_1) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{S}^{n-2})$, on the right hand side of (3.40) we obtain

$$\int_0^D \int_{\mathbb{S}^{n-1}} \frac{d\Omega_1}{|re_1 - r_1\Omega_1|^m} dr_1 = c(n) \int_0^D \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos(\theta_1)^{n-2} d\theta_1}{(r^2 + r_1^2 - 2rr_1 \sin(\theta_1))^{\frac{m}{2}}} dr_1, \tag{3.42}$$

where $c(n) := |\mathbb{S}^{n-2}|$ (by convention $c(2) := 2$).

Consider the case $n = 2$ and $m = 1$. Using (3.42) and the estimate $(r^2 + r_1^2 - 2rr_1 \sin(\theta_1))^{\frac{1}{2}} \geq 2^{-\frac{1}{2}}(|r_1 - r\sin(\theta_1)| + r|\cos(\theta_1)|)$, we obtain

$$S(r) := \int_0^D \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta_1}{(r^2 + r_1^2 - 2rr_1 \sin(\theta_1))^{\frac{m}{2}}} dr_1 \tag{3.43}$$

$$\leq \int_0^{2\pi} \int_0^D \frac{\sqrt{2} dr_1}{|r_1 - r\sin(\theta_1)| + r|\cos(\theta_1)|} d\theta_1$$

$$\leq \int_0^{2\pi} \int_{-2D}^{2D} \frac{\sqrt{2} dr_1}{|r_1| + r|\cos(\theta_1)|} d\theta_1 \leq \int_0^{2\pi} \int_0^{2D} \frac{2^{\frac{3}{2}} dr_1}{r_1 + r|\cos(\theta_1)|} d\theta_1$$

$$\leq 2^{\frac{3}{2}} \int_0^{2\pi} \ln\left(\frac{2D + r|\cos(\theta_1)|}{r|\cos(\theta_1)|}\right) d\theta_1 \leq 2^{\frac{3}{2}} \int_0^{2\pi} \ln\left(\frac{4D}{r|\cos(\theta_1)|}\right) d\theta_1 \leq C_1 - 2^{\frac{5}{2}}\pi \ln(r), \tag{3.44}$$

where $C_1 := 2^{\frac{3}{2}} \int_0^{2\pi} (\ln(4D) - \ln(|\cos(\theta_2)|)) d\theta_2 < \infty$. Estimate (3.38) follows from (3.40), (3.42) and (3.44).

Consider the case $n \geq 3$ and $2 \leq m \leq n-1$. Note that

$$r\cos(\theta_1) \leq \sqrt{r^2 + r_1^2 - 2rr_1 \sin(\theta_1)}, \tag{3.45}$$

for $(r, r_1, \theta_1) \in (0, +\infty)^2 \times (-\frac{\pi}{2}, \frac{\pi}{2})$. Combining (3.42) and (3.45), we obtain

$$\int_0^D \int_{\mathbb{S}^{n-1}} \frac{d\Omega_1}{|re_1 - r_1\Omega_1|^m} dr \leq \frac{c(n)}{r^{m-2}} \int_0^D \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos(\theta_1)^{n-m} d\theta_1}{(r^2 + r_1^2 - 2rr_1 \sin(\theta_1))} dr_1$$

$$\leq \frac{c(n)}{r^{m-2}} \int_0^D \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos(\theta_1) d\theta_1}{(r^2 + r_1^2 - 2rr_1 \sin(\theta_1))} dr_1 = \frac{c(n)}{r^{m-1}} \int_0^D \frac{\ln\left(\frac{r+r_1}{r-r_1}\right)}{r_1} dr_1$$

$$= \frac{c(n)}{r^{m-1}} \int_0^{\frac{D}{r}} \frac{\ln\left(\frac{1+\eta}{1-\eta}\right)}{\eta} d\eta \tag{3.46}$$

(we perform the change of variables $r_1 = r\eta$, $dr_1 = rd\eta$). Finally (3.39) follows from (3.46) and (3.40) and the estimate $\int_0^{+\infty} \frac{\ln\left(\frac{1+\eta}{1-\eta}\right)}{\eta} d\eta < +\infty$.

We are now ready to prove (3.14). Let $n \geq 2$. From (3.8), it follows that

$$|\gamma_{n+1}(z_0, z_{n+1}, v_{n+1})| \leq \|\sigma_a\|_\infty \|k\|^{n+1}_{L^\infty(X \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1})} R(z_0, z_{n+1}), \tag{3.47}$$

for a.e. $(z_0, z_{n+1}, v_{n+1}) \in X \times X \times \mathbb{S}^{n-1}$, where

$$R(z_0, z_{n+1}) = \int_{X^n} \frac{1}{\Pi_{i=1}^{n+1}|z_i - z_{i-1}|^{n-1}} dz_1 \dots dz_n, \tag{3.48}$$

for $(z_0, z_{n+1}) \in X \times X$.

Assume $n = 2$. Combining (3.38) and (3.48) we obtain

$$R(z_0, z_3) \le \int_X \frac{C - C' \ln(|z_3 - z_1|)}{|z_1 - z_0|} dz_1. \tag{3.49}$$

Performing the change of variables $z_1 = z_0 + r_1\Omega_1$, on the right hand side of (3.49), we obtain

$$R(z_0, z_3) \le \int_0^D \int_{\mathbb{S}^1} (C - C' \ln(|r_1\Omega_1 + z_0 - z_3|)) d\Omega_1 dr_1. \tag{3.50}$$

Assume $n \ge 3$. Using (3.48) and (3.39), we obtain

$$R(z_0, z_{n+1}) \le C \int_{X^2} \frac{1}{|z_{n+1} - z_n|^{n-1}|z_n - z_{n-1}|^{n-1}|z_{n-1} - z_0|} dz_n dz_{n-1}, \tag{3.51}$$

where C does not depend on (z_0, z_{n+1}). Performing the change of variables $z_i = z_{i+1} + r_i\Omega_i$, $(r_i, \Omega_i) \in (0, +\infty) \times \mathbb{S}^{n-1}$, $i = n-1, n$, we obtain

$$\begin{aligned} R(z_0, z_{n+1}) \le & \int_{(0,D)^2 \times (\mathbb{S}^{n-1})^2} \frac{dr_{n-1} dr_n d\Omega_{n-1} d\Omega_n}{|z_{n+1} - z_0 + r_n\Omega_n + r_{n-1}\Omega_{n-1}|} \\ = & \int_{(0,D)\times\mathbb{S}^{n-1}} \int_{(0,D)\times\mathbb{S}^{n-1}} \frac{dr_{n-1} d\Omega_{n-1}}{|\mathfrak{d}(r_n, \Omega_n) e_1 + r_{n-1}\Omega_{n-1}|} dr_n d\Omega_n, \end{aligned} \tag{3.52}$$

where $\mathfrak{d}(r_n, \Omega_n) = |z_{n+1} - z_0 + r_n\Omega_n|$, $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$. Performing the change of variables $\Omega = (\sin(\theta), \cos(\theta)\Theta)$, $(\theta, \Theta) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{S}^{n-2}$, we obtain

$$\int_0^D \int_{\mathbb{S}^{n-1}} \frac{d\Omega}{|\sigma e_1 - r\Omega|} dr \le c(n) S(\sigma), \tag{3.53}$$

for $\sigma \in (0, +\infty)$, where $c(n) := |\mathbb{S}^{n-2}|$ and $S(\sigma)$ is defined by (3.43). Therefore from (3.53), (3.44) and (3.52), it follows that

$$R(z_0, z_{n+1}) \le \int_{(0,D)\times\mathbb{S}^{n-1}} (C - C' \ln(|z_{n+1} - z_0 + r_n\Omega_n|)) dr_n d\Omega_n, \tag{3.54}$$

where the positive constants C, C' do not depend on (z_0, z_{n+1}).

Finally taking account of (3.50), we obtain that (3.54) holds for any dimension $n \ge 2$ and for any $(z_0, z_{n+1}) \in X^2$, $z_0 \ne z_{n+1}$, where the positive constants C, C' do not depend on (z_0, z_{n+1}). Let $n \ge 2$ and $(z_0, z_{n+1}) \in X^2$, $z_0 \ne z_{n+1}$. From (3.54) and the estimate $|r_n + (z_0 - z_{n+1}) \cdot \Omega_n| \le |r_n\Omega_n + z_0 - z_{n+1}|$ it follows that

$$\begin{aligned} R(z_0, z_{n+1}) \le & c(n)CD - C' \int_{\mathbb{S}^{n-1}} \int_0^D \ln(|r_n + (z_0 - z_{n+1}) \cdot \Omega_n|) dr_n d\Omega_n \\ \le & c(n)(CD - C' \int_{-2D}^{2D} \ln(|r_n|) dr_n) = c(n)(CD - 2C' \int_0^{2D} \ln(r_n) dr_n), \end{aligned} \tag{3.55}$$

where $c(n) := |\mathbb{S}^{n-1}|$. Statement (3.14) follows from (3.47) and (3.55).

We prove (3.15). We first obtain

$$\bar{K}^{n+1}(I-K)^{-1}J\phi(x) = \int_{X\times\mathbb{S}^{n-1}} \gamma_{n+1}(x,x',v')(I-K)^{-1}J\phi(x,v)dv, \tag{3.56}$$

for a.e. $x \in X$ and for $\phi \in L^1(\Gamma_-, d\xi)$. Therefore using (3.14) we obtain

$$\|\bar{K}^{n+1}(I-K)^{-1}J\phi(x)\|_{L^\infty(X)} \le C\|\phi\|_{L^1(\Gamma_-,d\xi)}, \tag{3.57}$$

for $\phi \in L^1(\Gamma_-, d\xi)$, where $C := \|\gamma_{n+1}\|_{L^\infty(X\times X\times\mathbb{S}^{n-1})}\|(I-K)^{-1}J\|_{\mathcal{L}(L^1(\Gamma_-,d\xi),L^1(X\times\mathbb{S}^{n-1}))}$. Therefore there exists a (unique) function $\Phi \in L^\infty(X\times\Gamma_-)$ such that

$$\bar{K}^{n+1}(I-K)^{-1}J\phi(x) = \int_{\Gamma_-} \Phi(x,x',v')\phi(x',v')d\xi(x',v'). \tag{3.58}$$

Set $\Gamma_{n+1}(x,x',v') := |\nu(x')\cdot v'|\Phi(x,x',v')$ for a.e. $(x,x',v') \in X\times\Gamma_-$ and recall the definition of $d\xi$. Then (3.15) follows from (3.58).□

4. Derivation of stability estimates

4.1. Estimates for ballistic part

Proof of Theorem 2.3. Let $\phi \in L^\infty(X)$, $\|\phi\|_{L^\infty(X)} \le 1$ and $\psi \in L^1(\Gamma_-, d\xi)$, $\|\psi\|_{L^1(\Gamma_-,d\xi)} \le 1$. We have :

$$\left|\int_X \phi(x)\left[\left(A-\widetilde{A}\right)\psi\right](x)dx\right| \le \|A-\widetilde{A}\|_{\mathcal{L}(L^1(\Gamma_-,d\xi),L^1(X))}. \tag{4.1}$$

Using (4.1) and the decomposition of the albedo operator (see Theorem 2.1) we obtain

$$|\Delta_0(\phi,\psi)| \le \|A-\widetilde{A}\|_{\mathcal{L}(L^1(\Gamma_-,d\xi),L^1(X))} + |\Delta_1(\phi,\psi)|, \tag{4.2}$$

where

$$\begin{aligned}\Delta_0(\phi,\psi) = &\int_{X\times\mathbb{S}^{n-1}} \phi(x)\left(\sigma_a(x,v')E(x,x-\tau_-(x,v')v')\right.\\ &\left.-\widetilde{\sigma}_a(x,v')\tilde{E}(x,x-\tau_-(x,v')v')\right)\psi(x-\tau_-(x,v')v',v')dv'dx, \end{aligned}\tag{4.3}$$

$$\Delta_1(\phi,\psi) = \int_X \phi(x)\int_{\Gamma_-}(\Gamma_1-\tilde{\Gamma}_1)(x,x',v')\psi(x',v')d\mu(x')dv'dx, \tag{4.4}$$

and where $\Gamma_1 = \alpha_1 + \Gamma_2$ (and $\tilde{\Gamma}_1 = \tilde{\alpha}_1 + \tilde{\Gamma}_2$).

Note that performing the change of variables $x = x' + tv'$ on the right-hand side of (4.3), we obtain

$$\Delta_0(\phi,\psi) = \int_{\Gamma_-}\int_0^{\tau_+(x',v')} \phi(x'+tv')(\eta-\tilde{\eta})(t;x',v')dt\psi(x',v')d\xi(x',v'), \tag{4.5}$$

where

$$\eta(t;x',v') = \sigma_a(x'+tv',v')e^{-\int_0^t \sigma(x'+sv',v')ds},\ \tilde\eta(t;x',v') = \tilde\sigma_a(x'+tv',v')e^{-\int_0^t \tilde\sigma(x'+sv',v')ds}. \tag{4.6}$$

We use the following result: For any function $G \in L^1(\Gamma_-, d\xi)$ and for a.e. $(x'_0, v'_0) \in \Gamma_-$ there exists a sequence of functions $\psi_{\varepsilon,x'_0,v'_0} \in L^1(\Gamma_-, d\xi)$ (which does not depend on the function G), $\|\psi_{\varepsilon,x'_0,v'_0}\|_{L^1(\Gamma_-,d\xi)} = 1$, $\psi_{\varepsilon,x'_0,v'_0} \geq 0$ and $\mathrm{supp}\psi_{\varepsilon,x'_0,v'_0} \subseteq \{(x',v') \in \Gamma_- \mid |x'-x'_0| + |v-v'_0| < \varepsilon\}$ such that the following limit holds

$$\int_{\Gamma_-} G(x',v')\psi_{\varepsilon,x'_0,v'_0}(x',v')d\xi(x',v') \to G(x'_0,v'_0),\ \text{as } \varepsilon \to 0^+ \tag{4.7}$$

(we refer the reader to [14, Corollary 4.2] for the proof of this statement). In particular, the limits (4.7) holds for some sequence $\psi_{\varepsilon,x'_0,v'_0}$ when (x'_0,v'_0) belongs to the Lebesgue set of G denoted by $\mathcal{L}(G)$ and the complement of $\mathcal{L}(G)$ is a negligible subset of Γ_- (see [14]).

Let $\mathcal{L} := \cap_{m\in\mathbb{N}\cup\{0\}}\mathcal{L}(G_m)$ where G_m is the measurable function on Γ_- defined by

$$G_m(x',v') = \int_0^{\tau_+(x',v')} t^m(\eta-\tilde\eta)(t;x',v')dt, \tag{4.8}$$

for $m \in \mathbb{N}\cup\{0\}$. Note that the complement of $\mathcal{L}$ is still negligible. Let $(x'_0,v'_0) \in \mathcal{L}$ and $\phi \in C_0(0,\tau_+(x'_0,v'_0))$, where $C_0(0,\tau_+(x'_0,v'_0))$ denotes the set of continuous and compactly supported functions on $(0,\tau_+(x'_0,v'_0))$. Consider the sequence $(\phi_m) \in (L^\infty(X))^{\mathbb{N}}$ defined by

$$\phi_m(x) = \chi_{[0,\frac{1}{m+1})}(|x'|)\phi(t), \tag{4.9}$$

for $x \in X$ and $m \in \mathbb{N}$ where $x = x'_0 + tv'_0 + x'$, $x'\cdot v'_0 = 0$, where $\chi_{[0,\frac{1}{m+1})}(t) = 0$ when $t \geq \frac{1}{m+1}$ and $\chi_{[0,\frac{1}{m+1})}(t) = 1$ otherwise. The support of ϕ_m concentrates around the line which passes through x'_0 with direction v'_0. From (4.5), (4.7), (4.8) (and the Stone-Weierstrass theorem), we obtain that

$$\lim_{\varepsilon\to 0^+} \Delta_0(\phi_m, \psi_{\varepsilon,x'_0,v'_0}) = \int_0^{\tau_+(x'_0,v'_0)} \phi_m(x'_0+tv'_0)(\eta-\tilde\eta)(t;x'_0,v'_0)dt. \tag{4.10}$$

For the single and multiple scattering part, using (2.22) and (3.9) and (3.11) for $m=1$ we obtain:

$$|\Delta_1(\phi_m,\psi_{\varepsilon,x'_0,v'_0})| \leq C\int_{\Gamma_-} \Phi_m(x',v')\psi_{\varepsilon,x'_0,v'_0}(x',v')d\xi(x',v') \tag{4.11}$$

where $C = \|\frac{(\Gamma_1-\tilde\Gamma_1)(x,x',v')}{|\nu(x')\cdot v'|w_n(x,x',v')}\|_\infty$ and Φ is the function from Γ_- to $\mathbb{R}$ defined by

$$\Phi_m(x',v') = \int_X w_n(x,x',v')|\phi_m(x)|dx \tag{4.12}$$

(where $w_n(x,x',v')$ is defined in Theorem 2.5). From the definition of w_n, it follows that Φ_m is a bounded and continuous function on Γ_-. Therefore using Lebesgue dominated convergence theorem, we obtain

$$\lim_{\varepsilon\to 0^+}\int_{\Gamma_-}\Phi_m(x',v')\psi_{\varepsilon,x_0',v_0'}(x',v')d\xi(x',v')=\Phi_m(x_0',v_0')=\int_X w_n(x,x_0',v_0')|\phi_m(x)|dx. \tag{4.13}$$

Then using (4.9), (4.13) and Lebesgue convergence theorem, we obtain $\lim_{m\to+\infty}\int_X w_n(x,x_0',v_0')|\phi_m(x)|dx=0$. Therefore taking account of (4.11) and (4.13), we obtain

$$\lim_{m\to+\infty}\lim_{\varepsilon\to 0^+}\Delta_1(\phi_m,\psi_{\varepsilon,x_0',v_0'})=0. \tag{4.14}$$

Combining (4.2) (with $\psi=\psi_{\varepsilon,x',v'}$), (4.10), (4.9), (4.14), we obtain

$$\left|\int_0^{\tau_+(x',v')}\phi(t)(\eta-\tilde\eta)(t;x',v')dt\right|\le\|\mathcal{A}-\widetilde{\mathcal{A}}\|_{\mathcal{L}(L^1(\Gamma_-,d\xi),L^1(X))}, \tag{4.15}$$

for $(x',v')\in\mathcal{L}$ and $\phi\in C_0(0,\tau_+(x',v'))$. From the density of $C_0(0,\tau_+(x',v'))$ in $L^1(0,\tau_+(x',v'))$, it follows that (4.15) also holds for $\phi\in L^1(0,\tau_+(x',v'))$. Applying (4.15) on $\phi(t):=\mathrm{sign}(\eta-\tilde\eta)(t;x',v')$ for a.e. $t\in(0,\tau_+(x',v'))$ (where $\mathrm{sign}(s)=1$ when $s\ge 0$ and $\mathrm{sign}(s)=-1$ otherwise), we obtain (2.24). □

Proof of Theorem 2.6. Let $\varepsilon:=\|A-\tilde A\|_{\mathcal{L}(L^1(\Gamma_-,d\xi);L^1(X))}$. Let us define

$$\zeta(t):=\zeta(t;x',v')=e^{-\int_0^t\sigma_a(x'+sv',v')ds},\quad \eta(t):=\eta(t;x',v')=\sigma_a(x'+tv',v')\zeta(t;x',v').$$

Note that $\eta(t)=\frac{d}{dt}\zeta(t)$. Then,

$$|\zeta(t)-\tilde\zeta(t)|=\Big|\int_0^t\frac{d}{dt}(\zeta-\tilde\zeta)(s)ds\Big|\le\int_0^t|\eta-\tilde\eta|(s)ds\le\varepsilon,$$

thanks to (2.24). The point-wise (in t) control on $\zeta(t)$ and the estimate (2.24) for $\eta(t)$ show by application of the triangle inequality and the fact that $\zeta(t)$ is bounded from below by the positive constant e^{-Mt} that

$$\int_0^{\tau_+(x',v')}\frac{e^{-Mt}}{1+M\tau_+(x',v')}\Big|\sigma_a(x'+tv',v')-\tilde\sigma_a(x'+tv',v')\Big|dt\le\varepsilon. \tag{4.16}$$

Reconstructions degrade as t increases but are still quite accurate for small values of t, i.e., in the vicinity of ∂X. This concludes the proof of the theorem. The improved bound (2.32), which is independent of optical thickness $\tau_+(x',v')$ for the stronger norm $\|A\|_*$ is proved similarly. □

Proof of Theorem 2.8. Let $\varepsilon := \|A-\tilde A\|_{\mathcal{L}(L^1(\Gamma_-);L^1(X))}$. Let $\mathcal{G} := \{(t,x',v') \in (0,+\infty)\times\Gamma_- \mid t\in(0,\tau_+(x',v'))\}$ and $h\in L^\infty(\mathcal{G})$ be defined by

$$h(t;x',v') = -\int_0^t \sigma(x'+sv',v')ds + \int_t^{\tau_+(x',v')} \sigma(x'+sv',v')ds, \tag{4.17}$$

for $(t,x',v')\in\mathcal{G}$. The function $\tilde h\in L^\infty(\mathcal{G})$ is defined similarly.

We first prove a stability estimate (4.22) on h, $\tilde h$. Let $(x',v')\in\Gamma_-$, $t\in(0,\tau_+(x',v'))$. Set $(y_1,v_1) := (x'+\tau_+(x',v')v',-v')$ and $t_1=\tau_+(x',v')-t$. Due to the symmetry of σ_a, σ with respect to the speed variable v (σ_a, $\widetilde\sigma_a$ bounded from below), we obtain:

$$\frac{\eta(t;x',v')}{\eta(t_1;y_1,v_1)} = e^{h(t;x',v')},\ \frac{\tilde\eta(t;x',v')}{\tilde\eta(t_1;y_1,v_1)} = e^{\tilde h(t;x',v')}, \tag{4.18}$$

where η is defined by (4.6). Note that from (4.17), it follows that $h(t;x',v')\ge -\|\sigma\|_\infty t\ge -Mt\ge -MD$ where D denotes the diameter of X. A similar estimate is valid for $\tilde h$ and $\tilde\sigma$. Therefore using the fact that $|a-\tilde a|\le e^{-\min(a,\tilde a)}|e^a-e^{\tilde a}|$ for $a=h(t)$, we obtain

$$|h-\widetilde h|(t;x',v') \le e^{\max(-h(t),-\tilde h(t))}\left|\frac{\eta(t;x',v')}{\eta(t_1;y_1,v_1)} - \frac{\widetilde\eta(t;x',v')}{\widetilde\eta(t_1;y_1,v_1)}\right|. \tag{4.19}$$

Note that from (4.6) it follows that

$$0<\sigma_0 e^{-\int_0^t\sigma(x-sv,v)ds} \le \eta(t;y,v)\le\sigma_a(y+tv,v), \tag{4.20}$$

for $(y,v)\in\Gamma_-$ and $t\in(0,\tau_+(y,v))$. A similar estimate is valid for $(\tilde\sigma_a,\tilde\eta)$. Therefore using the equality $\frac{a}{b}-\frac{\tilde a}{\tilde b}=\frac{a-\tilde a}{b}+\frac{\tilde a}{b\tilde b}(\tilde b-b)$ (for $a=\eta(t;x',v')$, $b=\eta(t_1;y_1,v_1)$), we obtain

$$\begin{aligned}
&e^{\max(-h(t),-\tilde h(t))}\left|\frac{\eta(t;x',v')}{\eta(t_1;y_1,v_1)} - \frac{\widetilde\eta(t;x',v')}{\widetilde\eta(t_1;y_1,v_1)}\right| \\
&\le e^{\max(-h(t),-\tilde h(t))}\Big(\frac{|\eta-\tilde\eta|(t;x',v')}{\eta(t_1,y_1,v_1)} + \frac{\tilde\eta(t;x',v')}{\eta(t_1,y_1,v_1)\tilde\eta(t_1,y_1,v_1)}|\eta-\tilde\eta|(t_1;y_1,v_1)\Big) \\
&\le \frac{e^{M\tau_+(x',v')}}{\sigma_0}\Big(|\eta-\tilde\eta|(t;x',v') + |\eta-\tilde\eta|(t_1;y_1,v_1)\Big).
\end{aligned} \tag{4.21}$$

Using (4.21), integrating in the t variable ($t_1=\tau_+(x',v')-t$) and using (2.24) and (4.19), we obtain

$$\int_0^{\tau_+(x',v')} e^{-M\tau_+(x',v')}|h-\widetilde h|(t;x',v')dt \le \frac{2\varepsilon}{\sigma_0} \tag{4.22}$$

for a.e. $(x',v')\in\Gamma_-$. We now prove the following estimate

$$\sup_{(x',v')\in\Gamma_-}\int_0^{\tau_+(x',v')} |\sigma_a-\tilde\sigma_a|(x'+tv',v')dt \le 2e^{M\tau_+(x',v')}(1+\frac{\sigma_1\tau_+(x',v')}{\sigma_0\min(\delta,\tau_+(x',v'))})\varepsilon. \tag{4.23}$$

The estimate on $\sigma_a(x,v')$ comes from the estimate (2.24) and an estimate on $\exp\Big(-\int_0^{\tau_+(x',v')}\sigma(x'+sv',v')ds\Big)$. It thus remains to control the constant term $\exp(-\int_0^{\tau_+(x',v')}\sigma(x'+sv',v')ds)$.

Note that the negative of the logarithm of latter term is nothing but the X-ray transform (Radon transform when $n = 2$) of σ along the line of direction v' passing through x'. In the setting of measurements that are supposed to be accurate in $\mathcal{L}(L^1(\Gamma_-, d\xi); L^1(X))$, the line integral is not directly captured as it corresponds to a measurement performed at a point $x = x' + \tau_+(x', v')v'$. This is the reason why we assume that σ is known in the δ_0-vicinity of ∂X.

Knowledge of σ and $\tilde\sigma$ in the δ_0-vicinity of ∂X allows one to control $\int_0^{\tau_+(x',v')} \sigma(x' + sv', v')ds$ by $\|A - \tilde{A}\|_{\mathcal{L}(L^1(\Gamma_-,d\xi);L^1(X))}$. When the X-ray transform of σ is well captured by available measurements, as for instance in the presence of boundary measurements [2], then δ_0 can be set to 0.

More precisely, using (2.24) and the estimate $\tilde\sigma \geq \sigma_0$ we obtain

$$\int_{\max(\tau_+(x',v')-\delta_0,0)}^{\tau_+(x',v')} \frac{|\eta - \tilde\eta|}{\tilde\sigma_a}(t, x', v') \leq \frac{\varepsilon}{\sigma_0}, \tag{4.24}$$

$$\int_{\max(\tau_+(x',v')-\delta_0,0)}^{\tau_+(x',v')} \frac{|\sigma_a - \tilde\sigma_a|}{\tilde\sigma_a}(x' + tv', v')e^{-\int_t^{\tau_+} \sigma(x'+sv',v')ds} \leq \frac{\varepsilon}{\sigma_0}, \tag{4.25}$$

where we used the equality $\sigma(x' + tv', v') = \tilde\sigma(x' + tv', v')$ for $t \in (\max(\tau_+(x', v') - \delta_0, 0), \tau_+(x', v'))$ to obtain (4.25) . Thus we obtain by triangle inequality

$$\left| e^{-\int_0^{\tau_+} \sigma(x'+sv',v')ds} - e^{-\int_0^{\tau_+} \sigma(x'+sv',v')ds} \right| dt \leq \frac{2\varepsilon}{\sigma_0 \min(\delta_0, \tau_+(x', v'))}. \tag{4.26}$$

Now note that

$$\begin{aligned}(\sigma_a - \tilde\sigma_a)(x' + tv', v') &= (\eta - \tilde\eta)(t, x', v')\frac{\eta(t_1, y_1, v_1)e^{\int_0^{\tau_+(x',v')} \sigma(x'+sv',v')ds}}{(\sigma_a + \tilde\sigma_a)(x' + tv', v')} \\ &+ (\eta - \tilde\eta)(t_1, y_1, v_1)\frac{\tilde\eta(t, x', v')e^{\int_0^{\tau_+(x',v')} \sigma(x'+sv',v')ds}}{(\sigma_a + \tilde\sigma_a)(x' + tv', v')} \\ &+ \frac{\tilde\sigma_a^2(x' + tv', v')e^{\int_0^{\tau_+(x',v')} \sigma(x'+sv',v')ds}}{(\sigma_a + \tilde\sigma_a)(x' + tv', v')}\left(e^{-\int_0^{\tau_+(x',v')} \tilde\sigma(x'+sv',v')ds} - e^{-\int_0^{\tau_+(x',v')} \sigma(x'+sv',v')ds}\right).\end{aligned} \tag{4.27}$$

Then we use (2.24) to estimate the integral over $(0, \tau_+(x', v'))$ of the first and second terms of the sum on the right hand side of (4.27) and we use (4.26) to estimate the integral over $(0, \tau_+(x', v'))$ of the third term. This yields (4.23).

We prove

$$\sup_{(x',v')\in\Gamma_-} \|\sigma - \widetilde\sigma\|_{W^{-1,1}(0,\tau_+(x',v'))} \leq \frac{e^{M\tau_+(x',v')}}{\sigma_0}\|\mathcal{A} - \widetilde{\mathcal{A}}\|_{\mathcal{L}(L^1(\Gamma_-,d\xi),L^1(X))}. \tag{4.28}$$

Combining (4.28) and (4.23), we obtain (2.35), which completes the proof of Theorem 2.8. Estimate (4.28) is, in particular, a consequence of the identities

$$\frac{dh(s; y_0, v_0)}{dt} = -2\sigma(y_0 + sv_0, v_0), \;\; \frac{d\widetilde{h}(s; y_0, v_0)}{dt} = -2\widetilde\sigma(y_0 + sv_0, v_0),$$

for a.e. $(y_0, v_0) \in \Gamma_-$ and $t \in (0, \tau_+(y_0, v_0))$. Indeed integrating by part we have:

$$\int_0^{\tau_+(x',v')} \phi(t)(\sigma - \tilde{\sigma})(x' + tv', v')dt = \frac{1}{2}\int_0^{\tau_+(x',v')} (\tilde{h} - h)(t; x', v')\frac{d\phi}{dt}(t)dt \qquad (4.29)$$

for a.e. $(x', v') \in \Gamma_-$ and $\phi \in C^1(\mathbb{R})$, $\mathrm{supp}\phi \subseteq (0, \tau_+(x', v'))$. Hence using (4.22), we obtain

$$\left|\int_0^{\tau_+(x',v')} \phi(t)(\sigma - \tilde{\sigma})(x' + tv', v')dt\right| \le \frac{e^{M\tau_+(x',v')}}{\sigma_0}\|\mathcal{A} - \widetilde{\mathcal{A}}\|_{\mathcal{L}(L^1(\Gamma_-,d\xi),L^1(X))}\|\phi\|_{W_0^{1,\infty}(0,\tau_+(x',v'))},$$

for a.e. $(x', v') \in \Gamma_-$ and $\phi \in C^1(\mathbb{R})$, $\mathrm{supp}\phi \subseteq (0, \tau_+(x', v'))$. This proves (4.28).□

Proof of Corollary 2.9. Let $p > 1$. Let $\|f\|_{s,p} := \sup_{(x',v')\in\Gamma_-} \|f(x' + tv', v')\|_{W_t^{s,p}(0,\tau_+(x',v'))}$ for any $s \in [-1, r]$. We first derive (2.36) from the following estimate

$$\|\sigma - \widetilde{\sigma}\|_{-1,p} \le C\|\mathcal{A} - \widetilde{\mathcal{A}}\|^{\frac{1}{p}}_{\mathcal{L}(L^1(\Gamma_-,d\xi),L^1(X))}, \qquad (4.30)$$

where $C := \frac{(M\tau_+(x',v'))^{\frac{p-1}{p}} e^{\frac{M\tau_+(x',v')}{p}}}{2^{1-\frac{1}{p}}\sigma_0^{\frac{1}{p}}}$. Assuming that $\|\sigma - \widetilde{\sigma}\|_{r,p}$ is bounded by C_0, then using (4.30) and using the complex interpolation result [6, Theorem 6.4.5 pp. 153], we obtain

$$\|\sigma - \tilde{\sigma}\|_{s,p} \le \|\sigma - \tilde{\sigma}\|^{1-\theta}_{-1,p}\|\sigma - \tilde{\sigma}\|^{\theta}_{r,p} \le C'\|\mathcal{A} - \widetilde{\mathcal{A}}\|^{\frac{1-\theta}{p}}_{\mathcal{L}(L^1(\Gamma_-,d\xi),L^1(X))}, \qquad (4.31)$$

for $-1 \le s \le r$, where $s = (1 - \theta) \times (-1) + r\theta$ and $C' = C^{1-\theta}C_0^{\theta}$. This proves (2.36).

We prove (4.30). From (4.17), it follows that $|h - \tilde{h}|(t; x', v') \le M\tau_+(x', v')$. Hence we obtain

$$\left|h - \widetilde{h}\right|^p (t; x', v') \le (M\tau_+(x', v'))^{p-1}|h - \tilde{h}|(t; x', v'), \qquad (4.32)$$

for $(x', v') \in \Gamma_-$ and $t \in (0, \tau_+(x', v'))$. Using (4.29) and Hölder inequality, and using (4.32), we obtain

$$\left|\int_0^{\tau_+(x',v')} (\sigma - \tilde{\sigma})(x' + tv', v')\phi(t)dx\right| \le \frac{1}{2}\left(\int_0^{\tau_+(x',v')} |h - \widetilde{h}|^p(t; x', v')dt\right)^{\frac{1}{p}} \|\frac{d\varphi}{dt}\|_{L^{p'}(0,\tau_+(x',v'))}$$

$$\le \frac{(M\tau_+(x', v'))^{\frac{p-1}{p}} e^{\frac{M\tau_+(x',v')}{p}}}{2^{1-\frac{1}{p}}\sigma_0^{\frac{1}{p}}}\|\mathcal{A} - \widetilde{\mathcal{A}}\|^{\frac{1}{p}}_{\mathcal{L}(L^1(\Gamma_-,d\xi),L^1(X))}\|\varphi\|_{W^{1,p'}(0,\tau_+(x',v'))}, \qquad (4.33)$$

for a.e. $(x', v') \in \Gamma_-$ and for $\phi \in W_0^{1,p'}(0, \tau_+(x', v')) := \{\psi \in L^{p'}(0, \tau_+(x', v')) \mid \mathrm{supp}\psi \subset (0, \tau_+(x', v')),\ \frac{d\psi}{dt}(t) \in L^{p'}((0, \tau_+(x', v')), \mathbb{C}^n)\}$ where $p'^{-1} + p^{-1} = 1$. Estimate (4.33) proves (4.30).□

4.2. Estimates for single scattering

Proof of Theorem 2.5. Let $(x,x')\in X\times\partial X$. Set $v'=\widehat{x-x'}$ and let $v'^{\perp}\in\mathbb{S}^{n-1}$ be such that $v'\cdot v'^{\perp}=0$. Let $t_0'=|x-x'|$, then $x=x'+t_0'v'$. First assume $n=2$. Using the equality $\alpha_1=\Gamma_1-\Gamma_2$, and using (2.22), (3.9), we obtain

$$\frac{|\alpha_1-\tilde\alpha_1|(x+\varepsilon v'^{\perp},x',v')}{|\nu(x')\cdot v'|w_2(x+\varepsilon v'^{\perp},x',v')}\le\left\|\frac{(\Gamma_1-\tilde\Gamma_1)(z,z',w')}{|\nu(z')\cdot w'|w_2(z,z',w')}\right\|_\infty+\frac{\left\|\frac{(\Gamma_2-\tilde\Gamma_2)(z,z',w')}{|\nu(z')\cdot w'|}\right\|_\infty}{w_2(x+\varepsilon v'^{\perp},x',v')}\quad(4.34)$$

for $\varepsilon>0$. Therefore using (4.34) as $\varepsilon\to0^+$ and (2.23) (and $w_2(x+\varepsilon v'^{\perp},x',v')=\ln\Big(\varepsilon^{-1}\Big)+o\Big(\ln\Big(\varepsilon^{-1}\Big)\Big)$ as $\varepsilon\to0^+$), we obtain (2.27).

Assume $n=3$. Using the equality $\alpha_1=\Gamma_1-\Gamma_2$, and using (2.22), (3.11) (for "$m=1$"), we obtain

$$\begin{aligned}\varepsilon\frac{|\alpha_1-\tilde\alpha_1|(x+\varepsilon v'^{\perp},x',v')}{|\nu(x')\cdot v'|}&\le\left\|\frac{(\Gamma_1-\tilde\Gamma_1)(z,z',w')}{|\nu(z')\cdot w'|w_n(z,z',w')}\right\|_{L^\infty(X\times\Gamma_-)}\\&-\varepsilon\ln(\varepsilon)\left\|\frac{(\Gamma_2-\tilde\Gamma_2)(z,z',w')}{|\nu(z')\cdot w'|\ln(|z-z'-((z-z')\cdot w')w'|)}\right\|_{L^\infty(X\times\Gamma_-)}\end{aligned}\quad(4.35)$$

for $\varepsilon>0$. Therefore using (4.35) as $\varepsilon\to0^+$ and (2.23), we obtain (2.28). Assume $n\ge4$. Using the equality $\alpha_1=\Gamma_1-\Gamma_2$, and using (2.22), (3.11) (for "$m=1$"), we obtain

$$\begin{aligned}\varepsilon^{n-2}\frac{|\alpha_1-\tilde\alpha_1|(x+\varepsilon v'^{\perp},x',v')}{|\nu(x')\cdot v'|}&\le\left\|\frac{(\Gamma_1-\tilde\Gamma_1)(z,z',w')}{|\nu(z')\cdot w'|w_n(z,z',w')}\right\|_{L^\infty(X\times\Gamma_-)}\\&+\varepsilon\left\|\frac{|z-z'-((z-z')\cdot w')w'|^{n-3}(\Gamma_2-\tilde\Gamma_2)(z,z',w')}{|\nu(z')\cdot w'|}\right\|_{L^\infty(X\times\Gamma_-)}\end{aligned}\quad(4.36)$$

for $\varepsilon>0$. Therefore using (4.36) as $\varepsilon\to0^+$ and (2.23), we obtain (2.28). □

Proof of Theorem 2.13 We prove (2.47). From (2.27), (2.28), (2.40) and (2.41), it follows that

$$|E(x,x')\sigma_g(x)-\tilde E(x,x')\sigma_g(x)|\le\left\|\frac{(\Gamma_1-\tilde\Gamma_1)(x,x',v')}{|\nu(x')\cdot v'|w_n(x,x',v')}\right\|_\infty,\quad(4.37)$$

for $(x,x')\in X\times\Gamma_-$. Using (4.37) and the estimate $\min(E(x,x'),\tilde E(x,x'))\ge e^{-D\max(\|\sigma\|_\infty,\|\tilde\sigma\|_\infty)}$ (see (2.19)) where D is the diameter of X, we obtain

$$|\sigma_g-\tilde\sigma_g|(x)\le e^{M|x-x'|}\Big(\left\|\frac{(\Gamma_1-\tilde\Gamma_1)(x,x',v')}{|\nu(x')\cdot v'|w_n(x,x',v')}\right\|_\infty+|E-\tilde E|(x,x')\Big),\quad(4.38)$$

for $(x,x')\in X\times\Gamma_-$ (we used the identity $ab-\tilde a\tilde b=(a-\tilde a)b+(b-\tilde b)\tilde a$ for $a:=\sigma_g(x)$ and $b:=E(x,x')$). Let $(x',v')\in\Gamma_-$. Integrating (4.38) over the line which passes through

x' with direction v' and using the stability estimates (4.22) and (4.26), and using (2.19), we obtain

$$\int_0^{\tau_+(x',v')} |\sigma_g - \tilde{\sigma}_g|(x' + tv')dt \le C\varepsilon, \tag{4.39}$$

where C is a constant which depends on δ_0, σ_0, M, $\mathrm{diam}(X)$. Integrating (4.39) over $\{x' \in \partial X \mid v' \cdot \nu(x') < 0\}$ with measure $|\nu(x') \cdot v'| d\mu(x')$, we obtain (2.47). We now prove (2.48)–(2.49). We first prove the stability estimate (4.42) given below. Assume $\tilde{g} \in W^{1,\infty}(X)$, $\|\nabla \tilde{g}\|_{L^\infty(X)} \le c$ for some nonnegative constant c. Using (2.44) and $\min(\sigma_s, \tilde{\sigma}_s) \ge \sigma_{s,0}$, we obtain

$$\|(1-\tilde{g})(h(g) - h(\tilde{g}))\|_{L^1(X)} \le \frac{1}{\sigma_{s,0}} \Big(\|(\sigma_s - \tilde{\sigma}_s) f(\tilde{g})\|_{L^1(X)} + \|\sigma_g - \tilde{\sigma}_g\|_{L^1(X)} \Big) \tag{4.40}$$

where $f(t) := (1-t)h(t)$, $t \in [0,1)$ (we used the identity $(1-\tilde{g})(ab - \tilde{a}\tilde{b}) = (1-\tilde{g})((a-\tilde{a})b + (b-\tilde{b})\tilde{a})$ for $a = \sigma_s$ and $b = h(\tilde{g})$, and we used the estimate $\tilde{g} \le 1$). Using the identity $\sigma_s = \sigma - \sigma_a$, we have

$$\begin{aligned} &\|(\sigma_s - \tilde{\sigma}_s) f(\tilde{g})\|_{L^1(X)} \le \|\sigma_a - \tilde{\sigma}_a\|_{L^1(X)} \|f(\tilde{g})\|_{L^\infty(X)} + \|(\sigma - \tilde{\sigma}) f(\tilde{g})\|_{L^1(X)} \\ &\le \|\sigma_a - \tilde{\sigma}_a\|_{L^1(X)} \|f(\tilde{g})\|_{L^\infty(X)} + \|\sigma - \tilde{\sigma}\|_{W^{-1,1}(X)} \|\nabla f(\tilde{g}))\|_{L^\infty(X)} \end{aligned} \tag{4.41}$$

(we used the fact that $\sigma = \tilde{\sigma}$ at the vicinity of the boundary ∂X). Extend $f(t)$ by $c(n)$ at $t = 1$, where $c(n)$ is given in Lemma 2.12. Then note that $f \in C^1([0,1])$ (see (2.45) and (A.2)) so that $\|\nabla f(\tilde{g}))\|_{L^\infty(X)} \le C \|\nabla \tilde{g}\|_{L^\infty(X)}$ for some positive constant C independent of $\tilde{g}$. Combining (4.40), (4.41), (2.47) and (2.35), we obtain

$$\|(1-\tilde{g})(h(g) - h(\tilde{g}))\|_{L^1(X)} \le C\varepsilon, \tag{4.42}$$

where ε is defined in Theorem 2.13 and C depends on δ_0, σ_0, $\sigma_{s,0}$, M and c. On one hand we use the estimates $\|(1-\tilde{g})(h(g) - h(\tilde{g}))\|_{L^1(X)} \ge (1 - \inf_{x \in X} \tilde{g}(x)) \|h(g) - h(\tilde{g})\|_{L^1(X)}$ and (4.42), and we use the estimate $h(t) - h(t') \ge C_1(t^2 - t'^2)$ for $(t, t') \in [0,1)^2$ where C_1 is a positive constant which follows from $\dot{h}(0) = 0$, $\ddot{h}(0) > 0$, $\dot{h}(t) > 0$ for $t \in (0,1)$ and $\lim_{t \to 1^-} \dot{h}(t) = +\infty$ (see Lemma 2.12 and its proof), and we obtain

$$\|g^2 - \tilde{g}^2\|_{L^1(X)} \le \frac{C\varepsilon}{C_1 \Big(1 - \inf_{x \in X} \tilde{g}(x) \Big)}. \tag{4.43}$$

On the other hand, using the equality $(1-\tilde{g})(h(g) - h(\tilde{g})) = h(g)(g - \tilde{g}) + f(g) - f(\tilde{g})$, and using the estimates $|f(g) - f(\tilde{g})| \le \sup_{t \in (0,1)} |\dot{f}(t)| |g - \tilde{g}|$ and (4.42) we obtain

$$\|g - \tilde{g}\|_{L^1(X)} \le \frac{2C\varepsilon}{\inf_{x \in X} h(g)} \le \frac{2C(1 - \inf_{x \in X} g(x))}{c(n)} \varepsilon, \tag{4.44}$$

when $h(g) \ge 2 \sup_{t \in (0,1)} |\dot{f}(t)|$ (we also used the estimate $h(t) \ge \frac{c(n)}{1-t}$ for $t \in [0,1)$, see (2.45) and (A.2)). Now assume also that $g \in W^{1,\infty}(X)$, $\|\nabla g\|_{L^\infty(X)} \le c$. Then the derivation of (4.43) and (4.44) also gives

$$\|g^2-\tilde g^2\|_{L^1(X)} \le \frac{C\varepsilon}{C_1\Big(1-\inf_{x\in X} g(x)\Big)}, \tag{4.45}$$

$$\|g-\tilde g\|_{L^1(X)} \le \frac{2C\varepsilon}{\inf_{x\in X} h(\tilde g)} \le \frac{2C(1-\inf_{x\in X}\tilde g(x))}{c(n)}\varepsilon,\ \text{when } h(\tilde g)\ge 2\sup_{t\in(0,1)}|\dot f(t)|. \tag{4.46}$$

Combining (4.43)–(4.46) and the estimates $|g-\tilde g|\inf_{x\in X}(g+\tilde g)\le|g^2-\tilde g^2|\le(g+\tilde g)|g-\tilde g|\le 2|g-\tilde g|$, one obtain (2.48)–(2.49). □

Appendix A.

In this appendix, we prove the technical lemmas 2.12, 3.1, and 3.2.

Proof of Lemma 2.12. First consider the case $n=2$. The statement in Lemma 2.12 is a straightforward consequence of (2.45) and we have the following inversion formula:

$$g=\Big(\frac{\pi h(g)-1}{\pi h(g)+1}\Big)^{\frac12},\ g\in[0,1). \tag{A.1}$$

Let now $n=3$. Using (2.46) and the identity $\cos(\theta)=1-2\sin^2(\frac{\theta}{2})$, $\theta\in\mathbb{R}$, we obtain

$$2\pi h(g)=\frac{1+g}{(1-g)^2}\int_0^\pi \frac12\Big(1+\frac{4g}{(1-g)^2}\sin^2\big(\frac{\theta}{2}\big)\Big)^{-\frac32}d\theta,$$

for $g\in[0,1)$. Performing the change of variables $\theta=2\arcsin(\frac{t}{\sqrt{1+t^2}})$ $(d\theta=\frac{2dt}{1+t^2})$, we obtain

$$2\pi h(g)=\frac{1+g}{(1-g)^2}\int_0^{+\infty}(1+t^2)^{\frac12}\Big(1+\Big(\frac{1+g}{1-g}\Big)^2t^2\Big)^{-\frac32}dt.$$

Performing the change of variables $v=\frac{1+g}{1-g}t$, we obtain

$$2\pi h(g)=\int_0^{+\infty}\sqrt{\frac{\Big(\frac{1}{1-g}\Big)^2+\Big(\frac{1}{1+g}\Big)^2v^2}{(1+v^2)^3}}dv=\frac{1}{1-g}\int_0^{+\infty}\sqrt{\frac{1+\Big(\frac{1-g}{1+g}\Big)^2v^2}{(1+v^2)^3}}dv, \tag{A.2}$$

for $g\in[0,1)$. Note that from the above, it follows that $2\pi h(g)=\frac{1}{1-g}+o(\frac{1}{1-g})$ as $g\to1^-$ (where we used the integral value $\int_0^{+\infty}\frac{dv}{(1+v^2)^{\frac32}}=\Big[\frac{v}{\sqrt{1+v^2}}\Big]_0^{+\infty}=1$).

Differentiating (A.2) with respect to g, we obtain

$$2\pi\dot h(g)=\frac{1}{(1-g)^3}\int_0^{+\infty}\frac{dv}{(1+v^2)^{\frac32}\sqrt{\omega_1^2(g)+\omega_2^2(g)v^2}}-\frac{1}{(1+g)^3}h_2(g), \tag{A.3}$$

for $g\in[0,1)$, where $\dot h=\frac{dh}{dg}$, $\omega_1(g):=\frac{1}{1-g}$, $\omega_2(g):=\frac{1}{1+g}$, and

$$h_2(g):=\int_0^{+\infty}\frac{v^2dv}{(1+v^2)^{\frac32}\sqrt{\omega_1^2(g)+\omega_2^2(g)v^2}}. \tag{A.4}$$

Integrating by parts, we obtain

$$h_2(g) = \int_0^{+\infty} \frac{\omega_1^2(g)dv}{(1+v^2)^{\frac{1}{2}} \left(\omega_1^2(g)+\omega_2^2(g)v^2\right)^{\frac{3}{2}}}, \tag{A.5}$$

for $g \in [0,1)$, where we use that a primitive of the function $r(v) := v(1+v^2)^{-\frac{3}{2}}$ is given by the function $R(v) := -(1+v^2)^{-\frac{1}{2}}$ and where we used that the derivative of the function $s(v) := \left(\omega_1^2(g)+\omega_2^2(g)v^2\right)^{-\frac{1}{2}}$ is given by $\dot{s}(v) = \omega_1^2(g)\left(\omega_1^2(g)+\omega_2^2(g)v^2\right)^{-\frac{3}{2}}$. Performing the change of variables "v"$= \frac{1-g}{1+g}v$ on the right hand side of (A.5), we obtain

$$h_2(g) = \omega_2^2(g) \int_0^{+\infty} (1+v^2)^{-\frac{3}{2}} \left(1+\left(\frac{1+g}{1-g}\right)^2 v^2\right)^{-\frac{1}{2}} dv, \tag{A.6}$$

for $g \in [0,1)$. Combining (A.3) and (A.6), we obtain

$$\begin{aligned} 2\pi \dot{h}(g) = \frac{1}{(1-g)^2}\Big[&\int_0^{+\infty} (1+v^2)^{-\frac{3}{2}}\left(1+\left(\frac{1-g}{1+g}\right)^2 v^2\right)^{-\frac{1}{2}} dv \\ &- \left(\frac{1-g}{1+g}\right)^2 \int_0^{+\infty} (1+v^2)^{-\frac{3}{2}}\left(1+\left(\frac{1+g}{1-g}\right)^2 v^2\right)^{-\frac{1}{2}} dv\Big], \end{aligned} \tag{A.7}$$

for $g \in [0,1)$. Using the estimate $1+\left(\frac{1+g}{1-g}\right)^2 v^2 > 1+\left(\frac{1-g}{1+g}\right)^2 v^2$ for $v \in (0,+\infty)$ and $g \in (0,1)$, we obtain that the second integral on the right hand side of (A.7) is less than the first integral on the right-hand side of (A.7) for $g \in (0,1)$. Therefore using also that the second integral is multiplied by $\left(\frac{1-g}{1+g}\right)^2 (< 1)$, we obtain $\dot{h}(g) > 0$ for $g \in (0,1)$. □

Proof of Lemma 3.1 For $m \geq 2$, let β_m denotes the distributional kernel of the operator K^m where K is defined by (3.2). We first give the explicit expression of β_2, β_3. Then by induction we give the explicit expression of the kernel β_m. Finally we prove Lemma 3.1.

From (3.2) it follows that

$$\begin{aligned} K^2\psi(x,v) = &\int_0^{\tau_-(x,v)} \int_{\mathbb{S}^{n-1}} k(x-tv,v_1,v) \int_0^{\tau_-(x-tv,v_1)} E(x,x-tv,x-tv-t_1v_1) \\ &\times \int_{\mathbb{S}^{n-1}} k(x-tv-t_1v_1,v',v_1)\psi(x-tv-t_1v_1,v')dv'dt_1dv_1dt, \end{aligned} \tag{A.8}$$

for a.e. $(x,v) \in X \times \mathbb{S}^{n-1}$ and for $\psi \in L^1(X \times \mathbb{S}^{n-1})$. Performing the change of variables $x' = x - tv - t_1v_1$ ($dx' = t_1^{n-1}dt_1dv_1$) on the right hand side of (A.8), we obtain

$$\begin{aligned} K^2\psi(x,v) = &\int_0^{\tau_-(x,v)} \int_X k(x-tv, \widehat{x-tv-x'}, v)E(x,x-tv,x') \\ &\times \int_{\mathbb{S}^{n-1}} k(x',v',\widehat{x-tv-x'})\psi(x',v')dv'dx'dt, \end{aligned} \tag{A.9}$$

for a.e. $(x,v) \in X \times \mathbb{S}^{n-1}$ and for $\psi \in L^1(X \times \mathbb{S}^{n-1})$. Therefore

$$\beta_2(x,v,x',v') = \int_0^{\tau_-(x,v)} E(x,x-tv,x')\frac{k(x-tv,v_1,v)k(x',v',v_1)_{|v_1=\widehat{x-tv-x'}}}{|x-tv-x'|^{n-1}}dt, \tag{A.10}$$

for a.e. $(x,v,x',v')\in X\times\mathbb{S}^{n-1}\times X\times\mathbb{S}^{n-1}$. From (A.10) and (3.2) it follows that

$$K^3\psi(x,v)=\int_{(0,\tau_-(x,v))\times\mathbb{S}^{n-1}}E(x,x-tv)k(x-tv,v_1,v)\int_{X\times\mathbb{S}^{n-1}}\beta_2(x-tv,v_1,x',v')\psi(x',v')dx'dv'dv_1dt$$

$$=\int_{X\times\mathbb{S}^{n-1}}\psi(x',v')\int_0^{\tau_-(x,v)}\int_{\mathbb{S}^{n-1}}k(x-tv,v_1,v)\int_0^{\tau_-(x-tv,v_1)}\frac{E(x,x-tv,x-tv-t_1v_1,x')}{|x-tv-t_1v_1-x'|^{n-1}}$$
$$\times k(x-tv-t_1v_1,v_2,v_1)k(x',v',v_2)_{v_2=\widehat{x-tv-t_1v_1-x'}}dt_1dv_1dtdx'dv', \quad \text{(A.11)}$$

for a.e. $(x,v)\in X\times\mathbb{S}^{n-1}$ and for $\psi\in L^1(X\times\mathbb{S}^{n-1})$. Therefore performing the change of variables $z=x-tv-t_1v_1$ ($dz=t_1^{n-1}dt_1dv_1$) on the right hand side of (A.10) we obtain

$$\beta_3(x,v,x',v')=\int_0^{\tau_-(x,v)}\int_X\frac{E(x,x-tv,z,x')k(x',v',\widehat{z-x'})}{|x-tv-z|^{n-1}|z-x'|^{n-1}}$$
$$\times k(x-tv,\widehat{x-tv-z},v)k(z,\widehat{z-x'},\widehat{x-tv-z})dtdz, \quad \text{(A.12)}$$

for a.e. $(x,v,x',v')\in X\times\mathbb{S}^{n-1}\times X\times\mathbb{S}^{n-1}$. Then by induction we have

$$\beta_m(x,v,z_m,v_m)=\int_0^{\tau_-(x,v)}\int_{X^{m-2}}\frac{E(x,x-tv,z_2,\ldots,z_m)}{|x-tv-z_2|^{n-1}\Pi_{i=2}^{m-1}|z_i-z_{i+1}|^{n-1}} \quad \text{(A.13)}$$
$$\times k(x-tv,v_1,v)\Pi_{i=2}^m k(z_i,v_i,v_{i-1})_{|v_1=\widehat{x-tv-z_2},\ v_i=\widehat{z_i-z_{i+1}},\ i=2\ldots m-1}dtdz_2\ldots dz_{m-1},$$

for a.e. $(x,v,x',v')\in X\times\mathbb{S}^{n-1}\times X\times\mathbb{S}^{n-1}$.

We prove (3.7) for $m=2$. From (A.10), (3.6) and (3.3) it follows that

$$\bar{K^2}J\psi(x)=\int_{\mathbb{S}^{n-1}}\sigma_a(x,v)\int_{X\times\mathbb{S}^{n-1}}\int_0^{\tau_-(x,v)}E(x,x-tv,y)\frac{k(x-tv,v_1,v)k(y,v',v_1)_{|v_1=\widehat{x-tv-y}}}{|x-tv-y|^{n-1}}dt$$
$$\times E(y,y-\tau_-(y,v')v')\psi(y-\tau_-(y,v')v',v')dydv'dv, \quad \text{(A.14)}$$

for a.e. $x\in X$ and for $\psi\in L^1(\Gamma_-,d\xi)$. Then performing the change of variable $z=x-tv$ ($t=|x-z|$, $v=\widehat{x-z}$ and $dz=t^{n-1}dtdv$) on the right hand side of (A.14), we obtain

$$\bar{K^2}J\psi(x)=\int_{X\times X\times\mathbb{S}^{n-1}}\sigma_a(x,\widehat{x-z})E(x,z,y)\frac{k(z,v_1,v)k(y,v',v_1)_{|v_1=\widehat{z-y}}}{|x-z|^{n-1}|z-y|^{n-1}}$$
$$\times E(y,y-\tau_-(y,v')v')\psi(y-\tau_-(y,v')v',v')dzdydv', \quad \text{(A.15)}$$

for a.e. $x\in X$ and for $\psi\in L^1(\Gamma_-,d\xi)$. Performing the change of variables $y=x'+t'v'$ ($x'\in\partial X$, $t'>0$, $dz=|\nu(x')\cdot v'|d\mu(x')dt'$) on the right hand side of (A.15), we obtain $\bar{K^2}J\psi(x)=\int_{\Gamma_-}\alpha_2(x,x',v')\psi(x',v')d\mu(x')dv'$ for a.e. $x\in X$ and for $\psi\in L^1(\Gamma_-,d\xi)$, which proves (3.7) for $m=2$.

Then we prove (3.8) before proving (3.7) for $m\ge 3$. Let $m\ge 3$. From (A.13) and the definition of the operator $\bar{K^m}$ (3.6), it follows that

$$\bar{K^m}\psi(z_0)=\int_{\mathbb{S}^{n-1}}\sigma_a(z_0,v_0)\int_{X\times\mathbb{S}^{n-1}}\beta_m(z_0,v_0,z_m,v_m)\psi(z_m,v_m)dz_mdv_mdv_0$$
$$=\int_{\mathbb{S}^{n-1}}\sigma_a(z_0,v_0)\int_{X\times\mathbb{S}^{n-1}}\int_0^{\tau_-(z_0,v_0)}\int_{X^{m-2}}E(z_0,z_0-tv_0,z_2,\ldots,z_m)$$
$$\times\frac{k(z_0-tv_0,v_1,v_0)\Pi_{i=2}^m k(z_i,v_i,v_{i-1})_{|v_1=\widehat{z_0-tv_0-z_2},\ v_i=\widehat{z_i-z_{i+1}},\ i=2\ldots m-1}}{|z_0-tv_0-z_2|^{n-1}\Pi_{i=2}^{m-1}|z_i-z_{i+1}|^{n-1}}$$
$$\times dtdz_2\ldots dz_{m-1}\psi(z_m,v_m)dz_mdv_mdv_0, \quad \text{(A.16)}$$

for a.e. $z_0 \in X$ and for $\psi \in L^1(X \times \mathbb{S}^{n-1})$. Therefore performing the change of variables $z_1 = z_0 - tv_0$ ($dz_1 = t^{n-1}dtdv_0$, $t = |z_0 - z_1|$ and $v_0 = \widehat{z_0 - z_1}$) on the right hand side of (A.16) we obtain $\bar{K^m}\psi(z_0) = \int_{X\times\mathbb{S}^{n-1}} \gamma_m(z_0, z_m, v_m)\psi(z_m, v_m)\, dz_m dv_m$, for a.e. $z_0 \in X$ and for $\psi \in L^1(X \times \mathbb{S}^{n-1})$, which proves (3.8).

We prove (3.7). Let $m \geq 3$. From (3.8), (3.6) and (3.3), it follows that

$$\begin{aligned}
&\bar{K^m}J\psi(z_0) = \int_{X\times\mathbb{S}^{n-1}} \gamma_m(z_0, z_m, v_m) \\
&\times E(z_m, z_m - \tau_-(z_m, v_m)v_m)\psi(z_m - \tau_-(z_m, v_m)v_m, v_m)dz_m dv_m \\
&= \int_{X\times\mathbb{S}^{n-1}} \int_{X^{m-1}} \frac{E(z_0, \ldots, z_m, z_m - \tau_-(z_m, v_m)v_m)}{\Pi_{i=1}^m |z_i - z_{i-1}|^{n-1}} \psi(z_m - \tau_-(z_m, v_m)v_m, v_m) \\
&\times [\sigma_a(z_0, v_0)\Pi_{i=1}^m k(z_i, v_i, v_{i-1})]_{v_i = \widehat{z_i - z_{i+1}},\ i=0\ldots m-1}\, dz_1 \ldots dz_{m-1}dz_m dv_m,
\end{aligned} \tag{A.17}$$

for a.e. $z_0 \in X$ and for $\psi \in L^1(\Gamma_-, d\xi)$. Then performing the change of variables $z_m =$ "z_m" $+t'v_m$ ("z_m"$\in \partial X$, $t' > 0$, $dz_m = |\nu(\text{“}z_m\text{”}) \cdot v_m| d\mu(\text{“}z_m\text{”})dt'$), we obtain $\bar{K^m}J\psi(z_0) = \int_{\Gamma_-} \alpha_m(z_0, z_m, v_m)\psi(z_m, v_m)d\mu(z_m)dv_m$ for a.e. $z_0 \in X$ and for $\psi \in L^1(\Gamma_-, d\xi)$, which proves (3.7).□

Proof of Lemma 3.2. First note that from (2.21) and (3.7), it follows that

$$\alpha_m(x, x', v') \leq \|\sigma_a\|_\infty \|k\|_\infty^m |\nu(x') \cdot v'| I_{m,n}(x, x', v'), \tag{A.18}$$

for a.e. $(x, x', v') \in X \times \Gamma_-$ and for $m \in \mathbb{N}$, $m \geq 1$, where

$$I_{1,n}(z_0, x', v') = \int_0^{\tau_+(x',v')} \frac{dt'}{|z_0 - x' - t'v'|^{n-1}}, \tag{A.19}$$

$$I_{m+1,n}(z_0, x', v') = \int_{X^m} \int_0^{\tau_+(x',v')} \frac{dt' dz_1 \ldots dz_m}{|x' + t'v' - z_m|^{n-1} \Pi_{i=1}^m |z_i - z_{i-1}|^{n-1}}, \tag{A.20}$$

for $(z_0, x', v') \in X \times \Gamma_-$ and $m \geq 1$.

We prove (3.9) and (3.10). Let $n = 2$. Let $(x, x', v') \in X \times \Gamma_-$ be such that $x \neq x' + \lambda v'$ for any $\lambda \in \mathbb{R}$. Set $(w)_\perp := w - (w \cdot v')v'$ for any $w \in \mathbb{R}^n$. Using (A.19) and using the equality $|x - x' - t'v'|^2 = (t' - (x - x') \cdot v')^2 + |(x - x')_\perp|^2$, we obtain

$$I_{1,n}(x, x', v') = \int_{-(x-x')\cdot v'}^{\tau_+(x',v')-(x-x')\cdot v'} \frac{1}{(|(x - x')_\perp|^2 + t^2)^{\frac{n-1}{2}}} dt, \tag{A.21}$$

$$= \ln\Big(\frac{|x - x' - \tau_+(x', v')v'| - (x - x' - \tau_+(x', v')v') \cdot v'}{|x - x'| - (x - x') \cdot v'}\Big), \tag{A.22}$$

where we used that $\int_0^x \frac{dt}{\sqrt{1+t^2}} = \ln\big(x + \sqrt{1 + x^2}\big)$. Estimate (3.9) follows from (A.18) and (A.22).

Let $(x, x', v') \in X \times \Gamma_-$ be such that $x \neq x' + \lambda v'$ for any $\lambda \in \mathbb{R}$. Using (A.22) (with "$x = z$") and (A.20), we obtain

$$\begin{aligned}
I_{2,n}(x, x', v') &= \int_X \frac{\ln\Big(\frac{|z-x'-\tau_+(x',v')v'|-(z-x'-\tau_+(x',v')v')\cdot v'}{|z-x'|-(z-x')\cdot v'}\Big)}{|x - z|} dz \\
&= I'_{2,n}(x, x' + \tau_+(x', v')v', v') - I'_{2,n}(x, x', v'),
\end{aligned} \tag{A.23}$$

where

$$I'_{2,n}(x,a,v') = \int_X \frac{\ln(|z-a|-(z-a)\cdot v')}{|x-z|}dz, \tag{A.24}$$

for $a \in \bar{X}$. We prove

$$\sup_{(x,a,v')\in X\times\bar{X}\times\mathbb{S}^{n-1}} |I'_{2,n}(x,a,v')| < \infty. \tag{A.25}$$

Then estimate (3.10) follows from (A.18), (A.23) and (A.25).

Let $(x,a,v') \in X\times\bar{X}\times\mathbb{S}^{n-1}$. Note that by using (A.24), we obtain

$$\begin{aligned}|I'_{2,n}(x,a,v')| &\leq C\int_X \frac{(|z-a|+(z-a)\cdot v')^{\frac{1}{8}}}{|x-z|(|z-a|^2-((z-a)\cdot v')^2)^{\frac{1}{8}}}dz \\ &\leq (2D)^{\frac{1}{8}}C\int_X \frac{1}{|x-z|(|z-a|^2-((z-a)\cdot v')^2)^{\frac{1}{8}}}dz, \end{aligned} \tag{A.26}$$

where $C := \sup_{r\in(0,6D)} r^{\frac{1}{8}}|\ln(r)| < \infty$ and D denotes the diameter of X. Consider $v'^{\perp} \in \mathbb{S}^{n-1}$ a unit vector orthogonal to v' and perform the change of variable $z = a+\lambda_1 v'+\lambda_2 v'^{\perp}$, then we obtain

$$|I'_{2,n}(x,a,v')| \leq (2D)^{\frac{1}{8}}C\int_{[-D,D]^2} \frac{d\lambda_1 d\lambda_2}{((\lambda_{1,x}-\lambda_1)^2+(\lambda_{2,x}-\lambda_2)^2)^{\frac{1}{2}}|\lambda_2|^{\frac{1}{4}}}, \tag{A.27}$$

where $\lambda_{1,x}v'+\lambda_{2,x}v'^{\perp} = x-a$. Then note that

$$\int_{-D}^{D} \frac{d\lambda_1}{((\lambda_{1,x}-\lambda_1)^2+(\lambda_{2,x}-\lambda_2)^2)^{\frac{1}{2}}} = \ln\left(\frac{D-\lambda_{1,x}+((\lambda_2-\lambda_{2,x})^2+(D-\lambda_{1,x})^2)^{\frac{1}{2}}}{-D-\lambda_{1,x}+((\lambda_2-\lambda_{2,x})^2+(D+\lambda_{1,x})^2))^{\frac{1}{2}}}\right), \tag{A.28}$$

Combining (A.27) and (A.28), we obtain

$$\begin{aligned}|I'_{2,n}(x,a,v')| &\leq C'\int_{-D}^{D} \frac{(D-\lambda_{1,x}+((\lambda_2-\lambda_{2,x})^2+(D-\lambda_{1,x})^2)^{\frac{1}{2}})^{\frac{1}{8}}d\lambda_2}{|\lambda_2|^{\frac{1}{4}}(-(D+\lambda_{1,x})+\sqrt{(\lambda_2-\lambda_{2,x})^2+(D+\lambda_{1,x})^2})^{\frac{1}{8}}} \\ &\leq C''\int_{-D}^{D} \frac{(D+\lambda_{1,x}+((\lambda_2-\lambda_{2,x})^2+(D+\lambda_{1,x})^2)^{\frac{1}{2}})^{\frac{1}{8}}}{|\lambda_2|^{\frac{1}{4}}|\lambda_2-\lambda_{2,x}|^{\frac{1}{4}}}d\lambda_2 \\ &\leq C'''\int_{-D}^{D} \frac{d\lambda_2}{|\lambda_2|^{\frac{1}{4}}|\lambda_2-\lambda_{2,x}|^{\frac{1}{4}}} = C'''|\lambda_{2,x}|^{\frac{1}{2}}\int_{-\frac{D}{|\lambda_{2,x}|}}^{\frac{D}{|\lambda_{2,x}|}} \frac{ds}{|s|^{\frac{1}{4}}|s-1|^{\frac{1}{4}}}, \end{aligned} \tag{A.29}$$

where $C' := (2D)^{\frac{1}{8}}C^2$, $C'' := (12D^2)^{\frac{1}{8}}C^2$ and $C''' := (72D^3)^{\frac{1}{8}}C^2$. Finally note that

$$|\lambda_{2,x}|^{\frac{1}{2}}\int_{|s|\leq\frac{D}{|\lambda_{2,x}|}} \frac{1}{|s|^{\frac{1}{4}}|s-1|^{\frac{1}{4}}}ds \leq C_1(\lambda_{2,x})+C_2(\lambda_{2,x}), \quad \text{where} \tag{A.30}$$

$$C_1(\lambda_{2,x}) := |\lambda_{2,x}|^{\frac{1}{2}}\int_{|s|\leq 2}|s|^{-\frac{1}{4}}|s-1|^{-\frac{1}{4}}ds \leq D^{\frac{1}{2}}\int_{|s|\leq 2}|s|^{-\frac{1}{4}}|s-1|^{-\frac{1}{4}}ds \tag{A.31}$$

$$\begin{aligned}C_2(\lambda_{2,x}) &:= |\lambda_{2,x}|^{\frac{1}{2}}\int_{2\leq|s|\leq\max(2,\frac{D}{|\lambda_{2,x}|})}|s|^{-\frac{1}{4}}|s-1|^{-\frac{1}{4}}ds \leq |\lambda_{2,x}|^{\frac{1}{2}}2^{\frac{1}{4}}\int_{2\leq|s|\leq\max(2,\frac{D}{|\lambda_{2,x}|})}|s|^{-\frac{1}{2}}ds \\ &\leq\ 2^{\frac{9}{4}}\Big(|\lambda_{2,x}|\max(2,\frac{D}{|\lambda_{2,x}|})\Big)^{\frac{1}{2}} \leq 2^{\frac{11}{4}}D^{\frac{1}{2}}. \end{aligned} \tag{A.32}$$

Combining (A.29)–(A.32) we obtain (A.25).

The statements (3.11)–(3.13) follow from (A.18) and the following statements (A.33)–(A.35)

$$|x - x' - ((x - x') \cdot v')v'|^{n-1-m} I_{m,n}(x, x', v') \in L^\infty(X \times \Gamma_-), \tag{A.33}$$

$$\frac{I_{n-1,n}(x, x', v')}{\ln\left(|x - x' - ((x - x') \cdot v')v'|\right)} \in L^\infty(X \times \Gamma_-), \tag{A.34}$$

$$I_{n,n}(x, x', v') \in L^\infty(X \times \Gamma_-), \tag{A.35}$$

for $(m, n) \in \mathbb{N} \times \mathbb{N}$, $n \geq 3$, $1 \leq m \leq n - 2$ and where $I_{m,n}$ is defined by (A.19) and (A.20).

We prove (A.33)–(A.35), which will complete the proof of Lemma 3.2. We proceed by induction on m. We prove (A.33) for $m = 1$. Let $n \geq 3$. Note that formula (A.21) still holds. Note also that

$$|w - \lambda v'|^2 = |w_\perp|^2 + |w \cdot v' - \lambda|^2 \geq 2^{-1}(|w_\perp| + |w \cdot v' - \lambda|)^2, \tag{A.36}$$

for $(w, \lambda) \in \mathbb{R}^n \times (-D, D)$, $|w| \leq 2D$. Therefore

$$\int_{-2D}^{2D} \frac{d\lambda}{|w - \lambda v'|^{n-1}} \leq \int_{-2D}^{2D} \frac{2^{\frac{n-1}{2}} d\lambda}{(|w_\perp| + |\lambda|)^{n-1}} \leq \frac{2^{\frac{n+1}{2}}}{(n-2)|w_\perp|^{n-2}}. \tag{A.37}$$

Thus (A.33) for $m = 1$ follows from (A.37) and (A.21).

Let $m \in \mathbb{N}$, $m \geq 1$ be such that (A.33)–(A.35) hold for any $n \geq 3$. We prove that (A.33)–(A.35) hold for any $n \geq 3$ and for "m"$= m + 1$. Let (x, x', v') be such that $x \neq x' + \lambda v'$ for any $\lambda \in \mathbb{R}$. From (A.19) and (A.20) it follows that

$$I_{m+1,n}(x, x', v') \leq \int_X \frac{I_{m,n}(z, x', v')}{|x - z|^{n-1}} dz. \tag{A.38}$$

Assume that $m + 1 \leq n - 1$. Then from (A.38) and (A.33) for (m, n) it follows that there exists a constant C (which does not depend on (x, x', v')) such that

$$I_{m+1,n}(x, x', v') \leq C \int_X \frac{1}{|x - z|^{n-1} |(z - x')_\perp|^{n-1-m}} dz. \tag{A.39}$$

Performing the change of variables $z - x' = z' + \lambda v'$, $z' \cdot v' = 0$, we obtain

$$I_{m+1,n}(x, x', v') \leq C \int_{\substack{z' \cdot v' = 0 \\ |z'| \leq D}} \left(\int_{-D}^{D} \frac{d\lambda}{|x - x' - z' - \lambda v'|^{n-1}} \right) \frac{dz'}{|z'|^{n-1-m}}. \tag{A.40}$$

Combining (A.40) and (A.37) (with "$w = x - x' - z'$"), we obtain

$$I_{m+1,n}(x, x', v') \leq \frac{2^{\frac{n+1}{2}} C}{n-2} I''_{m+1,n}(x, x', v'), \quad \text{where} \tag{A.41}$$

$$\begin{aligned} I''_{m+1,n}(x, x', v') :=& \int_{\substack{z' \cdot v' = 0 \\ |z'| \leq D}} \frac{dz'}{|(x - x')_\perp - z'|^{n-2} |z'|^{n-1-m}} \\ =& \int_{z' \in B_{n-1}(0,D)} \frac{dz'}{|\mathrm{d}(x, x')e_1 - z'|^{n-2} |z'|^{n-1-m}}, \end{aligned} \tag{A.42}$$

with $\mathrm{d}(x,x') = |(x-x')_\perp|$, $e_1 = (1,0,\ldots,0) \in \mathbb{R}^{n-1}$ and $B_{n-1}(0,D)$ denotes the Euclidean ball of $\mathbb{R}^{n-1}$ of center 0 and radius D. Using spherical coordinates $z' = \mathrm{d}(x,x')e_1 + r\Omega$, $(r,\Omega) \in (0,+\infty)\times\mathbb{S}^{n-2}$ and $\Omega = (\sin(\theta),\cos(\theta)\Theta)$ $((\theta,\Theta) \in (-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{S}^{n-3})$ we obtain

$$\begin{aligned} I''_{m+1,n}(x,x',v') &\le |\mathbb{S}^{n-3}| \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n-3}(\theta) \\ &\times \Big(\int_0^{2D} \frac{dr}{(|r+\mathrm{d}(x,x')\sin(\theta)|^2 + \mathrm{d}(x,x')^2\cos(\theta)^2)^{\frac{n-1-m}{2}}} \Big) d\theta \end{aligned} \tag{A.43}$$

(by convention $|\mathbb{S}^0| := 2$). Note that by performing the change of variables "$r = r + \mathrm{d}(x,x')\sin(\theta)$" and using the estimate $a^2+b^2 \ge 2^{-1}(a+b)^2$, we obtain

$$\begin{aligned} &\int_0^{2D} \frac{dr}{(|r+\mathrm{d}(x,x')\sin(\theta)|^2 + \mathrm{d}(x,x')^2\cos(\theta)^2)^{\frac{n-1-m}{2}}} \\ &\le \int_{-3D}^{3D} \frac{2^{\frac{n-1}{2}}dr}{(|r| + \mathrm{d}(x,x')\cos(\theta))^{n-1-m}} = \int_0^{3D} \frac{2^{\frac{n+1}{2}}dr}{(r + \mathrm{d}(x,x')\cos(\theta))^{n-1-m}} \\ &\le \begin{cases} \dfrac{2^{\frac{n+1}{2}}}{(n-m-2)(\mathrm{d}(x,x')\cos(\theta))^{n-2-m}}, \text{ if } m+1 < n-1, \\ 2^{\frac{n+1}{2}} \ln\left(\dfrac{3D + \mathrm{d}(x,x')\cos(\theta)}{\mathrm{d}(x,x')\cos(\theta)}\right), \text{ if } m+1 = n-1, \end{cases} \end{aligned} \tag{A.44}$$

for $\theta \in (-\frac{\pi}{2},\frac{\pi}{2})$ (we also used the estimate $\mathrm{d}(x,x') \le D$). Assume $m+1 < n-1$. Then combining (A.43) and (A.44), we obtain

$$I''_{m+1,n}(x,x',v') \le \frac{2^{\frac{n+1}{2}}|\mathbb{S}^{n-3}|}{(n-2-m)\mathrm{d}(x,x')^{n-2-m}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{m-1}(\theta)d\theta. \tag{A.45}$$

Therefore using also (A.41) we obtain that (A.33) holds for "m"$= m+1 < n-1$. Assume $m+1 = n-1$. Then note that

$$\begin{aligned} \ln\left(\frac{3D+\mathrm{d}(x,x')\cos(\theta)}{\mathrm{d}(x,x')\cos(\theta)}\right) &\le \ln\left(\frac{4D}{\mathrm{d}(x,x')\cos(\theta)}\right) \\ &\le \ln(4D) - \ln(\mathrm{d}(x,x')) - \ln(\cos(\theta)), \end{aligned} \tag{A.46}$$

for $\theta \in (-\frac{\pi}{2},\frac{\pi}{2})$, where we recall that $\mathrm{d}(x,x') = |(x-x')_\perp|$. Combining (A.43), (A.44) and (A.46), we obtain

$$I''_{m+1,n}(x,x',v') \le 2^{\frac{n+1}{2}}|\mathbb{S}^{n-3}|\left(C_1 - C_2\ln(\mathrm{d}(x,x'))\right), \tag{A.47}$$

where $C_1 := \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n-3}(\theta)(\ln(4D) - \ln(\cos(\theta)))d\theta < \infty$ and $C_2 := \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n-3}(\theta)d\theta$. Therefore using also (A.41), we obtain that (A.34) holds for "m"$= m+1 = n-1$.

Assume that $m+1 = n$. From (A.38) and (A.34) for $(n-1,n)$ it follows that there exists a constant C (which does not depend on (x,x',v')) such that

$$I_{m+1,n}(x,x',v') \le C \int_X \frac{|\ln(|(z-x')_\perp|)|}{|x-z|^{n-1}} dz. \tag{A.48}$$

Performing the change of variables $z - x' = z' + \lambda v'$, $z' \cdot v' = 0$, we obtain

$$I_{m+1,n}(x,x',v') \leq C \int_{\substack{z'\cdot v'=0\\ |z'|\leq D}} \left(\int_{-D}^{D} \frac{d\lambda}{|x - x' - z' - \lambda v'|^{n-1}} \right) |\ln(|z'|)|\, dz'. \tag{A.49}$$

Combining (A.49) and (A.37) (with "$w = x - x' - z'$"), we obtain

$$I_{m+1,n}(x,x',v') \leq \frac{2^{\frac{n+1}{2}} C}{n-2} I''_{n,n}(x,x',v'), \tag{A.50}$$

where

$$I''_{n,n}(x,x',v') := \int_{\substack{z'\cdot v'=0\\ |z'|\leq D}} \frac{|\ln(|z'|)|}{|(x-x')_\perp - z'|^{n-2}} dz' \tag{A.51}$$

$$= \int_{z' \in B_{n-1}(0,D)} \frac{|\ln(|z'|)|}{|\mathrm{d}(x,x')e_1 - z'|^{n-2}} dz' \leq \int_{z' \in B_{n-1}(0,D)} \frac{C'}{|z'|^{\frac{1}{2}} |\mathrm{d}(x,x')e_1 - z'|^{n-2}} dz',$$

and $\mathrm{d}(x,x') = |(x-x')_\perp|$ and $C' := \sup_{r\in(0,D)} r^{\frac{1}{2}} |\ln(r)|$, $e_1 = (1,0,\ldots,0) \in \mathbb{R}^{n-1}$, and where $B_{n-1}(0,D)$ denotes the Euclidean ball of $\mathbb{R}^{n-1}$ of center 0 and radius D. Using spherical coordinates $z' = \mathrm{d}(x,x')e_1 + r\Omega$, $(r,\Omega) \in (0,+\infty) \times \mathbb{S}^{n-2}$ and $\Omega = (\sin(\theta), \cos(\theta)\Theta)$ $((\theta,\Theta) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{S}^{n-3})$, and using the estimate $|\mathrm{d}(x,x')e_1 + r\Omega| \geq |r + \mathrm{d}(x,x')\sin(\theta)|$, we obtain

$$I''_{n,n}(x,x',v') \leq |\mathbb{S}^{n-3}| \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n-3}(\theta) \left(\int_0^{2D} \frac{C'}{|r + \mathrm{d}(x,x')\sin(\theta)|^{\frac{1}{2}}} dr \right) d\theta. \tag{A.52}$$

Note that

$$\int_0^{2D} \frac{C'}{|r + \mathrm{d}(x,x')\sin(\theta)|^{\frac{1}{2}}} dr \leq \int_{-3D}^{3D} \frac{C'}{|r|^{\frac{1}{2}}} dr < \infty, \tag{A.53}$$

for $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Combining (A.50), (A.52) and (A.53), we obtain

$$I_{m+1,n}(x,x',v') \leq \frac{2^{\frac{n+1}{2}} C C' |\mathbb{S}^{n-3}|}{n-2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n-3}(\theta) d\theta \int_{-3D}^{3D} r^{-\frac{1}{2}} dr. \tag{A.54}$$

Therefore (A.35) holds for "m"$= m + 1 = n$. □

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