Corrector theory for elliptic equations in random media with singular Green's function. Application to random boundaries

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Abstract

We consider the problem of the random fluctuations in the solutions to elliptic PDEs with highly oscillatory random coefficients. In our setting, as the correlation length of the fluctuations tends to zero, the heterogeneous solution converges to a deterministic solution obtained by averaging. When the Green's function to the unperturbed operator is sufficiently singular (i.e., not square integrable locally), the leading corrector to the averaged solution may be either deterministic or random, or both in a sense we shall explain.

Our main application is the solution of an elliptic problem with random boundary condition that may be used to model diffusion of signaling molecules through a layer of cells into a bulk of extracellular medium. The problem is then described by an elliptic pseudo-differential operator (a Dirichlet-to-Neumann operator) on the boundary of the domain with random potential.

In the physical setting of a three dimensional extracellular medium on top of a two-dimensional surface of cells forming a layer of epithelium, we show that the approximate corrector to averaging consists of a deterministic correction plus a Gaussian field of amplitude proportional to the correlation length of the random medium. The result is obtained under some assumptions on the four-point correlation function in the medium. We provide examples of such random media based on Gaussian and Poisson statistics.

Key words: Boundary homogenization, Robin problem, Fluctuation theory, Central limits, PDEs with random coefficients, Dirichlet to Neumann map.

Mathematics subject classifications: 35B40, 35B27, 35R60; Secondary: 60G60, 60F05.

1 Introduction

We consider elliptic pseudo-differential equations with random potential of the form

$$P(x,D)u_\varepsilon + \tilde{q}_\varepsilon(x,\frac{x}{\varepsilon},\omega)u_\varepsilon = f(x), \tag{1}$$

for x in an open subset $X \subset \mathbb{R}^d$ with appropriate boundary conditions on ∂X if necessary. The equations are parametrized by $0 < \varepsilon \ll 1$ modeling the correlation length of the random medium. Here, $\tilde{q}_\varepsilon(x,\frac{x}{\varepsilon},\omega)$ consists of a low frequency part $q_0(x)$ and a high frequency part $q(\frac{x}{\varepsilon},\omega)$, which is a re-scaled version of $q(x,\omega)$, a stationary mean zero random field defined on some abstract

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probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with (possibly multi-dimensional) parameter $x \in \mathbb{R}^d$. We denote by $\mathbb{E}$ the mathematical expectation with respect to the probability measure $\mathbb{P}$. Equations with coefficients varying at a much smaller scale than the scale at which the phenomenon is observed have many practical applications in the physical modeling of complex media. In this paper, we primarily consider the particular application of diffusion of signaling molecules through a three dimensional extracellular medium on top of a two dimensional layer of cells while the interaction between the molecules and the cells are modeled as a random boundary condition.

It is both mathematically and practically interesting to develop asymptotic theories for solutions to (1) if only because numerical solutions become prohibitively expensive when $\varepsilon \to 0$. Homogenization theory or averaging theory aims at finding an effective or homogenized equation whose solution u is the limit of u_ε as ε goes to zero. Corrector theory aims at further approximating the heterogeneous solution by capturing the leading terms in the corrector $u_\varepsilon - u$.

The homogenization/averaging of such a problem, where randomness appears as a potential, is easier than the case where the randomness interacts with derivatives as in, e.g., problems with random diffusion coefficients where $P(x, D) = -\nabla \cdot A(\frac{x}{\varepsilon}, \omega) \cdot \nabla$. Unlike the latter case whose homogenized equation involves nontrivial expressions of $A(x, \omega)$, cf. [13, 16], the homogenized/averaged equation for (1) is obtained simply by averaging $\tilde{q}_\varepsilon$, cf. [1, 11]. At this step, only mild conditions such as stationarity and ergodicity of the random fields are required.

Corrector theory for the problems with random diffusion coefficients is much more difficult in arbitrary dimensions. In one space dimension, the correctors are asymptotically Gaussian in some settings as shown in [1, 7]. Such results are obtained with an additional requirement that the random fields are strongly mixing with mixing coefficients decaying sufficiently fast, see below for the notion of mixing. In higher dimensions, corrector theory for problems with random potential is also available [1, 11] under similar mixing conditions. In particular, the procedure in [1] applies for elliptic PDE that admits a Green's function whose singularity at the origin is square integrable, and says that weakly in space the corrector has random fluctuations of order $\varepsilon^{d/2}$. This covers the case of diffusion equation with random potential in dimension $d \leq 3$.

The main objective of this paper is to consider the case where the Green's function of (1) is more *singular* in the sense that it fails to be square integrable near the origin. In this case, a deterministic corrector may be comparable or larger than the random corrector.

We consider the case that the random field $q(x, \omega)$ is stationary with integrable correlation function $R(x) := \mathbb{E}\{q(0)q(x)\}$, and show that the homogenized/averaged equation is again obtained by averaging $\tilde{q}_\varepsilon$. We then consider the corrector $u_\varepsilon - u$ weakly in space, i.e., consider the random variable $\langle u_\varepsilon - u, M\rangle$ for arbitrary smooth test function M. The fluctuation of this variable is again of order $\varepsilon^{d/2}$ as before. The main difference with the case of square integrable Green's function is that the mean of the corrector is of size larger than or equal to $\varepsilon^{d/2}$. Hence a complete approximation of u_ε should include a characterization of the deterministic term $\mathbb{E}\{u_\varepsilon - u\}$, at least its components that are of size larger than the random fluctuation. As we demonstrate in this paper, the sizes of these components depend on the singular structure of the Green's function and the dimension d. Moreover, the limit of these components can be calculated explicitly using the procedure developed here. These results are obtained under a further assumption that we can estimate sufficiently high order moments of the random fields.

Although our approach can be carried out for general equations of the form (1), we state and prove the main theorems for the following specific model to simplify notation. It is a diffusion equation with a random Robin boundary condition posed on the half space $\mathbb{R}^n_+$, i.e. $\{x \in \mathbb{R}^n \mid x_n >$

$0\}$ whose boundary is identified with $\mathbb{R}^d$ where $d = n - 1$,

$$\begin{cases} (-\Delta + \lambda^2) u_\varepsilon(x,\omega) = 0, & x = (x', x_n) \in \mathbb{R}^n_+, \\ \dfrac{\partial}{\partial \nu} u_\varepsilon + (q_0 + q(\dfrac{x'}{\varepsilon}, \omega)) u_\varepsilon = f(x'), & x = (x', 0) \in \partial \mathbb{R}^n_+. \end{cases} \tag{2}$$

Here, the outward normal direction, i.e., the $-x_n$ direction, is denoted by ν. We show below that this equation is equivalent to the following elliptic pseudo-differential equation of the form (1):

$$(\sqrt{-\Delta_\perp + \lambda^2} + q_0 + q_\varepsilon) u_\varepsilon = f, \tag{3}$$

where $\Delta_\perp$ is the Laplacian on $\mathbb{R}^d$, obtained from the Laplacian on $\mathbb{R}^n$ with $\partial^2_{x_n}$ eliminated. Here, $\sqrt{-\Delta_\perp + \lambda^2}$ is a pseudo-differential operator defined by (20). Also, we used q_ε as short-hand notation for $q(\frac{x'}{\varepsilon})$. In the sequel and to simplify notation, we will use either $q_\varepsilon(x)$ or simply q_ε to denote the scaled function $q(\frac{x}{\varepsilon}, \omega)$.

This type of boundary problems have applications in chemical physics and biology. For instance, in the context of cell communication by diffusing signals, the equation in (2) models the diffusion of signaling molecules in a bulk of extracellular medium which is covered at the bottom by a monolayer of cells forming a layer of epithelium. The cells on the epithelium layer can secrete signaling molecules and they can absorb them as well, depending on levels of gene expression in the cells. The boundary condition in (2) models the actions between the cells and the signaling molecules.

The authors of [4, 5] have investigated a similar diffusion process of particles through a heterogeneous surface which reflects particles except on some periodically or randomly located patches that absorb particles. Hence, in their setting, the boundary conditions in (2) are: on the patches Dirichlet conditions are imposed while otherwise Neumann conditions are imposed. Analyzing the data obtained from Brownian dynamics simulations, they find that when the patchy surface is sufficiently fine-grained, e.g., the periodicity of the locations of the patches being $\varepsilon \ll 1$, the diffusion equation with such heterogeneous boundary conditions can be well approximated by an effective equation with a homogeneous boundary which absorbs particles in a uniform rate over the entire surface. Formal asymptotic analysis of the homegenization mechanism and different numerical homogenization procedures have been developed in e.g., [15, 17].

Rigorous mathematical proof of homogenization in the above setting is challenging. We consider here a Robin boundary condition with random impedance modeling a random coupling of secreting and absorption of signaling molecules by the cells on the surface. We derive a rigorous averaging and corrector theory in this setting. As ε goes to zero, our result implies that the cells with random impedance can be replaced by cells with constant, averaged, impedance. The next-order approximation consists of a random fluctuation which is weakly Gaussian and of size $\varepsilon^{d/2}$ and a deterministic corrector of order ε. These deterministic and random correctors can be expressed in terms of statistical quantities of the random field $q(x, \omega)$.

The rest of this paper is structured as follows. We state the main results for the random Robin problem in dimension $n = 3$ (hence $d = 2$) in section 2 after introducing preliminary material on the Robin problem and assumptions on the random fields. In section 3, we write the Robin problem on $\mathbb{R}^n$ as a pseudo-differential equation on $\mathbb{R}^d$ and derive some properties of its solution operator $\mathcal{G}$. In section 4, we present examples of random fields that satisfy the imposed assumptions. The proofs of the main results are shown in section 5. Generalization to higher dimensions and concluding remarks are presented in section 6. Some technical lemmas are postponed to Appendix A.

2 Problem setting and main results

2.1 Diffusion equation with Robin boundary

We first analyze the Robin problem introduced above. In particular, we consider the homogenized equation of (2), which is obtained by averaging $q_\varepsilon(x,\omega)$:

$$\begin{cases} (-\Delta+\lambda^2)u(x)=0, & x\in\mathbb{R}^n_+, \\ \dfrac{\partial}{\partial\nu}u(x')+q_0u(x')=f(x'), & x'\in\mathbb{R}^d. \end{cases} \tag{4}$$

We also impose that the solution decays sufficiently fast as $|x|$ tends to infinity. Above, we identified the boundary $\partial\mathbb{R}^n_+$ with $\mathbb{R}^d$ where $d=n-1$. For simplicity we assume that the damping coefficient λ^2 is a constant with $\lambda>0$, and the impedance q_0 in the Robin boundary condition is also a positive constant. Theory for the above equation is presented in section 3. Both (4) and (2) are well-posed for almost all realizations assuming that $q_0+q(x/\varepsilon,\omega)$ is positive a.e. In the sequel and to simplify notation, we still use x, instead of x', to denote a point in $\mathbb{R}^d$.

Let us define the standard Dirichlet to Neumann (DtN) operator Λ as follows:

$$\Lambda g(x):=\frac{\partial}{\partial\nu}\tilde g(x). \tag{5}$$

Here, the function $g(x)$ is defined on the boundary $\mathbb{R}^d$ and $\tilde g$ is the solution of the volume problem (4) with a Dirichlet boundary condition $\tilde g|_{\partial\mathbb{R}^n_+}=g$. Hence, Λ maps the boundary value to the boundary flux. Either by calculating the symbol of Λ or by verifying it directly, we observe that $\Lambda=\sqrt{-\Delta+\lambda^2}$; see section 3. Note that Δ here is the Laplacian on $\mathbb{R}^d$, i.e., the surface Laplacian $\Delta_\perp$ in (3). To simplify notation, we will use Δ to denote both of the Laplacians on $\mathbb{R}^n$ and $\mathbb{R}^d$. The volume problem (4) is then equivalent to the following pseudo-differential equation posed on the whole space $\mathbb{R}^d$,

$$(\sqrt{-\Delta+\lambda^2}+q_0)u=f. \tag{6}$$

Indeed by definition, the trace of the solution to (4) satisfies equation (6), and the lift $\tilde u$ of solution to (6) solves equation (4). In fact, we show in section 3 that (6) admits a well defined solution operator $\mathcal{G}$ and consequently the diffusion equation in the volume is also well-posed.

Let $G(x,y)$ be the corresponding Green's function, i.e., the Schwartz kernel of $\mathcal{G}$. By homogeneity, we observe that G is of the form $G(|x-y|)$. This Green's function will be investigated further in section 3. The latter function decays exponentially at infinity and behaves like $|x|^{-d+1}$ near the origin when $d\ge 2$. The exponential decay allows us to easily work in infinite domain. The singularity at the origin shows that G fails to be locally square integrable and hence is of the type that this paper aims to analyze. In the presence of a random impedance, we denote the corresponding Green's operator by $\mathcal{G}_\varepsilon$.

Considering the application of (4) in biology, the physical domain is $n=3$ and hence $d=2$. Our results are presented in that setting of practical interest.

2.2 Assumptions on the random fields

We recall that the random impedance $q_\varepsilon(x,\omega)$ in (2) is of the form $q(x/\varepsilon,\omega)$. The assumptions on the random impedance are imposed on $q(x,\omega)$. We assume that $q(x,\omega)$ is a stationary and strong mixing process with integrable mixing coefficient. These are standard assumptions on random fields modeling heterogeneous media in mathematical physics, and are enough for homogenization theory. To analyze the limiting distribution of the random fluctuation in the setting of non-square-integrable Green's functions, we need additional assumptions which take the form of estimates on fourth-order moments of q. Details are described below.

Stationarity. We assume that $q(x,\omega)$ is *stationary*, i.e., for any $n \in \mathbb{N}$ and any n-tuple $(x_1, \cdots, x_n)$, the joint distribution of $(q(x_1,\omega), \cdots, q(x_n,\omega))$ is conserved under (spatial) translation. In particular $\mathbb{E}q(x)$ is a constant independent of x. Without loss of generality we assume this constant is zero, i.e., $q(x,\omega)$ is mean-zero.

Strong mixing. We assume $q(x,\omega)$ is *strong mixing* or α*-mixing* in the following sense. For any Borel sets $A, B \subset \mathbb{R}^d$, the sub-$\sigma$-algebras $\mathcal{F}_A$ and $\mathcal{F}_B$ generated by the process restricted on A and B respectively decorrelate so rapidly that there exists some function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ with $\alpha(r)$ vanishing to zero as r tends to infinity, and for any $\mathcal{F}_A$ measurable set U and $\mathcal{F}_B$ measurable set V, we have

$$|\mathbb{P}(U)\mathbb{P}(V) - \mathbb{P}(U \cap V)| \le \alpha\big(d(A,B)\big). \tag{7}$$

Here $d(A,B)$ is the distance between the sets A and B. What this means is that (functionals of) the random fields restricted on disjoint spatial domains A and B become more and more independent as the distance between the sets A and B increases. The function α quantifies that decay. We further assume that $\alpha(r)$ has the following asymptotic behavior for some real number $\delta > 0$:

$$\alpha(r) \sim \frac{1}{r^{d+\delta}}, \text{ for } r \text{ sufficiently large.} \tag{8}$$

This implies in particular that $\phi(r) \in L^1(\mathbb{R}, r^{d-1}dr)$, i.e., $\alpha(|x|)$ as a function of $x \in \mathbb{R}^d$ is integrable.

There are in fact several different definitions of mixing coefficients; the $\alpha(r)$ defined above is among the least restricted ones. For additional information on the notion of mixing, we refer the reader to [8].

Fourth order cumulants. A further assumption on $q(x,\omega)$ is imposed so that we have an approximate formula for the fourth order cross-moment of the process. To formulate this condition, we need to introduce some terminologies.

Let $F = \{1,2,3,4\}$ and $\mathcal{U}$ be the collections of two pairs of unordered numbers in F, i.e.,

$$\mathcal{U} = \big\{p = \{\big(p(1),p(2)\big), \big(p(3),p(4)\big)\} \mid p(i) \in F, p(1) \neq p(2), p(3) \neq p(4)\big\}. \tag{9}$$

As members in a set, the pairs $(p(1),p(2))$ and $(p(3),p(4))$ are required to be distinct; however, they can have one common index. There are three elements in $\mathcal{U}$ whose indices $p(i)$ are all different. They are precisely $\{(1,2),(3,4)\}$, $\{(1,3),(2,4)\}$ and $\{(1,4),(2,3)\}$. Let us denote by $\mathcal{U}_*$ the subset formed by these three elements, and its complement by $\mathcal{U}^*$.

Intuitively, we can visualize $\mathcal{U}$ in the following manner. Draw four points with indices 1 to 4. There are six line segments connecting them. The set $\mathcal{U}$ can be visualized as the collection of all possible ways to choose two line segments among the six. $\mathcal{U}_*$ corresponds to choices so that the two segments have disjoint ends, and $\mathcal{U}^*$ corresponds to choices such that the segments share one common end.

We assume that $q(x,\omega)$ has *controlled fourth order cumulants* in the sense that the following holds: For each $p \in \mathcal{U}^*$, there exists a real valued nonnegative function ϕ_p in $L^1 \cap L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, so that for any four point set $\{x_i\}_{i=1}^4$, $x_i \in \mathbb{R}^d$, we have the following condition on the fourth order cross-moment of $\{q(x_i,\omega)\}$:

$$\begin{aligned}&\Big|\mathbb{E}\prod_{i=1}^4 q(x_i) - \sum_{p\in\mathcal{U}_*}\mathbb{E}\{q(x_{p(1)})q(x_{p(2)})\}\mathbb{E}\{q(x_{p(3)})q(x_{p(4)})\}\Big|\\ &\le \sum_{p\in\mathcal{U}^*}\phi_p(x_{p(1)}-x_{p(2)},x_{p(3)}-x_{p(4)}).\end{aligned}\tag{10}$$

Observe that since $\mathbb{E}q(x,\omega)\equiv 0$, the left hand side is the (joint) *cumulant* of $\{q(x_i,\omega)\}$, and hence the notation for this property. In the sequel, we will denote the cumulant of $\{q(x_i)\}_{i=1}^4$ by $\vartheta(q(x_1),\cdots,q(x_4))$.

Remark 2.1. This condition is motivated by Gaussian random fields for which all but two cumulants vanish and hence we can set ϕ_p to be zero for all p in (10). Although it satisfies the condition above, a Gaussian random field is not bounded and large negative values of q_ε in equation (3) may yield non-uniqueness. The above condition on the cumulants hence provides a "decomposition" of fourth order moments into pairs just as Gaussian random fields up to an error we wish to control.

Uniform boundedness. Suppose $q_0(x)$ is a positive and uniformly bounded function. We assume for simplicity that $q(x,\omega)$ is uniformly bounded in the space $\Omega\times\mathbb{R}^d$ by the infimum of q_0. That is to say,

$$\|q(x,\omega)\|_{L^\infty(\Omega\times\mathbb{R}^d)}\le \inf_{x\in\mathbb{R}^d} q_0(x).\tag{11}$$

Furthermore, the condition above implies that the impedance $q_0+q_\varepsilon(x)$ in (3) is non-negative a.e. and therefore by Corollary 3.2 below (3) is well-posed.

Remark 2.2. We observe that by scaling, $q_\varepsilon(x,\omega)$ is also stationary, mean zero, α-mixing and has controlled cumulants. Nevertheless, we need to scale the spatial variable appropriately when using (7) or (10).

2.3 Main results

With those assumptions above, we are ready to state the main results of this paper. Before doing so, we introduce some notation.

We define the (auto-)correlation function, also known as the covariance function, of the random field $q(x,\omega)$ as

$$R(x) := \mathbb{E}\{q(0)q(x)\} = \mathbb{E}\{q(y)q(y+x)\}.\tag{12}$$

The last equality holds since q is stationary. As a correlation function, R is a positive semi-definite in the sense of (40) below. By Bochner's theorem its Fourier transform is a positive finite measure. Hence we can define the *strength* of the random field as follows;

$$\sigma^2 := \int_{\mathbb{R}^d} R(x)dx.\tag{13}$$

Since σ^2 is the Fourier transform of R evaluated at zero, it is non-negative. We consider the nontrivial case and set $\sigma>0$. We observe also that the random field $q(x,\omega)$ has *short range correlation* in the sense that $R\in L^1(\mathbb{R}^d)$. Indeed, we have

$$R(x) = \mathrm{Corr}\big(q(0),q(x)\big)\mathrm{Var}\big(q(0)\big)\le \rho(|x|)\|q\|_{L^\infty}^2\le C\|q\|_{L^\infty}^2\alpha(|x|),\tag{14}$$

and the last member is in $L^1(\mathbb{R}^d)$ thanks to (8). Throughout the paper, we use C to denote various constants. The function ρ above is the ρ-mixing coefficient defined as in (7) with its left hand side replaced by $\text{Corr}(\xi, \eta)$ where ξ and η are arbitrary square integrable random variables measurable with respect to $\mathcal{F}_A$ and $\mathcal{F}_B$ respectively. The ρ-mixing coefficient is stronger than the α-mixing coefficient and hence the last inequality above hold; see [8, p.4].

Now we state the main theorems in the version of $d = 2$ which is the physical dimension of the Robin problem concerning the biological application.

Theorem 2.3. *Let u_ε and u solve (3) and (6) respectively and $d = 2$. Suppose λ, q_0 in those equations are positive constants and f is in $L^2(\mathbb{R}^2)$. Assume that the random field $q(x, \omega)$ is stationary and mean-zero with correlation function $R(x) \in L^1(\mathbb{R}^2)$. Assume also that $q(x, \omega)$ is uniformly bounded as in (11). Then we have*

$$\mathbb{E}\|u_\varepsilon - u\|^2_{L^2(\mathbb{R}^2)} \le C\varepsilon^2 |\log \varepsilon| \|f\|^2_{L^2}, \tag{15}$$

where the constant C only depends on the parameter λ, $\|q\|_{L^\infty}$, dimension d and $\|R\|_{L^1}$, but not on ε.

We will prove this theorem in section 5. The proof works for $d \ge 3$ as well, and in that case the $\varepsilon^2|\log \varepsilon|$ above should be replaced by ε^2. The above theorem says u_ε and u are close in the energy norm $L^2(\Omega, L^2(\mathbb{R}^2))$. Let us denote the corrector by ξ_ε. We can decompose it into two parts as follows:

$$\xi_\varepsilon = (\mathbb{E}\{u_\varepsilon\} - u) + (u_\varepsilon - \mathbb{E}\{u_\varepsilon\}). \tag{16}$$

We call them the *deterministic corrector* and the *stochastic corrector*, respectively.

For the deterministic corrector, we can calculate its limit explicitly. Let us define

$$\tilde{R} := \int_{\mathbb{R}^2} \frac{R(y)}{2\pi|y|} dy. \tag{17}$$

Since R is integrable and bounded, this integral is finite. With this notation, we have the following theorem on the limit of the deterministic corrector.

Theorem 2.4. *Let u_ε and u solve (3) and (6) respectively and $d = 2$. Let $q(x, \omega)$ satisfy the same conditions as in the previous theorem. Then we have,*

$$\lim_{\varepsilon \to 0} \frac{\mathbb{E}\{u_\varepsilon\} - u}{\varepsilon} = \tilde{R}\mathcal{G}u. \tag{18}$$

Here the limit is taken in the weak *sense. That is, for an arbitrary test function $M \in C_c^\infty(\mathbb{R}^2)$, the real number $\varepsilon^{-1}\langle M, \mathbb{E}\{\xi_\varepsilon\}\rangle$ converges to $\langle \mathcal{G}M, \tilde{R}u\rangle$.*

Note that $\mathcal{G}$ as the solution operator of (6) is self-adjoint. In general, the solution operator of (1) is not self-adjoint, and the term $\mathcal{G}M$ above should be replaced by $\mathcal{G}^*M$ where $\mathcal{G}^*$ denotes the adjoint operator.

For the stochastic corrector, we have the following central limit theorem.

Theorem 2.5. *Let u_ε and u solve (3) and (6) respectively and $d = 2$. Let $q(x, \omega)$ be stationary and mean-zero with strong mixing coefficient $\alpha(r)$ satisfying (8), and be uniformly bounded as in (11). Assume further that the joint fourth order cumulant of q satisfies (10). Then:*

$$\frac{u_\varepsilon - \mathbb{E}\{u_\varepsilon\}}{\varepsilon} \xrightarrow{\text{dist.}} -\sigma \int_{\mathbb{R}^2} G(x - y)u(y)dW_y, \tag{19}$$

where σ is defined in (13) and W_y is the standard multivariate Wiener process in $\mathbb{R}^2$. The convergence here is weakly in $\mathbb{R}^2$ and in probability distribution.

Remark 2.6. From Theorem 2.4, it is clear that we can replace $\mathbb{E}\{u_\varepsilon\}$ in the theorem above by $u + \varepsilon\tilde{R}\mathcal{G}u$ since the rest is of order smaller than ε.

3 Properties of the Green's function

In this section, we first show that the Robin problem (4) is equivalent to the pseudo-differential equation (6) by calculating the symbol of the Dirichlet to Neumann map Λ. Using this symbol we show that (6) admits a well defined solution operator $\mathcal{G}$ and derive an expression for the corresponding Green's function G.

3.1 Symbol of the Dirichlet to Neumann map

We now verify the claim that the DtN map Λ equals the pseudo-differential operator $\sqrt{-\Delta+\lambda^2}$ defined as

$$\sqrt{-\Delta+\lambda^2}f = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} e^{ix\cdot\xi}\sqrt{|\xi|^2+\lambda^2}\hat{f}(\xi)d\xi, \tag{20}$$

where $\hat{f}$ is the Fourier transform of f defined as

$$\hat{f}(\xi) := \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)dx. \tag{21}$$

We will also denote by $\mathscr{F}$ the Fourier transform operator, and by $\mathscr{F}^{-1}$ its inverse.

By definition (5), $\Lambda g(x)$ is the normal derivative of $\tilde{g}(x,x_n)$, the function satisfying:

$$\begin{cases} -\Delta\tilde{g}(x,x_n)+\lambda^2\tilde{g}(x,x_n)=0, & (x,x_n)\in\mathbb{R}^n_+, \\ \tilde{g}(x,0)=g(x), & x\in\mathbb{R}^d\equiv\partial\mathbb{R}^n_+. \end{cases} \tag{22}$$

Taking Fourier transform in the variable x, we obtain a second order ordinary differential equation in x_n, i.e.,

$$\begin{cases} -\partial^2_{x_n}\hat{\tilde{g}}(\xi,x_n)+(|\xi|^2+\lambda^2)\hat{\tilde{g}}=0, \\ \hat{\tilde{g}}(\xi,0)=\hat{g}(\xi). \end{cases} \tag{23}$$

Solve this ODE with the assumption that $\hat{\tilde{g}}$ decays for large frequency to get

$$\hat{\tilde{g}}(\xi,x_n)=\hat{g}(\xi)\exp(-x_n\sqrt{|\xi|^2+\lambda^2}).$$

Take derivative in the $-x_n$ direction, i.e. the outward normal direction and send x_n to zero to obtain Fourier transform of the function Λg. It has the form

$$\widehat{\Lambda g}(\xi)=\sqrt{|\xi|^2+\lambda^2}\hat{g}(\xi). \tag{24}$$

This verifies that the symbol of Λ is $\sqrt{|\xi|^2+\lambda^2}$. Compare this symbol with (20) and we see $\Lambda=\sqrt{-\Delta+\lambda^2}$. Therefore, (4) and (6) are equivalent by the argument below (6).

3.2 Solution of the pseudo-differential equation

As an immediate result, we show that (6) admits a solution operator $\mathcal{G}: H^{-\frac{1}{2}}(\mathbb{R}^d)\to H^{\frac{1}{2}}(\mathbb{R}^d)$ given by:

$$\mathcal{G}f(x) := \mathscr{F}^{-1}\frac{\hat{f}}{\sqrt{|\xi|^2+\lambda^2}+q_0} \equiv \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} e^{i\xi\cdot x}\frac{\hat{f}}{\sqrt{|\xi|^2+\lambda^2}+q_0}d\xi. \tag{25}$$

In particular, the map $\mathcal{G}: f\to\mathcal{G}f$ is continuous from $L^2(\mathbb{R}^d)$ to itself, and the operator norm is bounded by a constant that only depends on λ provided that the impedance is non-negative.

We recall some definitions. The Sobolev space H^s for $s \in \mathbb{R}$ is defined as

$$H^s(\mathbb{R}^d) := \left\{ v \in \mathcal{S}' \mid \hat{v}\langle\xi\rangle^s \in L^2(\mathbb{R}^d) \right\}, \tag{26}$$

where $\mathcal{S}'$ is the space of tempered distributions, i.e., linear functionals of the Schwartz space $\mathcal{S}$, and $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$. To simplify notation, we will denote $H^{\frac{1}{2}}$ by H, and the corresponding norm is

$$\|f\|_H := \left(\int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 \langle\xi\rangle d\xi \right)^{\frac{1}{2}}. \tag{27}$$

To prove that (6) is well-posed, we first write a variational formulation of it. To do so, multiply (6) by a smooth test function v, and integrate. We have

$$B[u, v] = \langle f, v\rangle, \tag{28}$$

where $B[u, v]$ is a bilinear form defined as

$$B[u, v] := \langle \Lambda u, v\rangle + \langle q(x)u, v\rangle. \tag{29}$$

¿From the symbol of Λ we see it maps $H^{1/2}$ to $H^{-1/2}$. As a result, the bilinear form $B[\cdot,\cdot]$ above is well defined on $H \times H$. We say u is a weak solution of (6) if (28) holds for arbitrary $v \in H$.

The following proposition states that the bilinear form B satisfies the conditions of the Lax-Milgram theorem and its corollary says (6) admits a unique solution in H. For the moment, we allow the impedance in (6) to be a non-negative function denoted by $q(x)$.

Proposition 3.1. *Let λ in (6) be a positive constant. Let $q(x)$ in (29) be a non-negative function and assume $\|q\|_{L^\infty}$ is finite. Set $\alpha = \|q\|_{L^\infty} + \max(1, \lambda)$, $\gamma = \min(1, \lambda)$. Then the bilinear form $B[u, v]$ in (29) satisfies the following:*

(i) $|B[u, v]| \leq \alpha \|u\|_H \|v\|_H$, for all $u, v \in H$, and

(ii) $\gamma\|u\|_H^2 \leq B[u, u]$, for all $u \in H$.

Proof: The following inequalities hold for all ξ.

$$\gamma \leq \sqrt{\frac{|\xi|^2 + \lambda^2}{|\xi|^2 + 1}} \leq \max(1, \lambda). \tag{30}$$

Using the inequality on the right, formula (24), and Cauchy-Schwarz, we get

$$|\langle \Lambda u, v\rangle| = \left| \int_{\mathbb{R}^d} \sqrt{\lambda^2 + |\xi|^2}\, \hat{u}\bar{\hat{v}} d\xi \right| \leq \max(1, \lambda) \left(\int_{\mathbb{R}^d} |\hat{u}|^2 \langle\xi\rangle d\xi \right)^{1/2} \left(\int_{\mathbb{R}^d} |\hat{v}|^2 \langle\xi\rangle d\xi \right)^{1/2}.$$

Since $\|u\|_{L^2} \leq \|u\|_H$ for all $u \in H$, we have

$$|B[u, v]| \leq \max(1, \lambda)\|u\|_H \|v\|_H + \|q\|_{L^\infty} \|u\|_{L^2} \|v\|_{L^2} \leq \alpha \|u\|_H \|v\|_H,$$

which verifies (i). For the second inequality, since $q(x)$ is non-negative, we have

$$B[u, u] \geq \langle \Lambda u, u\rangle = \int_{\mathbb{R}^d} |\hat{u}|^2 \sqrt{\lambda^2 + |\xi|^2} d\xi \geq \gamma \int_{\mathbb{R}^d} |\hat{u}|^2 \langle\xi\rangle d\xi.$$

In the last inequality we applied (30). This verifies (ii) and completes the proof. □

Corollary 3.2. *Let $\lambda, q(x)$ and γ be the same as in the preceding proposition. Assume also that f is in $H^{-1/2}$. Then (6) admits a weak solution $u \in H$ satisfying (28). In particular, if $f \in L^2$, then we have that*

$$\|u\|_{L^2} \le \gamma^{-1}\|f\|_{L^2}. \tag{31}$$

Proof: The first claim follows immediately from the preceding proposition and the Lax-Milgram theorem. The second one is due to the following estimate which is clear from (ii) of Proposition 3.1 and Cauchy-Schwarz inequality.

$$\gamma\|u\|_{L^2}^2 \le \gamma\|u\|_H^2 \le B[u,u] = \langle f, u\rangle \le \|f\|_{L^2}\|u\|_{L^2}.$$

This completes the proof. $\square$

Now it is a simple matter to check that $\mathcal{G}$ defined in (25) gives the solution operator. Therefore, the corollary above shows that the operator norm of $\mathcal{G}$ as a transformation on $L^2(\mathbb{R}^d)$ is bounded by the constant γ^{-1}.

Remark 3.3. The explicit bound γ^{-1} in estimate (31) is crucial for us when the random equation (3) is considered. It shows that $\mathcal{G}_\varepsilon$ is well defined as long as $q_0 + q_\varepsilon$ is non-negative (which is true thanks to (11)) and the operator norm $\|\mathcal{G}_\varepsilon\|_{\mathcal{L}(L^2)}$ is bounded uniformly for almost every realizations.

3.3 Decomposition of Green's function

Let $G(x,y)$ be the Green's function associated to the solution operator $\mathcal{G}$ of (6). By homogeneity $G(x,y) = G(x-y)$ and $G(x)$ solves

$$\left(\sqrt{-\Delta + \lambda^2} + q_0\right) G(x) = \delta_0(x).$$

Take Fourier transform on both sides. Our choice of the definition of Fourier transform (21) implies that $\mathscr{F}\delta_0(x) \equiv (2\pi)^{-d/2}$. Hence, $G(x)$ is recovered by the inversion formula as follows;

$$G(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\xi\cdot x}(\sqrt{|\xi|^2+\lambda^2} + q_0)^{-1} d\xi. \tag{32}$$

In dimension two, we have the following explicit characterization.

Lemma 3.4. *Let $d = 2$. Let λ, q_0 in (6) be positive constants and $d = 2$. The Green's function $G(x)$ defined above can be decomposed into three terms as follows:*

$$G(x) = \frac{1}{2\pi}\left(\frac{\exp(-\lambda|x|)}{|x|} - q_0 K_0(\lambda|x|) + G_r(|x|)\right). \tag{33}$$

Here K_0 is the modified Bessel function with index zero and the function $G_r(|x|)$ is smaller than $C_b\exp(-b|x|)$ for any positive real number $b < \lambda' \equiv \lambda/\sqrt{2}$.

Remark 3.5. In the sequel, we will call the first term on the right G_s and the second one G_b. Clearly, G_s has singularity of order $|x|^{-1}$ near the origin and has exponential decay at infinity; G_r is smooth near the origin and has exponential decay at infinity. Asymptotic analysis of Bessel functions shows that G_b has a logarithmic singularity near the origin and exponential decay at infinity, cf. [20]. In summary, we have

$$|G(x)| \leq C_\lambda \frac{\exp(-\lambda'|x|)}{|x|}, \tag{34}$$

where C_λ is a constant depending on λ.

PROOF: We first decompose the Fourier transform of G into three parts as follows.

$$2\pi\hat{G}(\xi) = \frac{1}{\sqrt{|\xi|^2+\lambda^2}} - \frac{q_0}{|\xi|^2+\lambda^2} + \frac{q_0^2}{(|\xi|^2+\lambda^2)[q_0+\sqrt{|\xi|^2+\lambda^2}]}. \tag{35}$$

Now the first two terms can be inverted explicitly. For instance, the second one is a standard example in textbooks on Fourier analysis or PDEs, cf. Taylor [19, Chap. 3], Evans [9, Chaper 4]. In our case the dimension equals two, and its inversion is the following.

$$-\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{q_0\ e^{ix\cdot\xi}}{|\xi|^2+\lambda^2} = -\frac{q_0}{2}\int_0^\infty \frac{e^{-\frac{|x|^2}{4t}-t}}{t}dt = -q_0 K_0(\lambda|x|). \tag{36}$$

Here K_0 is the modified Bessel function of the *second* kind with index 0. It has logarithmic singularity near the origin and decays exponentially at infinity.

In dimension two, the first term admits an explicit expression as well. Indeed, thanks to (24), $(\sqrt{|\xi|^2+\lambda^2})^{-1}$ can be viewed as the symbol of Λ^{-1}, i.e., the Neumann to Dirichlet operator which maps the Neumann boundary condition of a diffusion equation of the form (22) to its solution evaluated at the boundary. Therefore, G_s can be obtained by taking the trace of G_D, by which we denote the Green's function associated to (22) with Neumann boundary. Since $d=2$ and $n=3$, G_D can be calculated explicitly using the method of *images* as we show now. The fundamental solution of (22) posed on whole $\mathbb{R}^3$ is given by $\exp(-\lambda|x|)/4\pi|x|$, cf. Reed and Simon [18, Chap. IX.7]. By the method of images, the Green's function for the Neumann problem on the upper half space is given by

$$G_D(x,y) = \frac{1}{4\pi}\frac{\exp(-\lambda|y-x|)}{|y-x|} + \frac{1}{4\pi}\frac{\exp(-\lambda|y-\tilde{x}|)}{|y-\tilde{x}|},$$

for x in the upper space and $\tilde{x}$ denotes its image in the lower half space. Evaluating G_D for x on the boundary, we obtain that

$$G_s(x,y) = \frac{1}{2\pi}\frac{\exp(-\lambda|y-x|)}{|y-x|}.$$

Clearly, it has singularity of order $|x-y|^{-1}$ near the origin and decays exponentially at infinity.

Now we are left with the third term of (35). We won't give an explicit formula for its Fourier inversion. Nevertheless, we can show that its inversion decays exponentially at infinity and has no singularity near the origin. The proof is a little more involved and hence postponed to the appendix as Lemma A.2. It essentially uses the Paley-Wiener theorem. Now the proof is complete. $\square$

4 Two examples of random fields

In this section, we present two examples of random fields that satisfy the conditions in section 2.2, verifying that such random fields can indeed be constructed rather naturally.

4.1 Random field based on spatial Poisson point process

The first example is a random field based on the spatial Poisson point process. This model is analyzed in [3], to which we refer the reader for more details.

Consider a spatial Poisson process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with intensity ν. We can construct $q(x,\omega)$ as the mean zero part of $\tilde{q}(x,\omega)$ which is defined as follows.

$$\tilde{q}(x,\omega) = \sum_{j=1}^{\infty} \varphi(x - y_j), \tag{37}$$

where $\{y_j\}_{j=1}^{\infty}$ are the points in the spatial Poisson process. Here φ is some non-negative smooth function compactly supported in the unit ball. Intuitively, (37) models a superposition of bumps with profile function φ and centers $\{y_j\}$ randomly located on $\mathbb{R}^d$ with a spatial Poisson distribution. Clearly, $\tilde{q}$ and hence q are stationary.

Formulas for the cross-moments (of arbitrary order) of the random process $q(x,\omega)$ defined above are derived in [3]. In particular, the joint cumulant of $\{q(x_i,\omega)\}_{i=1}^{4}$ has the following expression;

$$\begin{aligned} \vartheta(q(x_1), \cdots, q(x_4)) &= \nu \int \varphi(z)\varphi(x_2 - x_1 + z)\varphi(x_3 - x_1 + z)\varphi(x_4 - x_1 + z)dz \\ &\le \nu \|\varphi\|_{L^\infty} \int \varphi(z)\varphi(x_2 - x_1 + z)\varphi(x_3 - x_1 + z)dz. \end{aligned} \tag{38}$$

We verify that the last integral above is bounded uniformly in the variables $x_2 - x_1$ and $x_3 - x_1$ since φ is bounded; it is also integrable for these variables. In other words, the cumulant function ϑ satisfies (10), for we can set ϕ_p to be the last integral in (38) for $p = \{(1,2),(1,3)\}$ and $\phi_p \equiv 0$ for all other p. This verifies that $q(x,\omega)$ defined above has controlled cumulants. One can check also that q is strong mixing with mixing coefficient satisfying (8); see [3, 8].

Unfortunately, $q(x,\omega)$ defined as such is not uniformly bounded due to possible clustering of the Poisson points; thus (11), which is required in the main theorems, is violated. Nevertheless, for this model, as in [3], a careful control of $\mathbb{E}\|q\|_{L^n}$ for n large allows us to remedy this issue. This procedure can be carried out as in [3] and so we do not dwell on the details here.

4.2 Composition of a function with a Gaussian random field

Our second example is constructed as function of a Gaussian random field. This model satisfies all the assumptions needed in the main theorems. A one-dimensional model of this type has been considered in [2].

We start with a stationary mean-zero and unit-variance Gaussian random field $g(x,\omega)$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$. As in (12) we define its correlation function as follows, which encodes essentially all information of g.

$$R_g(x) := \mathbb{E}\{g(0)g(x)\}. \tag{39}$$

As the correlation function of a stationary process R_g is symmetric, i.e. $R_g(x) = R_g(-x)$, and is *non-negative definite* in the sense that for any $x_j \in \mathbb{R}^d$, $\xi_j \in \mathbb{R}$, and $j = 1, \cdots, N$, we have

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \xi_i R_g(x_i - x_j) \xi_j \geq 0. \tag{40}$$

See [12, Chapter 5]. As a consequence, $|R_g(x)| \leq R_g(0) = 1$ and hence is uniformly bounded. As in (13) we define the strength of g as

$$\sigma_g := \hat{R}(0) \equiv \int_{\mathbb{R}^d} R_g(x) dx, \tag{41}$$

By Bochner's theorem, σ_g is a finite positive number. Since the mixing property of a Gaussian random field is related to its correlation function, we assume that R_g has a sufficiently smooth Fourier transform $\hat{R}_g$ so that $g(x, \omega)$ is strong mixing with mixing coefficient $\alpha(r)$ satisfying (8). In particular, $R_g \in L^1(\mathbb{R}^d)$ as seen in (14).

Our example of random field $q(x, \omega)$ is then defined as

$$q(x, \omega) := \Phi \circ g(x, \omega), \tag{42}$$

for some real valued deterministic function Φ defined on the real line. The following proposition provides a recipe of choosing Φ so that $q(x, \omega)$ constructed above satisfies all the desired properties listed in section 2.2.

Proposition 4.1. *Let $g(x, \omega)$ be the stationary mean-zero unit-variance Gaussian random field defined above with strong mixing coefficient $\alpha(r)$ satisfying (8). Let Φ be a real valued function on the real line satisfying*

1. *Φ is uniformly bounded by q_0, i.e.,*

$$|\Phi(s)| \leq \inf_{x \in \mathbb{R}^d} q_0. \tag{43}$$

2. *Φ integrates to zero with respect to the standard Gaussian measure, i.e.,*

$$\int_{\mathbb{R}} \Phi(t) e^{-\frac{t^2}{2}} dt = 0. \tag{44}$$

3. *The Fourier transform of Φ satisfies that*

$$\int_{\mathbb{R}} |\hat{\Phi}(\xi)| \big(1 + |\xi|^3\big) < \infty; \tag{45}$$

Denote by κ_c the value of this integral which is a finite positive real number.

Then $q(x, \omega)$ defined in (42) is a stationary mean-zero random field with the same strong mixing coefficient $\alpha(r)$ satisfying (8) and correlation function R in $L^\infty \cap L^1(\mathbb{R}^d)$; furthermore, it is uniformly bounded as in (11) and has controlled fourth order cumulants as in (10).

Proof: 1. From the definition of q and the bound (43) on $|\Phi|$ it is obvious that $q(x, \omega)$ is uniformly bounded and satisfies (11).

Also from the definition of q, we see that the σ-algebra $\mathcal{F}_A$ generated by variables $q(x, \omega), x \in A$ is in fact generated by the underlying Gaussian random variables $g(x, \omega), x \in A$. Hence q shares the same stationarity and strong mixing coefficient $\alpha(r)$ with g.

It is also easy to see that $q(0,\omega)$, hence $q(x,\omega)$ for all x, is mean-zero. Indeed, observe that $g(0)$ has normal distribution $\mathcal{N}(0,1)$, then (44) says exactly that $\mathbb{E}\{q(0)\}=0$.

2. From the definition of $R(x)$ and the bound (43), it is obvious that $|R|$ is uniformly bounded by $(\inf q_0)^2$. Thanks to strong mixing, $R(x)$ is integrable as seen in (14). Nevertheless, we show it by another method which provides a formula for R. In the Fourier domain, $R(x)$ has the following expression:

$$R(x) = \int_{\mathbb{R}^2} \hat{\Phi}(\xi_1)\hat{\Phi}(\xi_2)\exp\{-\frac{1}{2}(\xi_1^2 + 2R_g(x)\xi_1\xi_2 + \xi_2^2)\}d^2\xi; \tag{46}$$

Here we denote by ξ the vector $(\xi_1,\cdots,\xi_N)$, and by $d^N\xi$ the Lebesgue measure in $\mathbb{R}^N$. Recall that for any $s\in\mathbb{R}$, there exists $c(s)\in[0,1]$ so that

$$e^s - 1 = s + \frac{1}{2}s^2 e^{cs}. \tag{47}$$

Using this expansion, we rewrite (46) as

$$\mathbb{R}(x) = \int_{\mathbb{R}^2} \hat{\Phi}(\xi_1)\hat{\Phi}(\xi_2)\exp\{-\frac{1}{2}\xi^t\xi\}\Big(1 - R_g(x)\xi_1\xi_2 + \frac{1}{2}e^{-cR_g(x)\xi_1\xi_2}R_g^2(x)\xi_1^2\xi_2^2\Big)d^2\xi. \tag{48}$$

In the above equation, ξ^t is the transpose of ξ. The real number c above depends on ξ and x but is always in the interval $[0,1]$. Now (44) says that the constant one in the parenthesis above does not contribute to the integral. Hence we can write

$$R(x) = \kappa R_g(x) + R_g^2(x)\kappa_r(x),$$

where κ is a finite positive constant given by

$$\kappa := -\int_{\mathbb{R}^2} \hat{\Phi}(\xi_1)\hat{\Phi}(\xi_2)\xi_1\xi_2 e^{-\frac{1}{2}\xi^t\xi}d^2\xi = \left(\int_{\mathbb{R}} s\Phi(s)e^{-\frac{s^2}{2}}ds\right)^2.$$

and κ_r is a function given by

$$\kappa_r(x) := \frac{1}{2}\int_{\mathbb{R}^2} \hat{\Phi}(\xi_1)\hat{\Phi}(\xi_2)\xi_1^2\xi_2^2 e^{-\frac{1}{2}\xi^t(I+cD_0)\xi}d^2\xi,$$

where D_0 above is a symmetric two by two matrix whose off diagonal is $R_g(x)$ and whose diagonal entries are zeros. Since $c(x)$ is in $[0,1]$ and the matrix $I+D_0$ is non-negative definite due to (40), so is the matrix $I+cD_0$. Therefore we can ignore the exponential term in the expression of $\kappa_r(x)$ above and bound $\|\kappa_r\|_{L^\infty}$ by $\|\hat{\Phi}(\xi)\xi^2\|_{L^1}^2/2$. Consequently, we obtain

$$|R| \le \left(\kappa + \frac{\|\hat{\Phi}(\xi)\xi^2\|_{L^1}^2}{2}\right)|R_g|.$$

Thus $R\in L^1(\mathbb{R}^d)$ because $R_g(x)$ is integrable.

Moreover, the analysis above shows that as $|x|\to\infty$, R is roughly κR_g.

3. It remains to show that q has controlled fourth order cumulants. Fix any four points $\{x_i\}_{i=1}^4$ and let ϑ be the joint cumulant of $\{q(x_i)\}$; in the Fourier domain it can be expressed as

$$\vartheta = \int_{\mathbb{R}^4} \prod_{j=1}^4 \hat{\Phi}(\xi_j)e^{-\frac{\xi^t\xi}{2}}\Big(\prod_{i=1}^3 e^{-\frac{1}{2}\xi^t D_i\xi} - \sum_{i=1}^3 e^{-\frac{1}{2}\xi^t D_i\xi}\Big)d^4\xi. \tag{49}$$

Here the matrices $D_i, i = 1, 2, 3$ are defined as follows:

$$D_1 = \begin{pmatrix} 0 & \rho_{12} & 0 & 0 \\ \rho_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_{34} \\ 0 & 0 & \rho_{34} & 0 \end{pmatrix}, D_2 = \begin{pmatrix} 0 & 0 & \rho_{13} & 0 \\ 0 & 0 & 0 & \rho_{24} \\ \rho_{13} & 0 & 0 & 0 \\ 0 & \rho_{24} & 0 & 0 \end{pmatrix}, D_3 = \begin{pmatrix} 0 & 0 & 0 & \rho_{14} \\ 0 & 0 & \rho_{23} & 0 \\ 0 & \rho_{23} & 0 & 0 \\ \rho_{14} & 0 & 0 & 0 \end{pmatrix},$$

where $\rho_{ij} := R_g(x_i - x_j)$ is the covariance of $g(x_i)$ and $g(x_j)$. We apply the following identity to the product and the sum inside the parenthesis in (49).

$$abc - a - b - c = (a-1)(b-1)(c-1) + (a-1)(b-1) + (a-1)(c-1) + (b-1)(c-1) - 2,$$

We then use (44) to argue that the constant two above does not contribute to (49). Hence we have

$$\vartheta = \int_{\mathbb{R}^4} \prod_{j=1}^4 \hat{\Phi}(\xi_j) e^{-\frac{\xi^t \xi}{2}} \left(\prod_{i=1}^3 [e^{-\frac{1}{2}\xi^t D_i \xi} - 1] + \sum_{i<k} [e^{-\frac{1}{2}\xi^t D_i \xi} - 1][e^{-\frac{1}{2}\xi^t D_k \xi} - 1] \right).$$

For each fixed ξ, we use the Taylor expansion for exponential function as in (47) and write

$$e^{-\frac{1}{2}\xi^t D_i \xi} - 1 = -\frac{1}{2}\xi^t D_i \xi e^{-\frac{1}{2}\xi^t (c_i D_i)\xi},$$

where the real number c_i depends on ξ and D_i but is always an element in $[0, 1]$. Therefore, we have

$$\vartheta = \int_{\mathbb{R}^4} \prod_{j=1}^4 \hat{\Phi}(\xi_j) \Bigg(-e^{-\frac{1}{2}\xi^t (I + \sum_{i=1}^3 c_i D_i)\xi} \prod_{i=1}^3 \frac{1}{2}\xi^t D_i \xi + \\ + \sum_{i<k} e^{-\frac{1}{2}\xi^t (I + c_i D_i + c_k D_k)\xi} [\frac{1}{2}\xi^t D_i \xi][\frac{1}{2}\xi^t D_k \xi] \Bigg) d^4\xi.$$

Observe that $I + D_i$, $I + D_i + D_j$ with $(i < j)$ for $i, j = 1, 2, 3$, and $I + \sum_{i=1}^3 D_i$ are non-negative definite matrices. Since $c_i \in [0, 1]$, we deduce that $I + c_i D_i + c_k D_k$ for any $i < k$, and $I + \sum_{i=1}^3 c_i D_i$ are all non-negative definite. Indeed, we can rewrite them as a sum of non-negative definite matrices. For instance, without loss of generality we assume c_i is increasing in i, and then

$$I + \sum_{i=1}^3 c_i D_i = c_1 (I + \sum_{i=1}^3 D_i) + (c_2 - c_1)(I + \sum_{i=2}^3 D_i) + (c_3 - c_2)(I + D_3) + (1 - c_3) I.$$

Each of the matrices on the right hand side above is non-negative definite.

Therefore, we can bound the exponential terms in the integral by one, and conclude that

$$|\vartheta| \leq \int_{\mathbb{R}^4} \prod_{j=1}^4 |\hat{\Phi}(\xi_j)| \left(\prod_{i=1}^3 |\frac{1}{2}\xi^t D_i \xi| + \sum_{i<k} |\frac{1}{2}\xi^t D_i \xi| \cdot |\frac{1}{2}\xi^t D_k \xi| \right).$$

Now the products in the parenthesis above are just polynomials in the $|\xi_j|$ variables, and for each ξ_j, the highest possible power on it is three. The coefficients in those polynomials are products of two or three ρ_{ij} functions. Since $|\rho_{ij}| \leq 1$ by definition, we can bound the $\xi^t D_1 \xi$ of the first member in the parenthesis above by $|\xi_1 \xi_2| + |\xi_3 \xi_4|$. Then after evaluating the product, the coefficients in the polynomial of $|\xi_j|$ variables are products of two ρ_{ij} functions. With this in mind, it is easy to verify that

$$\begin{aligned}\big|\vartheta(q(x_1),\cdots,q(x_4))\big| \le \big(&|\rho_{12}\rho_{13}|+|\rho_{12}\rho_{24}|+|\rho_{34}\rho_{13}|+|\rho_{34}\rho_{24}|\\&+|\rho_{12}\rho_{14}|+|\rho_{12}\rho_{23}|+|\rho_{34}\rho_{14}|+|\rho_{34}\rho_{23}|\\&+|\rho_{13}\rho_{14}|+|\rho_{13}\rho_{23}|+|\rho_{24}\rho_{14}|+|\rho_{24}\rho_{23}|\big)\int_{\mathbb{R}^4}\prod_{j=1}^4\hat\Phi(\xi_j)\big(|\xi_j|^3+|\xi_j|^2+|\xi_j|+1\big)d^4\xi\end{aligned}$$

Thanks to (45), the last integral is finite and can be bounded by $3^4\kappa_c^4$. Compare the above inequality with the cumulant condition, i.e., (10); we see that all pairs of indices in the products of ρ functions above lie in $\mathcal{U}^*$ where $\mathcal{U}$ is defined in (9). Then for each $p\in\mathcal{U}^*$, we set $\phi_p := 81\kappa_c^4|R_g\otimes R_g|$, which is in $L^1\cap L^\infty(\mathbb{R}^d\times\mathbb{R}^d)$. We see (10) is indeed satisfied. This completes the proof. $\square$

5 Proof of the main results

In this section, we prove the main theorems in dimension $d=2$. Let us denote by $\xi_\varepsilon = u_\varepsilon - u$ the corrector. Now subtract (6) from (3) to get

$$(\sqrt{-\Delta+\lambda^2}+q_0+q_\varepsilon)\xi_\varepsilon = -q_\varepsilon u. \tag{50}$$

Recall that $\mathcal{G}$ is the solution operator $(\sqrt{-\Delta+\lambda^2}+q_0)^{-1}$, and $\mathcal{G}_\varepsilon$ is the solution operator with random impedance. Therefore, the above equation says $\xi_\varepsilon = -\mathcal{G}_\varepsilon q_\varepsilon u$. Unfortunately, $\mathcal{G}_\varepsilon$ is not as explicit as $\mathcal{G}$. Nevertheless, we will show shortly that $-\mathcal{G}q_\varepsilon u$ is the leading term of $-\mathcal{G}_\varepsilon q_\varepsilon u$ and hence it suffices to estimate the former. Let us assign it the following notation;

$$\chi_\varepsilon := -\mathcal{G}q_\varepsilon u. \tag{51}$$

We have the following estimate.

Lemma 5.1. *Let u solve (6) and χ_ε be defined as above and $d=2$. Assume that the coefficient λ, q_0, and the random field $q(x,\omega)$ satisfy the same conditions as in Theorem 2.3. Then we have*

$$\mathbb{E}\|\chi_\varepsilon\|_{L^2}^2 \le C\varepsilon^2|\log\varepsilon|\ \|u\|_{L^2}^2, \tag{52}$$

where the constant C depends on λ, q_0 and $\|R\|_{L^1}$ but not on u or ε.

Proof: 1. We first express $\|\chi_\varepsilon\|_{L^2}^2$ as a triple integral of the form

$$\int_{\mathbb{R}^{3d}} G(x-y)q_\varepsilon(y)u(y)G(x-z)q_\varepsilon(z)u(z)d[yzx].$$

Here and in the sequel, the short-hand notation $d[x_1\cdots x_n]$ is the same as $dx_1\cdots dx_n$. Take expectation and use the definition of $R(x)$ to obtain

$$\mathbb{E}\|\chi_\varepsilon\|_{L^2}^2 = \int_{\mathbb{R}^{3d}} G(x-y)G(x-z)R(\frac{y-z}{\varepsilon})u(y)u(z)d[yzx].$$

2. We integrate in x first. Use the estimate (34) to replace the Green's functions by potentials of the form $e^{-\lambda'|x-y|}/|x-y|$; then apply Lemma A.1 to bound the integration in x of these potentials. We obtain

$$\mathbb{E}\|\chi_\varepsilon\|_{L^2}^2 \le C \int_{\mathbb{R}^{2d}} e^{-\lambda'|y-z|}\big(\big|\log|y-z|\big|+1\big)\Big|R(\frac{y-z}{\varepsilon})u(y)u(z)\Big|d[yz]. \tag{53}$$

Now change variable $(y-z)/\varepsilon \to y$. Since $d=2$, the integral on the right hand side becomes

$$\varepsilon^2 \int_{\mathbb{R}^{2d}} e^{-\varepsilon\lambda'|y|}\big(|\log|y|+\log\varepsilon|+1\big)\Big|R(y)u(z+\varepsilon y)u(z)\Big|d[yz].$$

3. Now, bound the exponential term by 1, and integrate in z. Use Cauchy-Schwarz to get

$$\int_{\mathbb{R}^d} \big|u(z+\varepsilon y)u(z)\big|dz \le \|u\|_{L^2}\|u(\cdot+\varepsilon y)\|_{L^2} = \|u\|_{L^2}^2. \tag{54}$$

Therefore, we have

$$\mathbb{E}\|\chi_\varepsilon\|_{L^2}^2 \le C\varepsilon^2\|u\|_{L^2}^2 \int_{\mathbb{R}^d} \big(|\log|y|+1+|\log\varepsilon|\big)\big|R(y)\big|dy$$

Recall that $R(y)$ behaves like $|y|^{-d-\delta}$ for some positive δ; see (8) and (14). Hence the function $(|\log|y|+1)|R|$ is integrable. The integral above is then

$$C\varepsilon^2|\log\varepsilon|\cdot\|u\|_{L^2}^2\|R\|_{L^1} + O(\varepsilon^2).$$

This completes the proof. We also see that the constant C only depends on λ and $\|R\|_{L^1}$. $\square$

Theorem 2.3 now follows if we can control $\|\xi_\varepsilon - \chi_\varepsilon\|_{L^2}$. From (51) we see

$$(\sqrt{-\Delta+\lambda^2}+q_0+q_\varepsilon)\chi_\varepsilon = -q_\varepsilon u + q_\varepsilon \chi_\varepsilon.$$

Subtract this equation from (50); we get an equation for $\xi_\varepsilon - \chi_\varepsilon$. Apply $\mathcal{G}_\varepsilon$ on this equation to get

$$\xi_\varepsilon = \chi_\varepsilon - \mathcal{G}_\varepsilon q_\varepsilon \chi_\varepsilon. \tag{55}$$

The following proof relies on this expression and the fact that the operator $\mathcal{G}_\varepsilon$ is bounded uniformly in ε and ω as we have emphasized in Remark 3.3.

Proof of of Theorem 2.3: From the expression (55) we have,

$$\|u_\varepsilon - u\|_{L^2} \le \|\chi_\varepsilon\|_{L^2} + \sup_{\omega\in\Omega}\|\mathcal{G}_\varepsilon\|_{\mathcal{L}}\|q\|_{L^\infty(\Omega\times\mathbb{R}^d)}\|\chi_\varepsilon\|_{L^2}.$$

Due to (11) and Corollary 3.2, we have $\|q\|_{L^\infty} \le q_0$ and $\|\mathcal{G}_\varepsilon\|_{L^\infty(\Omega,\mathcal{L}(L^2))} \le \min\{1,\lambda\}^{-1}$. We will denote the products of the two constants by C. Then we have

$$\|u_\varepsilon - u\|_{L^2} \le (1+C)\|\chi_\varepsilon\|_{L^2}.$$

Square both sides and take expectation; then apply Lemma 5.1 to get

$$\mathbb{E}\{\|u_\varepsilon - u\|_{L^2}^2\} \le C\mathbb{E}\{\|\chi_\varepsilon\|_{L^2}^2\} \le C\varepsilon^2|\log\varepsilon|\cdot\|u\|_{L^2}^2.$$

Now use Corollary 3.2 to replace the L^2 norm of u by that of f. Again, all constants involved do not depend on ε. This completes the proof. $\square$

To prove Theorem 2.4 and 2.5, i.e., to characterize the limits of the deterministic and stochastic correctors, we first express ξ_ε as a sum of three terms with increasing order in q_ε. To this end, move the term $q_\varepsilon \xi_\varepsilon$ in (50) to the right hand side, and then apply $\mathcal{G}$ on it. We get

$$\xi_\varepsilon = -\mathcal{G} q_\varepsilon u - \mathcal{G} q_\varepsilon \xi_\varepsilon.$$

Iterate this formula one more time to get

$$\xi_\varepsilon = -\mathcal{G} q_\varepsilon u + \mathcal{G} q_\varepsilon \mathcal{G} q_\varepsilon u + \mathcal{G} q_\varepsilon \mathcal{G} q_\varepsilon \xi_\varepsilon. \tag{56}$$

Note that the limits in both theorems are taken weakly in space, so we consider an arbitrary test function M, e.g. in C_c^∞, and integrate the above formula with M. We get

$$\langle \xi_\varepsilon, M\rangle = -\langle \mathcal{G} q_\varepsilon u, M\rangle + \langle \mathcal{G} q_\varepsilon \mathcal{G} q_\varepsilon u, M\rangle + \langle \mathcal{G} q_\varepsilon \mathcal{G} q_\varepsilon \xi_\varepsilon, M\rangle. \tag{57}$$

Defining $m := \mathcal{G} M$, the last term can be written as $\langle q_\varepsilon \xi_\varepsilon, \mathcal{G} q_\varepsilon m\rangle$ since $\mathcal{G}$ is self-adjoint. Using this notation we now prove the second main theorem.

Proof of Theorem 2.4: Take expectation on the weak formulation (57). The first term vanishes since q_ε is mean zero. To estimate the thrid term, we observe that

$$\big|\langle \mathcal{G} q_\varepsilon \mathcal{G} q_\varepsilon \xi_\varepsilon, M\rangle\big| = \big|\langle q_\varepsilon \xi_\varepsilon, \mathcal{G} q_\varepsilon m\rangle\big| \leq \|q_\varepsilon\|_{L^\infty} \|\xi_\varepsilon\|_{L^2} \|\mathcal{G} q_\varepsilon m\|_{L^2}.$$

Thanks to the uniform bound (11) for $q(x,\omega)$, the term $\|q_\varepsilon\|_{L^\infty}$ is bounded by q_0. After taking expectations on both sides and using Cauchy-Schwarz on the right hand side, we obtain

$$\mathbb{E}\big|\langle \mathcal{G} q_\varepsilon \mathcal{G} q_\varepsilon \xi_\varepsilon, M\rangle\big| \leq C\big(\mathbb{E}\{\|\xi_\varepsilon\|^2\}\ \mathbb{E}\{\|\mathcal{G} q_\varepsilon m\|^2\}\big)^{1/2} \leq C\varepsilon^2 |\log\varepsilon| \cdot \|u\|_{L^2} \|m\|_{L^2}, \tag{58}$$

where the last inequality follows from Theorem 2.3 and Lemma 5.1. In the limit, this term is much smaller than ε.

Now we calculate the expectation of the second term in (57), which can be written as:

$$\mathbb{E}\langle q_\varepsilon u, \mathcal{G} q_\varepsilon m\rangle = \int_{\mathbb{R}^{2d}} G(x-y) R(\frac{x-y}{\varepsilon}) u(x) m(y) d[xy]. \tag{59}$$

As in the proof of Lemma 5.1, we change variable $(x-y)/\varepsilon$ to x. The integral above now becomes

$$\varepsilon^d \int_{\mathbb{R}^{2d}} G(\varepsilon x) R(x) u(y+\varepsilon x) m(y) d[xy] \leq \|u\|_{L^2} \|m\|_{L^2} \int_{\mathbb{R}^d} \varepsilon^d G(\varepsilon|x|) |R(x)| dx. \tag{60}$$

The last equality is obtained by integrating in y and applying the same technique as in (54). Recalling Lemma 3.4 and $d=2$, G can be decomposed into three terms. We have

$$\varepsilon^2 G(\varepsilon|x|) = \frac{\varepsilon^2}{2\pi}\left(\frac{\exp(-\lambda\varepsilon|x|)}{\varepsilon|x|} - q_0 K_0(\lambda\varepsilon|x|) + G_r(\varepsilon|x|)\right).$$

Since K_0 only has logarithmic singularity at the origin and G_r is uniformly bounded as we have seen in Lemma 3.4, the last two terms above are of order $\varepsilon^2|\log\varepsilon|$ and ε^2 respectively. Their contributions to (60) are neglectable.

Hence the leading term in (60) is

$$\varepsilon \int_{\mathbb{R}^2} \frac{e^{-\varepsilon\lambda|x|}}{2\pi|x|} R(x)u(y)m(y+\varepsilon x)dydx. \tag{61}$$

Taking the limit and recalling the definition of $\tilde{R}$ in (17), we see that this term is

$$\varepsilon\tilde{R}\langle u, m\rangle + o(\varepsilon) = \varepsilon\tilde{R}\langle \mathcal{G}u, M\rangle + o(\varepsilon).$$

This completes the proof. $\square$

Our poof of the third theorem also relies on the formula (57). The plan is as follows. First, we show that the leading term in the stochastic corrector $\xi_\varepsilon - \mathbb{E}\{\xi_\varepsilon\}$ is the first term in (57); this is done by showing that the variances of the other terms are small. Then we verify that the first term has a limiting distribution that can be written as the right hand side of (19); this step is rather standard and follows from a generalized central limit theorem in [1]; see below. For the moment, let us assume the following lemma and prove Theorem 2.5.

Lemma 5.2. *Let u solve (6) with $d = 2$ and M be a test function in $C_c^\infty(\mathbb{R}^d)$. Assume that the random field $q(x,\omega)$ satisfies the same conditions as in Theorem 2.5. Then we have the following estimate:*

$$\mathrm{Var}\ \ \langle \mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon u, M\rangle \le C\varepsilon^3, \tag{62}$$

where C depends on $\|M\|_{L^1}$, $\|M\|_{L^\infty}$, dimension d, $\|\phi_p\|_{L^1}$ and $\|\phi_p\|_{L^\infty}$ in (10), but not on ε.

Proof of Theorem 2.5: 1. From formula (57) we have that

$$\mathbb{E}\Big|\langle \frac{u_\varepsilon - \mathbb{E}\{u_\varepsilon\}}{\varepsilon} + \frac{\mathcal{G}q_\varepsilon u}{\varepsilon}, M\rangle\Big| \le \frac{1}{\varepsilon}\big(\ \mathrm{Var}\ \langle \mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon u, M\rangle\big)^{\frac12} + \frac{2}{\varepsilon}\mathbb{E}\{|\langle \mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\xi_\varepsilon, M\rangle|\}.$$

The last term is of order $\varepsilon|\log\varepsilon|$ thanks to the estimate (58), and the next-to-last is of order $\sqrt{\varepsilon}$ due to (62). Therefore the right hand side above vanishes in the limit. This shows convergence of $\varepsilon^{-1}\langle u_\varepsilon - \mathbb{E}\{u_\varepsilon\}, M\rangle$ to $-\varepsilon^{-1}\langle \mathcal{G}q_\varepsilon u, M\rangle$ in $L^1(\Omega)$ which in turn implies convergence in distribution. Hence, we only need to characterize the asymptotic distribution of the latter term.

2. The random variable $\varepsilon^{-1}\langle \mathcal{G}q_\varepsilon u, M\rangle$, which is the same as $\varepsilon^{-1}\langle q_\varepsilon u, m\rangle$ where $m = \mathcal{G}M$, is of the form of an oscillatory integral. Let $v(y)$ denote $u(y)m(y)$; it is an L^2 function. We want

$$\int_{\mathbb{R}^2} \frac{1}{\varepsilon} q(\frac{y}{\varepsilon})v(y)dy \xrightarrow{\text{dist.}} \sigma\int_{\mathbb{R}^2} v(y)dW_y, \tag{63}$$

where W_y is the standard two-variate Wiener process as in Theorem 2.5. This convergence result, with $\mathbb{R}^2$ replaced by a bounded domain and v continuous, was stated as (3.31) in [1] and was the main step in the proof of Theorem 3.7 there. The proof goes as follows. Break the integral on the left of (63) into integrals on small pieces, and on each piece write the integral as a properly scaled sum of weakly dependent random variables. Apply central limit theorem for such variables, e.g. [6], and show that each piece converges to a centered normal random variable with certain variance. At this stage, we need the strong mixing coefficient $\alpha(r)$ of q to satisfy (8). Then show that different pieces are independent in the limit. Consequently, the left side of (63) converges in distribution to a sum of independent normal random variables and hence is itself normal in the limit. The variance of this limiting normal random variable is then verified to be

$$\sigma^2\int v^2(y)dy,$$

the same as the variance of the right hand side of (63), closing the proof. For details, we refer the reader to [1].

Here, since we assumed that M is compactly supported, v decays fast and is in $L^2(\mathbb{R}^d)$, and we obtain (63) by using the known result on the ball with radius B and sending B to infinity. This completes the proof of the theorem. $\square$

It remains to prove the preceding lemma.

Proof of Lemma 5.2: We express random variable $\langle \mathcal{G}q_\varepsilon \mathcal{G}q_\varepsilon u, M\rangle$, which equals $\langle q_\varepsilon u, \mathcal{G}q_\varepsilon m\rangle$ where $m = \mathcal{G}M$, as the following integral.

$$I := \int_{\mathbb{R}^{2d}} u(x)m(y)G(x-y)q_\varepsilon(x)q_\varepsilon(y)d[xy].$$

Take the variance of this variable. Denote by ϑ the joint cumulant. We have the following expression for $\mathrm{Var}\{I\}$, i.e., $\mathbb{E}\{I^2\} - (\mathbb{E}\{I\})^2$;

$$\begin{aligned}\mathrm{Var}\{I\} = \int_{\mathbb{R}^{4d}} u(x)m(y)u(x')m(y')G(x-y)G(x'-y')\Big[\vartheta\{q_\varepsilon(x), q_\varepsilon(y), q_\varepsilon(x'), q_\varepsilon(y')\}\\ + R(\frac{x-x'}{\varepsilon})R(\frac{y-y'}{\varepsilon}) + R(\frac{x-y'}{\varepsilon})R(\frac{y-x'}{\varepsilon})\Big]d[xyx'y'].\end{aligned}$$

Then we identify x, y, x', y' with x_1, x_2, x_3, x_4. Let $\mathcal{U}$ and $\mathcal{U}^*$ be the sets defined in (9) and the paragraph below it. Recall that the joint cumulant $\vartheta\{q_\varepsilon(x_i)\}_{i=1}^4$ satisfies (10) with $\phi_p \in L^1 \cap L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$; we have the following bound for $\mathrm{Var}\{I\}$:

$$\begin{aligned}\int_{\mathbb{R}^{4d}} \big|u(x)m(y)u(x')m(y')G(x-y)G(x'-y')\Big(\sum_{p\in\mathcal{U}^*}\phi_p(\frac{x_{p(1)}-x_{p(2)}}{\varepsilon}, \frac{x_{p(3)}-x_{p(4)}}{\varepsilon})\\ + R(\frac{x-x'}{\varepsilon})R(\frac{y-y'}{\varepsilon}) + R(\frac{x-y'}{\varepsilon})R(\frac{y-x'}{\varepsilon})\Big)d[xyx'y'].\end{aligned} \tag{64}$$

Let us denote the contributions of the last two terms in the parenthesis above by J_2 and J_3 respectively, and denote the contribution of the other term by J_1. We observe that the variables in the $R \otimes R$ functions are independent with the variables in the Green's functions, while this is not the case for the variables in the ϕ_p functions.

We first estimate J_2. It has the following expression;

$$J_2 := \int_{\mathbb{R}^{4d}} \big|u(x)m(y)u(x')m(y')G(x-y)G(x'-y')R(\frac{x-x'}{\varepsilon})R(\frac{y-y'}{\varepsilon})\big|d[xyx'y'].$$

Perform a change of variables as follows:

$$x \to x,\ \frac{x-x'}{\varepsilon} \to x',\ \frac{y-y'}{\varepsilon} \to y',\ x-y \to y.$$

This change of variables yields a Jacobian ε^{2d} and the integral above becomes

$$\varepsilon^{2d}\int_{\mathbb{R}^{4d}} \big|u(x)m(x-y)u(x-\varepsilon x')m(y-\varepsilon y')G(y)G(y-\varepsilon(x'-y'))R(x')R(y')d[xyx'y']. \tag{65}$$

Now we observe that the function $m = \mathcal{G}M$ is uniformly bounded as follows;

$$\|m\|_{L^\infty} \le C(\|M\|_{L^\infty} + \|M\|_{L^1}). \tag{66}$$

Indeed, we use the estimate (34) for the Green's function and have

$$m(x) = \int_{\mathbb{R}^d} G(x-y)M(y)dy \le C\int_{\mathbb{R}^d} \frac{M(y)}{|x-y|^{d-1}}dy$$
$$\le C\left(\|M\|_{L^\infty}\int_{B_1(x)} \frac{1}{|x-y|^{d-1}}dy + \int_{B_1^c(x)} M(y)dy\right).$$

Here we denote by $B_1(x)$ the unit ball centered at x, and by $B_1^c(x)$ its complement. The integral inside $B_1(x)$ is bound by $\pi^{\lfloor \frac{d}{2}\rfloor}$, and the integral on $B_1^c(x)$ is bounded by $\|M\|_{L^1}$. Hence we obtain (66). Use this bound to control the m functions in (65). Integrate in x and use (54) to control the u functions. Integrate in y for the two Green's function and view the integration as a convolution. Use (34) to bound them by potentials of the form $e^{-\lambda'|x|}/|x|$, and use Lemma A.1 to get

$$\int_{\mathbb{R}^d} G(y)G(y-\varepsilon(x'-y'))dy \le Ce^{-\lambda'\varepsilon|x'-y'|}\Big(|\log(\varepsilon|x'-y'|)|\cdot\chi_{\{\varepsilon|x'-y'|\le 1\}}+1\Big),$$

where χ is the indicator function of a set. This estimate is a refined version of item two in (77) below, and it can be shown following the same proof while in (80) we perform the integration by parts only if $\rho \le 1$. Therefore, after controlling u, m, and G, we get

$$J_2 \le C\varepsilon^{2d}\|u\|_{L^2}^2\|m\|_{L^\infty}^2\int_{\mathbb{R}^{2d}}\Big(\big|\log(\varepsilon|x'-y'|)\big|\chi_{\{\varepsilon|x'-y'|\le 1\}}+1\Big)$$
$$\times|R(x')|\cdot|R(y')|d[x'y']. \tag{67}$$

The constant one in the parenthesis hence has a contribution of order ε^{2d} since $\|R\|_{L^1}$ is finite. For the logarithmic term, we observe that

$$\sup_{0<r\le 1} r^{d-1}|\log r| \le \frac{e^{-1}}{d-1}, \text{ for } d\ge 2. \tag{68}$$

Therefore, we have

$$\big|\log(\varepsilon|x'-y'|)\big|\chi_{\{\varepsilon|x'-y'|\le 1\}} \le \frac{e^{-1}}{(d-1)\varepsilon^{d-1}|x'-y'|^{d-1}}\chi_{\{\varepsilon|x'-y'|\le 1\}}.$$

The contribution of the logarithm term in (67) is bounded by

$$C\varepsilon^{d+1}\|u\|_{L^2}^2\|m\|_{L^\infty}^2\int_{\mathbb{R}^{2d}}\frac{|R(x')|\cdot|R(y')|}{|x'-y'|^{d-1}}d[x'y'].$$

Now apply the Hardy-Littlewood-Sobolev inequality, e.g. [14, §4.3], to get

$$\left|\int_{\mathbb{R}^{2d}}\frac{|R(x')|\cdot|R(y')|}{|x'-y'|^{d-1}}\right| \le C(\frac{2d}{d+1},d-1)\|R\|_{L^{\frac{2d}{d+1}}}^2. \tag{69}$$

Since $R\in L^1\cap L^\infty$, it is certainly in $L^{\frac{2d}{d+1}}$. Set $d=2$; we have proved that

$$J_2 \le C\varepsilon^3\|u\|_{L^2}^2\|m\|_{L^\infty}^2\|R\|_{L^\infty}^{\frac{3}{2}}\|R\|_{L^1}^{\frac{1}{2}} + O(\varepsilon^{2d}). \tag{70}$$

Similarly, J_3 can be shown to be of size smaller than ε^3 as well in dimension two.

Now we consider J_1. There are $C_6^2 - 3 = 12$ terms that appear in the sum over $p \in \mathcal{U}^*$ in (64), and they can be divided into two groups. In the first group, the function ϕ_p shares a variable with one of the Green's functions; in the second group, the variable of one of the Green's functions is a linear combination of the two variables of the ϕ_p function.

We first consider a typical term from the first group and still call it J_1; it has the following expression:

$$J_1 := \int_{\mathbb{R}^{4d}} \big|G(x-y)G(x'-y')\phi_p(\frac{x-y}{\varepsilon}, \frac{x-x'}{\varepsilon})u(x)m(y)u(x')m(y')\big| d[xyx'y'],$$

Note that the $x-y$ variable is shared by the first Green's function and ϕ_p. We perform the following change of variables:

$$x \to x,\ \frac{x-x'}{\varepsilon} \to x',\ \frac{x-y}{\varepsilon} \to y, x'-y' \to y'.$$

The Jacobian is again ε^{2d}, and then the integral becomes

$$\varepsilon^{2d} \int_{\mathbb{R}^{4d}} \big|u(x)m(x-\varepsilon y)u(x-\varepsilon x')m(x'-y')G(y')G(\varepsilon y)\phi_p(y,x')d[xyx'y'].$$

Use (66) to control the m functions; integrate in x and use (54) to control the u functions; integrate in y' to control the first Green's function. We obtain the following bound for J_2.

$$J_2 \le C\varepsilon^{2d}\|u\|_{L^2}^2 \|m\|_{L^\infty}^2 \|G\|_{L^1} \int_{\mathbb{R}^{2d}} \frac{1}{(\varepsilon|y|)^{d-1}} \phi_p(y,x')d[yx'], \tag{71}$$

where we have used (34) for the Green's function. The scaling ε^{-d+1} resulting from the Green's function combined with the Jacobian ε^{2d} indicates that J_2 is of size ε^{d+1} once we control the following integral:

$$\int_{\mathbb{R}^{2d}} \frac{\phi_p(y,x')}{|y|^{d-1}} d[yx'].$$

Indeed, this integral is finite since $|y|^{d-1}$ is integrable near the origin and ϕ_p is integrable at infinity. To summarize we have

$$J_2 \le C\varepsilon^{d+1}\|u\|_{L^2}^2 \|m\|_{L^\infty}^2 \|G\|_{L^1} \left\| \frac{\phi_p(y,x')}{|y|^{d-1}} \right\|_{L^1}. \tag{72}$$

For a typical term from the second group in the sum over $p \in \mathcal{U}^*$ in (64), we can apply the same procedure exactly and in (71) we will have $|x'-y|^{d-1}$ on the denominator in the integral, and we can control the integral as in (69). Therefore, the contributions of such terms are also of size ε^3 in dimension two. This completes the proof. $\square$

6 General setting with singular Green's function

In this section we explain how to apply the procedure of this paper to elliptic pseudo-differential equations of the form (3) in general dimensions. We consider the following pseudo-differential equation with random coefficient:

$$P(x,D)u_\varepsilon(x,\omega) + (q_0(x) + q_\varepsilon(x,\omega))u_\varepsilon = f(x), \tag{73}$$

posed on a subset X of $\mathbb{R}^d$ with appropriate boundary condition. As before, $q_\varepsilon(x,\omega) = q(x/\varepsilon,\omega)$ and $q(x,\omega)$ is a stationary, mean zero, finite variance, strong mixing random field defined on $(\Omega,\mathcal{F},\mathbb{P})$, with parameters $x \in \mathbb{R}^d$. Assume that the deterministic and random potentials, i.e., q_0 and q_ε, satisfy proper conditions so that the solution operators,

$$\mathcal{G} := \big(P(x,D) + q_0\big)^{-1}, \quad \mathcal{G}_\varepsilon := \big(P(x,D) + q_0 + q_\varepsilon\big)^{-1},$$

are well defined almost everywhere in Ω. Assume also that $\mathcal{G}$ and $\mathcal{G}_\varepsilon$ as transformations on $L^2(X)$ are bounded for all realizations, and the upper bound of the operator norm is independent of realizations. Assume further that the Green's function corresponding to $\mathcal{G}$ is singular, i.e., not square integrable near the origin, and is therefore of interest in this paper.

Using the same techniques developed in previous sections, we can show that u_ε converges to the solution of a homogenized equation denoted by u in the $L^2(X\times\Omega)$ norm. We can then show that the random corrector $u_\varepsilon - \mathbb{E}\{u_\varepsilon\}$ converges weakly and in probability distribution to a Gaussian process with variance of size ε^d. The large components, with size no less than $\varepsilon^{d/2}$, of the deterministic corrector $\mathbb{E}\{u_\varepsilon\} - u$ can also be captured. As in the main body of this paper, we need additional assumptions on some higher-order moments of the random field $q(x,\omega)$ to obtain the last two results.

To be precise, suppose the Green's function $G(x,y)$ has the following decomposition with decreasing singularities,

$$G(x,y) \sim \sum_{j=1}^{N} \frac{c_j(x,y)}{|x-y|^{\gamma_j}} + G_r(x,y). \tag{74}$$

Here, N is a finite integer and

$$d > \gamma_1 > \gamma_2 > \cdots > \gamma_N \geq \frac{d}{2}.$$

Let us denote the terms in the sum above as G_j. The functions $\{c_j(x,y)\}$ are uniformly bounded and decay fast enough so that $\{G_j\}$ are integrable if the domain X is unbounded. Further, $G_r(x,y)$ is a term that is both integrable and square integrable (with respect to one of the variables and uniformly in the other variable).

Then, the homogenized equation for (73) will be of the same form with q_ε averaged (or removed). In fact, we have the following as an analogy of Theorem 2.3.

$$\mathbb{E}\|u_\varepsilon - u\|_{L^2}^2 \leq \begin{cases} C\varepsilon^{2(d-\gamma_1)}\|u\|_{L^2}^2, & \text{if } 2\gamma_1 > d, \\ C\varepsilon^d|\log\varepsilon|\|u\|_{L^2}^2, & \text{if } 2\gamma_1 = d. \end{cases} \tag{75}$$

These estimates show that u_ε converges to the homogenized solution u in energy norm. At this stage, we do not need the mixing property or control of higher order moments of $q(x,\omega)$.

Under certain conditions on some moments of the random field, we know that the fluctuations in the corrector are approximately weakly Gaussian and of size $\varepsilon^{d/2}$. To further approximate u_ε, we would like to capture all the terms in the corrector whose means are larger. To do this, we expand u_ε as iterations of $\mathcal{G}$ on random potentials as follows.

$$u_\varepsilon(x) - u = -\mathcal{G}q_\varepsilon\mathcal{G}f + \mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\mathcal{G}f - \mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon\mathcal{G}f + \cdots + (-\mathcal{G}q_\varepsilon)^k\xi_\varepsilon. \tag{76}$$

The order k at which we terminate the iteration is chosen so that we can show $\mathbb{E}\{\|(\mathcal{G}q_\varepsilon)^{k-2}\mathcal{G}M\|_{L^2}^2\} \leq \varepsilon^\gamma$ with $\gamma > 2\gamma_1 - d$ for some test function M. Then weakly, the remainder term $(-\mathcal{G}q_\varepsilon)^k\xi_\varepsilon$ is of order less than $\varepsilon^{d/2}$. Hence, the finite terms in (76) before the remainder include all the components in the corrector whose means are weakly larger than the random fluctuations. Then it is a tedious routine as shown in the paper to calculate the large deterministic means of these terms and to check that their variances are less than ε^d. As a result, the limiting law of $u_\varepsilon - \mathbb{E}\{u_\varepsilon\}$ is given by the limiting law of $\frac{1}{\varepsilon^{d/2}}\mathcal{G}q_\varepsilon u$, which is Gaussian and admits a convenient stochastic integral representation.

As an example, we summarize and compare results for the diffusion equation (4) as the dimension n and hence d change.

When $n = 2$ and hence $d = 1$, the Green's function G has logarithmic singularity only and hence $G_j \equiv 0$ in (74). As a result, G is square integrable and the problem reduces to a case that is investigated in [1]. In particular, the deterministic corrector $\mathbb{E}\{u_\varepsilon - u\}$ is of size ε and does not show up in Theorem 2.5; in other words, the deterministic corrector is dominated by the random fluctuations, which are of size $\sqrt{\varepsilon}$.

When $n \geq 4$ and hence $d > 2$, then the leading term of the Green's function is given by a modified Bessel potential and has singularity of order $\gamma_1 = d-1$ at the origin, and $2\gamma_1 > d$. Then the leading term in the deterministic corrector will be of order $\varepsilon^{d-\gamma_1}$, which is larger that $\varepsilon^{d/2}$, In other words, the deterministic corrector dominates the fluctuations., which is of size $\varepsilon^{d/2}$

The physical dimension $n = 3$ considered in the main section turns out to be the critical case when the deterministic corrector is in fact of the same size as the fluctuations, and they are of size ε.

A Two technical lemmas

A.1 Convolution of potentials in the whole space

Lemma A.1. *Let us fix two distinct points $x, y \in \mathbb{R}^d$. Let α, β be positive numbers in $(0, d)$, and λ another positive number. We have the following convolution results.*

$$\int_{\mathbb{R}^d} \frac{e^{-\lambda|z-x|}}{|z-x|^\alpha} \frac{e^{-\lambda|z-y|}}{|z-y|^\beta} dz \leq \begin{cases} Ce^{-\lambda|x-y|}(|x-y|^{d-(\alpha+\beta)} + 1), & \text{if } \alpha+\beta > d; \\ Ce^{-\lambda|x-y|}(|\log|x-y|| + 1), & \text{if } \alpha+\beta = d; \\ Ce^{-\lambda|x-y|} & \text{if } \alpha+\beta < d. \end{cases} \tag{77}$$

Similarly, we also have that

$$\int_{\mathbb{R}^d} \frac{e^{-\lambda|z-x|}}{|z-x|^\alpha} e^{-\lambda|z-y|} \big|\log|z-y|\big| dz \leq Ce^{-\lambda|x-y|}. \tag{78}$$

The above constants depend only on the diam (X), *α, β, λ, and dimension d but not on $|x-y|$.*

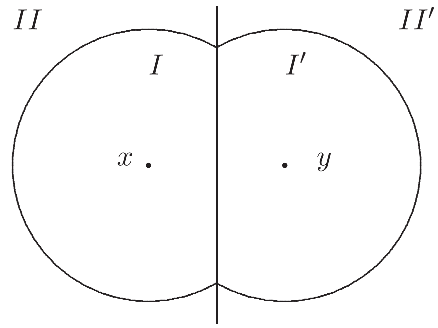
Proof: We use the partition of the integration domain as shown in Fig. 1. On I and similarly on I', we use $|z-x| + |z-y| \geq |x-y|$, and define $\rho = |x-y|$. Then we have

$$\int_I \frac{e^{-\lambda|z-x|}}{|z-x|^\alpha} \frac{e^{-\lambda|z-y|}}{|z-y|^\beta} dz \leq \frac{2\pi_d e^{-\lambda|x-y|}}{3\rho^\beta} \int_0^\rho \frac{r^{d-1}}{r^\alpha} dr.$$

The last integral can be calculated explicitly and yields $\rho^{d-\alpha}/(d-\alpha)$. Hence the integration over $I \cup I'$ can be bounded by

$$\frac{2(2d-\alpha-\beta)\pi_d e^{-\lambda|x-y|}}{3(d-\alpha)(d-\beta)|x-y|^{\alpha+\beta-d}}. \tag{79}$$

Figure 1: Integration region of the convolution of two potentials.



Now on the unbounded domain II, we observe that $|z-y|>\rho$ and $|z-y|>|z-x|$, and obtain similar relations on II'. Therefore the integration on $II\cup II'$ is bounded from above by

$$2e^{-\lambda|x-y|}\int_{II}\frac{e^{-\lambda|z-x|}}{|z-x|^{\alpha+\beta}}dz\le\frac{4\pi_d e^{-\lambda|x-y|}}{3}\int_\rho^\infty\frac{e^{-\lambda r}}{r^{\alpha+\beta-d+1}}dr.$$

Now, we estimate the last integral, which we call $A(\rho)$. When $\alpha+\beta<d$, the integrand is integrable over $\mathbb{R}_+$, the nonnegative real line. Therefore $A(\rho)$ is bounded by some constant, actually a multiple of $\Gamma(d-\alpha-\beta)$. This together with the bound (79) proves the third case in (77).

When $\alpha+\beta=d$, then an integration by parts yields

$$A(\rho)=\int_\rho^\infty\frac{e^{-\lambda r}}{r}=-e^{-\lambda\rho}\log\rho+\lambda\int_\rho^\infty e^{-\lambda r}\log r dr. \tag{80}$$

The last integral is finite over $\mathbb{R}_+$ and hence $|A(\rho)|\le Ce^{-\lambda\rho}(1+|\log\rho|)$. This together with the bound (79) proves the second case in (77).

When $\alpha+\beta>d$, let us denote $-\alpha-\beta+d-1=s$. Several integrations by parts yield

$$\begin{aligned}A(\rho)=\int_\rho^\infty e^{-\lambda r}r^s dr=\frac{\lambda^\gamma}{\prod_{j=1}^\gamma(s+j)}\int_\rho^\infty e^{-\lambda r}r^{s+\gamma}dr\\-e^{-\lambda\rho}\Big(\frac{\rho^{s+1}}{s+1}+\frac{\lambda\rho^{s+2}}{(s+1)(s+2)}+\cdots+\frac{\lambda^{\gamma-1}\rho^{s+\gamma}}{(s+1)\cdots(s+\gamma)}\Big).\end{aligned} \tag{81}$$

Here, γ is the largest integer that is smaller than or equal to $\alpha+\beta-d$. When they are equal, the right hand side above needs some slight modifications and the first integral involves a logarithmic function. In both cases, the first integral is finite and the second term is bounded by $Ce^{-\lambda\rho}(1+\rho^{d-\alpha-\beta})$. This together with the bound (79) proves the second case in (77).

The claim (78) follows from a similar and easier analysis which we omit. □

A.2 Fourier transform and exponential decay

Lemma A.2. *Let λ and q_0 be positive real numbers and let $\xi \in \mathbb{R}^2$. Set $\lambda' \equiv \lambda/\sqrt{2}$. Then, for any positive real number $b < \lambda'$, there exists a finite constant C_b such that*

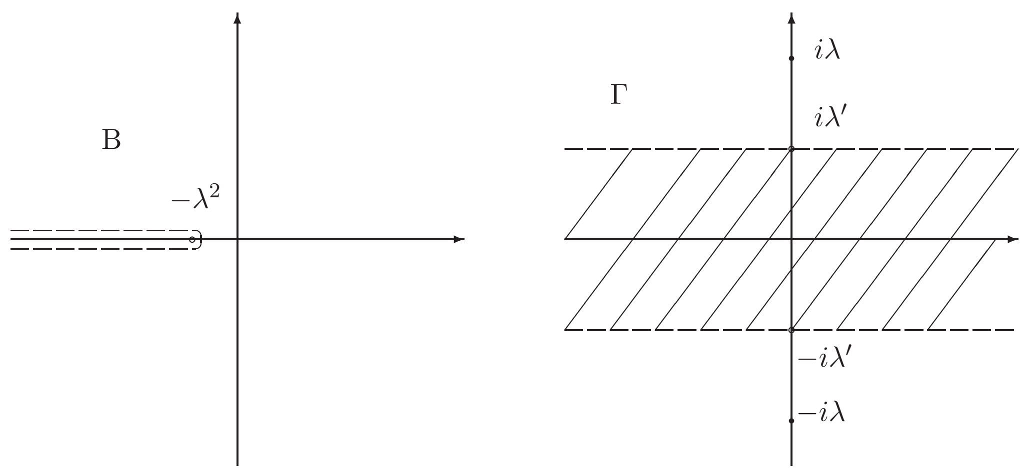
$$\left| \mathscr{F}^{-1} \frac{q_0^2}{(|\xi|^2 + \lambda^2)(q_0 + \sqrt{|\xi|^2 + \lambda^2})} \right| \leq C_b e^{-b|x|}. \tag{82}$$

Proof: 1. Let us denote by $h(\xi)$ the function whose inverse Fourier transform is considered in (82). Let us also define $h(z)$ to be the same function with ξ replaced by $z = (z_1, z_2) \in \mathbb{C}^2$, a complex valued function of two complex variables. Set

$$\Gamma := \{z \in \mathbb{C} | |\mathrm{Im}(z)| \leq \lambda'\}. \tag{83}$$

We claim that h is holomorphic on the region Γ^2, i.e. $\Gamma \times \Gamma$.

Figure 2: Holomorphic region of the function $h(z)$.



Left: holomorphic region of $g(w) = \sqrt{w + \lambda^2}$.
Right: holomorphic region of $g(z_1^2 + z_2^2)$; here $\lambda' = \lambda/\sqrt{2}$.

Indeed, let $w(z_1, z_2)$ be the function $z_1^2 + z_2^2$. It is clearly entire on $\mathbb{C}^2$. Define $g(w) := \sqrt{w + \lambda^2}$ as a function of one complex variable. It is holomorphic on the branched region $\mathrm{B} := \mathbb{C} \backslash (-\infty, -\lambda^2]$ as shown in Fig. 2. Now when $(z_1, z_2) \in \Gamma^2$, we verify that $w \in \mathrm{B}$ and hence $g(w(z))$ is holomorphic on Γ^2. This is because composition of holomorphic functions is again holomorphic; see [10]. Since $\lambda > q_0$, we verify that $g(w(z)) + q_0$ does not vanish. Thus, $h(z)$ is holomorphic on Γ^2.

The above arguments show that for any $\eta \in \mathbb{R}^2$ so that $|\eta_j| < \lambda'$, $i = 1, 2$, the function $h(\xi + i\eta)$ is analytic. Furthermore, it is easy to check that $\|h(\xi + i\eta)\|_{L^1}$ is bounded uniformly in η. Hence we apply Theorem IX.14 of [18], which says that under such conditions, for each $0 < b < \lambda'$, there exists C_b so that $|\mathscr{F}^{-1} h| \leq C_b e^{-b|x|}$. This completes the proof. □

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