Radiative transfer equations with varying refractive index: a mathematical perspective

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Abstract

This paper reviews established mathematical techniques to model the energy density of high frequency waves in random media by radiative transfer equations and to model the small mean free path limit of radiative transfer solutions by diffusion equations. It then applies these techniques to the derivation of radiative transfer and diffusion equations for the radiance, also known as specific intensity, of electromagnetic waves in situations where the refractive index of the underlying structure varies smoothly in space.

1 Introduction

Radiative transfer equations have long been used to model the energy density of high frequency waves propagating in highly heterogeneous media. Although they were first derived phenomenologically [6, 11], they can also be obtained as the high frequency limit (as the wavelength tends to zero) of solutions to quantum [8, 20] and classical [1, 11, 18] wave equations; see also the bibliography in the above references. In the references [1, 18], which we closely follow here, the radiative transfer equations model a phase space energy density for the propagating waves. Radiative transfer equations, which are posed in the phase space, are expensive to solve numerically and are thus often replaced by their diffusion approximation. There is a very large literature on this problem; see e.g. [5, 7, 11, 14]. The reference [5] includes the derivation of diffusion equations for radiative transfer equations with spatially varying refractive indices.

Quite a few works have recently concerned the extension of radiative transfer models for the specific intensity (also known as the radiance) of electromagnetic waves to the case of spatially varying refractive indices; see for instance [9, 12, 16, 17, 21]. The specific intensity, rather than the phase space density, is often used in the literature because it is more directly related to what can be measured experimentally. The above references obtain competing radiative transfer equations to model the radiance, and different corresponding diffusion approximations. It is not the objective of this paper

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to compare these different models and exhibit their relative strengths and weaknesses. Among these models however, there is only one that corresponds to the mathematically rigorous limit as the typical wavelength in the system tends to zero. Similarly, only one diffusive model is the mathematically rigorous limit of the radiative transfer equation as the mean free path tends to zero.

The references cited in the preceding paragraph all base their derivation of radiative transfer models with spatially varying refractive indices on geometric optics techniques [13]. Although geometric optics does address the propagation of high frequency waves in heterogeneous media, it does so one (WKB-) mode at a time, and only up to apparition of caustics unless special care is taken; see e.g. [22]. Since multiple scattering of waves quickly creates a superposition of an infinite number of fronts propagating in essentially every direction, geometric optics methods are not very well-adapted to analyzes of wave propagation in highly heterogeneous media. Rather, phase space based methods, such as those developed in [18] and also in [10, 15] are often preferable. The objective of this paper is to consider the radiative transfer models in [18] for the phase space energy density of electromagnetic waves, and to show that the radiance and the phase space energy density are related by a multiplicative factor that depends on the refractive index. This allows us to obtain the radiative transfer equation that the radiance satisfies in the limit of high frequencies. We then use the asymptotic methods developed e.g. in [5, 7] to derive the diffusion approximation for the radiance.

In this paper, we consider evolution equations with prescribed initial conditions. Volume source terms are neglected, although they can easily be incorporated. We also assume that the refractive index is isotropic, i.e., is a scalar quantity (isotropic tensor). Anisotropic tensors would require one to extend to this case the work done in [18]. This will not be considered in this paper. We also assume that the refractive index varies sufficiently smoothly so that its spatial gradient is defined. In the case of very abrupt changes in the refractive index (meaning changes over a fraction of a wavelength), radiative transfer equations need be augmented by interface conditions satisfying Snell's law along each surface where the refractive index jumps. This may be done as in e.g. [2], see also [13], and will not be considered in this paper.

The rest of the paper is structured as follows. Section 2 recalls the radiative transfer equation for the phase space energy density of electromagnetic waves propagating in random media. This equation is used in section 3 to obtain a radiative transfer model for the radiance. The diffusion approximation for both the phase space energy density and the radiance are presented in section 4. Some conclusions are offered in section 5.

2 High frequency limit and radiative transfer

In linearly magnetic and polarizable (dielectric) media and in the absence of volume source term, the Maxwell equations read

$$\begin{aligned} \epsilon_\varepsilon \frac{\partial \mathbf{E}_\varepsilon}{\partial t} &= \nabla \times \mathbf{H}_\varepsilon, \qquad & \nabla \cdot \epsilon_\varepsilon \mathbf{E}_\varepsilon &= 0, \\ \mu_\varepsilon \frac{\partial \mathbf{H}_\varepsilon}{\partial t} &= -\nabla \times \mathbf{E}_\varepsilon, \qquad & \nabla \cdot \mu_\varepsilon \mathbf{H}_\varepsilon &= 0, \end{aligned} \tag{1}$$

where $(\mathbf{E}_\varepsilon, \mathbf{H}_\varepsilon)$ is the electromagnetic field, $\epsilon_\varepsilon(\mathbf{x})$ is the dielectric constant and $\mu_\varepsilon(\mathbf{x})$ is the permeability. This evolution equation need be augmented by initial conditions $(\mathbf{E}_\varepsilon, \mathbf{H}_\varepsilon)(0, \mathbf{x}) = (\mathbf{E}_{0\varepsilon}, \mathbf{H}_{0\varepsilon})(\mathbf{x})$, where the functions $\mathbf{E}_{0\varepsilon}$ and $\mathbf{H}_{0\varepsilon}$ oscillate at frequencies of order ε^{-1}, where for λ the typical wavelength in the system and L the characteristic spatial size at which propagation is observed, $\varepsilon = \lambda/L \ll 1$. This is the high frequency regime, in which both the geometric optics and the radiative transfer models are accurate [11, 13, 18].

Conservation of the electromagnetic energy takes the form:

$$\mathcal{E}_\varepsilon(t) = \frac{1}{2}\int_{\mathbb{R}^3} \big(\epsilon_\varepsilon(\mathbf{x})|\mathbf{E}_\varepsilon|^2(t, \mathbf{x}) + \mu_\varepsilon(\mathbf{x})|\mathbf{H}_\varepsilon|^2(t, \mathbf{x})\big)d\mathbf{x} = \mathcal{E}_\varepsilon(0). \tag{2}$$

The role of kinetic models, such as radiative transfer equations, is to predict the spatial distribution of the energy density

$$\mathcal{E}_\varepsilon(t, \mathbf{x}) = \frac{1}{2}\big(\epsilon_\varepsilon(\mathbf{x})|\mathbf{E}_\varepsilon|^2(t, \mathbf{x}) + \mu_\varepsilon(\mathbf{x})|\mathbf{H}_\varepsilon|^2(t, \mathbf{x})\big). \tag{3}$$

For certain models of random underlying media (characterized by $\epsilon_\varepsilon(\mathbf{x})$ and $\mu_\varepsilon(\mathbf{x})$), this can be done in the high frequency (semiclassical) limit, i.e., as $\varepsilon \to 0$ in the above model, provided that the wave energy density is given a phase space interpretation. More specifically, and following the presentation in [18] and [1], let us assume to simplify that

$$\epsilon_\varepsilon(\mathbf{x}) = \epsilon_0, \qquad \mu_\varepsilon(\mathbf{x}) = \mu_0(\mathbf{x}) + \sqrt{\varepsilon}\mu_1\big(\frac{\mathbf{x}}{\varepsilon}\big). \tag{4}$$

Here $\mu_0(\mathbf{x})$ is the average permeability and $\mu_1(\mathbf{x}/\varepsilon)$ is a homogeneous random field with a correlation length comparable to the typical wavelength in the system. The average light speed and average refractive index given by

$$c(\mathbf{x}) = \frac{1}{\sqrt{\epsilon_0\mu_0(\mathbf{x})}}, \qquad \text{and} \qquad n(\mathbf{x}) = \frac{c}{c(\mathbf{x})}, \tag{5}$$

respectively, are thus spatially varying, where c is light speed in vacuum.

In the above so-called weak coupling regime, it is shown in [18] that in the limit of high frequencies, there exists a two-by-two matrix-valued function

$$\alpha(t, \mathbf{x}, \mathbf{k}) = \frac{1}{2}\begin{pmatrix} I+Q & U+iV \\ U-iV & I-Q \end{pmatrix}(t, \mathbf{x}, \mathbf{k}), \tag{6}$$

where (I, Q, U, V) are the Stokes parameters commonly used in the description of light polarization [6, 11], such that

$$\lim_{\varepsilon\to 0}\mathcal{E}_\varepsilon(t, \mathbf{x}) = \int_{\mathbb{R}^3} \operatorname{Tr}\alpha(t, \mathbf{x}, \mathbf{k})d\mathbf{k} = \int_{\mathbb{R}^3} I(t, \mathbf{x}, \mathbf{k})d\mathbf{k}. \tag{7}$$

Here Tr stands for Trace of the matrix. As a consequence, $I(t, \mathbf{x}, \mathbf{k})$ may be considered as a phase-space electromagnetic energy density. Furthermore, α satisfies the following *radiative transfer equation*:

$$\begin{aligned}&\frac{\partial\alpha}{\partial t} + \{\omega, \alpha\} + N\alpha - \alpha N + \frac{\pi\omega^2(\mathbf{x}, \mathbf{k})}{2(2\pi)^3} \\ &\times \int_{\mathbb{R}^3} \hat{R}(\mathbf{k}-\mathbf{q})T(\mathbf{k}, \mathbf{q})\big(\alpha(\mathbf{k}) - \alpha(\mathbf{q})\big)T(\mathbf{q}, \mathbf{k})\delta\big(\omega(\mathbf{x}, \mathbf{k}) - \omega(\mathbf{x}, \mathbf{q})\big)d\mathbf{q} = 0,\end{aligned} \tag{8}$$

with appropriate initial conditions (initial radiation source). The parameters appearing in the above equation are defined as follows. The dispersion relation is characterized by the following Hamiltonian

$$\omega(\mathbf{x},\mathbf{k}) = c(\mathbf{x})|\mathbf{k}|. \tag{9}$$

The propagation of the energy along the bicharacteristics associated to the above Hamiltonian [10, 13] is modeled by the following Poisson bracket

$$\begin{aligned} \{\omega,\alpha\}(\mathbf{x},\mathbf{k}) &= \nabla_{\mathbf{k}}\omega(\mathbf{x},\mathbf{k})\cdot\nabla_{\mathbf{x}}\alpha(\mathbf{x},\mathbf{k}) - \nabla_{\mathbf{x}}\omega(\mathbf{x},\mathbf{k})\cdot\nabla_{\mathbf{k}}\alpha(\mathbf{x},\mathbf{k}) \\ &= c(\mathbf{x})\hat{\mathbf{k}}\cdot\nabla_{\mathbf{x}}\alpha(\mathbf{x},\mathbf{k}) - |\mathbf{k}|\nabla c(\mathbf{x})\cdot\nabla_{\mathbf{k}}\alpha(\mathbf{x},\mathbf{k}), \qquad \hat{\mathbf{k}} = \frac{\mathbf{k}}{|\mathbf{k}|}. \end{aligned} \tag{10}$$

The effects of rotation of the polarization components as energy propagates along the bicharacteristics are characterized by the 2×2 skew-symmetric matrix $N = -N^t$, whose expression can be found in [1, 6, 18]. Finally, scattering caused by interaction of the propagating waves with the underlying heterogeneities in the permeability μ_1 is modeled by the last term on the left-hand side in (8). In this scattering operator, $\hat{R}(\mathbf{p})$ is the power spectrum of the fluctuations μ_1, and T is a 2×2 symmetric matrix that accounts for polarization rotation through scattering; see [1, 18]. Note that scattering is elastic thanks to the term $\delta\big(\omega(\mathbf{x},\mathbf{k}) - \omega(\mathbf{x},\mathbf{q})\big)$. When energy scatters, it is allowed to change direction but cannot change frequency $\omega(\mathbf{x},\mathbf{k}) = \omega$. Since the other operators in (8) preserve the Hamiltonian $\omega(\mathbf{x},\mathbf{k})$, we verify that the solutions $\alpha(t,\mathbf{x},\mathbf{k})$ at different values of $\omega = \omega(\mathbf{x},\mathbf{k})$ satisfy uncoupled equations.

The radiative transfer equation (8) fully characterizes energy propagation in the high frequency limit. It may be generalized to fluctuations in the dielectric constant $\epsilon_\varepsilon(\mathbf{x})$ [18] and to point scatterers [11] with a different expression for the scattering operator. If polarization effects are neglected by setting $U = V = Q \equiv 0$ in (6), a scalar radiative transfer equation can be derived for the phase-space energy density $I(t,\mathbf{x},\mathbf{k})$. If absorption effects, which were neglected in the derivation of (8), are included in the model, the scalar radiative transfer equation takes the following form:

$$\frac{\partial I}{\partial t} + \{\omega, I\} + \Sigma(\mathbf{x},\mathbf{k})I = \int_{\mathbb{R}^3} \sigma(\mathbf{x},\mathbf{k},\mathbf{q})I(t,\mathbf{x},\mathbf{q})\delta\big(\omega(\mathbf{x},\mathbf{k}) - \omega(\mathbf{x},\mathbf{q})\big)d\mathbf{q}, \tag{11}$$

where $\sigma(\mathbf{x},\mathbf{k},\mathbf{q})$ is the scattering function and $\Sigma(\mathbf{x},\mathbf{k}) = \Sigma_a(\mathbf{x}) + \Sigma_s(\mathbf{x},\mathbf{k})$ is the total extinction coefficient. The intrinsic attenuation is modeled by the coefficient $\Sigma_a(\mathbf{x})$, whereas the scattering coefficient $\Sigma_s(\mathbf{x},\mathbf{k})$ is defined by

$$\Sigma_s(\mathbf{x},\mathbf{k}) = \int_{\mathbb{R}^3} \sigma(\mathbf{x},\mathbf{k},\mathbf{q})\delta\big(\omega(\mathbf{x},\mathbf{k}) - \omega(\mathbf{x},\mathbf{q})\big)d\mathbf{q}. \tag{12}$$

Provided that the initial conditions $I(0,\mathbf{x},\mathbf{k})$ are prescribed, the evolution equation (11) uniquely characterizes the phase-space intensity $I(t,\mathbf{x},\mathbf{k})$ [7]. Any additional volume source term may be added to the right-hand side in (11). Note that the radiative transfer equation accounts for changes in the refractive index, or equivalently in the light speed, so long as the latter is sufficiently smooth so that its gradient is defined.

3 Radiance versus phase space energy density

The above equation (11) fully characterizes photon propagation in random media such as e.g. human tissues, as it is used in optical tomography. Yet, the radiative transfer equations mostly encountered in the literature to model photon propagation do not involve the phase space energy density introduced above but rather the *specific intensity*, also known as *radiance* $L(t, \mathbf{x}, \mathbf{\Omega}, \omega)$, where $\mathbf{\Omega} \in S^2$ is a unit vector and ω is a frequency.

In the SI system of units, the energy density $\mathcal{E}_\varepsilon$ has units of $J\,m^{-3}$, or Joules per cubic meter. Upon using the change of variables $d\mathbf{k} = |\mathbf{k}|^2 d|\mathbf{k}| d\hat{\mathbf{k}}$ in polar coordinates, we obtain that $I d\mathbf{k}$ is an energy density so that $I(t, \mathbf{x}, \mathbf{k})$ has units of $J\,sr^{-1}$, or Joules per steradian, the unit solid angle. On the other hand, the specific intensity has units of $J\,m^{-2}\,sr^{-1}$, or equivalently $W\,m^{-2}\,sr^{-1}\,Hz^{-1}$, where W is the units of Watts and Hz the unit of Hertz (frequency) [11]. It is defined so that the following conservation holds

$$\int_{S^2}\int_0^\infty \frac{1}{c(\mathbf{x})} L(t, \mathbf{x}, \mathbf{\Omega}, \omega) d\omega d\mathbf{\Omega} = \lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(t, \mathbf{x}) = \int_{\mathbb{R}^3} I(t, \mathbf{x}, \mathbf{k}) d\mathbf{k}. \tag{13}$$

Now the equation for $I(t, \mathbf{x}, \mathbf{k})$ can be solved at $\omega = \omega(\mathbf{x}, \mathbf{k}) = c_0(\mathbf{x})|\mathbf{k}|$ fixed as we observed earlier. We can thus perform the following change of variables:

$$\int_{\mathbb{R}^3} I(t, \mathbf{x}, \mathbf{k}) d\mathbf{k} = \int_{\mathbb{R}^3} I(t, \mathbf{x}, \mathbf{k}) |\mathbf{k}|^2 d|\mathbf{k}| d\hat{\mathbf{k}} = \int_{S^2}\int_0^\infty I(t, \mathbf{x}, \frac{\omega}{c(\mathbf{x})}\mathbf{\Omega}) \frac{\omega^2}{c^3(\mathbf{x})} d\omega d\mathbf{\Omega}. \tag{14}$$

Since both I and L have to represent the same physics in the presence of a geometric optics front for instance, we deduce that

$$\frac{\omega^2}{c^2(\mathbf{x})} I(t, \mathbf{x}, \frac{\omega}{c(\mathbf{x})}\mathbf{\Omega}) = L(t, \mathbf{x}, \mathbf{\Omega}, \omega). \tag{15}$$

This holds for all $\mathbf{x} \in \mathbb{R}^3$, $\mathbf{\Omega} \in S^2$, and $\omega \in \mathbb{R}^+$, i.e., in the whole phase space. This identification now allows us to derive an equation for the specific intensity L from (11). In the absence of spatial variations in the local light speed, both terms I and L are separated by a multiplicative constant and thus solve the same equation. However when $c(\mathbf{x})$ is not constant, the Poisson bracket (10) applied to L will differ from that applied to I.

Let us first introduce the quantity

$$\mathcal{I}(t, \mathbf{x}, \mathbf{\Omega}, \omega) = I(t, \mathbf{x}, \frac{\omega}{c(\mathbf{x})}\mathbf{\Omega}). \tag{16}$$

This corresponds to the change of variables $(\mathbf{x}, \mathbf{k}) \to (\mathbf{x}, \omega = c(\mathbf{x})|\mathbf{k}|, \mathbf{\Omega} = \mathbf{k}/|\mathbf{k}|)$. We thus find that

$$\nabla_{\mathbf{x}} \to \nabla_{\mathbf{x}} + \frac{\nabla c}{c}\omega\frac{\partial}{\partial\omega}, \qquad \nabla_{\mathbf{k}} \to c(\mathbf{x})\mathbf{\Omega}\frac{\partial}{\partial\omega} + \frac{c(\mathbf{x})}{\omega}(I_3 - \mathbf{\Omega}\otimes\mathbf{\Omega})\nabla_{\mathbf{\Omega}}.$$

Here, I_3 is the 3×3 identity matrix and $\big(I_3 - \mathbf{\Omega}\otimes\mathbf{\Omega}\big)\nabla_{\mathbf{\Omega}}$ is the projection of the usual gradient $\nabla_{\mathbf{\Omega}}$ defined in $\mathbb{R}^3$ onto the unit sphere S^2. We have thus found that

$$\{\omega, I\}(t, \mathbf{x}, \mathbf{k}) \to \Big(c(\mathbf{x})\mathbf{\Omega}\cdot\nabla_{\mathbf{x}}\mathcal{I} - \nabla c(\mathbf{x})\big(I_3 - \mathbf{\Omega}\otimes\mathbf{\Omega}\big)\nabla_{\mathbf{\Omega}}\mathcal{I}\Big)(t, \mathbf{x}, \mathbf{\Omega}, \omega).$$

Because $\omega(\mathbf{x},\mathbf{k})$ is invariant in the transport equation (11), the modified phase-space intensity $\mathcal{I}(t,\mathbf{x},\mathbf{\Omega},\omega)$ satisfies uncoupled equations for different values of ω. The change of variables in the scattering kernel in (11) is then straightforward and we obtain the equation

$$\begin{aligned}\frac{\partial \mathcal{I}}{\partial t} &+ \Big(c(\mathbf{x})\mathbf{\Omega}\cdot\nabla_{\mathbf{x}}\mathcal{I} - \nabla c(\mathbf{x})\cdot\big(I_3 - \mathbf{\Omega}\otimes\mathbf{\Omega}\big)\nabla_{\mathbf{\Omega}}\mathcal{I}\Big) + \tilde{\Sigma}(\mathbf{x},\mathbf{\Omega},\omega)\mathcal{I} \\ &= \int_{S^2} \tilde{\sigma}(\mathbf{x},\mathbf{\Omega},\mathbf{\Omega}',\omega)\mathcal{I}(t,\mathbf{x},\mathbf{\Omega}',\omega)d\mathbf{\Omega}'.\end{aligned} \tag{17}$$

Here, we have defined the coefficients

$$\tilde{\Sigma}(\mathbf{x},\mathbf{\Omega},\omega) = \Sigma(\mathbf{x},\frac{\omega\mathbf{\Omega}}{c(\mathbf{x})}), \qquad \tilde{\sigma}(\mathbf{x},\mathbf{\Omega},\mathbf{\Omega}',\omega) = \sigma(\mathbf{x},\frac{\omega}{c(\mathbf{x})}\mathbf{\Omega},\frac{\omega}{c(\mathbf{x})}\mathbf{\Omega}'). \tag{18}$$

The equation for $\mathcal{I}$ still preserves the same structure as (11): it follows the bicharacteristics of the Hamiltonian $\omega(\mathbf{x},\mathbf{k})$, where now $\mathbf{\Omega} = \hat{\mathbf{k}}$, between interactions with the underlying structure, which are either absorption or scattering events modeled by the extinction and scattering coefficient $\tilde{\Sigma}$ and $\tilde{\sigma}$.

It is now straightforward to obtain an equation for the specific intensity

$$L(t,\mathbf{x},\mathbf{\Omega},\omega) = \frac{\omega^2}{c^2(\mathbf{x})}\mathcal{I}(t,\mathbf{x},\mathbf{\Omega},\omega). \tag{19}$$

Since $\omega^2/(c^2(\mathbf{x}))$ is preserved by the scattering kernel, only the propagation along the bicharacteristics is affected. Upon performing the calculation

$$\frac{1}{c^3(\mathbf{x})}c(\mathbf{x})\mathbf{\Omega}\cdot\nabla c^2(\mathbf{x}) = 2\frac{\mathbf{\Omega}\cdot\nabla c(\mathbf{x})}{c(\mathbf{x})},$$

we deduce that the radiance solves the following equation

$$\begin{aligned}\frac{1}{c(\mathbf{x})}\frac{\partial L}{\partial t} &+ \Big(\mathbf{\Omega}\cdot\nabla_{\mathbf{x}}L - \frac{\nabla c(\mathbf{x})}{c(\mathbf{x})}\cdot\big(I_3 - \mathbf{\Omega}\otimes\mathbf{\Omega}\big)\nabla_{\mathbf{\Omega}}L\Big) + 2\frac{\mathbf{\Omega}\cdot\nabla c(\mathbf{x})}{c(\mathbf{x})}L \\ &= -\frac{\tilde{\Sigma}(\mathbf{x},\omega)}{c(\mathbf{x})}L + \int_{S^2}\frac{\tilde{\sigma}(\mathbf{x},\mathbf{\Omega},\mathbf{\Omega}',\omega)}{c(\mathbf{x})}L(t,\mathbf{x},\mathbf{\Omega}',\omega)d\mathbf{\Omega}'.\end{aligned} \tag{20}$$

In optical tomography, it is customary to use the notation

$$\mu_a(\mathbf{x}) = \frac{\Sigma_a(\mathbf{x})}{c(\mathbf{x})}, \qquad \mu_s(\mathbf{x},\omega)\theta(\mathbf{x},\mathbf{\Omega}\cdot\mathbf{\Omega}') = \frac{\tilde{\sigma}(\mathbf{x},\mathbf{\Omega},\mathbf{\Omega}',\omega)}{c(\mathbf{x})}, \tag{21}$$

where θ is a normalized scattering function (averaging to 1 over S^2), and to write the radiative transfer in terms of the refractive index introduced in (5). Noting that $n(\mathbf{x})^{-1}\nabla n(\mathbf{x}) = -c(\mathbf{x})^{-1}\nabla c(\mathbf{x})$, we deduce that the specific intensity takes with the above notation the following form

$$\begin{aligned}\frac{n(\mathbf{x})}{c}\frac{\partial L}{\partial t} &+ \mathbf{\Omega}\cdot\nabla_{\mathbf{x}}L + \frac{\nabla n(\mathbf{x})}{n(\mathbf{x})}\cdot\big(I_3 - \mathbf{\Omega}\otimes\mathbf{\Omega}\big)\nabla_{\mathbf{\Omega}}L - 2\frac{\mathbf{\Omega}\cdot\nabla n(\mathbf{x})}{n(\mathbf{x})}L \\ &= -(\mu_s(\mathbf{x},\omega) + \mu_a(\mathbf{x}))L + \mu_s(\mathbf{x},\omega)\int_{S^2}\theta(\mathbf{x},\mathbf{\Omega}\cdot\mathbf{\Omega}')L(t,\mathbf{x},\mathbf{\Omega}',\omega)d\mathbf{\Omega}'.\end{aligned} \tag{22}$$

Note that the radiance $L(t, \mathbf{x}, \mathbf{\Omega}, \omega)$ is not conserved along the bicharacteristics of the Hamiltonian $\omega(\mathbf{x}, \mathbf{k})$, although of course we have the global conservation in (13).

Let us conclude this section by a few remarks. Equation (22), with the above definition of the radiance, is the only model compatible with the semiclassical limit of high frequency waves, which is the regime of geometric optics and of radiative transfer. In the absence of randomness (i.e., $\sigma \equiv \Sigma \equiv 0$ or $\mu_a \equiv \mu_s \equiv 0$), techniques developed in e.g. [10, 15, 18] allow one to rigorously derive the Liouville equations (11) and (22). The Liouville equation may also be derived from geometric optics expansions, as was done for instance in [21]. However, the use of Wigner transforms as in [1, 10, 15, 18] significantly simplifies the derivation as it dispenses us from following geometric fronts. Moreover, since the radiative transfer solution involves an infinite number of fronts (energy radiates in every direction), the geometric optics formalism is not well-adapted to the derivation of radiative transfer models.

Because radiance is not conserved along the bicharacteristics of the Hamiltonian, it is more difficult to solve (22) than (17). It thus seems logical to propose that solving (17) numerically, for instance by a Monte Carlo method [4, 19], and using the change of variables (19) will yield better algorithms than methods based directly on (22). Note that the change of variables (15) could also be used for the matrix $\alpha(t, \mathbf{x}, \mathbf{k})$ that accounts for polarization effects. Thus using for instance the method developed in [3, 4], Monte Carlo methods can be used to solve radiative transfer equations for the vector-valued specific intensities (with components the appropriately normalized Stokes parameters); see [11, 18].

4 Diffusion limit

The diffusion approximation of transport equations has also long been analyzed in the mathematical and physical literatures. The diffusion approximation is typically derived in two ways; either as the $P1$ method, a truncation of the radiative transfer solution over the first $d+1$ spherical harmonics (in d space dimensions) in the angular variable, see e.g. [11]; or as an asymptotic expansion in the limit of vanishing mean free path, see [7, 14]. Although most derivations deal with the case of a constant refractive index, spatially varying indices have also been considered, see for instance [5] using asymptotic expansions, and more recently [12, 21] with the $P1$ method.

One advantage of the asymptotic method is that it provides an error estimate for the difference between the transport and diffusion solutions, at least in the idealized case of infinite media with no boundary. The treatment of boundaries in the diffusion approximation has also been extensively studied; see for instance [5, 7]. The analysis presented in these references is not modified by the spatial variations of the refractive index so we do not consider the difficulty further here.

The diffusion approximation holds when the scattering coefficient is large, the absorption coefficient is small, and the time scale is sufficiently large so that the diffusive regime sets in. This corresponds to replacing, in (22), μ_s by μ_s/η, μ_a by $\eta\mu_a$, and t by t/η, where $\eta \ll 1$. The (elastic) mean free path satisfies then:

$$l(\mathbf{x}, |\mathbf{k}|) = \frac{\eta}{\mu_t(\mathbf{x}, \omega)} \ll L, \qquad \mu_t(\mathbf{x}, \omega) = \mu_s(\mathbf{x}, \omega) + \eta^2 \mu_a(\mathbf{x}), \tag{23}$$

where $\mu_t(\mathbf{x},\omega)$ is the extinction coefficient and L a typical distance over which the specific intensity propagates. The limit as $\eta \to 0$, or equivalently $l/L \to 0$, is analyzed as follows. The phase space density $\mathcal{I}_\eta(t,\mathbf{x},\mathbf{\Omega},\omega)$ satisfies the following rescaled equation

$$\begin{aligned}\frac{\eta}{c_0(\mathbf{x})}\frac{\partial \mathcal{I}_\eta}{\partial t} + \Big(\mathbf{\Omega}\cdot\nabla_{\mathbf{x}}\mathcal{I}_\eta \; - \frac{\nabla c(\mathbf{x})}{c(\mathbf{x})}\cdot\big(I_3 - \mathbf{\Omega}\otimes\mathbf{\Omega}\big)\nabla_{\mathbf{\Omega}}\mathcal{I}_\eta\Big) + \eta\mu_a(\mathbf{x})\mathcal{I}_\eta \\ = \frac{\mu_s(\mathbf{x},\omega)}{\eta}\int_{S^2}\theta(\mathbf{x},\mathbf{\Omega}\cdot\mathbf{\Omega}')\Big(\mathcal{I}_\eta(t,\mathbf{x},\mathbf{\Omega}',\omega) - \mathcal{I}_\eta(t,\mathbf{x},\mathbf{\Omega},\omega)\Big)d\mathbf{\Omega}',\end{aligned} \tag{24}$$

with initial conditions $\mathcal{I}_\eta(0,\mathbf{x},\mathbf{\Omega},\omega) = \mathcal{I}_{\text{in}}(\mathbf{x},\omega)$ independent of $\mathbf{\Omega}$ to simplify (otherwise initial layers need be accounted for [7]).

Then plugging the asymptotic expansion $\mathcal{I}_\eta = \mathcal{I}_0 + \eta\mathcal{I}_1 + \eta^2\mathcal{I}_2$ into (24) and equating like powers of η yields at the order η^{-1} that $\mathcal{I}_0(t,\mathbf{x},\omega)$ is independent of $\mathbf{\Omega}$. The next order equation provides that

$$\mathbf{\Omega}\cdot\nabla_{\mathbf{x}}\mathcal{I}_0 = \mu_s(\mathbf{x},\omega)\int_{S^2}\theta(\mathbf{x},\mathbf{\Omega}\cdot\mathbf{\Omega}')\Big(\mathcal{I}_1(t,\mathbf{x},\mathbf{\Omega}',\omega) - \mathcal{I}_1(t,\mathbf{x},\mathbf{\Omega},\omega)\Big)d\mathbf{\Omega}', \tag{25}$$

because $\big(I_3 - \mathbf{\Omega}\otimes\mathbf{\Omega}\big)\nabla_{\mathbf{\Omega}}\mathcal{I}_0 = 0$. Let us define the anisotropy factor $\lambda_1(\mathbf{x})$ as the second eigenvalue of the scattering operator [7, 11, 18]:

$$\lambda_1(\mathbf{x})\mathbf{\Omega} = \int_{S^2}\theta(\mathbf{x},\mathbf{\Omega}\cdot\mathbf{\Omega}')\mathbf{\Omega}'d\mathbf{\Omega}'. \tag{26}$$

Then the unique solution to (25) orthogonal to $\mathbf{\Omega}-$independent functions is given by [7]

$$\mathcal{I}_1(t,\mathbf{x},\mathbf{\Omega},\omega) = \frac{-1}{\mu_s(\mathbf{x},\omega)(1-\lambda_1(\mathbf{x}))}\mathbf{\Omega}\cdot\nabla\mathcal{I}_0(t,\mathbf{x},\omega). \tag{27}$$

Up to a negligible term of order $O(\eta^2)$, we thus observe that $\mathcal{I}_1(t,\mathbf{x},\mathbf{\Omega},\omega) = -l^*(\mathbf{x},\omega)\mathbf{\Omega}\cdot\nabla\mathcal{I}_0(t,\mathbf{x},\omega)$, where the transport mean free path is defined by

$$l^*(\mathbf{x},\omega) = \frac{1}{\mu_t(\mathbf{x},\omega)(1-\lambda_1(\mathbf{x}))} = \frac{l(\mathbf{x},\omega)}{1-\lambda_1(\mathbf{x})}. \tag{28}$$

Finally, the average over S^2 of the equation of order η^1 yields:

$$\frac{1}{c(\mathbf{x})}\frac{\partial \mathcal{I}_0}{\partial t} + \int_{S^2}\Big(\mathbf{\Omega}\cdot\nabla_{\mathbf{x}}\mathcal{I}_1 \; - \frac{\nabla c(\mathbf{x})}{c(\mathbf{x})}\cdot\big(I_3 - \mathbf{\Omega}\otimes\mathbf{\Omega}\big)\nabla_{\mathbf{\Omega}}\mathcal{I}_1\Big)\frac{d\mathbf{\Omega}}{4\pi} + \mu_a(\mathbf{x})\mathcal{I}_0 = 0. \tag{29}$$

We then calculate that

$$\int_{S^2}\mathbf{\Omega}\cdot\nabla l^*(\mathbf{x},\omega)\mathbf{\Omega}\cdot\nabla\frac{d\mathbf{\Omega}}{4\pi} = \nabla\cdot\frac{l^*(\mathbf{x},\omega)}{3}\nabla = \nabla\cdot D(\mathbf{x},\omega)\nabla,$$

where we have defined the diffusion coefficient

$$D(\mathbf{x},\omega) = \frac{l^*(\mathbf{x},\omega)}{3} = \frac{1}{3\mu_t(\mathbf{x},\omega)(1-\lambda_1(\mathbf{x}))}. \tag{30}$$

Some algebra shows that

$$\int_{S^2}\big(I_3 - \mathbf{\Omega}\otimes\mathbf{\Omega}\big)\nabla_{\mathbf{\Omega}}\mathbf{\Omega}\frac{d\mathbf{\Omega}}{4\pi} = \int_{S^2}\big(I_3 - \mathbf{\Omega}\otimes\mathbf{\Omega}\big)\frac{d\mathbf{\Omega}}{4\pi} = I_3 - \frac{1}{3}I_3 = \frac{2}{3}I_3. \tag{31}$$

Finally, this yields the diffusion equation for $\mathcal{I}_0$:

$$\frac{1}{c(\mathbf{x})}\frac{\partial \mathcal{I}_0}{\partial t} - \nabla \cdot D(\mathbf{x},\omega)\nabla \mathcal{I}_0 - 2D(\mathbf{x})\frac{\nabla c(\mathbf{x})}{c(\mathbf{x})} \cdot \nabla \mathcal{I}_0 + \mu_a(\mathbf{x})\mathcal{I}_0 = 0, \tag{32}$$

with initial conditions $\mathcal{I}_0(0,\mathbf{x},\omega) = \mathcal{I}_{\rm in}(\mathbf{x},\omega)$. The equation may be recast in divergence form as

$$\frac{1}{c^3(\mathbf{x})}\frac{\partial \mathcal{I}_0}{\partial t} - \nabla \cdot \Big(\frac{D(\mathbf{x})}{c^2(\mathbf{x})}\nabla \mathcal{I}_0\Big) + \frac{\mu_a(\mathbf{x})}{c^2(\mathbf{x})}\mathcal{I}_0 = 0. \tag{33}$$

This shows that $\mathcal{I}_0 c^{-3}(\mathbf{x})$ is conserved when $\mu_a = 0$, which is consistent with the conservation law (13) and (14). As in e.g. [7], we can show that

$$\mathcal{I}_\eta = \mathcal{I}_0 - \eta l^*(\mathbf{x},\omega)\mathbf{\Omega} \cdot \nabla \mathcal{I}_0 + O(\eta^2), \tag{34}$$

which provides the degree of accuracy of the approximation when the mean free path is small. Here, $O(\eta^2)$ holds in any $L^p(\mathbb{R}^3 \times S^2)$ norm, $1 \le p \le \infty$, with proper assumptions.

As in the derivation of the radiative transfer equation, we can now use the relation (19) to obtain the approximation

$$L_\eta = L_0 - \eta\frac{l^*(\mathbf{x},\omega)}{c^2(\mathbf{x})}\mathbf{\Omega} \cdot \nabla[c^2(\mathbf{x})L_0] + O(\eta^2), \tag{35}$$

where L_0 satisfies the following diffusion equation in conservative form:

$$\frac{1}{c_0(\mathbf{x})}\frac{\partial L_0}{\partial t} - \nabla \cdot \Big(\frac{D(\mathbf{x},\omega)}{c^2(\mathbf{x})}\nabla[c^2(\mathbf{x})L_0]\Big) + \mu_a(\mathbf{x})L_0 = 0, \tag{36}$$

with appropriate initial conditions. We find that $L_0/c_0(\mathbf{x})$ is a conserved quantity when $\mu_a = 0$, which is consistent with the conservation relation (13). The above diffusion equation is consistent with that derived in [21].

5 Conclusions

We have recalled some results available in the literature on the derivation of radiative transfer equation for the phase space energy density of electromagnetic waves propagating in random media. Such radiative transfer equations account for spatial variations in the macroscopic light speed, or equivalently the macroscopic refractive index. We have then shown how these results may be readily adapted to the derivation of radiative transfer equations for the radiance by using the relationship (15) between the latter and the phase space energy density. Because the phase space energy density is conserved along the bicharacteristics of the Hamiltonian that describes high frequency waves propagation, it is easier to solve the radiative transfer equation for the phase space energy density (11) than that for the radiance (22).

A classical asymptotic expansion was used to derive the diffusion approximation of the radiative transfer equation for the phase space energy density. The same relationship (15) as above then allowed us to obtain a diffusion equation for the radiance in (36). Note that at the diffusion level, the diffusion equations for the energy density (33) and for the radiance (36) seem to be of equivalent computational complexity.

The models derived here may not be the most appropriate in specific physical settings. However they are the only models providing the correct limits as first the wavelength, and second the mean free path, converge to zero.

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