# Kinetics of scalar wave fields in random media

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**Abstract**

This paper concerns the derivation of kinetic models for high frequency scalar wave fields propagating in random media. The kinetic equations model the propagation in the phase space of the energy density of a wave field or the correlation function of two wave fields propagating in two possibly different media. Dispersive effects due to e.g. spatial and temporal discretizations, which are modeled as non-local pseudo-differential operators, are taken into account. The derivation of the models is based on a multiple scale asymptotic expansion of the spatio-temporal Wigner transform of two scalar wave fields.

*Key words:* High frequency wave equations, weak-coupling regime, radiative transfer, random media, spatio-temporal Wigner transform, discretized equations.

## 1 Introduction

High frequency wave propagation in highly heterogeneous media has long been modeled by radiative transfer equations in many fields: quantum waves in semiconductors, electromagnetic waves in turbulent atmospheres and plasmas, underwater acoustic waves, elastic waves in the Earth's crust. These kinetic models account for the wave energy transport in the phase space, i.e., in the space of positions and momenta. Their derivation may be either phenomenological or based on first principles for the wave field; see e.g. [9,11,16,17,19,25,26,28]. Such kinetic models account for the multiple interactions of wave fields with the fluctuations of the underlying medium. We consider here the so-called *weak-coupling* limit, whereby waves propagate over distances that are large compared to the typical wavelength in the system and the fluctuations have weak amplitude and correlation length comparable to the wavelength. A systematic method to derive kinetic equations from symmetric first-order hyperbolic systems, including systems of acoustics and elastic equations, in the weak-coupling limit has been presented in

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[24] and extended in various forms in [2–4,7,15,22]. In these papers, the energy density of waves is captured by the spatial *Wigner transform* (originally introduced by E.Wigner in [29]) of the wave field, which may be seen as the Fourier transform of the properly scaled two-point correlation function of the wave field. The method is based on formal multiple-scale asymptotic expansions of the Wigner transform and extends to fairly general equations the kinetic models rigorously derived in [13,27] for the Schrödinger equation.

This paper generalizes the derivation of kinetic equations to other wave phenomena. The main tool in the derivation is the asymptotic expansion of the *spatio-temporal Wigner transform* of the wave fields, which is defined in the phase space of the spatio-temporal domain, i.e., is a function of the primal variables $(t, \mathbf{x})$ and dual variables $(\omega, \mathbf{k})$. The additional variable $\omega$, compared to the spatial Wigner transform used in [24], allows for a more refined description of the dispersion relation between the temporal and spatial oscillations. This allows us to handle second-order scalar wave equations and dispersive effects such as e.g. those resulting from temporal and spatial discretizations of the wave equation, which cannot be addressed by the spatial Wigner transform. Although more general wave propagation models may be considered, we focus in this paper on non-symmetric two-by-two first-order systems and on scalar wave equations for concreteness. A practical application of the theory is to quantify the effect of spatial and temporal discretizations of the wave equation on the constitutive parameters in the kinetic model; see [12] for such effects in slowly varying media. As in [20,21], the spatial and temporal discretizations are modeled here by non-local pseudo-differential operators.

Whereas kinetic models often model the energy density of wave fields, more generally they model the correlation function of two wave fields possibly propagating in different media. Applications of the analysis of such correlations in imaging include probing in time the random fluctuations of a heterogeneous media. We also find a direct application in the analysis of the time reversal of waves in changing environment; see e.g. [6,8]. In this paper, we carefully derive the generalized transport equations resulting from the analysis of such correlations.

We do not consider the diffusion approximation of kinetic models in this paper and refer to e.g. [7,8,24] for extensions to that regime of the results presented here. We do not consider either kinetic models for waves that may have polarization effects, such as electromagnetic and elastic waves [24], except for a special case of electromagnetic waves, which can be modeled with scalar equations, in section 8. Although the methodology presented here extends to these vector-valued kinetic equations, we only consider *scalar* kinetic models in this paper.

The rest of the paper is structured as follows. Section 2 recalls the objective of kinetic models in the framework of acoustic wave propagation. The spatial Wigner transform is introduced in section 3. The pseudo-differential calculus used to derive the kinetic models is also recalled. Section 4 presents the derivation of a kinetic equation for the spatial Wigner transform associated to two fields solving two-by-two first-order systems of acoustic equations. Following [24], the multiple-scale asymptotic expansion of the Wigner transform is carefully revisited. The spatio-temporal Wigner transform is introduced in section 5. The analysis of section 4 is then extended in section 6 to the case of a dispersive system, which accounts e.g. for discretizations in time. Kinetic models for scalar equations, such as e.g. the second-order scalar wave equation, are derived in section 7. Fairly general discretizations in space and time, and arbitrary fluctuations in the density and compressibility parameters are allowed. Section 8 applies the theory developed in section 7 to several classical equations: acoustics, Schrödinger, Klein Gordon, and the equations of electromagnetism in a simplified setting. Some conclusions are offered in section 9.

## 2 Acoustic Waves

**Models of wave propagation.** The linear system of acoustic wave equations for the pressure $p(t,\mathbf{x})$ and the velocity field $\mathbf{v}(t,\mathbf{x})$ takes the form

$$\rho(\mathbf{x})\frac{\partial \mathbf{v}}{\partial t} + \nabla p = 0, \qquad \kappa(\mathbf{x})\frac{\partial p}{\partial t} + \nabla\cdot\mathbf{v} = 0, \qquad p(0,\mathbf{x}) = p_0(\mathbf{x}), \quad \mathbf{v}(0,\mathbf{x}) = \mathbf{v}_0(\mathbf{x}), \tag{2.1}$$

where $\rho(\mathbf{x})$ is density and $\kappa(\mathbf{x})$ compressibility of the underlying media. Here and below $t>0$ and $\mathbf{x}\in\mathbb{R}^d$, where $d\geq 2$ is spatial dimension. The velocity field is a $d-$dimensional vector. We thus have a $d+1$ dimensional system for the vector $\mathbf{u} = (\mathbf{v},p)$, which may be recast as $\frac{\partial \mathbf{u}}{\partial t} + B(\mathbf{x},\mathbf{D_x})\mathbf{u} = 0$, where $B(\mathbf{x},\mathbf{D_x})$ is a matrix-valued differential operator in the spatial variables. One verifies that the following energy is conserved:

$$\mathcal{E}_B(t) = \frac{1}{2}\int_{\mathbb{R}^d}\Big(\rho(\mathbf{x})|\mathbf{v}|^2(t,\mathbf{x}) + \kappa(\mathbf{x})p^2(t,\mathbf{x})\Big)d\mathbf{x} = \mathcal{E}_B(0). \tag{2.2}$$

First-order hyperbolic systems such as (2.1) are the starting point of the kinetic theories developed in [14,24] for acoustic and other waves.

The pressure $p(t,\mathbf{x})$ is known to satisfy the following closed form scalar equation

$$\frac{\partial^2 p}{\partial t^2} = \kappa^{-1}(\mathbf{x})\nabla\cdot\rho^{-1}(\mathbf{x})\nabla p, \tag{2.3}$$

with appropriate (non-vortical) initial conditions. We also have the conservation law

$$\mathcal{E}_H(t) = \frac{1}{2}\int_{\mathbb{R}^d}\Big(\kappa(\mathbf{x})\Big(\frac{\partial p}{\partial t}\Big)^2(t,\mathbf{x}) + \frac{|\nabla p|^2(t,\mathbf{x})}{\rho(\mathbf{x})}\Big)d\mathbf{x} = \mathcal{E}_H(0). \tag{2.4}$$

The conservation laws (2.2) and (2.4) are equivalent. Indeed let the potential $\phi(t,\mathbf{x})$ be a solution of (2.3) and define $\mathbf{v} = -\rho^{-1}\nabla\phi$ and $p = \partial_t\phi$. We then verify that $(\mathbf{v},p)$ solves (2.1) and that $\mathcal{E}_H[\phi](t) = \mathcal{E}_B[\mathbf{v},p](t)$. It is thus natural, as we show in section 7 of this paper, that the kinetic models for the energy distribution in the phase space associated to the conservations (2.2) and (2.4) agree.

A kinetic model for the scalar wave equation (2.3) is derived in section 7. In sections 4 and 6, we assume that $\rho = \rho_0$ is constant and recast (2.3) as $\frac{\partial^2 p}{\partial t^2} = c^2(\mathbf{x})\Delta p$, where $\Delta$ is the usual Laplacian and the sound speed $c(\mathbf{x}) = (\kappa(\mathbf{x})\rho_0)^{-1/2}$. Let us define

$$q(t,\mathbf{x}) = c^{-2}(\mathbf{x})\frac{\partial p}{\partial t}(t,\mathbf{x}). \tag{2.5}$$

We verify that $\mathbf{u} = (p, q)$ satisfies the following $2 \times 2$ system of equations

$$\frac{\partial \mathbf{u}}{\partial t} + A\mathbf{u} = 0, \qquad A = -\begin{pmatrix} 0 & c^2(\mathbf{x}) \\ \Delta & 0 \end{pmatrix}, \tag{2.6}$$

with appropriate initial conditions. The above system separates the interactions of the waves with the underlying medium fluctuations from the propagation in a homogeneous medium. This somewhat simplifies the derivation of the kinetic model.

**High frequency limit.** Kinetic models arise in the high frequency limit of wave propagation. We thus rescale $t \to \varepsilon^{-1}t$ and $\mathbf{x} \to \varepsilon^{-1}\mathbf{x}$ and obtain the following equation for $p_\varepsilon$ and $\mathbf{u}_\varepsilon$:

$$\varepsilon^2 \frac{\partial^2 p_\varepsilon}{\partial t^2} = \kappa_\varepsilon^{-1}(\mathbf{x})\varepsilon\nabla \cdot \rho_\varepsilon^{-1}(\mathbf{x})\varepsilon\nabla p_\varepsilon, \tag{2.7}$$

$$\varepsilon\frac{\partial \mathbf{u}_\varepsilon}{\partial t} + A_\varepsilon \mathbf{u}_\varepsilon = 0, \qquad A_\varepsilon = -\begin{pmatrix} 0 & c_\varepsilon^2(\mathbf{x}) \\ \varepsilon^2\Delta & 0 \end{pmatrix}, \tag{2.8}$$

with initial conditions of the form $p_\varepsilon(0, \mathbf{x}) = p_{0\varepsilon}(\varepsilon^{-1}\mathbf{x})$ and $\mathbf{u}_\varepsilon(0, \mathbf{x}) = \mathbf{u}_{0\varepsilon}(\varepsilon^{-1}\mathbf{x})$. We verify that acoustic energy conservation implies that

$$\mathcal{E}_H(t) = \frac{1}{2}\int_{\mathbb{R}^d} \Big(\kappa_\varepsilon(\mathbf{x})\Big(\varepsilon\frac{\partial p_\varepsilon}{\partial t}\Big)^2(t, \mathbf{x}) + \frac{|\varepsilon\nabla p_\varepsilon|^2(t, \mathbf{x})}{\rho_\varepsilon(\mathbf{x})}\Big)d\mathbf{x} = \mathcal{E}_H(0), \tag{2.9}$$

$$\mathcal{E}(t) = \frac{1}{2\rho_0}\int_{\mathbb{R}^d} \Big(|\varepsilon\nabla p_\varepsilon|^2(t, \mathbf{x}) + c_\varepsilon^2(\mathbf{x})q_\varepsilon^2(t, \mathbf{x})\Big)d\mathbf{x} = \mathcal{E}(0), \tag{2.10}$$

are independent of time. The above (equivalent) energy conservations are governed by quantities of the form $|\frac{\partial p_\varepsilon}{\partial t}|^2$, $|\varepsilon\nabla p_\varepsilon|^2$, and $q_\varepsilon^2$. Whereas such quantities do not solve closed-form equations in the high frequency limit $\varepsilon \to 0$, they can be decomposed in the phase space into a quantity that solves a transport equation; see e.g. (4.17) below. The role of kinetic models is to derive such a transport equation starting from (2.7), from (2.8), or from generalizations that account for dispersive effects. Whereas the spatial Wigner transform defined in section 3 may be used to this effect for (2.8) in the absence of temporal discretization, kinetic models derived from (2.7) or those that account for non local effects in time require us to use the spatio-temporal Wigner transform defined in section 5.

## 3 Spatial Wigner transform

Because wave vector fields oscillate at frequencies of order $\varepsilon^{-1}$, they converge at best only weakly to their limits as $\varepsilon \to 0$. Such limits are generally meaningless as most of the energy has been lost when passing to the high frequency limit. Much more interesting objects are the limits of the quadratic quantities in the vector field as they appear in the energy relations (2.9) and (2.10), because in fairly general contexts, no energy is lost while passing to the limit. The Wigner transform is adapted to describing such limits and to showing that no energy loss has occurred in the process; see e.g. [14,18]. Properly scaled so as to capture all the oscillations with frequencies of order $\varepsilon^{-1}$ (see [14]), the $d \times d$ matrix-valued Wigner transform of two spatially-dependent $d-$dimensional vector fields $\mathbf{u}(\mathbf{x})$ and $\mathbf{v}(\mathbf{x})$ is defined as

$$W[\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k}) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\mathbf{k}\cdot\mathbf{y}}\mathbf{u}(\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2})\mathbf{v}^*(\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2})d\mathbf{y}. \tag{3.1}$$

Here $^*$ means transposition and complex conjugation if necessary. We verify that

$$W[\mathbf{v},\mathbf{u}](\mathbf{x},\mathbf{k}) = W^*[\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k}). \tag{3.2}$$

The Wigner transform may thus be seen as the inverse Fourier transform of the (symmetrized) two-point correlation function of $\mathbf{u}(\mathbf{x})$ and $\mathbf{v}(\mathbf{x})$, where we define Fourier transforms using the convention $\hat{u}(\mathbf{k}) = \int_{\mathbb{R}^d} e^{-i\mathbf{k}\cdot\mathbf{x}}u(\mathbf{x})d\mathbf{x}$. We obtain by Fourier transformation in (3.1) the important property

$$\int_{\mathbb{R}^d} W[\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k})d\mathbf{k} = (\mathbf{u}\mathbf{v}^*)(\mathbf{x}), \tag{3.3}$$

which shows that $W[\mathbf{u},\mathbf{v}]$ may be seen as a phase-space description of the correlation $\mathbf{u}\mathbf{v}^*$.

Let $\mathbf{u}_\varepsilon$ be the solution of a wave equation such as (2.8). We define its spatial Wigner transform:

$$W_\varepsilon(t,\mathbf{x},\mathbf{k}) = W[\mathbf{u}_\varepsilon(t,\cdot),\mathbf{u}_\varepsilon(t,\cdot)](\mathbf{x},\mathbf{k}). \tag{3.4}$$

Whereas the high frequency limits of quadratic quantities of $\mathbf{u}_\varepsilon$ may not satisfy a closed-form equation (in the variables $(t,\mathbf{x})$), they are related to limits of $W_\varepsilon(t,\mathbf{x},\mathbf{k})$ thanks to (3.3). We will see that $W_\varepsilon(t,\mathbf{x},\mathbf{k})$ satisfies a closed-form equation and indeed admits a limit that can be characterized as $\varepsilon \to 0$ as the solution to a (phase-space) kinetic equation; see [24].

**Pseudo-differential calculus.** In order to obtain exact and approximate equations for the Wigner transform $W_\varepsilon(t,\mathbf{x},\mathbf{k})$ from e.g. the equations (2.8) verified by $\mathbf{u}_\varepsilon$, we need some pseudo-differential calculus e.g. as developed in [14]. We recall here the calculus needed in this paper. Let $P(\mathbf{x},\varepsilon\mathbf{D})$ be a matrix-valued pseudo-differential operator, defined by

$$P(\mathbf{x},\varepsilon\mathbf{D})\mathbf{u}(\mathbf{x}) = \int_{\mathbb{R}^d} e^{i\mathbf{x}\cdot\mathbf{k}}P(\mathbf{x},i\varepsilon\mathbf{k})\hat{\mathbf{u}}(\mathbf{k})\frac{d\mathbf{k}}{(2\pi)^d}. \tag{3.5}$$

We assume that $P(\mathbf{x},i\varepsilon\mathbf{k})$ is a smooth function and use the same mathematical symbol for the operator $P(\mathbf{x},\varepsilon\mathbf{D})$ and its symbol $P(\mathbf{x},i\varepsilon\mathbf{k})$.

Let $\hat{W}[\mathbf{u},\mathbf{v}](\mathbf{p},\mathbf{k})$ be the Fourier transform $\mathcal{F}_{\mathbf{x}\to\mathbf{p}}$ of the Wigner transform. We then verify that

$$\hat{W}[\mathbf{u},\mathbf{v}](\mathbf{p},\mathbf{k}) = \frac{1}{(2\pi\varepsilon)^d}\hat{\mathbf{u}}\Big(\frac{\mathbf{p}}{2}+\frac{\mathbf{k}}{\varepsilon}\Big)\hat{\mathbf{v}^*}\Big(\frac{\mathbf{p}}{2}-\frac{\mathbf{k}}{\varepsilon}\Big) = \frac{1}{(2\pi\varepsilon)^d}\hat{\mathbf{u}}\Big(\frac{\mathbf{p}}{2}+\frac{\mathbf{k}}{\varepsilon}\Big)\hat{\mathbf{v}}^*\Big(-\frac{\mathbf{p}}{2}+\frac{\mathbf{k}}{\varepsilon}\Big) \tag{3.6}$$

when $\mathbf{v}$ is real-valued. Thus for a homogeneous operator $P(\varepsilon\mathbf{D})$, we have:

$$W[P(\varepsilon\mathbf{D})\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k}) = P(i\mathbf{k}+\frac{\varepsilon\mathbf{D}}{2})W[\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k}). \tag{3.7}$$

The above calculation may be generalized to provide the following result:

$$W[P(\mathbf{x},\varepsilon\mathbf{D})\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k}) = \mathcal{L}_P W[\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k}), \tag{3.8}$$

$$\mathcal{L}_P W(\mathbf{x},\mathbf{k}) = \int_{\mathbb{R}^{2d}}\Big(\int_{\mathbb{R}^d} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\hat{P}(\boldsymbol{\xi}, i\mathbf{k}+i\varepsilon(\frac{\mathbf{p}}{2}-\boldsymbol{\xi}))\frac{d\mathbf{p}}{(2\pi)^d}\Big)e^{i\boldsymbol{\xi}\cdot\mathbf{y}}W(\mathbf{y},\mathbf{k}-\frac{\varepsilon\boldsymbol{\xi}}{2})\frac{d\boldsymbol{\xi}d\mathbf{y}}{(2\pi)^d}. \tag{3.9}$$

Throughout the text, we shall use the convention

$$P'(\mathbf{x},i\mathbf{k}) = \nabla_{i\mathbf{k}}P(\mathbf{x},i\mathbf{k}) = -i\nabla_{\mathbf{k}}P(\mathbf{x},i\mathbf{k}). \tag{3.10}$$

Assuming that $W(\mathbf{x},\mathbf{k})$ is sufficiently smooth in the $\mathbf{k}$ variable, Taylor expansions in $\hat{P}(\cdot - i\varepsilon\boldsymbol{\xi})$ in (3.9) yield that

$$\begin{aligned}
\mathcal{L}_P W(\mathbf{x},\mathbf{k}) &= \mathcal{M}_\varepsilon W(\mathbf{x},\mathbf{k}) + \varepsilon\mathcal{N}_\varepsilon W(\mathbf{x},\mathbf{k}) + O(\varepsilon^2)\\
\mathcal{M}_\varepsilon W(\mathbf{x},\mathbf{k}) &= P(\mathbf{x},i\mathbf{k}+\frac{\varepsilon\mathbf{D}}{2})W(\mathbf{x},\mathbf{k}) + \frac{i\varepsilon}{2}\nabla_{\mathbf{x}}P(\mathbf{x},i\mathbf{k}+\frac{\varepsilon\mathbf{D}}{2})\cdot\nabla_{\mathbf{k}}W(\mathbf{x},\mathbf{k})\\
\mathcal{N}_\varepsilon W(\mathbf{x},\mathbf{k}) &= i\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{k}}P(\mathbf{x},i\mathbf{k}+\frac{\varepsilon\mathbf{D}_{\mathbf{x}}}{2})W(\mathbf{x},\mathbf{k}).
\end{aligned} \tag{3.11}$$

In the examples considered below, the symbol $P(\mathbf{x},i\mathbf{k})$ depends on one of the two variables only so that $\mathcal{N}_\varepsilon \equiv 0$. The above calculations allow us to deduce that for functions $W(\mathbf{x},\mathbf{k})$ that are sufficiently smooth in both variables $\mathbf{x}$ and $\mathbf{k}$, we have

$$\begin{aligned}
\mathcal{L}_P W(\mathbf{x},\mathbf{k}) &= \mathcal{L}_\varepsilon W(\mathbf{x},\mathbf{k}) + \varepsilon\mathcal{N}_\varepsilon W(\mathbf{x},\mathbf{k}) + O(\varepsilon^2)\\
\mathcal{L}_\varepsilon W(\mathbf{x},\mathbf{k}) &= P(\mathbf{x},i\mathbf{k})W(\mathbf{x},\mathbf{k}) + \frac{i\varepsilon}{2}\{P,W\}(\mathbf{x},\mathbf{k}),
\end{aligned} \tag{3.12}$$

where we have defined the Poisson bracket

$$\{P,W\}(\mathbf{x},\mathbf{k}) = (\nabla_{\mathbf{x}}P\cdot\nabla_{\mathbf{k}}W - \nabla_{\mathbf{x}}W\cdot\nabla_{\mathbf{k}}P)(\mathbf{x},\mathbf{k}). \tag{3.13}$$

Similarly, we define

$$W[\mathbf{u},P(\mathbf{x},\varepsilon\mathbf{D})\mathbf{v}](\mathbf{x},\mathbf{k}) = \mathcal{L}^* W(\mathbf{x},\mathbf{k}). \tag{3.14}$$

We verify that when $\mathbf{v}$ and $P(\mathbf{x},\varepsilon\mathbf{D})\mathbf{v}$ are real-valued,

$$\begin{aligned}
\mathcal{L}^* W(\mathbf{x},\mathbf{k}) &= \big(\mathcal{M}^*_\varepsilon + \varepsilon\mathcal{N}^*_\varepsilon\big)W(\mathbf{x},\mathbf{k}) + O(\varepsilon^2) = \big(\mathcal{L}^*_\varepsilon + \varepsilon\mathcal{N}^*_\varepsilon\big)W(\mathbf{x},\mathbf{k}) + O(\varepsilon^2)\\
\mathcal{M}^*_\varepsilon W(\mathbf{x},\mathbf{k}) &= W(\mathbf{x},\mathbf{k})P^*(\mathbf{x},i\mathbf{k}-\frac{\varepsilon\mathbf{D}}{2}) - \frac{i\varepsilon}{2}\nabla_{\mathbf{k}}W(\mathbf{x},\mathbf{k})\cdot\nabla_{\mathbf{x}}P^*(\mathbf{x},i\mathbf{k}-\frac{\varepsilon\mathbf{D}}{2})\\
\mathcal{L}^*_\varepsilon W(\mathbf{x},\mathbf{k}) &= W(\mathbf{x},\mathbf{k})P^*(\mathbf{x},i\mathbf{k}) - \frac{i\varepsilon}{2}\{W,P^*\}(\mathbf{x},\mathbf{k}).
\end{aligned} \tag{3.15}$$

In the latter expressions, we use the convention that the differential operator $\mathbf{D}$ acts on $W[\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k})$, and thus for instance $W(\mathbf{x},\mathbf{k})P^*(\mathbf{x},i\mathbf{k}-\frac{\varepsilon\mathbf{D}}{2})$ should be interpreted as the inverse Fourier transform of the matrix $\hat{W}[\mathbf{u},\mathbf{v}](\mathbf{p},\mathbf{k})P^*(i\mathbf{k}-\frac{\varepsilon i\mathbf{p}}{2})$.

We need to consider one more item of calculus to account for fast varying coefficient. Let $V(\mathbf{x})$ be a real-valued matrix-valued function. Then we find that

$$\begin{aligned}
W[V(\frac{\mathbf{x}}{\varepsilon})\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k}) &= \int_{\mathbb{R}^d} e^{i\frac{\mathbf{x}\cdot\mathbf{p}}{\varepsilon}}\hat{V}(\mathbf{p})W[\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k}-\frac{\mathbf{p}}{2})\frac{d\mathbf{p}}{(2\pi)^d},\\
W[\mathbf{u},V(\frac{\mathbf{x}}{\varepsilon})\mathbf{v}](\mathbf{x},\mathbf{k}) &= \int_{\mathbb{R}^d} e^{i\frac{\mathbf{x}\cdot\mathbf{p}}{\varepsilon}}W[\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k}+\frac{\mathbf{p}}{2})\hat{V}^t(\mathbf{p})\frac{d\mathbf{p}}{(2\pi)^d}.
\end{aligned} \tag{3.16}$$

Here $\hat{V}(\mathbf{p})$ is the Fourier transform of $V(\mathbf{x})$ component by component. This may be generalized as follows. Let $V(\mathbf{x},\mathbf{y})$ be a real-valued matrix function, with Fourier transform $\check{V}(\mathbf{q},\mathbf{p})$ and Fourier transform with respect to the second variable $\hat{V}(\mathbf{x},\mathbf{p})$. We then find that

$$\begin{aligned} W[V(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k}) &= \int_{\mathbb{R}^{2d}} e^{i\frac{\mathbf{x}\cdot\mathbf{p}}{\varepsilon}} e^{i\mathbf{x}\cdot\mathbf{q}} \check{V}(\mathbf{q},\mathbf{p}) W[\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k}-\frac{\mathbf{p}}{2}-\frac{\varepsilon\mathbf{q}}{2})\frac{d\mathbf{p}d\mathbf{q}}{(2\pi)^{2d}}, \\ &= \int_{\mathbb{R}^{d}} e^{i\frac{\mathbf{x}\cdot\mathbf{p}}{\varepsilon}} \hat{V}(\mathbf{x},\mathbf{p}) W[\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k}-\frac{\mathbf{p}}{2})\frac{d\mathbf{p}}{(2\pi)^{d}} + O(\varepsilon), \end{aligned} \tag{3.17}$$

provided that $W[\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k})$ is sufficiently smooth in the $\mathbf{k}$ variable.

**Multiple scale expansion.** The error terms in (3.11) and (3.12), although both deduced from Taylor expansions, have different expressions. While the former involves second-order derivatives in $\mathbf{k}$ of $W(\mathbf{x},\mathbf{k})$, the latter involves second-order derivatives in both the $\mathbf{k}$ and $\mathbf{x}$ variables. When $W(\mathbf{x},\mathbf{k})$ has bounded second-order derivatives in $\mathbf{x}$ and $\mathbf{y}$, then $(\mathcal{L}_P-\mathcal{L}_\varepsilon)W = O(\varepsilon^2)$ and $(\mathcal{L}_P-\mathcal{M}_\varepsilon)W = O(\varepsilon^2)$. In the sequel however, we will need to apply the operator $\mathcal{M}_\varepsilon$ to functions that oscillate in the $\mathbf{x}$ variable and are smooth in the $\mathbf{k}$ variable. Such functions will have the form $W(\mathbf{x},\frac{\mathbf{x}}{\varepsilon},\mathbf{k})$. The differential operator $\mathbf{D}$ acting on such functions then takes the form $\mathbf{D}=\mathbf{D}_\mathbf{x}+\varepsilon^{-1}\mathbf{D}_\mathbf{y}$. We then verify that

$$\mathcal{M}_\varepsilon[W(\mathbf{x},\frac{\mathbf{x}}{\varepsilon},\mathbf{k})](\mathbf{x},\mathbf{k}) = P(\mathbf{x},i\mathbf{k}+\frac{\mathbf{D}_\mathbf{y}}{2})W(\mathbf{x},\mathbf{y},\mathbf{k})|_{\mathbf{y}=\frac{\mathbf{x}}{\varepsilon}} + O(\varepsilon). \tag{3.18}$$

We will not need higher-order terms. Note that on such functions, $(\mathcal{L}_P-\mathcal{L}_\varepsilon)W = O(1)$, which implies that $\mathcal{L}_\varepsilon$ cannot be used as an approximation of $\mathcal{L}_P$.

## 4 Kinetic models for the spatial Wigner transform

We now develop a kinetic theory for the acoustic system (2.8) and generalizations that account for some (spatial) dispersive effects. The derivation of the kinetic equations mostly follows the methodology introduced in [24] and generalized in [15,22]. More general dispersive systems, that cannot be handled by the methods developed in [15,22,24] are considered in section 6. The main objective of this section is to introduce the method of two-scale expansions that will also be used later and to show that this method can be used to analyze the correlation function of wave fields propagating in possibly different media as in [8].

In the weak coupling regime, the medium is characterized by the sound speed:

$$c_\varepsilon^2(\mathbf{x}) = c_0^2 - \sqrt{\varepsilon}V(\frac{\mathbf{x}}{\varepsilon}), \tag{4.1}$$

where $c_0$ is the background speed assumed to be constant to simplify and $V(\mathbf{x})$ accounts for the random fluctuations. More general models are considered in section 7. The correlation length of the random heterogeneities of order $\varepsilon$ is here to ensure maximum interaction between the waves and the underlying media. The scaling $\sqrt{\varepsilon}$ is the unique scaling that allows the energy to be significantly modified by the fluctuations while still solving a transport equation. It is known [26] that larger fluctuations lead to localization of the wave energy, which cannot be accounted for by kinetic models. Since the localization length is always smaller than the diffusive (kinetic) length in spatial dimension $d=1$, we restrict ourselves to the case $d \geq 2$.

Let us consider the correlation of two fields $\mathbf{u}_\varepsilon^1$ and $\mathbf{u}_\varepsilon^2$ propagating in random media with the same background velocity $c_0$ but possibly different heterogeneities modeled by $V^\varphi$, $\varphi = 1, 2$. We also replace the Laplacian in (2.8) by the more general smooth, real-valued, positive Fourier multiplier operator $p(\varepsilon \mathbf{D})$, which may account for (spatial) dispersive effects. We assume moreover that $p(-i\mathbf{k}) = p(i\mathbf{k})$. We retrieve $p(\varepsilon \mathbf{D}) = \Delta$ for $p(i\boldsymbol{\xi}) = (i\boldsymbol{\xi}) \cdot (i\boldsymbol{\xi}) = -|\boldsymbol{\xi}|^2$.

We thus consider the two equations

$$\varepsilon \frac{\partial \mathbf{u}_\varepsilon^\varphi}{\partial t} + A_\varepsilon^\varphi \mathbf{u}_\varepsilon^\varphi = 0, \qquad \varphi = 1, 2, \tag{4.2}$$

and assume the following structure for $A_\varepsilon^\varphi$:

$$A_\varepsilon^\varphi = -\begin{pmatrix} 0 & c_0^2 \\ p(\varepsilon \mathbf{D}) & 0 \end{pmatrix} + \sqrt{\varepsilon} V^\varphi(\frac{\mathbf{x}}{\varepsilon}) K, \qquad K = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{4.3}$$

The correlation of two signals propagating in two different media may be of interest in probing the temporal variations in the statistics of random media and has also found recent applications in the analysis of time reversed waves [6,8].

**Structure of the random fluctuations.** The random inhomogeneities of the underlying media are modeled by the functions $V^\varphi(\mathbf{x})$. We assume that $V^\varphi(\mathbf{x})$ for $\varphi = 1, 2$ is a statistically homogeneous mean-zero random field. Because higher-order statistical moments of the heterogeneous fluctuations do not appear in kinetic models, all we need to know about the statistics of the random media in the high frequency limit are the two-point correlation functions, or equivalently their Fourier transform the power spectra, defined by

$$c_0^4 R^{\varphi\psi}(\mathbf{x}) = \langle V^\varphi(\mathbf{y}) V^\psi(\mathbf{y}+\mathbf{x}) \rangle, \qquad 1 \leq \varphi, \psi \leq 2, \tag{4.4}$$

$$(2\pi)^d c_0^4 \hat{R}^{\varphi\psi}(\mathbf{p}) \delta(\mathbf{p}+\mathbf{q}) = \langle \hat{V}^\varphi(\mathbf{p}) \hat{V}^\psi(\mathbf{q}) \rangle. \tag{4.5}$$

Here $\langle \cdot \rangle$ means ensemble average (mathematical expectation). We verify that $\hat{R}^{\varphi\psi}(-\mathbf{p}) = \hat{R}^{\varphi\psi}(\mathbf{p})$. In section 7, we will consider more general random fluctuations of the form $V^\varphi(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})$, where for each $\mathbf{x} \in \mathbb{R}^d$, $V^\varphi(\mathbf{x}, \mathbf{y})$ is a statistically homogeneous mean-zero random field.

**Equation for the Wigner transform.** We define the Wigner transform of the two fields as

$$W_\varepsilon(t, \mathbf{x}, \mathbf{k}) = W[\mathbf{u}_\varepsilon^1(t, \cdot), \mathbf{u}_\varepsilon^2(t, \cdot)](\mathbf{x}, \mathbf{k}), \tag{4.6}$$

and deduce from (4.2) and (4.6) that

$$\varepsilon \frac{\partial W_\varepsilon}{\partial t} + W[A_\varepsilon^1 \mathbf{u}_\varepsilon^1, \mathbf{u}_\varepsilon^2] + W[\mathbf{u}_\varepsilon^1, A_\varepsilon^2 \mathbf{u}_\varepsilon^2] = 0. \tag{4.7}$$

The pseudo-differential calculus recalled in section 3 is used to obtain the equation:

$$\varepsilon\frac{\partial W_\varepsilon}{\partial t} + P(i\mathbf{k}+\frac{\varepsilon\mathbf{D}}{2})W_\varepsilon + W_\varepsilon P^*(i\mathbf{k}-\frac{\varepsilon\mathbf{D}}{2}) + \sqrt{\varepsilon}\Big(\mathcal{K}^1_\varepsilon K W_\varepsilon + \mathcal{K}^{2*}_\varepsilon W_\varepsilon K^*\Big) = 0, \qquad (4.8)$$

$$P(i\mathbf{k}+\frac{\varepsilon\mathbf{D}}{2}) = -\begin{pmatrix} 0 & c_0^2 \\ p(i\mathbf{k}+\frac{\varepsilon\mathbf{D}}{2}) & 0\end{pmatrix}, \qquad \mathcal{K}^\varphi_\varepsilon W = \int_{\mathbb{R}^d} e^{i\frac{\mathbf{x}\cdot\mathbf{p}}{\varepsilon}}\hat{V}^\varphi(\mathbf{p})W(\mathbf{k}-\frac{\mathbf{p}}{2})\frac{d\mathbf{p}}{(2\pi)^d}. \qquad (4.9)$$

Note that (4.8) is an exact evolution equation for the two-by-two Wigner transform $W_\varepsilon(t,\mathbf{x},\mathbf{k})$. Its initial conditions are obtained by evaluating (4.6) at $t=0$, and thus depend on the initial conditions for $\mathbf{u}^1_\varepsilon$ and $\mathbf{u}^2_\varepsilon$.

**Multiple scale expansion.** We are now interested in the high-frequency limit as $\varepsilon\to 0$ of $W_\varepsilon$. Because of the presence of a highly-oscillatory phase $\exp(\varepsilon^{-1}i\mathbf{x}\cdot\mathbf{k})$ in the operator $\mathcal{K}^\varphi_\varepsilon$, direct asymptotic expansions on $W_\varepsilon$ and (4.8) cannot provide the correct limit. Rather, as is classical in the homogenization of equations in highly oscillatory media [10,24], we introduce the following two-scale version of $W_\varepsilon$:

$$W_\varepsilon(t,\mathbf{x},\mathbf{k}) = W_\varepsilon(t,\mathbf{x},\frac{\mathbf{x}}{\varepsilon},\mathbf{k}), \qquad (4.10)$$

and still use the symbol $W_\varepsilon$ for the function on $\mathbb{R}^{3d+1}$ in the new variables $(t,\mathbf{x},\mathbf{y},\mathbf{k})$. We then find that the differential operator $\mathbf{D}$ acting on the spatial variables should be replaced by $\mathbf{D}_\mathbf{x}+\frac{1}{\varepsilon}\mathbf{D}_\mathbf{y}$. The equation for $W_\varepsilon$ thus becomes

$$\varepsilon\frac{\partial W_\varepsilon}{\partial t} + P(i\mathbf{k}+\frac{\mathbf{D}_\mathbf{y}}{2}+\frac{\varepsilon\mathbf{D}_\mathbf{x}}{2})W_\varepsilon + W_\varepsilon P^*(i\mathbf{k}-\frac{\mathbf{D}_\mathbf{y}}{2}-\frac{\varepsilon\mathbf{D}_\mathbf{x}}{2}) + \sqrt{\varepsilon}\Big(\mathcal{K}^1 K W_\varepsilon + \mathcal{K}^{2*} W_\varepsilon K^*\Big) = 0, \qquad (4.11)$$

where we have defined $\mathcal{K}^\varphi W = \int_{\mathbb{R}^d} e^{i\mathbf{y}\cdot\mathbf{p}}\hat{V}^\varphi(\mathbf{p})W(\mathbf{k}-\frac{\mathbf{p}}{2})\frac{d\mathbf{p}}{(2\pi)^d}$. Asymptotic expansions in the new set of variables can now account for the fast oscillations of the heterogeneous medium. Using the asymptotic expansion $P = P_0+\varepsilon P_1 + O(\varepsilon^2)$ in (4.9) and

$$W_\varepsilon(t,\mathbf{x},\mathbf{y},\mathbf{k}) = W_0(t,\mathbf{x},\mathbf{k}) + \sqrt{\varepsilon}W_1(t,\mathbf{x},\mathbf{y},\mathbf{k}) + \varepsilon W_2(t,\mathbf{x},\mathbf{y},\mathbf{k}), \qquad (4.12)$$

we equate like powers of $\varepsilon$ in (4.11) to obtain a sequence of three equations.

**Leading order and dispersion relation.** The leading equation in the above expansion yields

$$\mathcal{L}_0 W_0 \equiv P_0(i\mathbf{k})W_0 + W_0 P_0^*(i\mathbf{k}) = 0; \quad P_0 = J\Lambda_0,\ \Lambda_0 = \begin{pmatrix} p(i\mathbf{k}) & 0 \\ 0 & -c_0^2\end{pmatrix},\ J = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}. \qquad (4.13)$$

Let us define $q_0(i\mathbf{k}) = \sqrt{-p(i\mathbf{k})}$. The diagonalization of the dispersion matrix $P_0$ yields

$$\lambda_\pm(\mathbf{k}) = \pm i c_0 q_0(i\mathbf{k}),\ \mathbf{b}_\pm(\mathbf{k}) = \frac{1}{\sqrt{2}}\begin{pmatrix}\pm i q_0^{-1}(i\mathbf{k}) \\ c_0^{-1}\end{pmatrix},\ \mathbf{c}_\pm(\mathbf{k}) = -\Lambda_0\mathbf{b}_\pm(\mathbf{k}) = \frac{1}{\sqrt{2}}\begin{pmatrix}\pm i q_0(i\mathbf{k}) \\ c_0\end{pmatrix}. \qquad (4.14)$$

The vectors are normalized such that $-\mathbf{b}_\pm^*\Lambda_0\mathbf{b}_\pm = \mathbf{b}_\pm^*\mathbf{c}_\pm = 1$ and we verify the spectral decomposition $P_0 = \lambda_+\mathbf{b}_+\mathbf{c}_+^* + \lambda_-\mathbf{b}_-\mathbf{c}_-^*$. Since $(\mathbf{b}_+(\mathbf{k}),\mathbf{b}_-(\mathbf{k}))$ forms a basis of $\mathbb{R}^2$ for any $\mathbf{k}\in\mathbb{R}^d_*$, any matrix $W$ may thus be decomposed as $W=\sum_{i,j=\pm}\alpha_{ij}\mathbf{b}_i\mathbf{b}_j^*$ where $\alpha_{ij} = \mathbf{c}_i^*W\mathbf{c}_j = \mathrm{tr}\Big(W\mathbf{c}_i\mathbf{c}_j^*\Big)$, and a straightforward calculation shows that $\mathbf{c}_k^*(P_0W+WP_0^*)\mathbf{c}_m = \alpha_{km}(\lambda_k+\overline{\lambda_m})$.

Using the above decomposition for the matrix $W_0 = \sum_{i,j=\pm} a_{ij}\mathbf{b}_i\mathbf{b}_j^*$, equation (4.13) implies that $a_{+-} = a_{-+} = 0$ so that

$$W_0 = a_+\mathbf{b}_+\mathbf{b}_+^* + a_-\mathbf{b}_-\mathbf{b}_-^*; \qquad a_\pm = \mathbf{c}_\pm^* W_0 \mathbf{c}_\pm. \tag{4.15}$$

Because all the components of $\mathbf{u}_\varepsilon^\varphi$ are real-valued, we verify that $\bar{W}(-\mathbf{k}) = W(\mathbf{k})$. Here $^-$ means complex conjugation component by component. From the above expression for $a_\pm$ and the fact that $\mathbf{c}(-\mathbf{k}) = \mathbf{c}(\mathbf{k})$, we deduce that

$$\bar{a}_\pm(-\mathbf{k}) = a_\mp(\mathbf{k}). \tag{4.16}$$

It is thus sufficient to find an equation for $a_+(\mathbf{k})$. We verify that $\int_{\mathbb{R}^d} a_+ d\mathbf{k} = \frac{1}{2}\int_{\mathbb{R}^d} \mathrm{tr}(-\Lambda_0 W_0) d\mathbf{k}$, so that in the case where (4.2) is (2.8) and $W_\varepsilon$ is the Wigner transform of $\mathbf{u}_\varepsilon$, we have

$$\frac{1}{2}\int_{\mathbb{R}^{2d}} \mathrm{tr}(-\Lambda_0 W_0) d\mathbf{k} d\mathbf{x} = \mathcal{E}(t) = \int_{\mathbb{R}^{2d}} a_+(t,\mathbf{x},\mathbf{k}) d\mathbf{k} d\mathbf{x}, \tag{4.17}$$

at least in the limit $\varepsilon \to 0$, where $\mathcal{E}$ is defined in (2.10). Thus $a_+$ can be given the interpretation of an energy density in the phase-space.

**First-order corrector.** The next equation in the asymptotic expansion in powers of $\varepsilon$ is

$$P_0(i\mathbf{k} + \frac{\mathbf{D_y}}{2})W_1 + W_1 P_0^*(i\mathbf{k} - \frac{\mathbf{D_y}}{2}) + \theta W_1 + \mathcal{K}^1 K W_0 + \mathcal{K}^{2*} W_0 K^* = 0. \tag{4.18}$$

The parameter $0 < \theta \ll 1$ is a regularization (limiting absorption) parameter that will be sent to 0 at the end. It is required to ensure the causality of wave propagation [24]. We denote by $\hat{W}_1(t,\mathbf{x},\mathbf{p},\mathbf{k})$ the Fourier transform $\mathbf{y} \to \mathbf{p}$ of $W_1$. Since $W_0$ is independent of $\mathbf{y}$ so that $\hat{W}_0 = (2\pi)^d \delta(\mathbf{p}) W_0$, we verify that $\hat{W}_1$ satisfies the equation

$$P_0(i\mathbf{k}+i\frac{\mathbf{p}}{2})\hat{W}_1 + \hat{W}_1 P_0^*(i\mathbf{k}-i\frac{\mathbf{p}}{2}) + \theta\hat{W}_1 + \hat{V}^1(\mathbf{p}) K W_0(\mathbf{k}-\frac{\mathbf{p}}{2}) + \hat{V}^2(\mathbf{p}) W_0(\mathbf{k}+\frac{\mathbf{p}}{2}) K^* = 0. \tag{4.19}$$

Since the vectors $\mathbf{b}_i(\mathbf{k})$ form a complete basis of $\mathbb{R}^2$ for all $\mathbf{k}$, we can decompose $\hat{W}_1$ as

$$\hat{W}_1(\mathbf{p},\mathbf{k}) = \sum_{i,j=\pm} \alpha_{ij}(\mathbf{p},\mathbf{k}) \mathbf{b}_i(\mathbf{k}+\frac{\mathbf{p}}{2}) \mathbf{b}_j^*(\mathbf{k}-\frac{\mathbf{p}}{2}). \tag{4.20}$$

Multiplying (4.19) by $\mathbf{c}_m^*(\mathbf{k}+\frac{\mathbf{p}}{2})$ on the left and by $\mathbf{c}_n(\mathbf{k}-\frac{\mathbf{p}}{2})$ on the right, recalling $\lambda_n^* = -\lambda_n$, and calculating $\mathbf{b}_n^*(\mathbf{p}) K^* \mathbf{c}_m(\mathbf{q}) = \frac{1}{2c_0^2}\lambda_m(\mathbf{q})$ and $\mathbf{c}_m^*(\mathbf{p}) K \mathbf{b}_n(\mathbf{q}) = \frac{-1}{2c_0^2}\lambda_m(\mathbf{p})$, we get

$$\alpha_{mn}(\mathbf{p},\mathbf{k}) = \frac{1}{2c_0^2} \frac{\hat{V}^1(\mathbf{p})\lambda_m(\mathbf{k}+\frac{\mathbf{p}}{2}) a_n(\mathbf{k}-\frac{\mathbf{p}}{2}) - \hat{V}^2(\mathbf{p})\lambda_n(\mathbf{k}-\frac{\mathbf{p}}{2}) a_m(\mathbf{k}+\frac{\mathbf{p}}{2})}{\lambda_m(\mathbf{k}+\frac{\mathbf{p}}{2}) - \lambda_n(\mathbf{k}-\frac{\mathbf{p}}{2}) + \theta}. \tag{4.21}$$

Note that $W_1$ is linear in the random fields $V^\varphi$.

**Transport equation.** Finally the third equation in the expansion in powers of $\varepsilon$ yields

$$\begin{aligned} &P_0(i\mathbf{k} + \frac{\mathbf{D_y}}{2})W_2 + W_2 P_0^*(i\mathbf{k} - \frac{\mathbf{D_y}}{2}) + \mathcal{K}_1 K W_1 + \mathcal{K}_2^* W_1 K^* \\ &+\frac{\partial W_0}{\partial t} + P_1(i\mathbf{k}) W_0 + W_0 P_1^*(i\mathbf{k}) = 0. \end{aligned} \tag{4.22}$$

We consider ensemble averages in the above equation and thus look for an equation for $\langle a_+\rangle$, which we still denote by $a_+$. We may assume that $W_2$ is orthogonal to $W_0$ in order to justify the expansion in $\varepsilon$, so that $\langle \mathbf{c}_+^*\mathcal{L}_0W_2\mathbf{c}_+\rangle = 0$. This part cannot be justified rigorously and may be seen as a reasonable closure argument [24], which provides the correct limit as $\varepsilon\to 0$ in cases that can be analyzes rigorously [13]. We multiply the above equation on the left by $\mathbf{c}_+^*(\mathbf{k})$ and on the right by $\mathbf{c}_+(\mathbf{k})$. Recalling the convention in (3.10), we obtain that since $p(i\mathbf{k}) = -q_0^2(i\mathbf{k})$, we have $p'(i\mathbf{k}) = -i\nabla_{\mathbf{k}}p(i\mathbf{k}) = 2iq_0(i\mathbf{k})\nabla_{\mathbf{k}}q_0(i\mathbf{k})$. This implies that $\mathbf{c}_+^*P_1W_0\mathbf{c}_+ = \mathbf{c}_+^*W_0P_1^*\mathbf{c}_+ = \frac{c_0}{2}\nabla_{\mathbf{k}}q_0(i\mathbf{k})\cdot\nabla_{\mathbf{x}}a_+(\mathbf{x},\mathbf{k})$. Let us define

$$\omega_+(\mathbf{k}) = -c_0q_0(i\mathbf{k}) = i\lambda_+(i\mathbf{k}). \tag{4.23}$$

Our convention for the frequencies in the acoustic case are then $\omega_\pm \pm c_0|\mathbf{k}| = 0$. We thus find that $\mathbf{c}_+^*\mathcal{L}_2W_0\mathbf{c}_+ = \{\omega_+, a_+\}(\mathbf{x},\mathbf{k})$, where the Poisson bracket is defined in (3.13). When $p(i\boldsymbol{\xi}) = (i\boldsymbol{\xi})^2$, we obtain that $c_0\nabla_{\mathbf{k}}q_0(i\mathbf{k}) = c_0\hat{\mathbf{k}}$. Upon taking ensemble averages and still denoting by $a_+$ the ensemble average $\langle a_+\rangle$, we get the equation

$$\frac{\partial a_+}{\partial t} + \{\omega_+, a_+\}(\mathbf{x},\mathbf{k}) + \langle \mathbf{c}_+^*\mathcal{L}_1W_1\mathbf{c}_+\rangle = 0.$$

Here we have defined $\mathcal{L}_1W = \mathcal{K}^1KW + \mathcal{K}^{2*}WK^*$. It remains to evaluate $\langle \mathbf{c}_+^*\mathcal{L}_1W_1\mathbf{c}_+\rangle$. Let us define $\hat{W}_1(\mathbf{p},\mathbf{k}) = \hat{V}^1(\mathbf{p})W_1^1(\mathbf{p},\mathbf{k}) + \hat{V}^2(\mathbf{p})W_1^2(\mathbf{p},\mathbf{k})$ with obvious notation. Using the symmetry $\hat{R}^{ij}(-\mathbf{p}) = \hat{R}^{ij}(\mathbf{p})$, we deduce that

$$\begin{aligned}\langle\widehat{\mathcal{L}_1W_1}(\mathbf{p},\mathbf{k})\rangle = \delta(\mathbf{p})c_0^4\int_{\mathbb{R}^d}\Big(&\hat{R}^{11}(\mathbf{k}-\mathbf{q})KW_1^1(\mathbf{q}-\mathbf{k},\frac{\mathbf{k}+\mathbf{q}}{2}) + \hat{R}^{12}(\mathbf{k}-\mathbf{q})KW_1^2(\mathbf{q}-\mathbf{k},\frac{\mathbf{k}+\mathbf{q}}{2})\\ &+\hat{R}^{21}(\mathbf{k}-\mathbf{q})W_1^1(\mathbf{k}-\mathbf{q},\frac{\mathbf{k}+\mathbf{q}}{2})K^* + \hat{R}^{22}(\mathbf{k}-\mathbf{q})W_1^2(\mathbf{k}-\mathbf{q},\frac{\mathbf{k}+\mathbf{q}}{2})K^*\Big)d\mathbf{q}.\end{aligned}$$

Using the convention of summation over repeated indices, we obtain after some algebra that

$$\begin{aligned}\langle\mathbf{c}_+^*(\mathbf{k})\mathcal{L}_1W_1(\mathbf{k})\mathbf{c}_+(\mathbf{k})\rangle = \frac{\lambda_+(\mathbf{k})}{4(2\pi)^d}\int_{\mathbb{R}^d}\Big(&\frac{-\hat{R}^{11}(\mathbf{k}-\mathbf{q})\lambda_i(\mathbf{q})a_+(\mathbf{k})}{\lambda_i(\mathbf{q})-\lambda_+(\mathbf{k})+\theta} + \frac{\hat{R}^{12}(\mathbf{k}-\mathbf{q})\lambda_+(\mathbf{k})a_i(\mathbf{q})}{\lambda_i(\mathbf{q})-\lambda_+(\mathbf{k})+\theta}\\ &+\frac{\hat{R}^{12}(\mathbf{k}-\mathbf{q})\lambda_+(\mathbf{k})a_j(\mathbf{q})}{\lambda_+(\mathbf{k})-\lambda_j(\mathbf{q})+\theta} + \frac{-\hat{R}^{22}(\mathbf{k}-\mathbf{q})\lambda_j(\mathbf{q})a_+(\mathbf{k})}{\lambda_+(\mathbf{k})-\lambda_j(\mathbf{q})+\theta}\Big)d\mathbf{q}.\end{aligned} \tag{4.24}$$

Since $\lambda_j(\mathbf{k})$ is purely imaginary, we deduce from the relation $\frac{1}{ix+\varepsilon}\to\frac{1}{ix}+\pi\mathrm{sign}(\varepsilon)\delta(x)$, as $\varepsilon\to 0$, which holds in the sense of distributions, that

$$\lim_{0<\theta\to 0}\Big(\frac{1}{\lambda_j(\mathbf{q})-\lambda_+(\mathbf{k})+\theta} + \frac{1}{\lambda_+(\mathbf{q})-\lambda_j(\mathbf{k})+\theta}\Big) = 2\pi\delta(i\lambda_j(\mathbf{q})-i\lambda_+(\mathbf{k})).$$

This implies that $j=+$ in order for the delta function not to be restricted to the point $\mathbf{k}=0$ (we assume $\lambda_+(\mathbf{k})=0$ implies $\mathbf{k}=0$). So using (4.23), we obtain that

$$\langle\mathbf{c}_+^*(\mathbf{k})\mathcal{L}_1W_1(\mathbf{k})\mathbf{c}_+(\mathbf{k})\rangle = \Big(\Sigma(\mathbf{k})+i\Pi(\mathbf{k})\Big)a_+(\mathbf{k}) - \int_{\mathbb{R}^d}\sigma(\mathbf{k},\mathbf{q})a_+(\mathbf{q})\delta\Big(\omega_+(\mathbf{q})-\omega_+(\mathbf{k})\Big)d\mathbf{q},$$

where we have defined the scattering coefficients:

$$
\begin{aligned}
\Sigma(\mathbf{k}) &= \frac{\pi\omega_+^2(\mathbf{k})}{2(2\pi)^d}\int_{\mathbb{R}^d}\frac{\hat{R}^{11}+\hat{R}^{22}}{2}(\mathbf{k}-\mathbf{q})\delta\Big(\omega_+(\mathbf{q})-\omega_+(\mathbf{k})\Big)d\mathbf{q},\\
i\Pi(\mathbf{k}) &= \frac{1}{4(2\pi)^d}\ \text{p.v.}\int_{\mathbb{R}^d}(\hat{R}^{11}-\hat{R}^{22})(\mathbf{k}-\mathbf{q})\sum_{i=\pm}\frac{\lambda_+(\mathbf{k})\lambda_i(\mathbf{q})}{\lambda_+(\mathbf{k})-\lambda_i(\mathbf{q})}d\mathbf{q},\\
\sigma(\mathbf{k},\mathbf{q}) &= \frac{\pi\omega_+^2(\mathbf{k})}{2(2\pi)^d}\hat{R}^{12}(\mathbf{k}-\mathbf{q}).
\end{aligned}
\tag{4.25}
$$

The radiative transfer equation for $a_+$ is thus

$$
\frac{\partial a_+}{\partial t}+\{\omega_+,a_+\}(\mathbf{x},\mathbf{k})+(\Sigma(\mathbf{k})+i\Pi(\mathbf{k}))a_+=\int_{\mathbb{R}^d}\sigma(\mathbf{k},\mathbf{q})a_+(\mathbf{q})\delta\Big(\omega_+(\mathbf{q})-\omega_+(\mathbf{k})\Big)d\mathbf{q}. \tag{4.26}
$$

In the case where the two media are identical and $p(i\mathbf{k}) = -|\mathbf{k}|^2$ so that $q_0(i\mathbf{k}) = |\mathbf{k}|$ and $\omega_+(\mathbf{k}) = -c_0|\mathbf{k}|$, we retrieve the classical radiative transfer equation for acoustic wave propagation [24], whereas (4.26) generalizes the kinetic model obtained in [8].

## 5 Spatio-temporal Wigner transform

The Wigner transform defined in (3.4) captures the spatial oscillations of frequency of order $\varepsilon^{-1}$ of the vector field $\mathbf{u}_\varepsilon$. The temporal oscillations are not considered explicitly although the dispersion relation associated to (2.8) implies that $i\omega$ is an eigenvalue of $P_0$.

When the dispersion relation is rendered more complicated by the presence of a non-local pseudo-differential operator in the time variable instead of $\varepsilon\partial_t$, the Wigner transform defined in (3.4) fails to account for the connection between the temporal and spatial oscillations. In order to capture the correct dispersion relation, a larger phase space needs to be introduced. It is parameterized by the variables $(t,\omega,\mathbf{x},\mathbf{k})$ where $\mathbf{k}$ is the dual variable to $\mathbf{x}$ as before and $\omega$ is now the dual variable to $t$. The spatio-temporal Wigner transform, which allows one to capture oscillations of frequency $\omega = O(\varepsilon^{-1})$ and wavenumber $|\mathbf{k}| = O(\varepsilon^{-1})$ is then defined for two vector fields $\mathbf{u}(t,\mathbf{x})$ and $\mathbf{v}(t,\mathbf{x})$ as

$$
W[\mathbf{u},\mathbf{v}](t,\omega,\mathbf{x},\mathbf{k})=\int_{\mathbb{R}^{d+1}}e^{i\mathbf{k}\cdot\mathbf{y}+i\tau\omega}\mathbf{u}(t-\frac{\varepsilon\tau}{2},\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2})\mathbf{v}^*(t+\frac{\varepsilon\tau}{2},\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2})\frac{d\mathbf{y}d\tau}{(2\pi)^{d+1}}. \tag{5.1}
$$

Let us illustrate the use of the spatio-temporal Wigner transform by considering the following constant coefficient equation

$$
R(\varepsilon D_t)\mathbf{u}_\varepsilon(t,\mathbf{x})+P(\varepsilon\mathbf{D}_\mathbf{x})\mathbf{u}_\varepsilon(t,\mathbf{x})=0, \tag{5.2}
$$

with appropriate conditions at infinity in the time variable, where $R(i\omega)$ is a Fourier multiplier. For $R(i\omega) = i\omega$, we retrieve the Cauchy problem considered in the preceding section. For such an equation as (5.2), the relation (4.7) for $W_\varepsilon$ defined in (4.6) no longer holds and it is unclear how it could be extended. Finding an equation for $W[\mathbf{u}_\varepsilon,\mathbf{u}_\varepsilon](t,\omega,\mathbf{x},\mathbf{k})$ as defined in (5.1) is however straightforward. Indeed we deduce from (5.2) that

$$W[R(\varepsilon D_t)\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon] + W[P(\varepsilon \mathbf{D}_\mathbf{x})\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon] = 0. \tag{5.3}$$

Now the differential calculus recalled in (3.7) allows us to infer that

$$\Big(R(i\omega + \frac{\varepsilon D_t}{2}) + P(i\mathbf{k} + \frac{\varepsilon \mathbf{D}_\mathbf{x}}{2})\Big) W_\varepsilon(t, \omega, \mathbf{x}, \mathbf{k}) = 0, \tag{5.4}$$

where $W_\varepsilon = W[\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon]$. We therefore obtain a closed equation for $W_\varepsilon(t, \omega, \mathbf{x}, \mathbf{k})$. In the limit $\varepsilon \to 0$, we deduce that $R(i\omega) + P(i\mathbf{k}) = 0$ on the support of the Wigner transform. This is the dispersion relation for the temporal and spatial oscillations of the vector field $\mathbf{u}_\varepsilon$.

**Relationship to the spatial Wigner transform.** We verify from the definition of both Wigner transforms that

$$W_\varepsilon(t, \mathbf{x}, \mathbf{k}) = \int_{\mathbb{R}} W_\varepsilon(t, \omega, \mathbf{x}, \mathbf{k}) d\omega. \tag{5.5}$$

In general situations, it does not seem possible to directly obtain an equation for $W_\varepsilon(t, \mathbf{x}, \mathbf{k})$ from the equation for $W_\varepsilon(t, \omega, \mathbf{x}, \mathbf{k})$. This is however possible when $R(i\omega) = i\omega$. Indeed, we also deduce from (5.2) that $W_\varepsilon(t, \omega, \mathbf{x}, \mathbf{k})(\bar{R}(i\omega - \frac{\varepsilon D_t}{2}) + P^*(i\mathbf{k} - \frac{\varepsilon \mathbf{D}_\mathbf{x}}{2})) = 0$. Upon summing the above equation with (5.4) when $R(i\omega) = i\omega$, we obtain

$$\varepsilon \frac{\partial W_\varepsilon}{\partial t}(t, \omega, \mathbf{x}, \mathbf{k}) + P(i\mathbf{k} + \frac{\varepsilon \mathbf{D}_\mathbf{x}}{2}) W_\varepsilon(t, \omega, \mathbf{x}, \mathbf{k}) + W_\varepsilon(t, \omega, \mathbf{x}, \mathbf{k}) P^*(i\mathbf{k} - \frac{\varepsilon \mathbf{D}_\mathbf{x}}{2}) = 0. \tag{5.6}$$

It remains to integrate the above equation in $\omega$ to obtain an equation for $W_\varepsilon(t, \mathbf{x}, \mathbf{k})$.

**Approximating initial conditions.** The spatio-temporal Wigner transform satisfies a first-order evolution equation in (5.6) and we will show in subsequent sections that its high frequency limit satisfies a kinetic evolution equation as well. It is therefore useful to estimate its initial value, at least approximatively, from the wave fields.

Let $\phi(t, \mathbf{x}) \in \mathcal{C}_0^\infty(\mathbb{R}^{d+1})$ be a smooth compactly supported function of unit mass so that $\int_{\mathbb{R}^{d+1}} |\phi|^2(t, \mathbf{x}) dt d\mathbf{x} = 1$. We define the rescaled version

$$\phi_\varepsilon(t, \mathbf{x}) = \frac{1}{\varepsilon^{\alpha \frac{d+1}{2}}} \phi(\frac{t}{\varepsilon^\alpha}, \frac{\mathbf{x} - \mathbf{x}_0}{\varepsilon^\alpha}), \quad 0 < \alpha < 1. \tag{5.7}$$

Using (5.1), we verify that

$$W[\phi_\varepsilon \mathbf{u}, \phi_\varepsilon \mathbf{v}](t, \omega, \mathbf{x}, \mathbf{k}) = |\phi_\varepsilon|^2(t, \mathbf{x}) W[\mathbf{u}, \mathbf{v}](t, \omega, \mathbf{x}, \mathbf{k}) + O(\varepsilon |\nabla \phi_\varepsilon|), \tag{5.8}$$

and that the second term is smaller than the first one. Upon sending $\varepsilon$ to 0 in the high frequency limit, we observe that $\frac{1}{\varepsilon^{\alpha(d+1)}} |\phi|^2(\frac{t}{\varepsilon^\alpha}, \frac{\mathbf{x}-\mathbf{x}_0}{\varepsilon^\alpha}) \to \delta(t)\delta(\mathbf{x} - \mathbf{x}_0)$. The right-hand side in (5.8) allows us to estimate the "initial condition" $W[\mathbf{u}, \mathbf{v}](0, \omega, \mathbf{x}_0, \mathbf{k})$ up to an error of order $\varepsilon^{1-\alpha} \to 0$ from the left-hand side, which probes the fields $\mathbf{u}$ and $\mathbf{v}$ in the $\varepsilon^\alpha$ vicinity of $t = 0$ and $\mathbf{x} = \mathbf{x}_0$.

Note that if one starts from initial conditions for a problem of the form (2.8), it is easy to verify that propagation over times $\varepsilon^\alpha \ll \varepsilon^{1/2}$ is well-approximated by propagation in a homogeneous medium (up to an error of order $\varepsilon^{\alpha - 1/2}$). So for $\alpha > 1/2$, we can easily estimate the left-hand side in (5.8) from prescribed initial conditions for the wave fields.

## 6 Kinetic model for the spatio-temporal Wigner transform

In this section we consider the following generalization of (4.2):

$$R(\varepsilon D_t)\mathbf{u}_\varepsilon^\varphi + A_\varepsilon^\varphi \mathbf{u}_\varepsilon^\varphi = 0, \qquad \varphi = 1,2, \tag{6.1}$$

where $A_\varepsilon^\varphi$ is defined in (4.3) and $R(i\omega)$ is a smooth function satisfying the symmetry

$$\bar{R}(i\omega) = -R(i\omega). \tag{6.2}$$

Although other dispersive effects may also be accounted for, $R(i\omega)$ can model the (non-local) temporal discretization of $\partial_t$ and $p(i\mathbf{k})$ the spatial discretization of $\Delta_\mathbf{x}$. It is then important to quantify how the constitutive parameters of the radiative transfer equation (4.26) are modified by spatial and temporal discretizations of the wave field. The use of pseudo-differential operators to understand finite difference schemes has a long history [20,21,23]. Consider for instance the simplest example of a second-order discretization in time and space of (2.8) in dimension $d = 2$ [12], with $\varepsilon\Delta$ and $\varepsilon h$ the time step and grid size, respectively. The resulting finite difference equation may be recast in the framework of (6.1) with operators $R(D_t)$ and $q(\mathbf{D_x})$ having the respective symbols:

$$R(i\omega) = i\frac{\sin\omega\Delta}{\Delta}, \qquad q(i\mathbf{k}) = \frac{2\cos hk_x + 2\cos hk_y - 4}{h^2}, \qquad \mathbf{k} = (k_x, k_y). \tag{6.3}$$

Note that $R^2(\varepsilon D_t)p_\varepsilon = c_\varepsilon^2(\mathbf{x})q(\varepsilon\mathbf{D})p_\varepsilon$ is the second-order discretization of the wave equation with time step $2\varepsilon\Delta$ and spatial grid $\varepsilon h$, whose kinetic model will be treated in section 7.

**Equation for the Wigner transform.** We consider the Wigner transform

$$W_\varepsilon(t,\omega,\mathbf{x},\mathbf{k}) = W[\mathbf{u}_\varepsilon^1, \mathbf{u}_\varepsilon^2](t,\omega,\mathbf{x},\mathbf{k}). \tag{6.4}$$

Using the calculus and notation of section 4, we deduce that $W_\varepsilon$ solves the following equations

$$\begin{aligned} R(i\omega + \frac{\varepsilon}{2}D_t)W_\varepsilon + P(i\mathbf{k} + \frac{\varepsilon\mathbf{D_x}}{2})W_\varepsilon + \sqrt{\varepsilon}\mathcal{K}_\varepsilon^1 K W_\varepsilon &= 0 \\ \bar{R}(i\omega - \frac{\varepsilon}{2}D_t)W_\varepsilon + W_\varepsilon P^*(i\mathbf{k} - \frac{\varepsilon\mathbf{D_x}}{2}) + \sqrt{\varepsilon}\mathcal{K}_\varepsilon^2 W_\varepsilon K^* &= 0. \end{aligned} \tag{6.5}$$

The operators $\mathcal{K}_\varepsilon^\varphi$ involve the fast scale $\mathbf{y} = \varepsilon^{-1}\mathbf{x}$. However the fast scale $\varepsilon^{-1}t$ does not appear and thus need not be introduced. We thus consider the following two-scale version $W_\varepsilon(t,\omega,\mathbf{x},\mathbf{k}) = W_\varepsilon(t,\omega,\mathbf{x},\frac{\mathbf{y}}{\varepsilon},\mathbf{k})$ (still using the notation $W_\varepsilon$ in the new variables $(t,\omega,\mathbf{x},\mathbf{y},\mathbf{k})$). The latter satisfies the following equations

$$\begin{aligned} R(i\omega + \frac{\varepsilon}{2}D_t)W_\varepsilon + P(i\mathbf{k} + \frac{\mathbf{D_y}}{2} + \frac{\varepsilon\mathbf{D_x}}{2})W_\varepsilon + \sqrt{\varepsilon}\mathcal{K}^1 K W_\varepsilon &= 0 \\ \bar{R}(i\omega - \frac{\varepsilon}{2}D_t)W_\varepsilon + W_\varepsilon P^*(i\mathbf{k} - \frac{\mathbf{D_y}}{2} - \frac{\varepsilon\mathbf{D_x}}{2}) + \sqrt{\varepsilon}\mathcal{K}^2 W_\varepsilon K^* &= 0. \end{aligned} \tag{6.6}$$

We now plug into (6.6) the following expansion

$$W_\varepsilon(t,\mathbf{x},\mathbf{y},\mathbf{k}) = W_0(t,\mathbf{x},\mathbf{k}) + \sqrt{\varepsilon}W_1(t,\mathbf{x},\mathbf{y},\mathbf{k}) + \varepsilon W_2(t,\mathbf{x},\mathbf{y},\mathbf{k}), \tag{6.7}$$

and equate like powers of $\varepsilon$. The three leading equations provide us with a kinetic model for the leading term $W_0(t, \mathbf{x}, \mathbf{k})$ as follows.

**Leading order and dispersion relation.** The equations of order $O(1)$ are given by

$$\Big(R(i\omega) + P(i\mathbf{k})\Big)W_0 = 0, \qquad W_0\Big(\bar{R}(i\omega) + P^*(i\mathbf{k})\Big) = 0. \tag{6.8}$$

Upon summing these equations and recalling the symmetry properties of $R$, we obtain that

$$P(i\mathbf{k})W_0 + W_0 P^*(i\mathbf{k}) = 0.$$

This implies as in section 4 that

$$W_0(t, \omega, \mathbf{x}, \mathbf{k}) = a_+(t, \omega, \mathbf{x}, \mathbf{k})\mathbf{b}_+(\mathbf{k})\mathbf{b}_+^*(\mathbf{k}) + a_-(t, \omega, \mathbf{x}, \mathbf{k})\mathbf{b}_-(\mathbf{k})\mathbf{b}_-^*(\mathbf{k}). \tag{6.9}$$

The relations in (6.8) implies that

$$W_0(t, \omega, \mathbf{x}, \mathbf{k}) = a_+(t, \mathbf{x}, \mathbf{k})\delta(\omega - \omega_+(\mathbf{k}))\mathbf{b}_+(\mathbf{k})\mathbf{b}_+^*(\mathbf{k}) + a_-(t, \mathbf{x}, \mathbf{k})\delta(\omega - \omega_-(\mathbf{k}))\mathbf{b}_-(\mathbf{k})\mathbf{b}_-^*(\mathbf{k}), \tag{6.10}$$

where the values $\omega_\pm(\mathbf{k})$ are obtained by solving the dispersion relation:

$$R(i\omega_\pm(\mathbf{k})) + \lambda_\pm(\mathbf{k}) = 0. \tag{6.11}$$

We assume in this section that these equations are uniquely solvable. This can be verified for many classical discretizations of the wave equation provided that sufficiently high frequencies are neglected. Non-uniquely solvable dispersion relations will be considered in detail in section 7. Since $\lambda_-(\mathbf{k}) = -\lambda_+(\mathbf{k})$, we verify then that $\omega_+(\mathbf{k}) = -\omega_-(\mathbf{k})$ since they satisfy the same uniquely solvable equation.

It thus remains to derive equations for the propagating modes $a_\pm(t, \mathbf{x}, \mathbf{k})$. Because all the components of $\mathbf{u}_\varepsilon^\varphi$ are real-valued, (4.16) holds and it is sufficient to find an equation for $a_+(t, \mathbf{x}, \mathbf{k})$. The rest of the derivation is similar to that of section 4 and we outline the differences.

**First-order corrector.** The equations of order $O(\sqrt{\varepsilon})$ provide

$$\begin{aligned} \Big(R(i\omega) + P(i\mathbf{k} + \frac{\mathbf{D}_y}{2})\Big)W_1 + \tfrac{1}{2}\theta(\omega)W_1 + \mathcal{K}_1 K W_0 &= 0 \\ W_1\Big(\bar{R}(i\omega) + P^*(i\mathbf{k} - \frac{\mathbf{D}_y}{2})\Big) + \tfrac{1}{2}\theta(\omega)W_1 + \mathcal{K}_2^* W_0 K^* &= 0. \end{aligned} \tag{6.12}$$

The regularization parameter $\theta(\omega)$ is allowed to depend on $\omega$. To preserve causality we need $|\theta(\omega)| \ll 1$ and $\operatorname{sign}(\theta(\omega)) = -\operatorname{sign}(\omega)$. Upon summing the latter two equations we obtain

$$P(i\mathbf{k} + \frac{\mathbf{D}_y}{2})W_1 + W_1 P^*(i\mathbf{k} - \frac{\mathbf{D}_y}{2}) + \theta W_1 + \mathcal{K}_1 K W_0 + \mathcal{K}_2^* W_0 K^* = 0. \tag{6.13}$$

Except for the presence of the frequency variable $\omega$, this is the same equation as in section 4. We can thus decompose the Fourier transform $\mathbf{y} \to \mathbf{p}$ of $W_1$ as

$$\hat{W}_1(\omega, \mathbf{p}, \mathbf{k}) = \sum_{i,j=\pm} \alpha_{ij}(\mathbf{p}, \mathbf{k}, \omega)\mathbf{b}_i(\mathbf{k} + \frac{\mathbf{p}}{2})\mathbf{b}_j^*(\mathbf{k} - \frac{\mathbf{p}}{2}), \tag{6.14}$$

and obtain that the coefficients $\alpha_{ij}$ are given by

$$\begin{aligned}
&\Big(\lambda_m(\mathbf{k}+\frac{\mathbf{p}}{2})-\lambda_n(\mathbf{k}-\frac{\mathbf{p}}{2})+\theta\Big)\alpha_{mn}(\omega,\mathbf{p},\mathbf{k})\\
&=\frac{1}{2c_0^2}\Big(\hat{V}^1(\mathbf{p})a_n(\mathbf{k}-\frac{\mathbf{p}}{2})\delta\Big(\omega-\omega_n(\mathbf{k}-\frac{\mathbf{p}}{2})\Big)\lambda_m(\mathbf{k}+\frac{\mathbf{p}}{2})\\
&\qquad-\hat{V}^2(\mathbf{p})a_m(\mathbf{k}+\frac{\mathbf{p}}{2})\delta\Big(\omega-\omega_m(\mathbf{k}+\frac{\mathbf{p}}{2})\Big)\lambda_n(\mathbf{k}-\frac{\mathbf{p}}{2})\Big).
\end{aligned} \tag{6.15}$$

**Second-order corrector and equation for $a_+$.** The equations of order $\varepsilon$ finally yield

$$\begin{aligned}
&\Big(R(i\omega)+P(i\mathbf{k}+\frac{\mathbf{D}_y}{2})\Big)W_2+\mathcal{K}_1KW_1+\Big(R'(i\omega)\frac{D_t}{2}+p'(i\mathbf{k})\cdot\frac{\mathbf{D}_x}{2}K^*\Big)W_0=0\\
&W_2\Big(\bar{R}(i\omega)+P^*(i\mathbf{k}-\frac{\mathbf{D}_y}{2})\Big)+\mathcal{K}_2^*W_1K^*-W_0\Big(\bar{R}'(i\omega)\frac{D_t}{2}+\bar{p}'(i\mathbf{k})\cdot\frac{\mathbf{D}_x}{2}K\Big)=0.
\end{aligned} \tag{6.16}$$

Summing the above two equations provides the following constraint

$$\begin{aligned}
&P(i\mathbf{k}+\frac{\mathbf{D}_y}{2})W_2+W_2P^*(i\mathbf{k}-\frac{\mathbf{D}_y}{2})+\mathcal{K}_1KW_1+\mathcal{K}_2^*W_1K^*\\
&\quad+R'(i\omega)D_tW_0+p'(i\mathbf{k})\cdot\frac{\mathbf{D}_x}{2}K^*W_0-W_0\Big(\bar{p}'(i\mathbf{k})\cdot\frac{\mathbf{D}_x}{2}K\Big)=0.
\end{aligned} \tag{6.17}$$

Except for the presence of $R'(i\omega)$, this is the same equation as (4.22) in section 4. The dispersion relation (4.23) is now replaced by (6.11). We verify that the term $\delta(\omega-\omega_+(\mathbf{k}))$, which is equal to $\delta(\omega-\omega_+(\mathbf{q}))$ on the shell $\delta(c_0q_0(i\mathbf{q})-c_0q_0(i\mathbf{k}))$, factorizes in the above expression. Evaluating the density of the equation (6.17) at $\omega=\omega_+(\mathbf{q})$ and following the same steps as in section 4, we find that the mode $a_+(t,\mathbf{x},\mathbf{k})$ satisfies the equation

$$\begin{aligned}
&R'(i\omega_+)\frac{\partial a_+}{\partial t}+\{i\lambda_+,a_+\}(\mathbf{x},\mathbf{k})+\Big(\operatorname{sign}(-\omega_+)\Sigma(\mathbf{k})+i\Pi(\mathbf{k})\Big)a_+\\
&=\operatorname{sign}(-\omega_+)\int_{\mathbb{R}^d}\sigma(\mathbf{k},\mathbf{q})a_+(\mathbf{q})\delta\Big(-i\lambda_+(i\mathbf{q})+i\lambda_+(i\mathbf{k})\Big)d\mathbf{q},
\end{aligned} \tag{6.18}$$

where the constitutive parameters in the above equation are defined in (4.25).

This equation may be simplified as follows. We deduce from the dispersion relation (6.11) and our convention on $R(i\omega)$ that

$$\{i\lambda_+,a_+\}=R'(i\omega_+)\{\omega_+,a_+\},\ \ \delta(R(i\omega)-R(i\omega_0))=\frac{\delta(\omega-\omega_0)}{|R'(i\omega_0)|},\ \ \frac{\operatorname{sign}(-\omega_+)}{R'(i\omega_+)}=\frac{1}{|R'(i\omega_+)|}.$$

This allows us to recast (6.18) as

$$\frac{\partial a_+}{\partial t}+\{\omega_+,a_+\}(\mathbf{x},\mathbf{k})+(\tilde{\Sigma}(\mathbf{k})+i\tilde{\Pi}(\mathbf{k}))a_+=\int_{\mathbb{R}^d}\tilde{\sigma}(\mathbf{k},\mathbf{q})a_+(\mathbf{q})\delta\Big(\omega_+(\mathbf{q})-\omega_+(\mathbf{k})\Big)d\mathbf{q}, \tag{6.19}$$

where the above coefficients are related to those in (4.25) by

$$\tilde{\Sigma}(\mathbf{k})=\frac{\Sigma(\mathbf{k})}{|R'(i\omega_+(\mathbf{k}))|^2},\qquad \tilde{\sigma}(\mathbf{k},\mathbf{q})=\frac{\sigma(\mathbf{k},\mathbf{q})}{|R'(i\omega_+(\mathbf{k}))|^2},\qquad \tilde{\Pi}(\mathbf{k})=\frac{\Pi(\mathbf{k})}{R'(i\omega_+(\mathbf{k}))}. \tag{6.20}$$

The above relations explicitly quantify the influence of dispersive effects caused by e.g. spatial and temporal discretizations. Two main differences may be observed: firstly, the dispersion relation (6.11) replaces (4.23). Secondly, the waves interactions with the underlying medium are governed by the constitutive parameters given in (6.20) rather than (4.25). The latter only involve the coefficient $R'(i\omega_+(\mathbf{k}))$. We find for instance for the second-order discretization in time that

$$R'(i\omega) = \cos(\omega\Delta) \qquad \text{when} \qquad R(i\omega) = i\frac{\sin \omega\Delta}{\Delta},$$

at least for sufficiently low frequencies so that the dispersion relation is uniquely solvable. This provides us with a complete picture of how dispersive effects including discretizations of the wave equation modify energy transport and two-field correlations.

## 7 Kinetic model for the scalar wave equation

The methodology developed in the preceding section to obtain a kinetic equation for the spatio-temporal Wigner transform may also be directly applied to the scalar wave equation (2.3), which we recast more generally as

$$R(\varepsilon D_t)p_\varepsilon + \mathcal{H}_\varepsilon p_\varepsilon = 0, \qquad \mathcal{H}_\varepsilon = b_\varepsilon(\mathbf{x})\beta(\varepsilon\mathbf{D}_\mathbf{x})d_\varepsilon(\mathbf{x})\gamma(\varepsilon\mathbf{D}_\mathbf{x}). \tag{7.1}$$

Note that kinetic models for scalar second-order equations, whether dispersive effects are present or not, cannot be obtained using the methods developed in [15,22,24]. We need to use the spatio-temporal Wigner transform introduced in section 5. The principal symbol of the high frequency limit of $\mathcal{H}_\varepsilon$ is defined by

$$H(\mathbf{x},\mathbf{k}) = b_0(\mathbf{x})d_0(\mathbf{x})\beta(i\mathbf{k})\gamma(i\mathbf{k}). \tag{7.2}$$

The conditions on the operators appearing in the above equation are as follows. To preserve the second-order character of (7.1), we consider the case

$$R(i\omega) = \bar{R}(i\omega), \tag{7.3}$$

for instance when $R(i\omega) = S(\omega^2)$ in the case of the scalar wave equation. This relation is to be compared with (6.2) in the preceding section. We denote by $\omega_n(\mathbf{x},\mathbf{k})$ the solutions of the dispersion relation

$$R(i\omega) + H(\mathbf{x},\mathbf{k}) = 0. \tag{7.4}$$

We assume that all solutions $\omega_n(\mathbf{x},\mathbf{k})$ are real-valued. When $R(i\omega) = S(\omega^2)$, they come in pairs $\omega_{-n} = \omega_n$. The (even) number of modes indexed by $n$ may be finite or infinite but is assumed to be independent of $(\mathbf{x},\mathbf{k})$. This generalizes the case considered in the preceding section where $n = \pm 1$. We also assume that the operator $\gamma(\varepsilon D_\mathbf{x})$ is vector-valued (such as $\gamma(i\mathbf{k}) = i\mathbf{k}$) and $\beta(\varepsilon\mathbf{D}_\mathbf{x}) = \gamma^*(\varepsilon\mathbf{D}_\mathbf{x})$ is its adjoint operator (such that e.g. the Fourier transform of $\beta(\varepsilon\mathbf{D}_\mathbf{x})\mathbf{u}(\mathbf{x})$ is $-i\mathbf{k}\cdot\hat{\mathbf{u}}(\mathbf{k})$). Both $\beta$ and $\gamma$ are real-valuedness preserving and the above assumptions imply that $(\beta\gamma)(i\mathbf{k}) \geq 0$. Classical scalar equations and their discretizations by finite differences that fit into the above framework include the scalar wave equation with $R(i\omega) = -\omega^2$, the linear Klein-Gordon equation with $R(i\omega) = -\omega^2 + \alpha^2$ for some $\alpha > 0$, and the Schrödinger equation with $R(i\omega) = \omega$. In all cases, $\gamma(i\mathbf{k}) = i\mathbf{k}$ and $\beta(i\mathbf{k})$ is the symbol of the adjoint operator.

To be consistent with the weak-coupling regime considered earlier, we also assume that

$$b_\varepsilon(\mathbf{x}) = b_0(\mathbf{x}) + \sqrt{\varepsilon} b_1(\frac{\mathbf{x}}{\varepsilon}), \qquad d_\varepsilon(\mathbf{x}) = d_0(\mathbf{x}) + \sqrt{\varepsilon} d_1(\frac{\mathbf{x}}{\varepsilon}), \tag{7.5}$$

where $b_0(\mathbf{x})$ and $d_0(\mathbf{x})$ are smooth functions and $b_1(\mathbf{x})$ and $d_1(\mathbf{x})$ are mean-zero stationary random fields. The power spectra are defined by

$$(2\pi)^d x_0 y_0 \hat{R}_{xy}(\mathbf{p})\delta(\mathbf{p}+\mathbf{q}) = \langle \hat{x}_1(\mathbf{p})\hat{y}_1(\mathbf{q})\rangle, \qquad (x,y) \in \{(b,b),(b,d),(d,d)\}. \tag{7.6}$$

Here $b_0$ and $d_0$ are normalizing constants equal to $b_0(\mathbf{x})$ and $d_0(\mathbf{x})$, respectively, when the latter are constant functions.

We finally define the Wigner transform:

$$W_\varepsilon(t,\omega,\mathbf{x},\mathbf{k}) = W[p_\varepsilon, p_\varepsilon](t,\omega,\mathbf{x},\mathbf{k}). \tag{7.7}$$

To avoid carrying through too many indices, we first consider the energy density associated to one single vector field and generalize the kinetic equation to the correlation of two fields propagating in two different media at the end of the section. We again have the two relations

$$W[R(\varepsilon D_t)p_\varepsilon, p_\varepsilon] + W[\mathcal{H}_\varepsilon p_\varepsilon, p_\varepsilon] = 0, \qquad W[p_\varepsilon, R(\varepsilon D_t)p_\varepsilon] + W[p_\varepsilon, \mathcal{H}_\varepsilon p_\varepsilon] = 0. \tag{7.8}$$

We verify that

$$\begin{aligned} W[R(\varepsilon D_t)p_\varepsilon, p_\varepsilon] &= R(i\omega)W_\varepsilon + \frac{\varepsilon R'(i\omega)}{2}\frac{\partial W_\varepsilon}{\partial t} + O(\varepsilon^2) \\ W[p_\varepsilon, R(\varepsilon D_t)p_\varepsilon] &= R(i\omega)W_\varepsilon - \frac{\varepsilon R'(i\omega)}{2}\frac{\partial W_\varepsilon}{\partial t} + O(\varepsilon^2). \end{aligned} \tag{7.9}$$

**Some pseudo-differential calculus.** Successive applications of (3.7) and (3.12) provide that

$$\begin{aligned} &W[b_0(\mathbf{x})\beta(\varepsilon\mathbf{D_x})d_0(\mathbf{x})\gamma(\varepsilon\mathbf{D_x})p_\varepsilon, p_\varepsilon] = \mathcal{L}_\mathcal{H} W_\varepsilon + O(\varepsilon^2) \\ &\mathcal{L}_\mathcal{H} = \Big(b_0(\mathbf{x}) + \frac{i\varepsilon}{2}\nabla_\mathbf{x} b_0(\mathbf{x})\cdot\nabla_\mathbf{k}\Big)\beta(i\mathbf{k}+\varepsilon\frac{\mathbf{D}}{2})\Big(d_0(\mathbf{x}) + \frac{i\varepsilon}{2}\nabla_\mathbf{x} d_0(\mathbf{x})\cdot\nabla_\mathbf{k}\Big)\gamma(i\mathbf{k}+\varepsilon\frac{\mathbf{D}}{2}). \end{aligned}$$

For functions $W$ that do not oscillate rapidly in the $\mathbf{x}$ variable, we obtain as in (3.12) that

$$\mathcal{L}_\mathcal{H} W = HW + \frac{i\varepsilon}{2}\{H,W\}(\mathbf{x},\mathbf{k}) + \frac{i\varepsilon}{2}[d_0\nabla_\mathbf{x} b_0(\mathbf{x})\cdot\nabla_\mathbf{k}(\beta\gamma)(i\mathbf{k})]W + O(\varepsilon^2). \tag{7.10}$$

Here we have used explicitly that $(\nabla_\mathbf{k}\beta)\gamma - \beta\nabla_\mathbf{k}\gamma = 0$, which holds because $\beta = \gamma^*$. When the latter does not hold, a term proportional to $((\nabla_\mathbf{k}\beta)\gamma - \beta\nabla_\mathbf{k}\gamma)W$ appears in (7.10). The reason for the presence of the zeroth order term $[d_0\nabla_\mathbf{x} b_0(\mathbf{x})\cdot\nabla_\mathbf{k}(\beta\gamma)(i\mathbf{k})]W$ in (7.10) is that $W$ as defined in (7.7) does not allow one to preserve energy (in the sense that $t \mapsto \int_{\mathbb{R}^{2d+1}} W(t,\omega,\mathbf{x},\mathbf{k})d\omega d\mathbf{x}d\mathbf{k}$ is constant). It is rather $b_0^{-1}(\mathbf{x})W$, or the latter expression multiplied by any function of the Hamiltonian $H(\mathbf{x},\mathbf{k})$, that preserves energy; see (7.15) and (8.1) below. The condition $\beta = \gamma^*$, which implies that the divergence and gradient operators in (2.3) should be discretized in a consistent manner, should then be interpreted as a sufficient condition for the kinetic model to preserve energy.

When $\mathcal{L}_{\mathcal{H}}$ is applied on highly oscillatory functions of the form $W(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}, \mathbf{k})$ we find that

$$\mathcal{L}_{\mathcal{H}}[W(\mathbf{x},\mathbf{y},\mathbf{k})](\mathbf{x},\mathbf{k}) = H(\mathbf{x},\mathbf{k}+\frac{-i\mathbf{D}_{\mathbf{y}}}{2})W(\mathbf{x},\mathbf{y},\mathbf{k})\Big|_{\mathbf{y}=\frac{\mathbf{x}}{\varepsilon}} + O(\varepsilon). \tag{7.11}$$

We will use both asymptotic expansions (7.10) and (7.11) in the sequel.

We now account for the highly oscillatory fluctuations in the random media. We verify using (3.16) that

$$\begin{aligned}
&W[b_1(\frac{\mathbf{x}}{\varepsilon})\beta(\varepsilon\mathbf{D}_{\mathbf{x}})d_0(\mathbf{x})\gamma(\varepsilon\mathbf{D}_{\mathbf{x}})p_\varepsilon, p_\varepsilon] = \mathcal{B}_\varepsilon W_\varepsilon + O(\varepsilon)\\
&\mathcal{B}_\varepsilon W = d_0(\mathbf{x})\int_{\mathbb{R}^d} e^{i\frac{\mathbf{x}\cdot\mathbf{p}}{\varepsilon}}\hat{b}_1(\mathbf{p})(\beta\gamma)(i\mathbf{k}-\frac{i\mathbf{p}}{2}+\frac{\varepsilon\mathbf{D}}{2})W(\mathbf{x},\mathbf{k}-\frac{\mathbf{p}}{2})\frac{d\mathbf{p}}{(2\pi)^d},\\
&W[b_0(\mathbf{x})\beta(\varepsilon\mathbf{D}_{\mathbf{x}})d_1(\frac{\mathbf{x}}{\varepsilon})\gamma(\varepsilon\mathbf{D}_{\mathbf{x}})p_\varepsilon, p_\varepsilon] = \mathcal{D}_\varepsilon W_\varepsilon + O(\varepsilon)\\
&\mathcal{D}_\varepsilon W = \beta(i\mathbf{k}+\frac{\varepsilon\mathbf{D}}{2})b_0(\mathbf{x})\int_{\mathbb{R}^d} e^{i\frac{\mathbf{x}\cdot\mathbf{p}}{\varepsilon}}\hat{d}_1(\mathbf{p})\gamma(i\mathbf{k}-\frac{i\mathbf{p}}{2}+\frac{\varepsilon\mathbf{D}}{2})W(\mathbf{x},\mathbf{k}-\frac{\mathbf{p}}{2})\frac{d\mathbf{p}}{(2\pi)^d}.
\end{aligned}$$

We have carefully kept the contributions of the form $\varepsilon\mathbf{D}/2$ in the above symbols. When $\mathcal{B}_\varepsilon$ and $\mathcal{D}_\varepsilon$ are applied to functions that do not oscillate rapidly in the $\mathbf{x}$ variable, then these contributions may be dropped without changing the $O(\varepsilon)$ accuracy, which will be sufficient in what follows. However, when applied to functions of the form $W(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}, \mathbf{k})$, we find that

$$\begin{aligned}
\mathcal{B}_\varepsilon[W(\mathbf{x},\mathbf{y},\mathbf{k})](\mathbf{x},\mathbf{k}) &= d_0(\mathbf{x})\int_{\mathbb{R}^d} e^{i\mathbf{y}\cdot\mathbf{p}}\hat{b}_1(\mathbf{p})(\beta\gamma)(i\mathbf{k}-\frac{i\mathbf{p}}{2}+\frac{\mathbf{D}_{\mathbf{y}}}{2})W(\mathbf{x},\mathbf{y},\mathbf{k}-\frac{\mathbf{p}}{2})\frac{d\mathbf{p}}{(2\pi)^d}\Big|_{\mathbf{y}=\frac{\mathbf{x}}{\varepsilon}},\\
\mathcal{D}_\varepsilon[W(\mathbf{x},\mathbf{y},\mathbf{k})](\mathbf{x},\mathbf{k}) &= \beta(i\mathbf{k}+\frac{\mathbf{D}_{\mathbf{y}}}{2})b_0(\mathbf{x})\int_{\mathbb{R}^d} e^{i\mathbf{y}\cdot\mathbf{p}}\hat{d}_1(\mathbf{p})\gamma(i\mathbf{k}-\frac{i\mathbf{p}}{2}+\frac{\mathbf{D}_{\mathbf{y}}}{2})W(\mathbf{x},\mathbf{y},\mathbf{k}-\frac{\mathbf{p}}{2})\frac{d\mathbf{p}}{(2\pi)^d}\Big|_{\mathbf{y}=\frac{\mathbf{x}}{\varepsilon}},
\end{aligned}$$

up to $O(\varepsilon)$ terms. Finally we check that

$$\begin{aligned}
&W[b_1(\frac{\mathbf{x}}{\varepsilon})\beta(\varepsilon\mathbf{D}_{\mathbf{x}})d_1(\frac{\mathbf{x}}{\varepsilon})\gamma(\varepsilon\mathbf{D}_{\mathbf{x}})u_\varepsilon, u_\varepsilon] = \mathcal{C}_\varepsilon W_\varepsilon\\
&\mathcal{C}_\varepsilon W(\mathbf{x},\mathbf{k}) = \beta(i\mathbf{k})\gamma(i\mathbf{k})\int e^{i\frac{\mathbf{p}+\mathbf{q}}{\varepsilon}\cdot\mathbf{x}}\hat{b}_1(\mathbf{p})\hat{d}_1(\mathbf{q})W(\mathbf{x},\mathbf{k}-\frac{\mathbf{p}+\mathbf{q}}{2})\frac{d\mathbf{p}d\mathbf{q}}{(2\pi)^{2d}} + O(\varepsilon).
\end{aligned}$$

The error estimate $O(\varepsilon)$ becomes an estimate of the form $O(1)$ when $\mathcal{C}_\varepsilon$ is applied to $W(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}, \mathbf{k})$. Such an estimate is however sufficient for what follows.

Expansions for $W[p_\varepsilon, \mathcal{H}_\varepsilon p_\varepsilon]$ are obtained in a similar manner. Formally, the adjoint operator are obtained from the above formulas by replacing $\mathbf{k}-\frac{\mathbf{p}}{2}$ by $\mathbf{k}+\frac{\mathbf{p}}{2}$ and any derivative $\mathbf{D}_{\mathbf{y}}$ by $-\mathbf{D}_{\mathbf{y}}$. The Poisson bracket is modified as shown in (3.15). These expansions allow us to obtain the the two constraints on $W_\varepsilon$:

$$\begin{aligned}
\Big[R(i\omega+\frac{\varepsilon D_t}{2})+\mathcal{L}_{\mathcal{H}}+\sqrt{\varepsilon}(\mathcal{B}_\varepsilon+\mathcal{D}_\varepsilon)+\varepsilon\mathcal{C}_\varepsilon\Big]W_\varepsilon &= O(\varepsilon^{3/2})\\
\Big[R(i\omega-\frac{\varepsilon D_t}{2})+\mathcal{L}^*_{\mathcal{H}}+\sqrt{\varepsilon}(\mathcal{B}^*_\varepsilon+\mathcal{D}^*_\varepsilon)+\varepsilon\mathcal{C}^*_\varepsilon\Big]W_\varepsilon &= O(\varepsilon^{3/2}).
\end{aligned} \tag{7.12}$$

We now use the previous asymptotic expansions and (6.7) to find an equation for $W_0(t,\omega,\mathbf{x},\mathbf{k})$.

**Dispersion relation.** The leading term in (7.8) or (7.12) provides that

$$\Big(R(i\omega) + H(\mathbf{x}, \mathbf{k})\Big) W(t, \omega, \mathbf{x}, \mathbf{k}) = 0. \tag{7.13}$$

This implies that $W(t, \omega, \mathbf{x}, \mathbf{k})$ is a distribution supported on the manifold given by (7.4), which admits $\omega_n(\mathbf{x}, \mathbf{k})$ as distinct solutions by assumption. This allows us to decompose $W_0$ as

$$W_0(t, \omega, \mathbf{x}, \mathbf{k}) = b_0(\mathbf{x}) \sum_n a_n(t, \mathbf{x}, \mathbf{k}) \delta(\omega - \omega_n(\mathbf{x}, \mathbf{k})). \tag{7.14}$$

The presence of $b_0(\mathbf{x})$ in the above equation will appear clearly later on and is related to energy conservation. Note that $a_n(t, \mathbf{x}, \mathbf{k})$ is the density of $W_0(t, \omega, \mathbf{x}, \mathbf{k})$ in the $\omega$ variable in the vicinity of $\omega - \omega_n(\mathbf{x}, \mathbf{k}) = 0$. Thus the values of $a_n(t, \mathbf{x}, \mathbf{k})$ are uniquely determined from knowledge of $W_0(t, \omega, \mathbf{x}, \mathbf{k})$.

**Corrector $W_1$.** Upon subtracting the two equations in (7.8) and (7.9), we obtain that

$$\varepsilon R'(i\omega) \frac{\partial W_\varepsilon}{\partial t} + \mathcal{L}_0 W_\varepsilon + \sqrt{\varepsilon} \mathcal{L}_1 W_\varepsilon + \varepsilon \mathcal{L}_2 W_\varepsilon = O(\varepsilon^2),$$

where the operators $\mathcal{L}_k$ are explicitly deduced from (7.12). Let us recast $W_\varepsilon(t, \omega, \mathbf{x}, \mathbf{k}) = b_0(\mathbf{x}) \tilde{W}(t, \omega, \mathbf{x}, \frac{\mathbf{x}}{\varepsilon}, \mathbf{k})$ and introduce the decomposition

$$\tilde{W}(t, \omega, \mathbf{x}, \mathbf{y}, \mathbf{k}) = \tilde{W}_0(t, \omega, \mathbf{x}, \mathbf{k}) + \sqrt{\varepsilon} \tilde{W}_1(t, \omega, \mathbf{x}, \mathbf{y}, \mathbf{k}) + \varepsilon \tilde{W}_2(t, \omega, \mathbf{x}, \mathbf{y}, \mathbf{k}). \tag{7.15}$$

We deduce from (7.12) that

$$\begin{aligned}&\varepsilon R'(i\omega) \frac{\partial \tilde{W}}{\partial t} + \Big(H(\mathbf{x}, \mathbf{k} + \frac{-i\mathbf{D}_\mathbf{y}}{2}) - H(\mathbf{x}, \mathbf{k} - \frac{-i\mathbf{D}_\mathbf{y}}{2})\Big) \tilde{W} + i\varepsilon \{H(\mathbf{x}, \mathbf{k}), \tilde{W}\}(\mathbf{x}, \mathbf{k}) \\ &+ \sqrt{\varepsilon} \mathcal{L}_1 \tilde{W} + \varepsilon \mathcal{L}_2 \tilde{W} = O(\varepsilon^2).\end{aligned}$$

The equation for $\tilde{W}_1$ is thus

$$\Big(H(\mathbf{x}, \mathbf{k} + \frac{-i\mathbf{D}_\mathbf{y}}{2}) - H(\mathbf{x}, \mathbf{k} - \frac{-i\mathbf{D}_\mathbf{y}}{2})\Big) \tilde{W}_1 + i\theta(\omega) \tilde{W}_1 + \mathcal{L}_1 \tilde{W}_0 = 0. \tag{7.16}$$

Here, we have again introduced a small regularization parameter $\theta(\omega)$. The sign of $\theta(\omega)$ is chosen so that causality is preserved, which implies in the above equation that $\theta(\omega)$ has the same sign as $\omega$. Upon solving (7.16) we find that the Fourier transform of $\tilde{W}_1$ is given by

$$\begin{aligned}\hat{W}_1(\mathbf{p}, \mathbf{k}) &= \frac{\alpha(\mathbf{x}, \mathbf{p}, \mathbf{k} + \frac{\mathbf{p}}{2}, \mathbf{k} - \frac{\mathbf{p}}{2}) \tilde{W}_0(\mathbf{k} - \frac{\mathbf{p}}{2}) - \alpha(\mathbf{x}, \mathbf{p}, \mathbf{k} - \frac{\mathbf{p}}{2}, \mathbf{k} + \frac{\mathbf{p}}{2}) \tilde{W}_0(\mathbf{k} + \frac{\mathbf{p}}{2})}{H(\mathbf{x}, \mathbf{k} - \frac{\mathbf{p}}{2}) - H(\mathbf{x}, \mathbf{k} + \frac{\mathbf{p}}{2}) - i\theta(\omega)} \\ \alpha(\mathbf{x}, \mathbf{p}, \mathbf{k}, \mathbf{q}) &= d_0(\mathbf{x}) \hat{b}_1(\mathbf{p}) \beta(i\mathbf{q}) \gamma(i\mathbf{q}) + b_0(\mathbf{x}) \hat{d}_1(\mathbf{p}) \beta(i\mathbf{k}) \gamma(i\mathbf{q}).\end{aligned} \tag{7.17}$$

**Radiative transfer equation.** The equation for the ensemble average of $\tilde{W}_0$ (still denoted by $\tilde{W}_0$) is then

$$R'(i\omega) \frac{\partial \tilde{W}_0}{\partial t} + i\{H, \tilde{W}_0\} + \langle \mathcal{L}_1 \tilde{W}_1 \rangle + \langle \mathcal{L}_2 \rangle \tilde{W}_0 = 0. \tag{7.18}$$

We verify that $\langle \mathcal{L}_2 \rangle \tilde{W}_0 = 0$ since it involves the term $\beta(i\mathbf{k})\gamma(i\mathbf{k})R_{bd}(0)W_0(\mathbf{x},\mathbf{k})$, which cancels when its complex conjugate term is subtracted. Here $R_{bd}(\mathbf{y})$ is the two point correlation function $\langle b_1(\mathbf{x})d_1(\mathbf{x}+\mathbf{y})\rangle$. Since $\mathrm{sign}(\theta(\omega)) = \mathrm{sign}(\omega)$, we obtain that

$$
\begin{aligned}
\langle \mathcal{L}_1 \tilde{W}_1 \rangle &= i\,\mathrm{sign}(\omega)\int_{\mathbb{R}^d}\Big(B_\Sigma(\mathbf{x},\mathbf{k},\mathbf{q})\tilde{W}_0(\mathbf{k}) - B_\sigma(\mathbf{x},\mathbf{k},\mathbf{q})\tilde{W}_0(\mathbf{q})\Big)\frac{\delta\Big(H(\mathbf{x},\mathbf{k})-H(\mathbf{x},\mathbf{q})\Big)d\mathbf{q}}{(2\pi)^{d-1}}\\
B_\Sigma(\mathbf{x},\mathbf{k},\mathbf{q})\delta(0) &= (2\pi)^{-d}\langle \alpha(\mathbf{x},\mathbf{k}-\mathbf{q},\mathbf{k},\mathbf{q})\alpha(\mathbf{x},\mathbf{q}-\mathbf{k},\mathbf{q},\mathbf{k})\rangle\\
B_\sigma(\mathbf{x},\mathbf{k},\mathbf{q})\delta(0) &= (2\pi)^{-d}\langle \alpha^2(\mathbf{x},\mathbf{q}-\mathbf{k},\mathbf{k},\mathbf{q})\rangle.
\end{aligned}
$$

Note that $\beta(i\mathbf{k})\gamma(i\mathbf{q}) = (\beta(i\mathbf{k})\gamma(i\mathbf{q}))^* = \gamma^*(i\mathbf{q})\beta^*(i\mathbf{k}) = \beta(i\mathbf{q})\gamma(i\mathbf{k})$. This implies that $\alpha(\mathbf{x},\mathbf{k}-\mathbf{q},\mathbf{k},\mathbf{q}) = \alpha(\mathbf{x},\mathbf{q}-\mathbf{k},\mathbf{q},\mathbf{k})$. Since $H(\mathbf{x},\mathbf{k}) - H(\mathbf{x},\mathbf{q}) = 0$, we thus obtain that $B(\mathbf{x},\mathbf{k},\mathbf{q}) \equiv B_\Sigma(\mathbf{x},\mathbf{k},\mathbf{q}) = B_\sigma(\mathbf{x},\mathbf{k},\mathbf{q})$. The expression for $B$ is given more explicitly by

$$
\begin{aligned}
B(\mathbf{x},\mathbf{k},\mathbf{q}) = \Big(& b_0^2 d_0^2(\mathbf{x})\hat{R}_{bb}(\mathbf{k}-\mathbf{q})(\beta\gamma)^2(i\mathbf{k}) + b_0^2(\mathbf{x})d_0^2\hat{R}_{dd}(\mathbf{k}-\mathbf{q})\Big(\beta(i\mathbf{k})\gamma(i\mathbf{q})\Big)^2\\
&+2b_0d_0(b_0d_0)(\mathbf{x})\hat{R}_{bd}(\mathbf{k}-\mathbf{q})(\beta\gamma)(i\mathbf{q})\beta(i\mathbf{k})\gamma(i\mathbf{q})\Big).
\end{aligned}
\tag{7.19}
$$

This gives us the equation

$$
\begin{aligned}
&R'(i\omega)\frac{\partial \tilde{W}_0}{\partial t} + i\{H,\tilde{W}_0\}\\
&+i\,\mathrm{sign}(\omega)\int_{\mathbb{R}^d} B(\mathbf{x},\mathbf{k},\mathbf{q})\Big(\tilde{W}_0(\mathbf{k}) - \tilde{W}_0(\mathbf{q})\Big)\delta\Big(H(\mathbf{x},\mathbf{k})-H(\mathbf{x},\mathbf{q})\Big)\frac{d\mathbf{q}}{(2\pi)^{d-1}} = 0.
\end{aligned}
\tag{7.20}
$$

Now let $(\omega,\mathbf{x},\mathbf{k})$ be such that $\omega = \omega_m(\mathbf{x},\mathbf{k})$, which uniquely defines $m$. Then

$$
\tilde{W}_0(t,\omega,\mathbf{x},\mathbf{k}) = a_m(t,\mathbf{x},\mathbf{k})\delta(\omega - \omega_m(\mathbf{x},\mathbf{k})).
$$

Moreover, $H(\mathbf{x},\mathbf{k}) = -R(i\omega_m(\mathbf{x},\mathbf{k}))$ so that $i\{H,\tilde{W}_0\} = R'(i\omega_m)\{\omega_m,\tilde{W}_0\}$. With our assumptions on $R(i\omega)$, we have that $\delta(R(i\omega) - R(i\omega_0)) = |R'(i\omega_0)|^{-1}\delta(\omega-\omega_0)$ and $i\,\mathrm{sign}(\omega) = \mathrm{sign}(R'(i\omega))$. We next verify that for all $\mathbf{q}$ such that $H(\mathbf{x},\mathbf{q}) = -R(i\omega_n(\mathbf{x},\mathbf{q}))$ and $\omega_n(\mathbf{x},\mathbf{q}) = \omega = \omega_m(\mathbf{x},\mathbf{k})$, we have $\delta(H(\mathbf{x},\mathbf{k}) - H(\mathbf{x},\mathbf{q})) = |R'(i\omega_m(\mathbf{x},\mathbf{k}))|^{-1}\delta(\omega_m(\mathbf{x},\mathbf{k}) - \omega_n(\mathbf{x},\mathbf{q}))$. Collecting the above calculations, we obtain that the density of (7.20) in the variable $\omega$ in the vicinity of $(\omega,\mathbf{x},\mathbf{k})$ such that $\omega = \omega_m(\mathbf{x},\mathbf{k})$ is thus given by the equation

$$
\frac{\partial a_m}{\partial t} + \{\omega_m(\mathbf{x},\mathbf{k}),a_m\} + \Sigma_m(\mathbf{x},\mathbf{k})a_m = \int_{\mathbb{R}^d}\sigma(\mathbf{x},\mathbf{k},\mathbf{q})\sum_n a_n(\mathbf{q})\delta\Big(\omega_n(\mathbf{x},\mathbf{q}) - \omega_m(\mathbf{x},\mathbf{k})\Big)d\mathbf{q}, \tag{7.21}
$$

where we have defined

$$
\begin{aligned}
\Sigma_m(\mathbf{x},\mathbf{k}) &= \int_{\mathbb{R}^d}\sigma(\mathbf{x},\mathbf{k},\mathbf{q})\sum_n \delta\Big(\omega_n(\mathbf{x},\mathbf{q}) - \omega_m(\mathbf{x},\mathbf{k})\Big)d\mathbf{q}\\
\sigma(\mathbf{x},\mathbf{k},\mathbf{q}) &= \frac{B(\mathbf{x},\mathbf{k},\mathbf{q})}{|R'(i\omega_m(\mathbf{x},\mathbf{k}))|^2}\frac{1}{(2\pi)^{d-1}}.
\end{aligned}
\tag{7.22}
$$

Here, the index $m$ runs over all possible modes allowed by the dispersion relation. This provides us with a transport equation for the scalar wave equation, which accounts for fluctuations both in the density and the compressibility, for possible bending of the characteristics of propagation by variations in $\kappa_0(\mathbf{x}) = b_0^{-1}(\mathbf{x})$ and $\rho(\mathbf{x}) = d_0^{-1}(\mathbf{x})$, and for possible discretizations both in space (where $\gamma(i\mathbf{k})$ is an approximation of $i\mathbf{k}$ and $\beta(i\mathbf{k})$ an approximation of $-i\mathbf{k}\cdot$), and in time (where $R(i\omega)$ is an approximation of second-order differentiation in time). For low frequencies, we expect the number of modes in the dispersion relation to equal two, as in the continuous case $\omega_\pm(\mathbf{x},\mathbf{k}) = \pm c_0(\mathbf{x})|\mathbf{k}|$. We see however that strong dispersion may generate more than two modes, in which cases, mode coupling occurs according to the kinetic model (7.21); see also [7] for a similar behavior.

The computational advantages of kinetic models based on the scalar equation (7.1) rather than the system (4.2) are clear: by allowing multi-valuedness in the dispersion relation (7.4), we no longer need to perform diagonalizations as in (4.13) and decompositions as in (4.15) or (4.20).

**Correlations in different media.** The above theory may be generalized to the correlation of two vector fields propagating in different media as we did in sections 4 and 6. We also assume that the spatial operator is a sum of $M \geq 1$ operators of the form (7.1). This is for instance useful in the case of a Schrödinger equation with random potential. Let $p^\varphi$ be the solution of

$$\begin{aligned} &R(\varepsilon D_t)p^\varphi_\varepsilon + \mathcal{H}^\varphi_\varepsilon p^\varphi_\varepsilon = 0, \quad \mathcal{H}^\varphi_\varepsilon = \sum_{\mu=1}^M \mathcal{H}^\varphi_{\mu\varepsilon}, \quad 1 \leq \varphi \leq 2, \\ &\mathcal{H}^\varphi_{\mu\varepsilon} = b^\varphi_{\mu\varepsilon}(\mathbf{x})\beta_\mu(\varepsilon \mathbf{D_x}) d^\varphi_{\mu\varepsilon}(\mathbf{x})\gamma_\mu(\varepsilon\mathbf{D_x}), \quad 1 \leq \varphi \leq 2,\ 1 \leq \mu \leq M. \end{aligned} \tag{7.23}$$

We define the operators $H(\mathbf{x},\mathbf{k}) = \sum_{\mu=1}^N H_\mu(\mathbf{x},\mathbf{k})$, $H_\mu(\mathbf{x},\mathbf{k}) = b_{\mu0}(\mathbf{x}) d_{\mu0}(\mathbf{x})\beta_\mu(i\mathbf{k})\gamma_\mu(i\mathbf{k})$, for $1 \leq \mu \leq M$. We assume that the random fluctuations and the power spectra are given by

$$\begin{aligned} &b^\varphi_{\mu\varepsilon}(\mathbf{x}) = b_{\mu0}(\mathbf{x}) + \sqrt{\varepsilon} b^\varphi_{\mu1}(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}), \qquad d^\varphi_{\mu\varepsilon}(\mathbf{x}) = d_{\mu0}(\mathbf{x}) + \sqrt{\varepsilon} d^\varphi_{\mu1}(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}), \\ &(2\pi)^d x_{\mu0} y_{\mu0} \hat{R}^{\varphi\psi}_{\mu\nu xy}(\mathbf{x},\mathbf{p})\delta(\mathbf{p}+\mathbf{q}) = \langle \hat{x}^\varphi_{\mu1}(\mathbf{x},\mathbf{p}) \hat{y}^\psi_{\nu1}(\mathbf{x},\mathbf{q})\rangle \\ &(x,y) \in \{(b,b),(b,d),(d,d)\}, \quad 1 \leq \varphi,\psi \leq 2, \quad 1 \leq \mu,\nu \leq M. \end{aligned} \tag{7.24}$$

Let us now define the Wigner transform of the two wave fields:

$$W_\varepsilon(t,\omega,\mathbf{x},\mathbf{k}) = W[p^1_\varepsilon, p^2_\varepsilon](t,\omega,\mathbf{x},\mathbf{k}).$$

Thanks to (3.17), the presence of the slow scale $\mathbf{x}$ in the random fluctuations does not modify the asymptotic expansion used earlier in this section since the correction is of order $O(\varepsilon^{3/2})$. We do not repeat the details and present the final result. Let us define

$$\begin{aligned} \Sigma_m(\mathbf{x},\mathbf{k}) &= \int_{\mathbb{R}^d} \sum_{\mu,\nu=1}^M \frac{B^{11}_{\mu\nu}(\mathbf{x},\mathbf{k},\mathbf{q}) + B^{22}_{\mu\nu}(\mathbf{x},\mathbf{k},\mathbf{q})}{2|R'(i\omega_m(\mathbf{x},\mathbf{k}))|^2} \sum_n \delta\Big(\omega_n(\mathbf{x},\mathbf{q}) - \omega_m(\mathbf{x},\mathbf{k})\Big) \frac{d\mathbf{q}}{(2\pi)^{d-1}}, \\ \Pi_m(\mathbf{x},\mathbf{k}) &= \frac{1}{R'(i\omega_m(\mathbf{x},\mathbf{k}))} \text{p.v.} \int_{\mathbb{R}^d} \sum_{\mu,\nu=1}^M \frac{B^{11}_{\mu\nu}(\mathbf{x},\mathbf{k},\mathbf{q}) - B^{22}_{\mu\nu}(\mathbf{x},\mathbf{k},\mathbf{q})}{H(\mathbf{x},\mathbf{k}) - H(\mathbf{x},\mathbf{q})} \frac{d\mathbf{q}}{(2\pi)^d}, \\ \sigma_m(\mathbf{x},\mathbf{k},\mathbf{q}) &= \sum_{\mu,\nu=1}^M \frac{B^{12}_{\mu\nu}(\mathbf{x},\mathbf{k},\mathbf{q})}{|R'(i\omega_m(\mathbf{x},\mathbf{k}))|^2} \frac{1}{(2\pi)^{d-1}}, \\ \alpha^\varphi_\mu(\mathbf{x},\mathbf{p},\mathbf{k},\mathbf{q}) &= d_{\mu0}(\mathbf{x})\hat{b}^\varphi_{\mu1}(\mathbf{x},\mathbf{p})\beta_\mu(i\mathbf{q})\gamma_\mu(i\mathbf{q}) + b_{\mu0}(\mathbf{x})\hat{d}^\varphi_{\mu1}(\mathbf{x},\mathbf{p})\beta_\mu(i\mathbf{k})\gamma_\mu(i\mathbf{q}), \\ B^{\varphi\psi}_{\mu\nu}(\mathbf{x},\mathbf{k},\mathbf{q})\delta(0) &= (2\pi)^{-d}\langle \alpha^\varphi_\mu(\mathbf{x},\mathbf{k}-\mathbf{q},\mathbf{k},\mathbf{q})\alpha^\psi_\nu(\mathbf{x},\mathbf{q}-\mathbf{k},\mathbf{q},\mathbf{k})\rangle. \end{aligned} \tag{7.25}$$

Note that $B^{\varphi\psi}_{\mu\nu}(\mathbf{x},\mathbf{k},\mathbf{q})$ is a function of the power spectra defined in (7.24) as in (7.19). The modes $a_m(t,\mathbf{x},\mathbf{k})$ then satisfy the following equation

$$\begin{aligned}&\frac{\partial a_m}{\partial t}+\{\omega_m(\mathbf{x},\mathbf{k}),a_m\}+\Big(\Sigma_m(\mathbf{x},\mathbf{k})+i\Pi_m(\mathbf{x},\mathbf{k})\Big)a_m\\&=\int_{\mathbb{R}^d}\sigma_m(\mathbf{x},\mathbf{k},\mathbf{q})\sum_n a_n(\mathbf{q})\delta\Big(\omega_n(\mathbf{x},\mathbf{q})-\omega_m(\mathbf{x},\mathbf{k})\Big)d\mathbf{q}.\end{aligned}\tag{7.26}$$

where the frequencies $\omega_m$ are still the solutions of (7.4). The above expression generalizes (4.26) and (6.19) to the case of multiple mode solutions of the dispersion relation and of more general fluctuations of the density and compressibility.

## 8 Some applications of the theory

**Acoustic wave equation.** The typical example of application of the preceding theory is the treatment of acoustic wave propagation recalled in section 2. In the continuous (not discretized and non-dispersive) case of acoustic wave equations, $R(i\omega)=-\omega^2$, $\beta(i\mathbf{k})=-i\mathbf{k}\cdot$ and $\gamma(i\mathbf{k})=i\mathbf{k}$, $b_0(\mathbf{x})=\kappa^{-1}(\mathbf{x})$ and $d_0(\mathbf{x})=\rho^{-1}(\mathbf{x})$. We then have $H(\mathbf{x},\mathbf{k})=b_0(\mathbf{x})d_0(\mathbf{x})|\mathbf{k}|^2$ and $\omega_\pm(\mathbf{x},\mathbf{k})=\mp c_0(\mathbf{x})|\mathbf{k}|$, $c_0(\mathbf{x})=\sqrt{b_0(\mathbf{x})d_0(\mathbf{x})}$. The fluctuations $b_1$ and $d_1$ may thus be seen as fluctuations in the compressibility $\kappa$ and the density $\rho$, respectively. We then verify that

$$\sigma(\mathbf{x},\mathbf{k},\mathbf{q})=\frac{\pi c_0^2|\mathbf{k}|^2}{2(2\pi)^d}\Big(\frac{b_0^2}{b_0^2(\mathbf{x})}\hat{R}_{bb}(\mathbf{k}-\mathbf{q})+\frac{d_0^2}{d_0^2(\mathbf{x})}\hat{R}_{dd}(\mathbf{k}-\mathbf{q})\hat{\mathbf{k}}\cdot\hat{\mathbf{q}}+\frac{2b_0d_0}{b_0(\mathbf{x})d_0(\mathbf{x})}\hat{R}_{bd}(\mathbf{k}-\mathbf{q})\hat{\mathbf{k}}\cdot\hat{\mathbf{q}}\Big).$$

Here we use the notation $\mathbf{k}=\hat{\mathbf{k}}|\mathbf{k}|$. This expression agrees with [24, Eq (4.38)] and shows that the kinetic models associated to the conservations (2.2) and (2.4) are indeed the same as was announced in section 2. The rescaled energy $\tilde{a}_+(t,\mathbf{x},\mathbf{k})=H(\mathbf{x},\mathbf{k})a_+(t,\mathbf{x},\mathbf{k})$ satisfies the same equation as $a_+(t,\mathbf{x},\mathbf{k})$. Moreover we verify that

$$\mathcal{E}_H(t)=\int_{\mathbb{R}^d}\tilde{a}_+(t,\mathbf{x},\mathbf{k})d\mathbf{k},\tag{8.1}$$

where $\mathcal{E}_H(t)$ is defined in (2.4). Thus $\tilde{a}_+(t,\mathbf{x},\mathbf{k})$ describes the acoustic energy transport in the phase space.

**Schrödinger equations.** The phase function of a single particle in a heterogeneous potential may be written in the weak-coupling regime as

$$i\varepsilon\frac{\partial\psi_\varepsilon}{\partial t}+\frac{\varepsilon^2}{2}\Delta\psi_\varepsilon-\Big(V_0(\mathbf{x})+\sqrt{\varepsilon}V_1(\frac{\mathbf{x}}{\varepsilon})\Big)\psi_\varepsilon=0.\tag{8.2}$$

Choosing $R(i\omega)=\omega$, we then have the Hamiltonian $H_S(\mathbf{x},\mathbf{k})=\frac{|\mathbf{k}|^2}{2}+V_0(\mathbf{x})$, and $\omega(\mathbf{x},\mathbf{k})=-\frac{|\mathbf{k}|^2}{2}-V_0(\mathbf{x})$. The Wigner transform $W_\varepsilon(t,\omega,\mathbf{x},\mathbf{k})$ of $\psi_\varepsilon$ may be given the interpretation of the probability density of the particle in the phase space, at least in the limit $\varepsilon\to 0$, and the mode $a(t,\mathbf{x},\mathbf{k})$ satisfies the equation

$$\frac{\partial a}{\partial t}+\mathbf{k}\cdot\nabla_{\mathbf{x}}a-\nabla_{\mathbf{x}}V_0\cdot\nabla_{\mathbf{k}}a+\int_{\mathbb{R}^d}\hat{R}(\mathbf{p}-\mathbf{k})\Big(a(\mathbf{k})-a(\mathbf{p})\Big)\delta\Big(\frac{|\mathbf{p}|^2}{2}-\frac{|\mathbf{k}|^2}{2}\Big)\frac{d\mathbf{p}}{(2\pi)^{d-1}}=0,\tag{8.3}$$

where $\hat{R}(\mathbf{p})$ is the power spectrum of $V_1(\mathbf{y})$, with the normalization constant $V_0 = 1$.

**Klein Gordon equations.** Relativistic particles interacting with an electromagnetic field may be modeled in the weak-coupling regime by the following Klein Gordon equation

$$\varepsilon^2 \frac{\partial^2 \psi_\varepsilon}{\partial t^2} - \varepsilon^2 \Delta \psi_\varepsilon + \alpha^2 \psi_\varepsilon - \sqrt{\varepsilon} V_1(\frac{\mathbf{x}}{\varepsilon}) \psi_\varepsilon = 0, \tag{8.4}$$

where the potential $V_1$ represents the interactions with a highly oscillatory electromagnetic field (in a simplified model). We have the Hamiltonian and the dispersion relation

$$H_K(\mathbf{x}, \mathbf{k}) = |\mathbf{k}|^2 + \alpha^2, \qquad -\omega^2(\mathbf{x}, \mathbf{k}) + \alpha^2 + |\mathbf{k}|^2 = 0. \tag{8.5}$$

Then the kinetic models for the modes $a_\pm$, which correspond to the particles $(-\omega_+ > 0)$ and anti-particles $(-\omega_- < 0)$ in the system, are given by

$$\frac{\partial a_\pm}{\partial t} \pm \frac{\mathbf{k}}{\sqrt{\alpha^2 + |\mathbf{k}|^2}} \cdot \nabla_{\mathbf{x}} a_\pm + \int_{\mathbb{R}^d} \hat{R}(\mathbf{p} - \mathbf{k}) \Big( a_\pm(\mathbf{k}) - a_\pm(\mathbf{p}) \Big) \frac{\delta\Big( |\mathbf{p}|^2 - |\mathbf{k}|^2 \Big)}{2\sqrt{\alpha^2 + |\mathbf{k}|^2}} \frac{d\mathbf{p}}{(2\pi)^{d-1}} = 0. \tag{8.6}$$

More general models for the second-order spatial operator $\Delta$ can be accounted for as in the case of acoustic waves.

**Equations of electromagnetism.** In the absence of source terms, electromagnetic waves propagating in linearly magnetic and polarizable (dielectric) media are solutions of Maxwell's equations

$$\epsilon_\varepsilon \frac{\partial \mathbf{E}_\varepsilon}{\partial t} = \nabla \times \mathbf{H}_\varepsilon, \quad \nabla \cdot \epsilon_\varepsilon \mathbf{E}_\varepsilon = 0, \quad \mu_\varepsilon \frac{\partial \mathbf{H}_\varepsilon}{\partial t} = -\nabla \times \mathbf{E}_\varepsilon, \quad \nabla \cdot \mu_\varepsilon \mathbf{H}_\varepsilon = 0, \tag{8.7}$$

where $(\mathbf{E}, \mathbf{H})$ is the electromagnetic field, $\epsilon_\varepsilon(\mathbf{x})$ is the dielectric constant and $\mu_\varepsilon(\mathbf{x})$ is the permeability. We verify that

$$\mathcal{E}_M(t) = \frac{1}{2} \int_{\mathbb{R}^d} \Big( \epsilon_\varepsilon(\mathbf{x}) |\mathbf{E}_\varepsilon|^2(t, \mathbf{x}) + \mu_\varepsilon(\mathbf{x}) |\mathbf{H}_\varepsilon|^2(t, \mathbf{x}) \Big) d\mathbf{x} = \mathcal{E}_H(0), \tag{8.8}$$

is conserved. The full kinetic model obtained in [24] for (8.7) requires us to consider polarization effects that the scalar model developed in section 7 does not handle. The latter still allows us to find the kinetic structure of (8.7) when either $\epsilon_\varepsilon$ or $\mu_\varepsilon$ is constant. Consider for instance a magnetic media with

$$\epsilon_\varepsilon(\mathbf{x}) = \epsilon_0, \qquad \mu_\varepsilon(\mathbf{x}) = \mu_0(\mathbf{x}) + \sqrt{\varepsilon} \mu_1 \Big( \frac{\mathbf{x}}{\varepsilon} \Big). \tag{8.9}$$

When (8.9) holds, we find then that

$$\frac{\partial^2 \mathbf{E}_\varepsilon}{\partial t^2} - \nabla \cdot c_\varepsilon^2(\mathbf{x}) \nabla \mathbf{E}_\varepsilon = 0, \qquad \nabla \cdot \mathbf{E}_\varepsilon = 0, \tag{8.10}$$

where the light speed is defined as $c_\varepsilon^2(\mathbf{x}) = (\epsilon_0 \mu_\varepsilon(\mathbf{x}))^{-1}$. This implies that each component of $\mathbf{E}_\varepsilon$ satisfies a scalar wave equation. These components are not independent as the divergence condition $\nabla \cdot \mathbf{E}_\varepsilon = 0$ holds. The energy conservation for (8.10) is given by

$$\mathcal{E}_E(t) = \frac{\epsilon_0}{2} \int_{\mathbb{R}^d} \Big( |\varepsilon \frac{\partial \mathbf{E}_\varepsilon}{\partial t}|^2 + c_\varepsilon^2(\mathbf{x}) |\varepsilon \nabla \mathbf{E}_\varepsilon|^2 \Big) d\mathbf{x} = \mathcal{E}_E(0). \tag{8.11}$$

We verify that by introducing the magnetic vector potential $\mathbf{A}_\varepsilon(t, \mathbf{x})$ solution of (8.10) with the Coulomb gauge $\nabla \cdot \mathbf{A}_\varepsilon = 0$ and defining $\mathbf{E}_\varepsilon = -\partial_t \mathbf{A}_\varepsilon$ and $\mathbf{H}_\varepsilon = \mu_\varepsilon^{-1} \nabla \times \mathbf{A}_\varepsilon$, we obtain that both energies $\mathcal{E}_M[\mathbf{E}_\varepsilon, \mathbf{H}_\varepsilon] = \mathcal{E}_E[\mathbf{A}_\varepsilon]$ agree.

Let us now define the Wigner transform

$$W_\varepsilon(t, \omega, \mathbf{x}, \mathbf{k}) = W[\mathbf{E}_\varepsilon, \mathbf{E}_\varepsilon](t, \omega, \mathbf{x}, \mathbf{k}). \tag{8.12}$$

We decompose the light speed as in (4.1). Then each component of the $d \times d$ matrix satisfies in the limit the decomposition

$$W_{0mn}(t, \omega, \mathbf{x}, \mathbf{k}) = a_{+mn}(t, \mathbf{x}, \mathbf{k}) \delta(\omega - \omega_+(\mathbf{x}, \mathbf{k})) + a_{-mn}(t, \mathbf{x}, \mathbf{k}) \delta(\omega - \omega_-(\mathbf{x}, \mathbf{k})), \tag{8.13}$$

where $\omega_\pm(\mathbf{x}, \mathbf{k}) = \mp c_0(\mathbf{x}) |\mathbf{k}|$ and $a_{+mn}$ satisfies the equation (4.26). We thus obtain that each component $a_{+mn}$ satisfies the same kinetic equation. The $d \times d$ components of $a_+(t, \mathbf{x}, \mathbf{k})$, the density of $W_0(t, \omega, \mathbf{x}, \mathbf{k})$, are not independent however. The divergence relation $\nabla \cdot \mathbf{E}_\varepsilon = 0$ imposes in the limit $\varepsilon \to 0$ that

$$a_+(t, \mathbf{x}, \mathbf{k}) \mathbf{k} = 0, \qquad \mathbf{k}^t a(t, \mathbf{x}, \mathbf{k}) = 0. \tag{8.14}$$

Since $a_+(t, \mathbf{x}, \mathbf{k})$ is Hermitian thanks to (3.2), the above two constraints are equivalent. Nonetheless, it imposes $d$ constraints on a Hermitian matrix of dimension $d(d+1)/2$. The matrix $a_+(t, \mathbf{x}, \mathbf{k})$ is thus $d(d-1)/2$ dimensional, which corresponds to the number of *polarizations* that the system can generate. In dimension $d = 2$, we thus have one mode associated to $\omega_+ = c_0 |\mathbf{k}|$. In dimension $d = 3$, two independent modes are associated to $\omega_+ = c_0 |\mathbf{k}|$.

The kinetic model may then be expressed as follows. At each point $(\mathbf{x}, \mathbf{k})$ we assume that $(\hat{\mathbf{k}}, \hat{\mathbf{z}}_1(\mathbf{k}), \cdots, \hat{\mathbf{z}}_{d-1}(\mathbf{k}))$ form an orthonormal basis of $\mathbb{R}^d$ and decompose $a_+$ as

$$a_+(t, \mathbf{x}, \mathbf{k}) = \alpha_{mn}(t, \mathbf{x}, \mathbf{k}) \hat{\mathbf{z}}_m(\mathbf{k}) \hat{\mathbf{z}}_n^*(\mathbf{k}). \tag{8.15}$$

We denote by $\alpha$ the $(d-1) \times (d-1)$ matrix with entries $\alpha_{mn}$. We recall that in the case $d = 3$ with the appropriate choice of vectors $\hat{\mathbf{z}}_m$, the matrix $\alpha$ takes the form

$$\alpha = \frac{1}{2} \begin{pmatrix} I + Q & U + iV \\ U - iV & I - Q \end{pmatrix}, \tag{8.16}$$

where $(I, Q, U, V)$ are the Stokes parameters commonly used in the description of light polarization [11,24]. Furthermore define the matrix $N(\mathbf{x}, \mathbf{k})$ by (see [24]) $\{\hat{\mathbf{z}}_m(\mathbf{k}), \omega_+(\mathbf{x}, \mathbf{k})\} = N(\mathbf{x}, \mathbf{k}) \hat{\mathbf{z}}_m(\mathbf{k})$ (we verify that $N(\mathbf{x}, \mathbf{k})$ is skew-symmetric), and the matrix $T(\mathbf{x}, \mathbf{q})$ with components $T_{mn}(\mathbf{k}, \mathbf{q}) = \hat{\mathbf{z}}_m^*(\mathbf{k}) \hat{\mathbf{z}}_n(\mathbf{q})$. We then verify that the matrix $\alpha$ solves the following

$(d-1)\times(d-1)$ matrix-valued equation

$$\begin{aligned}&\frac{\partial\alpha}{\partial t}+\{\omega_+,\alpha\}+N\alpha-\alpha N+\frac{\pi\omega_+^2(\mathbf{x},\mathbf{k})}{2(2\pi)^d}\\&\times\int_{\mathbb{R}^d}\hat{R}(\mathbf{k}-\mathbf{q})T(\mathbf{k},\mathbf{q})\Big(\alpha(\mathbf{k})-\alpha(\mathbf{q})\Big)T(\mathbf{q},\mathbf{k})\delta\Big(\omega_+(\mathbf{x},\mathbf{k})-\omega_+(\mathbf{x},\mathbf{q})\Big)d\mathbf{q}=0.\end{aligned}\tag{8.17}$$

Here $\hat{R}(\mathbf{k})$ is the power spectrum of the fluctuations $V(\mathbf{x})$ in (4.1). The above kinetic equation is consistent with the results obtained in [24]. Spatial and temporal discretizations may be accounted for as we have done for acoustic wave equations.

In the case of linear dielectric media $\epsilon_\varepsilon(\mathbf{x})=\epsilon_0(\mathbf{x})+\sqrt{\varepsilon}\epsilon_1\Big(\frac{\mathbf{x}}{\varepsilon}\Big)$, and $\mu_\varepsilon=\mu_0$, we find that $\mathbf{H}_\varepsilon$ satisfies the scalar equation (8.10) as well as the divergence condition. The kinetic model for wave propagation in non-magnetic linear dielectric media is thus again given by (8.17).

It is interesting to observe that the components of the matrix $\alpha(t,\mathbf{x},\mathbf{k})$ are coupled in (8.17). Because the entries of the matrices $T(\mathbf{k},\mathbf{q})$ are not necessarily positive, the scattering operator in (8.17) cannot be given a probabilistic interpretation as in the scalar case, where $\sigma(\mathbf{k},\mathbf{q})$ properly normalized can be seen as the probability of scattering from the direction $\mathbf{q}$ into the direction $\mathbf{k}$. This led the authors in [5] to define an augmented phase space to obtain a probabilistic representation for solutions to (8.17). In the case treated in this paper, where $\alpha(t,\mathbf{x},\mathbf{k})$ is the projection onto the polarization modes of the matrix $a_+(t,\mathbf{x},\mathbf{k})$, such a probabilistic representation is trivial since each component of $a_+(t,\mathbf{x},\mathbf{k})$ satisfies a scalar equation.

## 9 Conclusions

We have shown that the spatial Wigner transform formalism could be used to derive kinetic models for two-by-two systems of acoustic waves. However, accounting for general dispersion relations obtained from e.g. time discretizations requires one to introduce the spatio-temporal Wigner transform. Such a transform can be used to derive fairly general kinetic models for acoustic equations in the form of first-order systems or second-order scalar equations. Dispersive effects generating multiple mode propagation and correlation of fields propagating in media with different heterogeneities are accounted for. This has interesting applications in the imaging of the heterogeneities of random media and in the refocusing properties of time reversed waves. It also allows one to explain how transport is modified by spatial and temporal discretizations of the wave equations in numerical simulations.

Several generalizations of the theory may be considered. The most natural extension of the theory consists of analyzing kinetic models for the second-order system of elastic equations for the displacement $\mathbf{u}(t,\mathbf{x})$ [1] and other system equations. In its full generality, such an extension requires one to deal with mode couplings and polarization effects that presumably cannot be handled by the scalar equations considered in this paper [24].

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## References


[1] J. Achenbach, Wave propagation in elastic solids, North Holland, Amsterdam, 1991.

[2] G. Bal, T. Chou, Water wave transport over spatially random drift, Wave Motion 35 (2002) 107–124.

[3] G. Bal, A. Fannjiang, G. Papanicolaou, L. Ryzhik, Radiative Transport in a Periodic Structure, Journal of Statistical Physics 95 (1999) 479–494.

[4] G. Bal, V. Freilikher, G. Papanicolaou, L. Ryzhik, Wave Transport along Surfaces with Random Impedance, Phys. Rev. B 62(10) (2000) 6228–6240.

[5] G. Bal, G. Papanicolaou, L. Ryzhik, Probabilistic Theory of Transport Processes with Polarization, SIAM J. Appl. Math. 60(5) (2000) 1639–1666.

[6] G. Bal, L. Ryzhik, Stability of time reversed waves in changing media, Disc. Cont. Dyn. Syst. A 12(5) (2005) 793–815.

[7] G. Bal, L. Ryzhik, Wave transport for a scalar model of the Love waves, Wave Motion 36(1) (2002) 49–66.

[8] G. Bal, R. Verástegui, Time Reversal in Changing Environment, SIAM Mult. Mod. Simul. 2(4) (2004) 639–661.

[9] Y. Barabanenkov, A. Vinogradov, Kravtsov.Yu, V. Tatarskii, Application of the theory of multiple scattering of waves to the derivation of the radiative transfer equation for a statistically inhomogeneous medium, Radiofizika 15 (1972) 1852–1860. English translation pp. 1420–1425.

[10] A. Bensoussan, J.-L. Lions, G. C. Papanicolaou, Asymptotic analysis for periodic structures, in: Studies in Mathematics and its Applications, 5, North-Holland Publishing Co., Amsterdam-New York, 1978.

[11] S. Chandrasekhar, Radiative Transfer, Dover Publications, New York, 1960.

[12] G. C. Cohen, Higher-Order numerical methods for transient wave equations, Scientific Computation, Springer Verlag, Berlin, 2002.

[13] L. Erdös, H. T. Yau, Linear Boltzmann equation as the weak coupling limit of a random Schrödinger Equation, Comm. Pure Appl. Math. 53(6) (2000) 667–735.

[14] P. Gérard, P. A. Markowich, N. J. Mauser, F. Poupaud, Homogenization limits and Wigner transforms, Comm. Pure Appl. Math. 50 (1997) 323–380.

[15] M. Guo, X.-P. Wang, Transport equations for a general class of evolution equations with random perturbations, J. Math. Phys. 40 (1999) 4828–4858.

[16] M. S. Howe, On the kinetic theory of wave propagation in random media, Phil. Trans. Roy. Soc. Lond. (274) (1973) 523–549.

[17] A. Ishimaru, Wave Propagation and Scattering in Random Media, New York: Academics, 1978.

[18] P.-L. Lions, T. Paul, Sur les mesures de Wigner, Rev. Mat. Iberoamericana 9 (1993) 553–618.

[19] L. Margerin, M. Campillo, B. A. van Tiggelen, Radiative Transfer and Diffusion of Waves in a Layered Medium: a New Insight into Coda $Q$, Geophys. Jour. Intern. 134(2) (1998) 596–612.

[20] P. Markowich, P. Pietra, C. Pohl, H. P. Stimming, A Wigner-measure analysis of the Dufort-Frankel scheme for the Schrödinger equation, SIAM J. Numer. Anal. 40 (2002) 1281–1310.

[21] P. Markowich, F. Poupaud, The pseudo-differential approach to finite differences revisited, Calcolo 3 (1999) 161–186.

[22] J. M. Powell, J. Vaneste, Transport equations for waves in randomly perturbed Hamiltonian systems, with application to Rossby waves, Wave Motion.

[23] R. D. Richtmyer, K. W. Morton, Difference Methods for Initial Value Problems, 2nd ed., Interscience, New York, 1967.

[24] L. Ryzhik, G. Papanicolaou, J. B. Keller, Transport equations for elastic and other waves in random media, Wave Motion 24 (1996) 327–370.

[25] H. Sato, M. C. Fehler, Seismic wave propagation and scattering in the heterogeneous earth, AIP series in modern acoustics and signal processing, AIP Press, Springer, New York, 1998.

[26] P. Sheng, Introduction to Wave Scattering, Localization and Mesoscopic Phenomena, Academic Press, New York, 1995.

[27] H. Spohn, Derivation of the transport equation for electrons moving through random impurities, J. Stat. Phys. 17 (1977) 385–412.

[28] K. Watson, Multiple scattering of electromagnetic waves in an underdense plasma, Jour. Math. Phys. 10 (1969) 688–702.

[29] E. Wigner, On the quantum correction for thermodynamic equilibrium, Phys. Rev. 40 (1932) 742–749.