# ON THE SELF-AVERAGING OF WAVE ENERGY IN RANDOM MEDIA

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**Abstract.** We consider the stabilization (self-averaging) and destabilization of the energy of waves propagating in random media. Propagation is modeled here by an Itô Schrödinger equation. The explicit structure of the resulting transport equations for arbitrary statistical moments of the wave field is used to show that wave energy density may be stable in the high frequency regime, in the sense that it only depends on the statistics of the random medium and not on the specific realization. Stability is conditional on having sufficiently smooth initial energy distributions. We show that wave energy is *not* stable, and instead scintillation is created by the wave dynamics, when the initial energy distribution is sufficiently singular. Application to time reversal of high frequency waves is also considered.

**Key words.** Waves in random media, self-averaging, Itô Schrödinger equations, transport equations, time reversal.

**AMS subject classifications.** 35L05, 60H25, 35Q40

## 1. Introduction.

Propagation of high frequency waves in random media has received a lot of attention in the past forty years. Classical (non-dispersive) wave analysis finds many applications, for instance in light propagating though turbulent atmospheres, microwaves in wireless communication, sound waves in underwater acoustics, or seismic waves generated by earthquakes. Dispersive waves find applications in atomic and high-energy physics for instance. We refer to [10, 18, 23, 32, 33, 36].

In the macroscopic description of the wave energy density, radiative transfer equations play an important role [10, 23, 31] when the fluctuations of the underlying heterogeneous medium are weak and have a correlation length comparable to the typical wavelength of the system. Although they seem to apply in a wide variety of propagation regimes, only very few rigorous mathematical results exist to derive them from first principles and only in the framework of quantum waves solution of a Schrödinger equation [15, 34]. Their derivation from full wave equations being quite challenging, simpler models of wave propagation have often been considered in the literature. There are at least two classes of models in which wave propagation greatly simplifies. The first class consists of replacing waves by particles in a geometrical optics regime. Particle dynamics are then easier to model; see for instance [2] for a recent application. The second class models wave propagation as a moving front and analyzes wave dynamics within this front. Wave propagation in random media is substantially simplified compared to the full wave equations in that we *know* a priori the front location. In this paper we consider an example in the latter class of models.

One of the most used models based on front propagation is the parabolic wave approximation. It singles out a main direction of propagation, say $z$, and analyzes wave dynamics in the transverse directions. It is accurate provided that backscattering can be neglected [35]. In this paper we further simplify the parabolic equation by assuming that the medium mean zero fluctuations are very fast in the direction $z$. In such a regime, fluctuations can be approximated by white noise. This results in an Itô Schrödinger equation to model wave propagation. This equation has been analyzed mathematically in [13]. It is shown in [1] that the Itô Schrödinger approximation can

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be rigorously derived from the wave equation in a one-dimensional setting. This wave propagation model is also referred to in the physical literature as the phase screen method [37, 38].

The main advantage of the Itô Schrödinger equation over other propagation models is that explicit equations can be obtained for arbitrary statistical moments of the wave field. Even though most of the resulting equations cannot be solved analytically, this is an important advantage over full wave or parabolic models. Although they may not apply in physical settings where backscattering cannot be neglected, these equations offer an interesting framework to understand macroscopic aspects of wave propagation. By macroscopic we mean here the description of a quantity that relies on the statistical properties of the underlying medium and not on its detailed structure. One such interesting macroscopic quantity is the *wave energy* density. One then aims at answering two types of questions. Is it possible to find an expression for the ensemble average of the energy density of the propagating waves? In the affirmative, is the energy density statistically *stable*, i.e. independent of the realization of the random medium in some sense to be described, or statistically *instable*, in which case *scintillation* can be introduced to quantify this instability?

Similar questions have been addressed in various regimes, including the parabolic approximation [2, 3, 4, 7, 29, 30]. We consider them here in the regime modeled by the Itô Schrödinger equation. The accessibility of equations for various moments of the wave field (here the second-order and fourth-order moments are utilized) is central to our results. After suitable scalings are introduced, we obtain that the phase space energy density (seen as the Fourier transform of the two-point correlation function of the wave field) may indeed be stable in a weak sense (i.e., after integration against a test function) provided that the "initial" energy density at $z = 0$ is sufficiently smooth. However we show that the energy density *does not* stabilize when the initial energy density is sufficiently singular. Instead we show that scintillation that may be absent at $z = 0$ is created by the wave dynamics and persists for all times $z > 0$. Important tools in the mathematical analysis of energy stability are some properties of the Wigner transform [20, 26, 31] and of the solutions to linear transport equations [12].

The results are then analyzed in the context of the time reversal of waves. Time reversed waves propagating in heterogeneous media have received a lot of attention recently [4, 5, 6, 7, 11, 17, 22, 29]. These works follow physical experiments performed by M. Fink showing that time reversed waves propagating in heterogeneous media enjoy focusing properties that the same waves propagating in homogeneous media do not. Following result in [4, 7], we show how time reversed waves behave in the Itô Schrödinger framework.

This paper is organized as follows. Section 2 presents the scalings that allow us to pass from the full wave equation first to the parabolic approximation and second to the Itô Schrödinger equation. Such a passage is not justified here. The equations for first, second, and fourth-order moments of the wave field and the equivalent equations for the Wigner transforms in the phase space are recalled in section 3. The proper scalings are then introduced in which one may expect stabilization of the energy density. The main scaling results are summarized in section 3.4. The main stabilization result is presented in section 4. Technical results on the transport equations for the second and fourth moment are postponed to the appendix. Section 5 shows that stability does not occur when the wave field at $z = 0$ is sufficiently singular. Section 6 briefly presents the theory of time reversal of high frequency waves and applies the results obtained in the preceding sections to this context.

**2. Parabolic wave and Itô Schrödinger equations.** The scalar wave equation for the pressure field $p(\mathbf{x}, z, t)$ is given by

$$\frac{1}{c^2(\mathbf{x}, z)} \frac{\partial^2 p}{\partial t^2} = \Delta_{\mathbf{x}} p + \frac{\partial^2 p}{\partial z^2}. \tag{2.1}$$

Here $c(\mathbf{x}, z)$ is the local sound speed and $\Delta_{\mathbf{x}}$ is the usual Laplacian operator in the transverse variable $\mathbf{x} \in \mathbb{R}^d$. Physically $d = 2$ and we consider more generally $d \geq 1$.

The objective is to understand the structure of the wave field $p(\mathbf{x}, z, t)$ when the wave speed $c(\mathbf{x}, z)$ is random. This is a quite difficult problem that has received a lot of attention in the mathematical and physical literatures. One main difficulty is that very few analytical calculations can be performed directly with (2.1). Here we make two classical simplifications: first we assume a beam-like structure to derive a wave parabolic equation; second we assume that the fluctuations in the beam direction are fast so as to obtain an Itô Schrödinger equation. The latter, which has already been analyzed in [13, 19] for instance, is more amenable to analytic calculations. Although its domain of physical validity is somewhat restricted, it enjoys many of the inherent difficulties of the wave equation (2.1), and thus offers an interesting framework for mathematical understanding of wave propagation in random media.

The first step is to assume that the wave field has a beam-like structure at $t = 0$ propagating in the $z$ direction and to neglect back scattering. Let us introduce the complex amplitude $\psi(\mathbf{x}, z; k)$ implicitly through the relation

$$p(\mathbf{x}, z, t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ik(z - c_0 t)} \psi(\mathbf{x}, z; k) c_0 dk, \tag{2.2}$$

where $c_0$ is the statistical mean of the sound speed $c(\mathbf{x}, z)$. We assume $c_0$ constant. The amplitude $\psi(\mathbf{x}, z; k)$ at position $(\mathbf{x}, z)$ of waves with frequency $\omega = c_0 k$ satisfies the equation

$$\frac{\partial^2 \psi}{\partial z^2} + 2ik \frac{\partial \psi}{\partial z} + \Delta_{\mathbf{x}} \psi + k^2 (n^2 - 1) \psi = 0, \tag{2.3}$$

where the index of refraction is defined by $n(\mathbf{x}, z) = c_0 / c(\mathbf{x}, z)$.

Approximations to the above equation for $\psi$ can be obtained in certain physical regimes of wave propagation. Let us introduce four physical scales. The first two scales are $L_x$ and $L_z$. They correspond to the transversal and longitudinal distances, respectively, at which we want to observe wave propagation. We thus rescale $\mathbf{x}$ and $z$ as $L_x \mathbf{x}$ and $L_z z$, where $\mathbf{x}$ and $z$ are now $O(1)$ quantities. The other scales we introduce are the lengths at which the underlying medium fluctuates. We denote by $l_x$ and $l_z$ the transversal and longitudinal *correlation lengths*, respectively. In the new variables we recast the refraction index as

$$(n^2 - 1)(\mathbf{x}, z) \to \nu \mu\Big(\frac{L_x \mathbf{x}}{l_x}, \frac{L_z z}{l_z}\Big).$$

Here $\nu$ is another scaling factor quantifying the amplitude of the fluctuations, and $\mu$ is a scaled random process with statistics of order $O(1)$. In the rescaled variables (2.3) becomes

$$\frac{-i}{2kL_z} \frac{\partial^2 \psi}{\partial z^2} + \frac{\partial \psi}{\partial z} + \frac{-iL_z}{2kL_x^2} \Delta_{\mathbf{x}} \psi - \frac{ikL_z}{2} \nu \mu\Big(\frac{L_x \mathbf{x}}{l_x}, \frac{L_z z}{l_z}\Big) \psi = 0. \tag{2.4}$$

The theory in this paper applies to high frequency waves, in the sense that the longitudinal propagation distance is much larger than the typical wavelength. We thus introduce the "small" parameter $\varepsilon$ and the scaling

$$kL_z = \frac{\kappa}{\varepsilon}, \qquad \text{so that} \qquad \frac{\lambda}{L_z} \sim \varepsilon, \tag{2.5}$$

where $\lambda = 2\pi/k$ is the typical wavelength of the system. To simplify notation we shall often assume that $\kappa = 1$ as frequency does not influence in any significant way wave propagation in our setting.

We are now ready to introduce our main assumptions on wave propagation. The first one is the small angle (small aperture) approximation:

$$L_x \ll L_z. \tag{2.6}$$

Formally we thus deduce that $\psi$ approximately solves the equation

$$\frac{\partial\psi}{\partial z} + \frac{-iL_z}{2kL_x^2}\Delta_{\mathbf{x}}\psi = \frac{ikL_z\nu}{2}\mu(\frac{L_x\mathbf{x}}{l_x}, \frac{L_z z}{l_z})\psi. \tag{2.7}$$

Our second main assumption is that the variations of the medium in the $z$ direction are faster than anything else in the system. This means that $l_z \ll \lambda$ and we introduce

$$kl_z \sim \varepsilon^{\alpha}, \qquad \text{or} \qquad \frac{l_z}{L_z} \sim \varepsilon^{1+\alpha}, \qquad \text{where} \qquad \alpha > 0. \tag{2.8}$$

Assuming that $\mu$ is a mean-zero process with $O(1)$ statistics, we know from the central limit theorem that the right-hand side in (2.7) will converge to a limit of order $O(1)$ provided that the size of the fluctuations scales as the square root of $L_z/l_z$ [1, 28], which means more specifically that

$$(kL_z\nu)^2 \sim \frac{L_z}{l_z}, \qquad \text{or equivalently} \qquad \nu \sim \varepsilon^{\frac{1-\alpha}{2}}. \tag{2.9}$$

In this regime we can formally replace

$$\frac{kL_z\nu}{2}\mu(\frac{L_x\mathbf{x}}{l_x}, \frac{L_z z}{l_z})dz \qquad \text{by} \qquad \kappa B(\frac{L_x\mathbf{x}}{l_x}, dz), \tag{2.10}$$

where $B(\mathbf{x}, dz)$ is the usual Wiener measure in $z$. Its statistics are described by

$$\langle B(\mathbf{x}, z)B(\mathbf{y}, z')\rangle = Q(\mathbf{y}-\mathbf{x})z \wedge z', \tag{2.11}$$

where $z \wedge z'$ is the minimum of $z$ and $z'$ and $Q$ is the correlation function. The notation $\langle X\rangle$ denotes statistical ensemble average of the quantity $X$.

The parabolic equation in this regime becomes then

$$d\psi(\mathbf{x}, z) = \frac{iL_z}{2kL_x^2}\Delta_{\mathbf{x}}\psi(\mathbf{x}, z)dz + i\kappa\psi(\mathbf{x}, z) \circ B(\frac{L_x\mathbf{x}}{l_x}, dz). \tag{2.12}$$

Here $\circ$ means that the stochastic equation is understood in the Stratonovich sense [19, 27]. In the Itô sense it becomes

$$d\psi(\mathbf{x}, z) = \frac{1}{2}\Big(\frac{iL_z}{kL_x^2}\Delta_{\mathbf{x}} - \kappa^2 Q(\mathbf{0})\Big)\psi(\mathbf{x}, z)dz + i\kappa\psi(\mathbf{x}, z)B(\frac{L_x\mathbf{x}}{l_x}, dz). \tag{2.13}$$

We shall not justify the derivation of (2.13). It has been shown in [1] that the parabolic approximation and the white noise limit can be taken consistently in the one-dimensional case following the formal scaling arguments presented above. Starting from (2.13) we analyze regimes in which the energy density of high frequency waves can be modeled by "macroscopic" equations in the form of radiative transfer equations. In such regimes, we wish to understand whether the energy density is stable with respect to the realization of the random medium, and when it is stable, in which sense. We also wish to understand the mechanisms that may destabilize the wave energy density and thus render it dependent on the random medium realization. In such a case, a scintillation function is introduced to characterize destabilization.

The main advantage of (2.13) over (2.1) is that many more explicit analytic calculations can be performed in the former than in the latter.

**3. Moment equations and spatial scalings.** It is shown in [13] that the stochastic partial differential equation (2.13) admits a unique solution as an infinite-dimensional martingale problem in the case of an initial condition $\psi(\mathbf{x},0) \in L^2(\mathbb{R}^d)$. All we need here in the sequel is that explicit equations for arbitrary-order statistical moments of the random field can be obtained; see for instance [13, 19, 37]. We recall in this section the derivation of these moment equations, present some relevant properties for later sections, and introduce the proper scalings in which statistical stability and instability of the wave energy can be obtained.

To simplify notation we assume that $\kappa = 1$ as the reduced wavenumber plays no role in the sequel. More precisely, all the moments considered in the sequel could be taken at different wavenumbers. The results would essentially remain identical. Since all wavenumbers visit the position $z$ at the same time as we consider non-dispersive waves, there is no qualitative gain to be obtained in this particular regime by looking at the correlation of fields at different wavenumbers. Restoring frequency dependence can be achieved by multiplying all instances of the correlation function $Q(\mathbf{x})$ by $\kappa^2$.

**3.1. Coherent field and first moment equation.** The first moment is defined by

$$m_1(\mathbf{x},z) = \langle \psi(\mathbf{x},z) \rangle. \tag{3.1}$$

The equation it satisfies is then

$$\frac{\partial m_1}{\partial z}(\mathbf{x},z) = \frac{1}{2}\Big(\frac{iL_z}{kL_x^2}\Delta_{\mathbf{x}} - Q(\mathbf{0})\Big) m_1(\mathbf{x},z). \tag{3.2}$$

It is obtained by taking ensemble averaging in (2.13) since $\psi(z)$ and $B(dz)$ are statistically independent by definition of the Itô formulation of the stochastic equation.

Let us consider the $L^2$ norm of the first moment:

$$M_2(z) = \Big(\int_{\mathbb{R}^d} |m_1(\mathbf{x},z)|^2 d\mathbf{x}\Big)^{1/2}. \tag{3.3}$$

Upon multiplying (3.2) by $m_1^*$ (the complex conjugate of $m_1$), adding the complex conjugate of (3.2) multiplied by $m_1$, and integrating by parts, we deduce that

$$M_2(z) = e^{-\frac{Q(\mathbf{0})}{2}z} M_2(0). \tag{3.4}$$

This shows that the coherent field $m_1$ decays exponentially in $z$. This exponential decay is not related to intrinsic absorption. Instead it describes the loss of coherence caused by multiple scattering; see [24].

More generally the solution to (3.2) is given by

$$m_1(\mathbf{x},z) = e^{-\frac{Q(\mathbf{0})}{2}z}\Big(\frac{-ikL_x^2}{2\pi L_z z}\Big)^{\frac{d}{2}}\int_{\mathbb{R}^d} e^{i\frac{kL_x^2}{2L_z z}|\mathbf{x}-\boldsymbol{\xi}|^2} m_1(\boldsymbol{\xi},0)d\boldsymbol{\xi}. \tag{3.5}$$

We observe that all terms in (3.2) have the same order provided that

$$kL_x^2 \sim L_z, \qquad \text{or equivalently} \qquad L_x \sim \varepsilon^{1/2} L_z. \tag{3.6}$$

This means that initial fluctuations in $\mathbf{x}$ at $z=0$ at the scale $\varepsilon^{1/2}$ will be modified smoothly from $z=0$ to $z>0$. Initial fluctuations at the scale $L_x/L_z \gg \varepsilon^{1/2}$ will essentially remain unaffected at $z=O(1)$ so that

$$m_1(\mathbf{x},z) \sim e^{-\frac{Q(\mathbf{0})}{2}z} m_1(\mathbf{x},0) + o(1).$$

Initial fluctuations at the scale $L_x/L_z \ll \varepsilon^{1/2}$ will be rapidly lost at distances $z=O(1)$ because of the dispersive effects of the Schrödinger equations:

$$\sup_{\mathbf{x}} |m_1(\mathbf{x},z)| = O\Big(\Big(\frac{kL_x^2}{L_z}\Big)^{d/2}\Big) \ll 1,$$

for $z=O(1)$ provided $m_1(\mathbf{x},0)$ is integrable for instance.

**3.2. Energy density and second moment equation.** The equation for the first moment is not satisfactory to describe wave propagation as it does not account for the energy transfer from coherent to incoherent states resulting from multiple scattering. How energy propagates is better understood by looking at the second moment

$$\tilde m_2(\mathbf{x}_1,\mathbf{x}_2,z) = \langle \psi(\mathbf{x}_1,z)\psi^*(\mathbf{x}_2,z)\rangle. \tag{3.7}$$

By application of the Itô calculus [27] we obtain that

$$d(\psi(\mathbf{x}_1,z)\psi^*(\mathbf{x}_2,z)) = \psi(\mathbf{x}_1,z)d\psi^*(\mathbf{x}_2,z) + d\psi(\mathbf{x}_1,z)\psi^*(\mathbf{x}_2,z) + d\psi(\mathbf{x}_1,z)d\psi^*(\mathbf{x}_2,z).$$

Using then (2.13), (2.11), and the fact that $\psi(\mathbf{x},z)\psi^*(\mathbf{y},z)$ and $B(\boldsymbol{\xi},dz)$ are independent in the Itô formulation, we obtain that

$$\frac{\partial \tilde m_2}{\partial z} = \frac{iL_z}{2kL_x^2}(\Delta_{\mathbf{x}_1} - \Delta_{\mathbf{x}_2})\tilde m_2 + \Big(Q\Big(\frac{L_x(\mathbf{x}_1-\mathbf{x}_2)}{l_x}\Big) - Q(\mathbf{0})\Big)\tilde m_2. \tag{3.8}$$

We want to consider regimes of wave propagation in which the above equation retains as many terms of order $O(1)$ as possible. We have three scaling parameters at our disposal: $L_x/L_z$, $l_x/L_z$, and $|\mathbf{x}_1-\mathbf{x}_2|$. Indeed we expect that the wave fields at $\mathbf{x}_1$ and $\mathbf{x}_2$ will be uncorrelated if the distance $|\mathbf{x}_2-\mathbf{x}_1|$ is $O(1)$ and thus need to rescale $|\mathbf{x}_2-\mathbf{x}_1|$ as well. All the terms in (3.8) are of order $O(1)$ provided that

$$\frac{L_x|\mathbf{x}_1-\mathbf{x}_2|}{l_x} \sim 1, \qquad \text{and} \qquad \frac{kL_x^2|\mathbf{x}_1-\mathbf{x}_2|}{L_z} \sim 1. \tag{3.9}$$

Since there are two constraints for three scaling parameters, we have a free parameter and assume that

$$L_x = \varepsilon^{\gamma} L_z, \tag{3.10}$$

for some $\gamma > 0$ since $L_x \ll L_z$. We then obtain that

$$\frac{l_x}{L_z} \sim \varepsilon^{1-\gamma}, \qquad \text{and} \qquad |\mathbf{x}_1 - \mathbf{x}_2| \sim \varepsilon^{1-2\gamma} \equiv \eta = \eta(\varepsilon). \tag{3.11}$$

We are interested in regimes such that $|\mathbf{x}_1 - \mathbf{x}_2| \ll 1$, which implies with the parabolic approximation that

$$0 < \gamma < \frac{1}{2}.$$

We have thus introduced a family of propagation regimes parameterized by $\gamma$. The limit $\gamma \to 0$ is interesting physically as it corresponds to the case $L_x/L_z \to O(1)$ and $l_x/L_z \to \varepsilon$. When moreover $\alpha \to 0$ so that $l_z/L_z \to \varepsilon$, we obtain in the limit the *weak coupling* regime. In this regime where the parabolic approximation no longer holds, wave energy in phase space is approximately given by the solution of a radiative transfer equation [31] of the form (3.16) below. The mathematical justification is however much more difficult and rigorous derivations have only been obtained for quantum waves [15, 34]. The other limit $\gamma \to 1/2$ corresponds to very narrow beam propagation $L_x/L_z \to \varepsilon^{1/2}$. Since $l_x/L_z \to \varepsilon^{1/2}$ as well, the lateral propagation distance is not sufficiently large that we can expect any self-averaging of the wave energy. This will be confirmed by the results presented in section 4.

It is convenient to introduce the rescaled variables

$$\mathbf{x} = \frac{\mathbf{x}_1 + \mathbf{x}_2}{2}, \quad \mathbf{y} = \frac{\mathbf{x}_1 - \mathbf{x}_2}{\eta}.$$

Defining now $m_2(\mathbf{x}, \mathbf{y}) = \tilde{m}_2(\mathbf{x}_1, \mathbf{x}_2)$ for the second moment in the new system of coordinates, we obtain that it solves:

$$\frac{\partial m_2}{\partial z} = \frac{iL_z}{kL_x^2\eta} \nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{y}} m_2(z) - \Big(Q(\mathbf{0}) - Q(\mathbf{y})\Big) m_2(z). \tag{3.12}$$

Notice that $kL_x^2\eta \sim L_z$ by choice of $\eta$ so all the terms in the above equation are of order $O(1)$. The analysis of (3.12) is more complicated than that of (3.2). However it is well-posed in the $L^2$ sense thanks to the following a priori bound:

LEMMA 3.1. *We have that* $\displaystyle\int_{\mathbb{R}^{2d}} |m_2(\mathbf{x}, \mathbf{y}, z)|^2 d\mathbf{x} d\mathbf{y} \leq \int_{\mathbb{R}^{2d}} |m_2(\mathbf{x}, \mathbf{y}, 0)|^2 d\mathbf{x} d\mathbf{y}$.

*Proof.* Indeed upon multiplying (3.12) by $m_2^*$ and multiplying the equation for $m_2^*$ by $m_2$, we obtain by integrations by parts and by the self-adjoint property of $\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{y}}$ that

$$\frac{\partial}{\partial z} \int_{\mathbb{R}^{2d}} |m_2(\mathbf{x}, \mathbf{y}, z)|^2 d\mathbf{x} d\mathbf{y} = -\int_{\mathbb{R}^{2d}} \Big(Q(\mathbf{0}) - Q(\mathbf{y})\Big) |m_2(\mathbf{x}, \mathbf{y}, z)|^2 d\mathbf{x} d\mathbf{y}. \tag{3.13}$$

Now we have from (2.11) that

$$0 \leq \frac{1}{2} \Big\langle \big(B(\mathbf{x}_1, z) - B(\mathbf{x}_2, z)\big)^2 \Big\rangle = Q(\mathbf{0}) - Q(\mathbf{x}_2 - \mathbf{x}_1).$$

Since the correlation function $Q(\mathbf{x})$ is maximal at $\mathbf{x} = \mathbf{0}$ we deduce that the right-hand side in (3.13) is non-positive, hence the bound. ∎

When the initial conditions of the wave field $\psi$ oscillate at the scale $\eta$, then $m_2(\mathbf{x}, \mathbf{y}, 0)$ oscillates at the scale $O(1)$ in both variables $\mathbf{x}$ and $\mathbf{y}$ and so does $m_2(\mathbf{x}, \mathbf{y}, z)$

given by (3.12) at $z \sim 1$. This is the regime we are interested in, and $\eta$ is the scale of the transverse wave fluctuations at $z = 0$.

It is both instructive and mathematically convenient to recast this equation in the phase space. We introduce the *Wigner transform* [20, 26, 31, 39] of the field $\psi(\mathbf{x}, z)$:

$$W(\mathbf{x},\mathbf{p},z) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\mathbf{p}\cdot\mathbf{y}}\psi(\mathbf{x}-\frac{\eta\mathbf{y}}{2},z)\psi^*(\mathbf{x}+\frac{\eta\mathbf{y}}{2},z)d\mathbf{y}. \tag{3.14}$$

Up to scaling factors, we observe that $\langle W\rangle$ is the Fourier transform of $m_2$ from the spatial variable $\mathbf{y}$ to the dual variable $\mathbf{p}$, and more precisely

$$m_2(\mathbf{x},\mathbf{y},z) = \int_{\mathbb{R}^d} e^{i\mathbf{p}\cdot\mathbf{y}}\langle W\rangle(\mathbf{x},\mathbf{p},z)d\mathbf{p}. \tag{3.15}$$

Thus $W$ can be interpreted as the energy density of waves (although it is positive only in the limit $\eta \to 0$ [20]) at $z$ and $\mathbf{x}$ propagating with wavenumber $\mathbf{p}/\eta$. We obtain by taking Fourier transforms in (3.12) that $\langle W\rangle$ satisfies the following *radiative transfer* equation:

$$\frac{\partial\langle W\rangle}{\partial z} + \frac{L_z}{kL_x^2\eta}\mathbf{p}\cdot\nabla_{\mathbf{x}}\langle W\rangle = \int_{\mathbb{R}^d}\Big[\hat{Q}(\mathbf{p}-\mathbf{p}') - Q(\mathbf{0})\delta(\mathbf{p}-\mathbf{p}')\Big]\langle W\rangle(\mathbf{p}')d\mathbf{p}'. \tag{3.16}$$

We deduce the existence of a unique solution to (3.16) in the $L^2$ sense from Lemma 3.1 and the Parseval identity. This equation is analyzed in detail in the appendix. Notice that the scattering coefficient $\hat{Q}$ satisfies

$$\hat{Q}(\mathbf{p}) \geq 0. \tag{3.17}$$

Indeed we deduce from (2.11) that

$$\begin{aligned}\langle \hat{B}(\mathbf{x}_1,z)\hat{B}^*(\mathbf{x}_2,z)\rangle &= \int_{\mathbb{R}^{2d}} e^{-i(\mathbf{x}_1-\mathbf{x}_2)\cdot\boldsymbol{\xi}_2+i\mathbf{x}_2(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)}Q(\mathbf{x}_1-\mathbf{x}_2)d\mathbf{x}_1d\mathbf{x}_2\\ &= \hat{Q}(\boldsymbol{\xi}_1)\delta(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2).\end{aligned}$$

Upon integrating both sides against any test function $\lambda(\boldsymbol{\xi}_1)\lambda^*(\boldsymbol{\xi}_2)$ we get that

$$0 \leq \big|(\hat{B}(\boldsymbol{\xi}_1,z),\lambda(\boldsymbol{\xi}_1))\big|^2 = \int_{\mathbb{R}^d}|\lambda|^2(\boldsymbol{\xi}_1)\hat{Q}(\boldsymbol{\xi}_1)d\boldsymbol{\xi}_1,$$

which implies (3.17). This can also be seen as an application of Bochner's theorem [8].

**3.3. Scintillation and fourth moment equations.** We can similarly obtain an equation for the fourth moment:

$$\tilde{m}_4(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4,z) = \langle\psi(\mathbf{x}_1,z)\psi^*(\mathbf{x}_2,z)\psi(\mathbf{x}_3,z)\psi^*(\mathbf{x}_4,z)\rangle. \tag{3.18}$$

From Itô calculus we get, denoting for brevity $\psi_k = \psi(\mathbf{x}_k, z)$,

$$d(\psi_1\psi_2^*\psi_3\psi_4^*) = \psi_2^*\psi_3\psi_4^*d\psi_1 + \cdots + \psi_1\psi_2^*\psi_3 d\psi_4^* + \psi_1\psi_2^*d\psi_3d\psi_4^* + \cdots + \psi_3\psi_4^*d\psi_1d\psi_2^*.$$

Using then (2.13), (2.11), and the fact that any functional of $\psi(z)$ and $B(dz)$ are independent in the Itô formulation, we obtain that

$$\begin{aligned}\frac{\partial\tilde{m}_4}{\partial z} &= \frac{iL_z}{2kL_x^2}(\Delta_{\mathbf{x}_1}-\Delta_{\mathbf{x}_2}+\Delta_{\mathbf{x}_3}-\Delta_{\mathbf{x}_4})\tilde{m}_4\\ &\quad -\Big(2Q(\mathbf{0}) + \sum_{1\leq m<n\leq 4}(-1)^{n-m}Q\Big(\frac{L_x(\mathbf{x}_m-\mathbf{x}_n)}{l_x}\Big)\Big)\tilde{m}_4.\end{aligned} \tag{3.19}$$

Again we want as many terms as possible of order $O(1)$. Up to some permutations in the indices, this implies that the same scaling as for the second order moment holds, namely

$$\frac{L_x}{L_z} \sim \varepsilon^{\gamma}, \quad \frac{l_x}{L_z} \sim \varepsilon^{1-\gamma}, \quad \mathbf{x}_1 - \mathbf{x}_2 \sim \varepsilon^{1-2\gamma}, \quad \mathbf{x}_3 - \mathbf{x}_4 \sim \varepsilon^{1-2\gamma}, \qquad \mathbf{x}_2 - \mathbf{x}_3 \sim 1.$$

All but four terms in (3.19) are then of order $O(1)$. The latter *cross terms* involve the correlation function taken at differences of points other than $\mathbf{x}_1 - \mathbf{x}_2$ and $\mathbf{x}_3 - \mathbf{x}_4$.

The fourth moment equation is easier to analyze after the change of variables

$$\mathbf{x} = \frac{\mathbf{x}_1 + \mathbf{x}_2}{2}, \quad \mathbf{y} = \frac{\mathbf{x}_1 - \mathbf{x}_2}{\eta}, \quad \boldsymbol{\xi} = \frac{\mathbf{x}_3 + \mathbf{x}_4}{2}, \quad \mathbf{t} = \frac{\mathbf{x}_3 - \mathbf{x}_4}{\eta}, \quad \eta = \frac{l_x}{L_x}.$$

Here we have imposed $l_x = \eta L_x$ to slightly simplify notation. Defining fourth moment

$$m_4(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t}, z) = \tilde{m}_4(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, z)$$

in the new variables we obtain that

$$\frac{\partial m_4}{\partial z} = \frac{iL_z}{kL_x^2\eta}(\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{y}} + \nabla_{\boldsymbol{\xi}} \cdot \nabla_{\mathbf{t}}) m_4(z) - \mathcal{Q} m_4(z), \tag{3.20}$$

where

$$\mathcal{Q}(\mathbf{x}, \mathbf{y}, \boldsymbol{\xi}, \mathbf{t}) = \Big(2Q(\mathbf{0}) - Q(\mathbf{y}) - Q(\mathbf{t}) + \sum_{\epsilon_i, \epsilon_j = \pm} \epsilon_i \epsilon_j Q\big(\frac{\mathbf{x} - \boldsymbol{\xi}}{\eta} + \epsilon_i \mathbf{y} - \epsilon_j \mathbf{t}\big)\Big). \tag{3.21}$$

This allows us to state the following a priori bound:

Lemma 3.2. *We have the following bound:*

$$\int_{\mathbb{R}^{4d}} |m_4(\mathbf{x}, \mathbf{y}, \boldsymbol{\xi}, \mathbf{t}, z)|^2 d\mathbf{x} d\mathbf{y} d\boldsymbol{\xi} d\mathbf{t} \le \int_{\mathbb{R}^{4d}} |m_4(\mathbf{x}, \mathbf{y}, \boldsymbol{\xi}, \mathbf{t}, 0)|^2 d\mathbf{x} d\mathbf{y} d\boldsymbol{\xi} d\mathbf{t}.$$

The proof is similar to that of Lemma 3.1 since $\mathcal{Q} \ge 0$. Indeed we verify that

$$\begin{aligned} 0 &\le \frac{1}{2}\Big\langle \big(B(\mathbf{x}_1, z) - B(\mathbf{x}_2, z) + B(\mathbf{x}_3, z) - B(\mathbf{x}_4, z)\big)^2 \Big\rangle \\ &= 2Q(\mathbf{0}) + \sum_{1 \le m < n \le 4} (-1)^{n-m} Q(\mathbf{x}_m - \mathbf{x}_n), \end{aligned}$$

which implies that $0 \le \mathcal{Q}$. We deduce that (3.20) admits a unique solution uniformly bounded for $z > 0$ in $L^2(\mathbb{R}^{4d})$.

We can also analyze fourth order moments in the phase space. Let us introduce

$$\mathcal{W}(\mathbf{x}, \mathbf{p}, \boldsymbol{\xi}, \mathbf{q}, z) = W(\mathbf{x}, \mathbf{p}, z) W(\boldsymbol{\xi}, \mathbf{q}, z). \tag{3.22}$$

This moment can easily be related to $m_4$ using (3.14). The equation satisfied by $\langle \mathcal{W} \rangle$ is then:

$$\frac{\partial \langle \mathcal{W} \rangle}{\partial z} + \frac{L_z}{kL_x^2\eta} T_2 \langle \mathcal{W} \rangle = \mathcal{R}_2 \langle \mathcal{W} \rangle + K_{12} \langle \mathcal{W} \rangle. \tag{3.23}$$

Here we have defined the following operators

$$\begin{aligned}
T_2\mathcal{W} &= (\mathbf{p}\cdot\nabla_{\mathbf{x}} + \mathbf{q}\cdot\nabla_{\boldsymbol{\xi}})\mathcal{W} \\
K_{12}\mathcal{W} &= \int_{\mathbb{R}^d} \hat{Q}(\mathbf{u}) e^{i\frac{(\mathbf{x}-\boldsymbol{\xi})\cdot\mathbf{u}}{\eta}} \Big(\mathcal{W}(\mathbf{p}-\tfrac{\mathbf{u}}{2},\mathbf{q}-\tfrac{\mathbf{u}}{2}) + \mathcal{W}(\mathbf{p}+\tfrac{\mathbf{u}}{2},\mathbf{q}+\tfrac{\mathbf{u}}{2}) \\
&\qquad\qquad -\mathcal{W}(\mathbf{p}-\tfrac{\mathbf{u}}{2},\mathbf{q}+\tfrac{\mathbf{u}}{2}) - \mathcal{W}(\mathbf{p}+\tfrac{\mathbf{u}}{2},\mathbf{q}-\tfrac{\mathbf{u}}{2})\Big) d\mathbf{u} \\
K_2\mathcal{W} &= \int_{\mathbb{R}^{2d}} \Big[\hat{Q}(\mathbf{p}-\mathbf{p}')\delta(\mathbf{q}-\mathbf{q}') + \hat{Q}(\mathbf{q}-\mathbf{q}')\delta(\mathbf{p}-\mathbf{p}')\Big]\mathcal{W}(\mathbf{p}',\mathbf{q}')d\mathbf{p}'d\mathbf{q}' \\
\mathcal{R}_2\mathcal{W} &= K_2\mathcal{W} - 2Q(\mathbf{0})\mathcal{W},
\end{aligned} \tag{3.24}$$

where $T_2$ is the transport operator, $\mathcal{R}_2$ is the product scattering operator and $K_{12}$ is the cross term scattering. We have also introduced the operator $K_2$ that will be used later. When $\mathbf{x}-\boldsymbol{\xi} \sim 1$, the term $K_{12}\langle\mathcal{W}\rangle$ is small since $\hat{Q}(\mathbf{p})$ decays to 0 as $|\mathbf{p}| \to \infty$. This means that in some integrated sense these cross terms become negligible as $\eta \to 0$. Energy stability is based on this observation.

Let us define the *energy fluctuation* or *scintillation* function

$$J_\eta(\mathbf{x},\mathbf{p},\boldsymbol{\xi},\mathbf{q},z) = \langle\mathcal{W}(\mathbf{x},\mathbf{p},\boldsymbol{\xi},\mathbf{q},z)\rangle - W_{22}(\mathbf{x},\mathbf{p},\boldsymbol{\xi},\mathbf{q},z), \tag{3.25}$$

$$W_{22}(\mathbf{x},\mathbf{p},\boldsymbol{\xi},\mathbf{q},z) = \langle W(\mathbf{x},\mathbf{p},z)\rangle\,\langle W(\boldsymbol{\xi},\mathbf{q},z)\rangle. \tag{3.26}$$

Then $J_\eta$ satisfies the equation

$$\begin{aligned}
&\Big(\frac{\partial}{\partial z} + \frac{L_z}{kL_x^2\eta}T_2 - \mathcal{R}_2 - K_{12}\Big)J_\eta = K_{12}\Big[\langle W(\mathbf{x},\mathbf{p},z)\rangle\langle W(\boldsymbol{\xi},\mathbf{q},z)\rangle\Big], \quad z>0, \\
&J_\eta(z=0) = \langle\mathcal{W}(\mathbf{x},\mathbf{p},\boldsymbol{\xi},\mathbf{q},0)\rangle - \langle W(\mathbf{x},\mathbf{p},0)\rangle\langle W(\boldsymbol{\xi},\mathbf{q},0)\rangle.
\end{aligned} \tag{3.27}$$

Our main task is then to show in which sense $J_\eta \to 0$ as $\eta \to 0$. The answer is that $J_\eta$ will converge strongly to 0 when $W(\mathbf{x},\mathbf{p},0)$ is sufficiently regular. However for sufficiently singular (but still physical) $W(\mathbf{x},\mathbf{p},0)$, some scintillation can appear during wave propagation, even if initially $J_\eta(z=0) \equiv 0$.

**3.4. High frequency scalings.** Summarizing the above calculations, we have obtained a family of wave fields and statistical moments parameterized in the high frequency regime by a parameter $0 < \gamma < 1/2$. The scale $\varepsilon$ in the direction of propagation $z$, the scale $\eta = \varepsilon^{1-2\gamma}$ in the transverse directions, and the other scaling parameters are related by

$$\frac{l_z}{L_z} = \varepsilon^{1+\alpha} \ll \frac{1}{kL_z} \sim \varepsilon \ll \frac{l_x}{L_z} \sim \varepsilon^{1-\gamma} \ll \eta = \varepsilon^{1-2\gamma} \ll \frac{L_x}{L_z} \sim \varepsilon^\gamma \ll 1.$$

To simplify notation, we assume from now on that $kL_x^2\eta = L_z$.

The Itô Schrödinger equation is then given by

$$d\psi_\eta = \frac{1}{2}(i\eta\Delta_{\mathbf{x}} - Q(\mathbf{0}))\psi_\eta dz + i\psi_\eta B(\frac{\mathbf{x}}{\eta}, dz). \tag{3.28}$$

Let us consider the Wigner transform of the above field

$$W_\eta(\mathbf{x},\mathbf{p},z) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\mathbf{p}\cdot\mathbf{y}}\psi_\eta(\mathbf{x}-\frac{\eta\mathbf{y}}{2},z)\psi_\eta^*(\mathbf{x}+\frac{\eta\mathbf{y}}{2},z)d\mathbf{y}. \tag{3.29}$$

Then its average $\langle W_\eta\rangle$ solves

$$\frac{\partial\langle W_\eta\rangle}{\partial z} + \mathbf{p}\cdot\nabla_{\mathbf{x}}\langle W_\eta\rangle = \int_{\mathbb{R}^d}\Big[\hat{Q}(\mathbf{p}-\mathbf{p}') - Q(\mathbf{0})\delta(\mathbf{p}-\mathbf{p}')\Big]\langle W_\eta\rangle(\mathbf{p}')d\mathbf{p}'. \tag{3.30}$$

Although $\eta$ no longer appears in the equation itself, it may appear in the initial conditions we impose on $W_\eta$. Finally we introduce

$$\mathcal{W}_\eta(\mathbf{x},\mathbf{p},\boldsymbol{\xi},\mathbf{q},z) = W_\eta(\mathbf{x},\mathbf{p},z)W_\eta(\boldsymbol{\xi},\mathbf{q},z), \tag{3.31}$$

whose average $\langle \mathcal{W}_\eta \rangle$ solves

$$\frac{\partial \langle \mathcal{W}_\eta \rangle}{\partial z} + (\mathbf{p}\cdot\nabla_{\mathbf{x}} + \mathbf{q}\cdot\nabla_{\boldsymbol{\xi}})\langle \mathcal{W}_\eta \rangle = \mathcal{R}_2\langle \mathcal{W}_\eta \rangle + K_{12}\langle \mathcal{W}_\eta \rangle. \tag{3.32}$$

It remains to address the choice of boundary conditions at $z = 0$ for the Wigner transform $W_\eta$. When $W_\eta$ is given by (3.29), the initial conditions $W_\eta(z = 0)$ are known to be quite singular in the limit $\eta \to 0$ [20]. More precisely, for initial wave fields uniformly bounded in $L^2(\mathbb{R}^d)$ as in the theory of the stochastic equation (2.13) [13], the Wigner transform is bounded in a space $\mathcal{A}' \supset \mathcal{M}_b(\mathbb{R}^{2d})$ and positive measures of the form $\delta(\mathbf{x}-\mathbf{x}_0)\delta(\mathbf{p}-\mathbf{p}_0)$ for $\mathbf{x}_0,\mathbf{p}_0 \in \mathbb{R}^d$ can be attained as limits of $W_\eta(\mathbf{x},\mathbf{p},0)$ as $\eta \to 0$ [26].

In several applications of wave propagation, one may not be interested in pure states given by (3.29), but in a *mixture of states*. It this case one is interested in a quantity of the form

$$W_\eta(\mathbf{x},\mathbf{p},z) = \frac{1}{(2\pi)^d}\int_S\int_{\mathbb{R}^d} e^{i\mathbf{p}\cdot\mathbf{y}}\psi_\eta(\mathbf{x}-\frac{\eta\mathbf{y}}{2},z;b)\psi_\eta^*(\mathbf{x}+\frac{\eta\mathbf{y}}{2},z;b)d\mathbf{y}\mu(db). \tag{3.33}$$

Here $S$ is a state space equipped with a non-negative bounded measure $\mu$ that may or may not depend on the scaling parameter $\eta$. The equations that $\langle W_\eta \rangle$ and $\langle \mathcal{W}_\eta \rangle$ satisfy still are (3.30) and (3.31), respectively. The principal difference with respect to (3.29) is that $W_\eta(z = 0)$ can now be arbitrarily smooth for appropriate choices of $S$ and $\mu$ (see also section 6).

The main question we want to answer in this paper is whether $\langle W_\eta \rangle$ is a good approximation to $W_\eta$. In the affirmative case, we have stabilization of the wave energy density as $W_\eta$ approximately solves a deterministic equation. In the negative case, the equation for $\langle W_\eta \rangle$ still provides a tool to estimate the energy density, but fluctuations and scintillation effects arise that need to be accounted for.

We shall see in the coming two sections that the statistical stability or instability of $W_\eta$ with respect to the statistical realization of the underlying medium very much depends on the regularity of the initial condition $W_\eta(z = 0)$, which we shall assume deterministic here since it is obviously independent of $B(\mathbf{x},z)$.

**4. Stabilization: deterministic energy densities.** The main result of this section is to show that $\langle W_\eta \rangle(\mathbf{x},\mathbf{p},z)$ introduced in the previous section indeed offers a good description of wave propagation in random media because statistical fluctuations $W_\eta - \langle W_\eta \rangle$ vanish in the limit $\eta = \eta(\varepsilon) \to 0$ weakly. This relies on assuming that $W_\eta(z = 0)$ is sufficiently regular.

Let us recall that $\langle W_\eta \rangle$ is the solution of (3.30) and $\langle \mathcal{W}_\eta \rangle$ the solution of (3.32). We still denote by $m_2 = m_2(\eta)$ the solution of (3.12) and by $W_{22} = W_{22}(\eta)$ the function given by (3.26). We define the following Banach spaces

$$X_{m,n} = L^m(\mathbb{R}^d_{\mathbf{p}}; L^n(\mathbb{R}^d_{\mathbf{x}})). \tag{4.1}$$

Our main result stating that the scintillation function $J_\eta$ defined in (3.27) is small when the initial condition $W_\eta(z = 0)$ is sufficiently smooth is the following.

THEOREM 4.1. *Let us assume that $W_\eta(\mathbf{x},\mathbf{p},0)$ is deterministic and in $X_{2,2}\cap X_{2,\infty}$, i.e.,*

$$\int_{\mathbb{R}^{2d}} |W_\eta(\mathbf{x},\mathbf{p},0)|^2 d\mathbf{x}d\mathbf{p} + \int_{\mathbb{R}^d} \sup_{\mathbf{x}} |W_\eta(\mathbf{x},\mathbf{p},0)|^2 d\mathbf{p} \leq C,$$

*where $C$ is a constant independent of $\eta$. Let us also assume that the correlation function $Q(\mathbf{x}) \in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$. Then the scintillation function $J_\eta(\mathbf{x},\mathbf{p},\boldsymbol{\xi},\mathbf{q},z)$ satisfies*

$$\|J_\eta\|_2(z) \leq C\eta^{d/2}, \tag{4.2}$$

*uniformly in $z$ on compact intervals.*

*Proof.* Here $\|\cdot\|_2$ is the usual $L^2$ norm on $\mathbb{R}^{4d}$. The proof is based on the results on the transport equations presented in the appendix. Since $W_\eta(z=0)$ is deterministic we obtain that $J_\eta(z=0)=0$. Thus from (3.27) and Theorem A.4 we deduce that

$$\|J_\eta\|_2(z) \leq C\int_0^z \|K_{12}W_{22}(s)\|_2 ds.$$

The Parseval identity yields that

$$\|J_\eta\|_2(z) \leq C\int_0^z \|\mathcal{Q}_{12}m_{22}(s)\|_2 ds,$$

where

$$m_{22}(\mathbf{x},\mathbf{y},\boldsymbol{\xi},\mathbf{t},z) = m_2(\mathbf{x},\mathbf{y},z)m_2(\boldsymbol{\xi},\mathbf{t},z)$$

and

$$\mathcal{Q}_{12}(\mathbf{x},\mathbf{y},\boldsymbol{\xi},\mathbf{t}) = \sum_{\epsilon_i,\epsilon_j=\pm} \epsilon_i\epsilon_j Q\Big(\frac{\mathbf{x}-\boldsymbol{\xi}}{\eta} + \epsilon_i\mathbf{y} - \epsilon_j\mathbf{t}\Big).$$

We deduce from Corollary A.3 that $m_2(s)\in X_{2,\infty}\cap X_{2,2}$ uniformly for $s\in(0,z)$. It thus remains to show that the $L^2$ norm of a function in $(X_{2,\infty}\cap X_{2,2})^2$ multiplied by $\mathcal{Q}_{12}$ is of order $\eta^{d/2}$ to conclude the proof of the theorem. More precisely we want to show that:

$$I^2 = \int_{\mathbb{R}^{4d}} Q^2(\frac{\mathbf{x}-\boldsymbol{\xi}}{\eta} + \epsilon_i\mathbf{y} - \epsilon_j\mathbf{t})|m_2|^2(\mathbf{x},\mathbf{y},s)|m_2|^2(\boldsymbol{\xi},\mathbf{t},s)d\mathbf{x}d\boldsymbol{\xi}d\mathbf{y}d\mathbf{t} = O(\eta^d),$$

uniformly for $s\in(0,z)$. We introduce the change of variables

$$\mathbf{r} = \frac{\mathbf{x}-\boldsymbol{\xi}}{\eta} - \epsilon_j\mathbf{t}$$

and get

$$\begin{aligned} I^2 &= \eta^d \int_{\mathbb{R}^{4d}} Q^2(\mathbf{r}+\epsilon_j\mathbf{y})|m_2|^2(\mathbf{x},\mathbf{y})|m_2|^2(\mathbf{x}-\eta(\mathbf{r}+\epsilon_j\mathbf{t}),\mathbf{t})d\mathbf{r}d\mathbf{x}d\mathbf{y}d\mathbf{t} \\ &\leq \|m_2\|_{X_{2,\infty}}(s)\eta^d \int_{\mathbb{R}^{3d}} Q^2(\mathbf{r}+\epsilon_j\mathbf{y})|m_2|^2(\mathbf{x},\mathbf{y})d\mathbf{r}d\mathbf{x}d\mathbf{y} \\ &\leq \eta^d\|m_2\|_{X_{2,\infty}}(s)\|m_2\|_2(s)\|Q\|_2^2. \end{aligned}$$

This concludes the proof of the theorem since $Q \in L^2(\mathbb{R}^d)$ by interpolation. □

We can use this result to show that $W_\eta$ is indeed stable in a weak sense. Let us consider a test function $\lambda \in L^2(\mathbb{R}^{2d})$. Then we obtain that $(W_\eta, \lambda)$, where $(\cdot,\cdot)$ is the usual inner product in $L^2(\mathbb{R}^{2d})$, is close to $(\langle W_\eta\rangle, \lambda)$. More precisely, we have

THEOREM 4.2. *Under the assumptions of Theorem 4.1 and $\lambda \in L^2(\mathbb{R}^{2d})$, we obtain that*

$$\left\langle \left\{ \Big( (W_\eta,\lambda) - (\langle W_\eta\rangle,\lambda) \Big)^2 \right\} \right\rangle \leq C\eta^{d/2}\|\lambda\|_2^2. \tag{4.3}$$

*Also $(W_\eta, \lambda)$ becomes deterministic in the limit of small values of $\eta$ as*

$$P\Big( \big|(W_\eta,\lambda) - (\langle W_\eta\rangle,\lambda)\big| \geq \alpha \Big) \leq \frac{C\eta^{d/2}\|\lambda\|_2^2}{\alpha^2} \to 0 \quad \text{as } \eta \to 0. \tag{4.4}$$

*Let us assume moreover that the initial conditions $W_\eta(\mathbf{x},\mathbf{p},0)$ converge strongly in $L^2(\mathbb{R}^{2d})$ to $\overline{W}(\mathbf{x},\mathbf{p},0)$ as $\eta \to 0$. Then the Wigner transform $W_\eta$ of the stochastic field $\psi_\eta$ solution of (3.28), converges weakly and in probability to the deterministic solution $\overline{W}(\mathbf{x},\mathbf{p},z)$ of (3.30) uniformly in $z$ on compact intervals.*

The proof of the theorem is based on the results of Theorem 4.1 and the calculation:

$$\left\langle \Big( (W_\eta,\lambda) - (\langle W_\eta\rangle,\lambda) \Big)^2 \right\rangle = (\langle \mathcal{W}_\eta\rangle, \lambda\otimes\lambda) - (\langle W_\eta\rangle,\lambda)^2 = (J_\eta, \lambda\otimes\lambda) = O(\eta^{d/2})\|\lambda\|_2^2.$$

By application of the Chebyshev inequality [9] we obtain (4.4).

The convergence result provides an error bound for the fluctuations (4.3). However, stability only occurs in a weak sense. The field $W_\eta(\mathbf{x},\mathbf{p},z)$ does not stabilize pointwise as $W_\eta(\mathbf{x},\mathbf{p},z)$ and $W_\eta(\boldsymbol{\xi},\mathbf{q},z)$ are uncorrelated provided that $|\mathbf{x}-\boldsymbol{\xi}| \gg \eta$. In the case where $|\mathbf{x}-\boldsymbol{\xi}| \sim \eta$ we do not expect $W_\eta(\mathbf{x},\mathbf{p},z)$ and $W_\eta(\boldsymbol{\xi},\mathbf{q},z)$ to be uncorrelated as the cross terms (the operator $K_{12}$) in (3.23) then become an $O(1)$ contribution and generate scintillation that will persist when $|\mathbf{p}-\mathbf{q}|$ is also of order $\eta$. Thanks to the error control in (4.3) these stability results are nevertheless stronger than the ones we can obtain in the parabolic approximation regime without passing to the White Noise limit [4]. For instance error control in (4.3) allows us to choose a test function that depends on $\eta$. For a test function of unit mass of the form:

$$\lambda_\eta(\mathbf{x},\mathbf{p}) = \frac{1}{\eta^{\alpha d}}\phi\Big(\frac{\mathbf{x}-\mathbf{x}_0}{\eta^\alpha}\Big)\frac{1}{\eta^{\beta d}}\varphi\Big(\frac{\mathbf{p}-\mathbf{p}_0}{\eta^\beta}\Big),$$

with $\alpha \geq 0$, $\beta \geq 0$, and $\phi$ and $\varphi$ smooth non-negative functions of compact support with unit mass in $\mathbb{R}^d$, we obtain that

$$\left\langle \Big( (W_\eta,\lambda_\eta) - (\langle W_\eta\rangle,\lambda_\eta) \Big)^2 \right\rangle \leq C\eta^{d/2-(\alpha+\beta)d}.$$

Let us assume for instance that $\overline{W} = \langle W_\eta\rangle$ is continuous and independent of $\eta$, which is the case if $\overline{W}(z=0) = \langle W_\eta\rangle(z=0)$ is continuous (we can apply Theorem A.1 with $X$ the Banach space of continuous functions equipped with the sup norm). Provided that $1/2 > \alpha+\beta$ we obtain that

$$(W_\eta(z),\lambda_\eta) \to \overline{W}(\mathbf{x}_0,\mathbf{k}_0,z), \qquad \text{as } \eta \to 0,$$

in mean square, hence in probability, and this uniformly in $z$ on compact intervals.

**5. Destabilization and scintillation effects.** We saw that the scintillation function $J_\eta$ was negligible in the limit $\eta \to 0$ when $W_\eta(z=0)$ was sufficiently smooth. This is no longer the case when $W_\eta(z=0)$ is sufficiently singular.

We show in the appendix that (3.32) is well-posed in the space of bounded measures. Let us assume that

$$W_\eta(\mathbf{x},\mathbf{p},0) = \delta(\mathbf{x}-\mathbf{x}_0)\delta(\mathbf{p}-\mathbf{p}_0), \tag{5.1}$$

which as we have already mentioned may happen as the limit Wigner transform of pure states bounded in $L^2(\mathbb{R}^d)$ [26]. It this case, the scintillation function $J_\eta$ does not converge to 0 as $\eta \to 0$. Instead, scintillation of order $O(1)$ is created and its total intensity does not decay as $z$ increases. More precisely we have the following result:

THEOREM 5.1. *Let us assume that the initial conditions for $W_\eta$ are given by (5.1). Then the scintillation function $J_\eta$ is composed of a singular term of the form*

$$\delta(\mathbf{x}-\boldsymbol{\xi})\delta(\mathbf{p}-\mathbf{q})\Big(\alpha(\mathbf{x},\mathbf{p},z) - e^{-2Qz}\alpha(\mathbf{x}-z\mathbf{p},\mathbf{p},0)\Big). \tag{5.2}$$

*The other contributions to $J_\eta$ are mutually singular with respect to this term. Moreover the density $\alpha(\mathbf{x},\mathbf{p},z)$ solves the radiative transfer equation*

$$\frac{\partial\alpha}{\partial z} + \mathbf{p}\cdot\nabla_{\mathbf{x}}\alpha + 2Q\alpha = \int_{\mathbb{R}^d}\hat{Q}(\mathbf{u})\Big(\alpha(\mathbf{x},\mathbf{p}+\frac{\mathbf{u}}{2},z) + \alpha(\mathbf{x},\mathbf{p}-\frac{\mathbf{u}}{2},z)\Big)d\mathbf{u}. \tag{5.3}$$

*Its initial condition is $a_0(\mathbf{x},\mathbf{p}) = \delta(\mathbf{x}-\mathbf{x}_0)\delta(\mathbf{p}-\mathbf{p}_0)$ so that $\alpha$ is a non-negative density that satisfies*

$$\int_{\mathbb{R}^{2d}}\alpha(\mathbf{x},\mathbf{p},z)d\mathbf{x}d\mathbf{p} = \int_{\mathbb{R}^{2d}}\alpha(\mathbf{x},\mathbf{p},0)d\mathbf{x}d\mathbf{p} = 1.$$

*The total intensity of this singular part of the scintillation function is thus given by*

$$(1-e^{-2Qz})\int_{\mathbb{R}^{2d}}\alpha(\mathbf{x},\mathbf{p},0)d\mathbf{x}d\mathbf{p} = (1-e^{-2Qz}).$$

In this theorem we have introduced the notation $Q = Q(\mathbf{0})$. We see that the intensity of scintillation, which vanishes at $z=0$, increases like $2Qz$ for small values of $z$ and reaches 1 exponentially fast for large values of $z$.

*Proof.* The result is based on an analysis of the solution in integral form (A.13) that $\langle\mathcal{W}_\eta\rangle$ satisfies, of the transport semigroup $\mathcal{G}_t = e^{-tT_2}$ and on the integral operators $K_2$ and $K_{12}$ defined in (3.24). The rules of creation and destruction of scintillation are essentially the following. The operator $K_{12}$ creates scintillation when it acts on functions supported on the subspace $\{\mathbf{x}=\boldsymbol{\xi}\}$. It is composed of two parts, one that preserves $\{\mathbf{p}=\mathbf{q}\}$ and one that does not. Moreover it destroys scintillation outside of the manifold $\{\mathbf{x}=\boldsymbol{\xi}\}$:

$$\begin{aligned}K_{12}[\delta(\mathbf{x}-\boldsymbol{\xi})\delta(\mathbf{p}-\mathbf{q})\alpha(\mathbf{x},\mathbf{p})] &= -\delta(\mathbf{x}-\boldsymbol{\xi})2\hat{Q}(\mathbf{p}-\mathbf{q})\alpha(\mathbf{x},\tfrac{\mathbf{p}+\mathbf{q}}{2})\\ &\quad+\delta(\mathbf{x}-\boldsymbol{\xi})\delta(\mathbf{p}-\mathbf{q})\int_{\mathbb{R}^d}\hat{Q}(\mathbf{u})(\alpha(\mathbf{x},\mathbf{p}+\frac{\mathbf{u}}{2})+\alpha(\mathbf{x},\mathbf{p}-\frac{\mathbf{u}}{2}))d\mathbf{u},\end{aligned}$$

$$\begin{aligned}K_{12}[\delta(\mathbf{x}-\boldsymbol{\xi}-\tau(\mathbf{p}-\mathbf{q}))\alpha(\mathbf{x},\mathbf{p},\mathbf{q})] &= \delta(\mathbf{x}-\boldsymbol{\xi}-\tau(\mathbf{p}-\mathbf{q}))\times\\ &\quad\int_{\mathbb{R}^d}e^{i\frac{\tau(\mathbf{p}-\mathbf{q})\cdot\mathbf{u}}{\eta}}\hat{Q}(\mathbf{u})\sum_{\epsilon_i,\epsilon_j=\pm}\epsilon_i\epsilon_j\,\alpha(\mathbf{x},\mathbf{p}+\epsilon_i\frac{\mathbf{u}}{2},\mathbf{q}+\epsilon_j\frac{\mathbf{u}}{2}).\end{aligned}$$

The latter term converges to 0 weakly as $\eta \to 0$ as soon as $\tau > 0$. In view of the above formula, we decompose $K_{12}$ as $K_{12}^+ + K_{12}^-$, where $K_{12}^+$ preserves $\{\mathbf{x} = \boldsymbol{\xi}\} \cap \{\mathbf{p} = \mathbf{q}\}$ (it is composed of the two terms such that $\epsilon_i = \epsilon_j$) and $K_{12}^-$ does not (it is composed of the two terms such that $\epsilon_i = -\epsilon_j$).

The operator $\mathcal{G}_t$ preserves $\{\mathbf{x} = \boldsymbol{\xi}\}$ only when acting on functions supported on the subspace $\{\mathbf{p} = \mathbf{q}\}$. However it does not preserve $\{\mathbf{x} = \boldsymbol{\xi}\}$ otherwise:

$$\begin{aligned}\mathcal{G}_s[\delta(\mathbf{x} - \boldsymbol{\xi})\delta(\mathbf{p} - \mathbf{q})\alpha(\mathbf{x}, \mathbf{p})] &= \delta(\mathbf{x} - \boldsymbol{\xi})\delta(\mathbf{p} - \mathbf{q})\alpha(\mathbf{x} - s\mathbf{p}, \mathbf{p})\\ \mathcal{G}_s[\delta(\mathbf{x} - \boldsymbol{\xi})\alpha(\mathbf{x}, \mathbf{p}, \mathbf{q})] &= \delta((\mathbf{x} - \boldsymbol{\xi}) - s(\mathbf{p} - \mathbf{q}))\alpha(\mathbf{x} - s\mathbf{p}, \mathbf{p}, \mathbf{q})\end{aligned}$$

The operator $K_2$ preserves $\{\mathbf{x} = \boldsymbol{\xi}\}$ but not $\{\mathbf{p} = \mathbf{q}\}$:

$$K_2[\delta(\mathbf{x} - \boldsymbol{\xi})\delta(\mathbf{p} - \mathbf{q})\alpha(\mathbf{x}, \mathbf{p})] = \delta(\mathbf{x} - \boldsymbol{\xi})\hat{Q}(\mathbf{p} - \mathbf{q})(\alpha(\mathbf{x}, \mathbf{p}) + \alpha(\mathbf{x}, \mathbf{q})).$$

The equation for $W_{22}$ is the same as that for $\langle \mathcal{W}_\eta \rangle$ except that the cross terms $K_{12}$ are replaced by 0. Thus $J_\eta$ is given by the terms in the expansion (A.13) that involve at least one time the operator $K_{12}$. Since the initial condition for both $W_{22}$ and $\langle \mathcal{W}_\eta \rangle$ is of the form

$$\delta(\mathbf{x} - \boldsymbol{\xi})\delta(\mathbf{p} - \mathbf{q})\alpha(\mathbf{x}, \mathbf{p}, 0),$$

where here $\alpha$ is a positive bounded measure, we deduce from the analysis of the above operators that the part of the solution that is supported on the subspace $\{\mathbf{x} = \boldsymbol{\xi}\}$ and $\{\mathbf{p} = \mathbf{q}\}$ is the part that never involves the operators $K_{12}^-$ and $K_2$. It is not difficult to realize that this part solves

$$\frac{\partial \alpha}{\partial z} + \mathbf{p} \cdot \nabla_{\mathbf{x}} \alpha + 2Q\alpha = \int_{\mathbb{R}^d} \hat{Q}(\mathbf{u})\Big(\alpha(\mathbf{x}, \mathbf{p} + \frac{\mathbf{u}}{2}, z) + \alpha(\mathbf{x}, \mathbf{p} - \frac{\mathbf{u}}{2}, z)\Big) d\mathbf{u}. \tag{5.4}$$

We then observe that the total mass

$$\int_{\mathbb{R}^{2d}} \alpha(\mathbf{x}, \mathbf{p}, z) d\mathbf{x} d\mathbf{p} = \int_{\mathbb{R}^{2d}} \alpha(\mathbf{x}, \mathbf{p}, 0) d\mathbf{x} d\mathbf{p} = 1.$$

Now the ballistic part of $\alpha(\mathbf{x}, \mathbf{p}, z)$, which is given by

$$e^{-2Qz}\alpha(\mathbf{x} - z\mathbf{p}, \mathbf{p}, 0),$$

is not scintillation as it is also present in $W_{22}$. This is the only part that does not involve the operator $K_{12}$. Hence the scintillation component is given by

$$\alpha(\mathbf{x}, \mathbf{p}, z) - e^{-2Qz}\alpha(\mathbf{x} - z\mathbf{p}, \mathbf{p}, 0).$$

There are other contributions of order $O(1)$ to the scintillation function. They are however mutually singular with respect to measures supported on $\{\mathbf{x} = \boldsymbol{\xi}\} \cap \{\mathbf{p} = \mathbf{q}\}$. It does not mean that these contributions are necessarily small. However since they are mutually singular with respect (5.2), the latter persists. This concludes the proof of the theorem. ∎

Notice that the above theorem singles out one specific contribution to the scintillation function. There are other contributions that do not decay exponentially in time, for instance:

$$K_{12}^- \mathcal{G}_s a_0 = \delta(\mathbf{x} - \boldsymbol{\xi})\delta(\mathbf{x} - s\mathbf{p} - \mathbf{x}_0)\hat{Q}(\mathbf{p} - \mathbf{q})\delta(\frac{\mathbf{p} + \mathbf{q}}{2} - \mathbf{p}_0). \tag{5.5}$$

This term is mutually singular with respect to (5.2) and yet will contribute a source term to $J_\eta$ whose total intensity is preserved by the evolution operator $\partial_z + 2Q - K_2$. Notice however that (5.5) will be regularized by $\partial_z + 2Q - K_2$ in the sense that the singular part of $(\partial_z + 2Q - K_2)(K_{12}^{-}\mathcal{G}_s a_0)$ decays exponentially in $z$. We have concentrated on the part described in Theorem 5.1 because its density $\alpha$ describes a closed-form equation (5.3).

We have seen that the scintillation function $J_\eta$ converges to 0 in the "generic" case, where the initial conditions of the Wigner transform $W_\eta(\mathbf{x}, \mathbf{p}, 0)$ are sufficiently smooth. However when the initial conditions are sufficiently singular, such as for instance $W_\eta(\mathbf{x}, \mathbf{k}, 0) = \delta(\mathbf{x} - \mathbf{x}_0)\delta(\mathbf{p} - \mathbf{p}_0)$, the scintillation function $J_\eta$ does not converge to 0 as $\eta \to 0$. Instead it increases as $z \to \infty$, which implies that the quantity $\langle W_\eta \rangle$ is no longer sufficient to completely characterize wave propagation in the random medium. A complete characterization would require to analyze higher moments of the wave field as is done in [19].

Let us conclude this section with a remark on the large $z$ behavior. All the estimates we have obtained in previous sections hold uniformly in $z$ on compact intervals. Since the $L^2$ norm is preserved by the equation (3.27) as shown by Lemma 3.2, we can show that $J_\eta$ is small in the $L^2$ sense uniformly for $z \ll \eta^{-d/2}$. Since the operator $K_{12}$ is of order $\eta^{-d}$ when applied to smooth functions, we may expect that $J_\eta$ is small in some sense for distances $z \ll \eta^{-d}$ although we have no rigorous argument to substantiate this claim.

Another quantity of interest to measure scintillation is the so-called *scintillation index*, which is an integrated version of the rescaled quantity $h = J_\eta / W_{22}$. There are several works on the numerical behavior of $h$ as $z \to \infty$. We refer to [21, 37, 38]. All numerical simulations suggest that $h$ converges to 1 as $z \to \infty$, which is incompatible with energy stability (where $h \approx 0$). Scintillation eventually dominates the dynamics of wave propagation. This would be consistent with other results obtained on the long "time" behavior of the Itô Schrödinger equation [19], where the energy density follows an exponential law, hence is by no means deterministic.

**6. Application to time reversal of waves in random media.** The propagation of time reversed waves in random media has received a lot of attention recently in the physical [14, 16, 17, 22, 25] and mathematical [4, 5, 6, 7, 11, 29] literatures. In the original experiments carried out by M. Fink, propagating waves emanating from a spatially localized source are recorded in time by an array of transducers. They are then time reversed and sent back into the medium, so that what is recorded last is sent back first. The striking aspect of time reversed waves is that they refocus better at the location of the initial source when propagation occurs in a heterogeneous medium than in a homogeneous medium.

The first quantitative explanation to this phenomenon for multidimensional waves was obtained in [7] and further studied in [4, 5, 29, 30]. An important question that arises in the context of time reversed waves is to understand how stable the back-propagated signal is with respect to the realizations of a random medium with given statistics. The framework introduced in [7] allows us to respond precisely to this question if wave propagation is modeled by the Itô Schrödinger equation and the scalings introduced in earlier sections are appropriate.

Let us briefly recall the mathematical framework for the propagation of time reversed waves. We follow the presentation in [4]. Let us denote by $\psi_\eta(\mathbf{x}, z = 0) = \psi_0((\mathbf{x} - \mathbf{x}_0)/\eta)$ the initial source term, where $\mathbf{x}_0$ is the center of the source term and $\psi_0$ is a given smooth function. We assume here that the reduced frequency $\kappa = 1$

throughout. We let the signal propagate till $z = L$ modeling wave propagation by the Itô Schrödinger equation (3.28). At $z = L$ the signal is truncated (multiplied by $\chi(\mathbf{x})$), filtered (convolved with $f(\mathbf{x})$), and time reversed (complex conjugated in this context). It is then sent back to $z = 0$. This corresponds to a *single time* time reversal experiment [5]. Following [4, 7] the back-propagated signal can be written as

$$\psi_\eta^B(\boldsymbol{\xi};\mathbf{x}_0) = \int_{\mathbb{R}^{2d}} e^{i\mathbf{k}\cdot(\boldsymbol{\xi}-\mathbf{y})} W_\eta(\mathbf{x}_0 + \eta\frac{\mathbf{y}+\boldsymbol{\xi}}{2}, \mathbf{k}, L)\psi_0(\mathbf{y})\frac{d\mathbf{y}d\mathbf{k}}{(2\pi)^d}. \tag{6.1}$$

The function $W_\eta(\mathbf{x},\mathbf{k},z)$ has a very similar expression to the mixture of states (3.33). More precisely it is given by

$$W_\eta(\mathbf{x},\mathbf{k},z) = \int_{\mathbb{R}^d} \hat{f}(\mathbf{q}) U_\eta(\mathbf{x},\mathbf{k},z;\mathbf{q}) d\mathbf{q}, \tag{6.2}$$

where

$$U_\eta(\mathbf{x},\mathbf{k},z;\mathbf{q}) = \int_{\mathbb{R}^d} e^{i\mathbf{k}\cdot\mathbf{y}} Q_\eta(\mathbf{x} - \frac{\eta\mathbf{y}}{2}, z) Q_\eta^*(\mathbf{x} + \frac{\eta\mathbf{y}}{2}, z)\frac{d\mathbf{y}}{(2\pi)^d}, \tag{6.3}$$

and $Q_\eta$ is the solution of (3.28) with initial condition

$$Q_\eta(\mathbf{x},0) = \chi(\mathbf{x}) e^{-\frac{i\mathbf{q}\cdot\mathbf{x}}{\eta}}.$$

We are thus in the framework described in section 3.4 and can apply the results obtained in sections 4 and 5. Our main result is the following:

THEOREM 6.1. *Let us assume that the initial condition $\psi_0(\mathbf{y}) \in L^2(\mathbb{R}^d)$, the filter $f(\mathbf{y}) \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, and the detector amplification $\chi(\mathbf{x})$ is sufficiently smooth. Let us also consider two test functions $\tilde\lambda(\mathbf{x}_0)$ and $\mu(\boldsymbol{\xi})$ in $L^2(\mathbb{R}^d)$. All functions here are supposed real-valued to simplify.*

*Then $\psi_\eta^B(\boldsymbol{\xi};\mathbf{x}_0)$ converges weakly and in probability to the deterministic signal*

$$\psi^B(\boldsymbol{\xi};\mathbf{x}_0) = \int_{\mathbb{R}^d} e^{i\mathbf{k}\cdot\boldsymbol{\xi}} \overline{W}(\mathbf{x}_0,\mathbf{k},L)\hat\psi_0(\mathbf{k}) d\mathbf{k},$$

*where $\overline{W}(\mathbf{x}_0,\mathbf{k},L)$ is the solution of (3.30) with initial conditions given by*

$$\overline{W}(\mathbf{x},\mathbf{k},0) = \hat{f}(\mathbf{k})|\chi(\mathbf{x})|^2.$$

*Moreover, introducing $\lambda(\boldsymbol{\xi},\mathbf{x}_0) = \tilde\lambda(\mathbf{x}_0)\mu(\boldsymbol{\xi})$ we have the following estimate*

$$\Big\langle (\psi_\eta^B - \langle\psi_\eta^B\rangle, \lambda)^2 \Big\rangle \le C\eta^d \|\psi_0\|_2^2 \|\lambda\|_2^2 = C\eta^d \|\psi_0\|_2^2 \|\mu\|_2^2 \|\tilde\lambda\|_2^2, \tag{6.4}$$

*uniformly in $L$ on compact intervals.*

*Proof.* The boundary condition at $z = 0$ for $W_\eta$ is given by

$$W_\eta(\mathbf{x},\mathbf{k},0) = \int_{\mathbb{R}^d} d\mathbf{y} e^{-i\mathbf{k}\cdot\mathbf{y}} f(\mathbf{y})\chi(\mathbf{x} + \frac{\eta\mathbf{y}}{2})\chi(\mathbf{x} - \frac{\eta\mathbf{y}}{2}).$$

As $\chi$ is a smooth function independent of $\eta$ and $f(\mathbf{y}) \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, we observe that $W_\eta(\mathbf{x},\mathbf{k},0)$ belongs to $X_{2,2} \cap X_{2,\infty}$. Thus the results of Theorem 4.1 apply. Let us now calculate

$$
\begin{aligned}
&\int_{\mathbb{R}^{2d}} (\psi_\eta^B - \langle\psi_\eta^B\rangle)(\boldsymbol{\xi};\mathbf{x}_0)\lambda(\mathbf{x}_0,\boldsymbol{\xi})d\mathbf{x}_0 d\boldsymbol{\xi}\\
&= \int_{\mathbb{R}^{4d}} e^{i\mathbf{k}\cdot(\boldsymbol{\xi}-\mathbf{y})}(W_\eta - \langle W_\eta\rangle)(\mathbf{x}_0+\eta\frac{\mathbf{y}+\boldsymbol{\xi}}{2},\mathbf{k},L)\psi_0(\mathbf{y})\mu(\boldsymbol{\xi})\tilde\lambda(\mathbf{x}_0)d\mathbf{x}_0 d\boldsymbol{\xi} d\mathbf{y} d\mathbf{k}\\
&= \int_{\mathbb{R}^{2d}} (W_\eta - \langle W_\eta\rangle)(\mathbf{x}_0,\mathbf{k},L)\int_{\mathbb{R}^d}\hat h(\mathbf{k},\mathbf{Y})\lambda(\mathbf{x}_0-\eta\mathbf{Y})d\mathbf{Y}d\mathbf{k}d\mathbf{x}_0.
\end{aligned}
$$

where $h(\boldsymbol{\xi}-\mathbf{y},\frac{\boldsymbol{\xi}+\mathbf{y}}{2}) = \psi_0(\mathbf{y})\mu(\boldsymbol{\xi})$ and $\hat h(\mathbf{k},\mathbf{Y})$ is its Fourier transform with respect to the first variable only. It remains to observe that

$$
\int_{\mathbb{R}^{2d}}\Big(\int_{\mathbb{R}^d}\hat h(\mathbf{k},\mathbf{Y})\lambda(\mathbf{x}_0-\eta\mathbf{Y})d\mathbf{Y}\Big)^2 d\mathbf{k}d\mathbf{x}_0 \le \|\lambda_0\|_2^2\|h\|_2^2 = \|\lambda_0\|_2^2\|\tilde\lambda\|_2^2\|\mu\|_2^2,
$$

to apply the result of Theorem 4.2 and obtain (6.4). The rest of the theorem follows from the strong convergence of $\langle W_\eta\rangle(z=0)$ to $\overline{W}(z=0)$ and the $L^2$ stability of (3.30). □

This uniform control makes the convergence stronger than the result obtained in the parabolic approximation of wave propagation in [4]. As in section 4 we can choose $\tilde\lambda(\mathbf{x})$ of the form

$$
\tilde\lambda(\mathbf{x}) = \frac{1}{\eta^{\alpha d}}\phi(\frac{\mathbf{x}-\mathbf{x}_0}{\eta^\alpha}),
$$

and obtain the convergence result

$$
\Big\langle(\psi_\eta^B - \langle\psi_\eta^B\rangle,\tilde\lambda\otimes\mu)^2\Big\rangle \le C\eta^{d(1/2-\alpha)}\|\psi_0\|_2^2\|\mu\|_2^2\|\phi\|_2^2 \to 0 \qquad (6.5)
$$

provided that $\alpha < 1/2$. This shows that the average in the center point $\mathbf{x}_0$ may be performed over a domain of diameter $\eta^\alpha$, much larger than the transverse scale $\eta$ but much smaller than $O(1)$.

We may also have destabilization when $W_\eta(z=0)$ is singular although this case is not very physical. In the limit $\eta\to 0$, the initial condition for $W_\eta(z=0)$ is given by $\hat f(\mathbf{k})|\chi(\mathbf{x})|^2$. This term is very singular if the support of $\chi(\mathbf{x})$ becomes concentrated at one point as $\eta\to 0$ and the filter $\hat f(\mathbf{k})$ also picks one given wavenumber $\mathbf{k}_0$ in the limit $\eta\to 0$. Whereas the former is physical if one assume that the volume covered by the transducers is small, the latter is not very physical as "good" detectors with minimal blurring correspond to $\hat f(\mathbf{k})$ close to a constant. In the setting where $W_\eta(z=0)$ converges to $\delta(\mathbf{x})\delta(\mathbf{p}-\mathbf{p}_0)$ however, the results of Theorem 5.1 apply and the refocused signal $\psi_\eta^B$ is no longer stable.

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**Appendix A. Results on radiative transfer equations.** This appendix presents some properties satisfied by the solutions of the transport equations defined in previous sections. Similar results can be found in [12] for instance. However they are not quite stated in the form that we need in this paper and are concerned specifically with transport equations of the form (3.16) and not (3.23).

**A.1. Analysis of second-order transport equation.** Let us consider the transport equation for $a(\mathbf{x},\mathbf{p},z)$

$$\begin{aligned}&\frac{\partial a}{\partial z}+\mathbf{p}\cdot\nabla_{\mathbf{x}}a=\int_{\mathbb{R}^d}\Big[\hat{Q}(\mathbf{p}-\mathbf{p}')-Q(\mathbf{0})\delta(\mathbf{p}-\mathbf{p}')\Big]a(\mathbf{x},\mathbf{p}',z)d\mathbf{p}'+f(\mathbf{x},\mathbf{p},z)\\&a(\mathbf{x},\mathbf{p},0)=a_0(\mathbf{x},\mathbf{p}),\end{aligned}\tag{A.1}$$

where $f(z)$ is an additional source term. With obvious notation we recast this equation as

$$\begin{aligned}&\frac{\partial a}{\partial z}+Ta+Qa=Ka+f(z)\\&a(\mathbf{x},\mathbf{p},0)=a_0(\mathbf{x},\mathbf{p}),\end{aligned}\tag{A.2}$$

where $Q=Q(\mathbf{0})$ is a constant and $K$ is a positive operator. The solution can be written in integral form

$$a(z)=e^{-zQ}\mathcal{H}_z a_0+\int_0^z e^{-(z-s)Q}\mathcal{H}_{z-s}(Ka+f)(s)ds,\tag{A.3}$$

where the free transport semigroup is defined by

$$\mathcal{H}_z=e^{-zT},\qquad \text{i.e.,}\qquad \mathcal{H}_z h(\mathbf{x},\mathbf{p})=h(\mathbf{x}-z\mathbf{p},\mathbf{p}).\tag{A.4}$$

Let $X$ be a Banach space of functions on $\mathbb{R}^{2d}$. We deduce from the fact that $\hat{Q}\geq 0$ and from the Minkowski inequality that for all function $g\in X$,

$$\|Kg\|_X=\Big\|\int_{\mathbb{R}^d}\hat{Q}(\mathbf{p}-\mathbf{p}')g(\mathbf{x},\mathbf{p}')d\mathbf{p}'\Big\|_X\leq\int_{\mathbb{R}^d}\hat{Q}(\mathbf{p}-\mathbf{p}')\|g\|_X d\mathbf{p}'=Q\|g\|_X.\tag{A.5}$$

We can then show the

Theorem A.1. *Let us assume that*

$$\|\mathcal{H}_z\|_{\mathcal{L}(X)}\leq 1\qquad \textit{for all } z\geq 0.$$

*Then the unique solution to the transport equation (A.1) satisfies that*

$$\|a\|_X(z)\leq\|a_0\|_X+\int_0^z\|f\|_X(s)ds.\tag{A.6}$$

*Proof.* Indeed when $f=0$ we obtain from the assumptions on $\mathcal{H}_z$ and $K$ that

$$\|a\|_X(z)\leq e^{-Qz}\|a_0\|_X+\int_0^z e^{-(z-s)Q}Q\|a\|_X(s)ds.$$

We see that $\|a\|_X(z)=\|a_0\|_X$ is a majorizing solution of the above equation so that

$$\|a\|_X(z)\leq\|a_0\|_X.$$

Similarly when $a_0=0$ we have

$$e^{Qz}\|a\|_X(z)\leq\int_0^z Q[e^{Qs}\|a\|_X(s)]ds+\int_0^z e^{Qs}\|f\|_X(s)ds.$$

This shows that

$$e^{Qz}\|a\|_X(z) \le \int_0^z e^{Q(z-s)}e^{Qs}\|f\|_X(s)ds = e^{Qz}\int_0^z \|f\|_X(t)dt.$$

This implies the result. □

COROLLARY A.2. *Let $1 \le m,n \le \infty$. Provided that $a_0$ and $f$ are uniformly in $X_{m,n} = L^m(\mathbb{R}^d_{\mathbf{p}}; L^n(\mathbb{R}^d_{\mathbf{x}}))$, we obtain that the solution $a(z)$ to (A.1) is also in $X_{m,n}$ uniformly in $z$ on compact intervals.*

The proof is based on the fact that

$$\int_{\mathbb{R}^d} |\mathcal{H}_s f(\mathbf{x},\mathbf{p})|^n d\mathbf{x} = \int_{\mathbb{R}^d} |f(\mathbf{x}-s\mathbf{p},\mathbf{p})|^n d\mathbf{x} = \int_{\mathbb{R}^d} |f(\mathbf{x},\mathbf{p})|^n d\mathbf{x}.$$

COROLLARY A.3. *Let us assume that $a_0 \in X_{2,2}\cap X_{2,\infty}$ and $f=0$. Then*

$$m_2(\mathbf{x},\mathbf{y},z) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\mathbf{y}\cdot\mathbf{p}} a(\mathbf{x},\mathbf{p},z)d\mathbf{p}, \tag{A.7}$$

*where $a(\mathbf{x},\mathbf{p},z)$ solves (A.1), is such that $m_2 \in X_{2,\infty}\cap X_{2,2}$, i.e.,*

$$\int_{\mathbb{R}^{2d}} |m_2(\mathbf{x},\mathbf{y},z)|^2 d\mathbf{x}d\mathbf{y} + \int_{\mathbb{R}^d} \sup_{\mathbf{x}} |m_2(\mathbf{x},\mathbf{y},z)|^2 d\mathbf{y} \le C, \tag{A.8}$$

*uniformly in $z$ on compact intervals.*

This follows from Corollary A.2 with $(m,n)=(2,2)$ and $(m,n)=(2,\infty)$ and the Parseval identity.

**A.2. Analysis of fourth-order transport equations.** We now consider the equation for $a(\mathbf{x},\mathbf{p},\boldsymbol{\xi},\mathbf{q},z)$:

$$\begin{aligned}
&\frac{\partial a}{\partial z} + (\mathbf{p}\cdot\nabla_{\mathbf{x}} + \mathbf{q}\cdot\nabla_{\boldsymbol{\xi}})a \\
&= \int_{\mathbb{R}^{2d}} \Big[\hat{Q}(\mathbf{p}-\mathbf{p}')\delta(\mathbf{q}-\mathbf{q}') + \hat{Q}(\mathbf{q}-\mathbf{q}')\delta(\mathbf{p}-\mathbf{p}')](a(\mathbf{p}',\mathbf{q}') - a(\mathbf{p},\mathbf{q}))d\mathbf{p}'d\mathbf{q}' \\
&\quad + \int_{\mathbb{R}^d} \hat{Q}(\mathbf{u})e^{i\frac{(\mathbf{x}-\boldsymbol{\xi})\cdot\mathbf{u}}{\eta}}\Big(a(\mathbf{p}-\tfrac{\mathbf{u}}{2},\mathbf{q}-\tfrac{\mathbf{u}}{2}) + a(\mathbf{p}+\tfrac{\mathbf{u}}{2},\mathbf{q}+\tfrac{\mathbf{u}}{2}) \\
&\qquad\qquad - a(\mathbf{p}-\tfrac{\mathbf{u}}{2},\mathbf{q}+\tfrac{\mathbf{u}}{2}) - a(\mathbf{p}+\tfrac{\mathbf{u}}{2},\mathbf{q}-\tfrac{\mathbf{u}}{2})\Big)d\mathbf{u} + f(z) \\
&a(\mathbf{x},\mathbf{p},\boldsymbol{\xi},\mathbf{p},0) = a_0(\mathbf{x},\mathbf{p},\boldsymbol{\xi},\mathbf{q}).
\end{aligned} \tag{A.9}$$

We recast the above equation as

$$\begin{aligned}
&\frac{\partial a}{\partial z} + T_2 a + 2Qa = K_2 a + K_{12} a + f \equiv S(z), \\
&a(\mathbf{x},\mathbf{p},\boldsymbol{\xi},\mathbf{p},0) = a_0(\mathbf{x},\mathbf{p},\boldsymbol{\xi},\mathbf{q}),
\end{aligned} \tag{A.10}$$

where $K_2$ is the scattering operator $K\otimes K$ and $K_{12}$ accounts for the cross terms.

In integral form we get

$$\begin{aligned}
a(\mathbf{x},\mathbf{p},\boldsymbol{\xi},\mathbf{q},z) &= e^{-2Qz}a_0(\mathbf{x}-z\mathbf{p},\mathbf{p},\boldsymbol{\xi}-z\mathbf{q},\mathbf{q}) \\
&\quad + \int_0^z e^{-2Qs}S(\mathbf{x}-s\mathbf{p},\mathbf{p},\boldsymbol{\xi}-s\mathbf{q},\mathbf{q},z-s)ds.
\end{aligned} \tag{A.11}$$

This allows us to obtain the following result:

THEOREM A.4. *Let $a_0 \in L_p(\mathbb{R}^{4d})$ for $1 \le p \le \infty$. Then the unique solution to (A.11) satisfies that*

$$\|a\|_p(z) \le e^{4Qz}\|a_0\|_p + e^{4Qz}\int_0^z e^{-4Qs}\|f\|_p(s)ds. \tag{A.12}$$

*Proof.* Following the same calculations as for Theorem A.1, we obtain that

$$\|a\|_p(z) \le e^{\|K_{12}\|_p z}\|a_0\|_p + e^{\|K_{12}\|_p z}\int_0^z e^{-\|K_{12}\|_p s}\|f\|_p(s)ds.$$

Now it follows from Hölder's inequality with $1/p + 1/p' = 1$ that

$$\Big\| \int_{\mathbb{R}^d} \hat{Q}(\mathbf{u})\big|a(\mathbf{p}-\frac{\mathbf{u}}{2},\mathbf{q}-\frac{\mathbf{u}}{2})\big|d\mathbf{u}\Big\|_p^p \le \|\hat{Q}\|_1^{\frac{p}{p'}} \Big\| \int_{\mathbb{R}^d} \hat{Q}(\mathbf{u})\big|a(\mathbf{p}-\frac{\mathbf{u}}{2},\mathbf{q}-\frac{\mathbf{u}}{2})\big|^p d\mathbf{u}\Big\|_1 \le Q^p\|a\|_p^p.$$

This shows that $\|K_{12}\|_p \le 4Q$. ☐

When $p = 2$ we deduce from Lemma 3.2 that the above theorem holds with $Q$ replaced by 0 in (A.12) since the $L^2$ norm is at most preserved by the evolution in $z$.

**A.3. Integral formulation of fourth-order transport.** Let us assume that $f = 0$ in (A.10). We define the semigroup

$$\mathcal{G}_z = e^{-zT_2}.$$

We can therefore recast (A.10) as

$$a(s_0) = e^{-2Qs_0}\mathcal{G}_{s_0}a_0 + \int_0^{s_0} e^{-2Q(s_0-s_1)}\mathcal{G}_{s_0-s_1}(K_2+K_{12})a(s_1)ds_1.$$

By induction we get that

$$\begin{aligned} a(s_0) &= e^{-2Qs_0}\mathcal{G}_{s_0}a_0 \\ &+e^{-2Qs_0}\sum_{k=0}^{\infty}\int_0^{s_0}\cdots\int_0^{s_k}\mathcal{G}_{s_0-s_1}(K_2+K_{12})\cdots\mathcal{G}_{s_{k+1}}a_0 ds_1\cdots ds_{k+1}. \end{aligned} \tag{A.13}$$

We deduce from this integral formulation that the equation (A.10) is bounded in $\mathcal{M}_b(\mathbb{R}^{4d})$, the space of bounded measures, for instance. Indeed we verify that

$$\|\mathcal{G}_s\mu\|_{\mathcal{M}_b} \le \|\mu\|_{\mathcal{M}_b}$$

for all $\mu \in \mathcal{M}_b(\mathbb{R}^{4d})$. This can be shown by duality since $\mathcal{G}_s$ preserves the supremum of bounded functions. We also verify that $K_2$ and $K_{12}$ are bounded in $\mathcal{L}(\mathcal{M}_b(\mathbb{R}^{4d}))$ since their kernels are bounded measures. This implies that the infinite sum in (A.13) converges in $\mathcal{L}(\mathcal{M}_b(\mathbb{R}^{4d}))$ and we have proved the:

LEMMA A.5. *Let us assume that $a_0 \in \mathcal{M}_b(\mathbb{R}^{4d})$. Then so is $a(z)$ uniformly on compact intervals.*

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