Random Homogenization and Convergence to Integrals with respect to the Rosenblatt Process

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Abstract

This paper concerns the random fluctuation theory of a one dimensional elliptic equation with highly oscillatory random coefficient. Theoretical studies show that the rescaled random corrector converges in distribution to a stochastic integral with respect to Brownian motion when the random coefficient has short-range correlation. When the random coefficient has long range correlation, it was shown for a large class of random processes that the random corrector converged to a stochastic integral with respect to fractional Brownian motion. In this paper, we construct a class of random coefficients for which the random corrector converges to a non-Gaussian limit. More precisely, for this class of random coefficients with long-range correlation, the properly rescaled corrector converges in distribution to a stochastic integral with respect to a Rosenblatt process.

1 Introduction

The equation of interest in this paper is the following one-dimensional elliptic equation with highly oscillatory coefficients:

$$-\frac{d}{dx}(a(\frac{x}{\varepsilon},\omega)\frac{d}{dx}u_\varepsilon(x,\omega)) = f(x), \quad x \in (0,1) \tag{1.1}$$

where the coefficient $a(x,\omega)$ is a stationary, bounded with bounded inverse, random potential. We are interested in the limiting behavior of the solution $u_\varepsilon(x,\omega)$ when $\varepsilon \to 0$ and more precisely in the size of the random fluctuations of $u_\varepsilon(x,\omega)$ and of their limiting distribution after proper rescaling.

Homogenization theory has been extensively studied in both periodic and random settings; see for instance [4, 6, 8, 9, 10, 14]. In the random setting, homogenization theory replace the random medium by a properly averaged effective, deterministic, medium. It is well known that u_ε converges to the deterministic, homogenized solution $\bar{u}(x)$ as $\varepsilon \to 0$. Fewer results are available concerning the theory of random fluctuations and in particular

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the theory of the random corrector $u_\varepsilon - \bar{u}$. In the one-dimensional setting, it has been shown that the property of the corrector strongly depends on the correlation property of the random potential. Using the explicit expressions of the solution to (1.1), it has been shown in [6] that when the random potential had short-range correlation and satisfies certain mixing properties, then the corrector's amplitude is of order $\sqrt{\varepsilon}$ and, after rescaling, converges in distribution to a stochastic integral with respect to Brownian motion. In [3], the result has been extended to a large class of random potential with long-range correlation, where the corrector's amplitude is of order $\varepsilon^{\alpha/2}$, with $\alpha \in (0,1)$ characterizing the decay of the correlation function of the random coefficient. Furthermore, the weak convergence limit of the rescaled corrector is then a stochastic integral with respect to fractional Brownian motion.

No theories of correctors are available for elliptic equations of the form (1.1) in higher spatial dimension; see [14] for estimates of the size of the random fluctuations. Similar results to the ones described above have been obtained in higher dimensions for Schrödinger-type equations with random potential. In [1], homogenization and corrector theory has been developed for a large class of second order elliptic equations with short-range correlated potentials, and it has been generalized into the long-range correlation case in [2]. In all of these cases, the limiting distributions of the rescaled correctors are Gaussian random fields, which admit convenient representations as a stochastic integral with respect to Brownian motion or fractional Brownian motion.

In this paper, we focus on (1.1) and follow the framework in [3] to obtain non-Gaussian limits for the random fluctuations. We construct a class of random potential with long-range correlation for which the limiting distribution of the rescaled corrector is no longer Gaussian as $\varepsilon \to 0$, but rather the so-called Rosenblatt distribution, which is an element in the second-order Wiener chaos. The corrector's amplitude is ε^α for $\alpha \in (0, 1/2)$. Our results are closely related to classical examples of non-central limits in probability theory; see [12, 7, 11]. Our approach is based on the explicit expression for the solution to (1.1) and a careful analysis of the oscillatory integral.

The paper is organized as follows. In section 2, we state the main theorem of the paper and compare it with previously established results. In section 3, we give a brief introduction to the Rosenblatt process and the corresponding stochastic integral. In section 4, we prove some key results concerning the weak convergence of oscillatory integral. In section 5, we prove the main theorem and briefly discuss possible extensions.

2 Main results

We formulate the problem as follows:

$$\begin{cases} -\dfrac{d}{dx}(a(\dfrac{x}{\varepsilon},\omega)\dfrac{d}{dx}u_\varepsilon(x,\omega)) = f(x), \quad x \in (0,1), \quad \omega \in \Omega, \\ u_\varepsilon(0,\omega) = 0, u_\varepsilon(1,\omega) = b \end{cases} \tag{2.1}$$

where $a(x,\omega) \in [a_0, a_0^{-1}]$ for some positive a_0, and is a stationary ergodic random process associated with the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. $f(x) \in \mathcal{C}[0,1]$ and $b \in \mathbb{R}$. Classical theory of elliptic equations shows the existence of a unique solution $u(.,\omega) \in H^1(0,1)$ $\mathbb{P}$-a.s.

It was shown that as the scale of the micro-structure $\varepsilon \to 0$, the solution $u_\varepsilon(x,\omega)$ to (2.1) converges $\mathbb{P}$-a.s. to the deterministic solution $\bar{u}(x)$ of the following equation:

$$\begin{cases} -\dfrac{d}{dx}(a^* \dfrac{d}{dx}\bar{u}(x)) = f(x), \quad x \in (0,1), \\ \bar{u}(0) = 0, \bar{u}(1) = b \end{cases} \tag{2.2}$$

where a^* is the harmonic mean of $a(x,\omega)$, i.e., $a^* = (\mathbb{E}\{a^{-1}(0,.)\})^{-1}$. See e.g. [8, 9, 10].

The analysis of the random corrector is based on the explicit expression for the solutions to (2.1) and (2.2). If we denote $a_\varepsilon(x) = a(\frac{x}{\varepsilon})$ and $F(x) = \int_0^x f(y)dy$, we have:

$$u_\varepsilon(x,\omega) = c_\varepsilon(\omega)\int_0^x \frac{1}{a_\varepsilon(y,\omega)}dy - \int_0^x \frac{F(y)}{a_\varepsilon(y,\omega)}dy, \quad c_\varepsilon(\omega) = \frac{b + \int_0^1 \frac{F(y)}{a_\varepsilon(y,\omega)}dy}{\int_0^1 \frac{1}{a_\varepsilon(y,\omega)}dy} \tag{2.3}$$

$$\bar{u}(x) = c^*\frac{x}{a^*} - \int_0^x \frac{F(y)}{a^*}dy, \quad c^* = a^*b + \int_0^1 F(y)dy. \tag{2.4}$$

We note that $u_\varepsilon(x,\omega) - \bar{u}(x)$ contains oscillatory integrals of the form

$$\int_{\mathbb{R}} \left(\frac{1}{a_\varepsilon(y,\omega)} - \frac{1}{a^*}\right) h(y)dy \tag{2.5}$$

for some function $h(y)$. Next we make some assumptions on the random process $a_\varepsilon(y,\omega)^{-1} - (a^*)^{-1}$.

2.1 Assumptions on the random process

Our goal is to analyze the statistical property of $u_\varepsilon - \bar{u}$ as $\varepsilon \to 0$ for a large class of random process $a(x,\omega)$, and show the existence of a non-Gaussian limiting corrector. To do this, we make the following assumptions on $a(x,\omega)$. Let

$$q(x,\omega) = \frac{1}{a(x,\omega)} - \frac{1}{a^*} \tag{2.6}$$

and assume that

$$q(x,\omega) = \Phi(g(x,\omega)) \tag{2.7}$$

for some function Φ and a random process $g(x,\omega)$ constructed explicitly as:

$$g(x) = \xi_{[x+U]} \tag{2.8}$$

where $\{\xi_k, k \in \mathbb{Z}\}$ is a centered stationary Gaussian sequence with unit variance, and its auto-correlation function $r(k) = \mathbb{E}\{\xi_0\xi_k\} \sim \kappa_g|k|^{-\alpha}$ for some $\kappa_g > 0$ and $\alpha \in (0, \frac{1}{2})$. U is uniformly distributed on $[0,1]$ and independent of $\{\xi_k, k \in \mathbb{Z}\}$. Through some elementary computation, we can show that $g(x)$ defined in (2.8) is a centered stationary Gaussian process with unit variance and long-range correlation: $R_g(x) = \mathbb{E}\{g(0)g(x)\} \sim \kappa_g|x|^{-\alpha}$.

Recall the definition of Hermite polynomials:

$$H_n(x) = (-1)^n \exp(\frac{x^2}{2}) \frac{d^n}{dx^n} \exp(-\frac{x^2}{2}), \tag{2.9}$$

assume $\mathbb{E}\{\Phi(g(x))^2\} < \infty$, and define $V_n = \mathbb{E}\{H_n(g(x))\Phi(g(x))\}$, we make the following key assumptions on $\Phi(x)$:

$$V_0 = V_1 = 0, \quad V_2 \neq 0. \tag{2.10}$$

The smallest integer n such that $V_n \neq 0$ is called the Hermite rank of Φ. So (2.10) states that the Hermite rank of Φ is 2. We will see later that $V_1 = 0$ is the key condition for the existence of a non-Gaussian corrector. When $V_1 \neq 0$, by Theorem 2.3 below, the rescaled corrector converges in distribution to some stochastic integral with respect to fractional Brownian motion.

The condition $V_0 = 0$ ensures that $q(x, \omega)$ is centered, and we have the following lemma concerning the correlation property of $q(x, \omega)$. For simplicity, we denote it as $q(x)$ from now on.

Lemma 2.1. *Let $R(x) = \mathbb{E}\{q(0)q(x)\}$, then*

$$R(x) \sim \kappa |x|^{-2\alpha} \tag{2.11}$$

where $\kappa = \frac{V_2^2 \kappa_g^2}{2}$.

Proof. We have $R(x) = \mathbb{E}\{\Phi(g(0))\Phi(g(x))\}$, and by Hermite expansion

$$\Phi(g(x)) = \sum_{n=0}^{\infty} \frac{V_n}{n!} H_n(g(x)) \tag{2.12}$$

Therefore,

$$\mathbb{E}\{\Phi(g(0))\Phi(g(x))\} = \sum_{n=0}^{\infty} \frac{V_n^2}{(n!)^2} \mathbb{E}\{H_n(g(0))H_n(g(x))\} = \sum_{n=0}^{\infty} \frac{V_n^2}{n!} R_g(x)^n \tag{2.13}$$

Since $V_0 = V_1 = 0$, we have

$$\mathbb{E}\{\Phi(g(0))\Phi(g(x))\} = R_g(x)^2 (\sum_{n=2}^{\infty} \frac{V_n^2}{n!} R_g(x)^{n-2}) \tag{2.14}$$

Because $\sum_{n=0}^{\infty} \frac{V_n^2}{n!} < \infty$ by assumption on Φ and $R_g(x) \sim \kappa_g |x|^{-\alpha}$, we verify that

$$R(x) = \mathbb{E}\{\Phi(g(0))\Phi(g(x))\} \sim \frac{V_2^2 \kappa_g^2}{2} |x|^{-2\alpha}. \tag{2.15}$$

The proof is complete. ☐

Since we assume $\alpha \in (0, \frac{1}{2})$, then $R(x) \notin L^1(\mathbb{R})$ so that $q(x)$ has long-range correlation and we have $|R(x)| \leq M|x|^{-2\alpha}$ for some constant M.

2.2 Main theorem

Now we state the main theorem and compare it with the previous results.

Theorem 2.2. *Let u_ε and $\bar{u}$ be the solutions in (2.3) and (2.4) and let $q(x,\omega)$ be a centered stationary random process of the form (2.7) with the Hermite rank of Φ being equal to 2. Then $u_\varepsilon - \bar{u}$ is a random process in $\mathcal{C}([0,1])$, and we have the following convergence in distribution in the space of continuous functions $\mathcal{C}([0,1])$:*

$$\frac{u_\varepsilon(x) - \bar{u}(x)}{\varepsilon^\alpha} \xrightarrow[\varepsilon\to 0]{\text{distribution}} \bar{U}(x) \tag{2.16}$$

where

$$\bar{U}(x) = \frac{V_2\kappa_g}{2}\int_{\mathbb{R}} F(x,y)dR_D(y) \tag{2.17}$$

$$F(x,y) = c^* 1_{[0,x]}(y) - F(y)1_{[0,x]}(y) + x(F(y) - \int_0^1 F(z)dz - a^*b)1_{[0,1]}(y). \tag{2.18}$$

Here $R_D(y)$ is a Rosenblatt process with $D = \alpha$.

It should be contrasted with the convergence results for processes with long-range correlation and the Hermite rank equals to 1 [3] or with short-range correlation [6].

Theorem 2.3. *Let u_ε and $\bar{u}$ be the solutions in (2.3) and (2.4), and let $q(x,\omega)$ be a centered stationary random process of the form (2.7) with the Hermite rank of Φ being equal to 1. Then $u_\varepsilon - \bar{u}$ is a random process in $\mathcal{C}([0,1])$, and we have the following convergence in distribution in the space of continuous functions $\mathcal{C}([0,1])$:*

$$\frac{u_\varepsilon(x) - \bar{u}(x)}{\varepsilon^{\alpha/2}} \xrightarrow[\varepsilon\to 0]{\text{distribution}} \bar{U}(x) \tag{2.19}$$

where

$$\bar{U}(x) = \sqrt{\frac{\kappa_g V_1^2}{H(2H-1)}}\int_{\mathbb{R}} F(x,y)dB^H(y) \tag{2.20}$$

Here $F(x,y)$ is given by (2.18) and $B^H(y)$ is a fractional Brownian motion with Hurst index $H = 1 - \frac{\alpha}{2}$.

Remark 2.4. In Theorem 2.3, we can assume $\alpha \in (0,1)$ instead of $\alpha \in (0,\frac{1}{2})$ in Theorem 2.2.

Theorem 2.5. *Let u_ε and $\bar{u}$ be the solutions in (2.3) and (2.4), and let $q(x,\omega)$ be a centered stationary random process of the form (2.7). If the correlation function R_g of g is integrable (instead of being equivalent to $|x|^{-\alpha}$ at infinity), then R is also integrable. The corrector $u_\varepsilon - \bar{u}$ is a random process in $\mathcal{C}([0,1])$ and we have the following convergence in distribution in the space of continuous functions $\mathcal{C}([0,1])$:*

$$\frac{u_\varepsilon(x) - \bar{u}(x)}{\sqrt{\varepsilon}} \xrightarrow[\varepsilon\to 0]{\text{distribution}} \bar{U}(x) \tag{2.21}$$

where

$$\bar{U}(x) = \left(2\int_0^\infty R(\tau)d\tau\right)^{\frac{1}{2}} \int_{\mathbb{R}} F(x,y)dB(y) \tag{2.22}$$

Here $F(x,y)$ is given by (2.18) and $B(y)$ is a standard Brownian motion.

We can see from the above theorems that the size of the corrector not only depends on the correlation property of the random process $q(x)$ but also the Hermite rank of Φ, and the limiting distribution of the properly rescaled corrector can be non-Gaussian.

The rest of this paper is devoted to the proof of Theorem 2.2. To do this, we first give a brief introduction of the Rosenblatt process and the stochastic integral with respect to it.

3 Rosenblatt process

In this section, we briefly recall some general facts about the Rosenblatt process and the stochastic integral with respect to it [13].

3.1 Non-central limit theorem and Wiener-Itô integral representation

The following theorem gives rise to the Rosenblatt process [11].

Theorem 3.1. *Assume X_n are centered stationary Gaussian sequence with unit variance $r(k) = \mathbb{E}\{X_0 X_k\} \sim k^{-D}L(k)$, where $D \in (0, 1/2)$ and $L(k)$ is a slowly varying function. Define*

$$Z_{N,2}(t) = \frac{1}{d_N}\sum_{i=1}^{[Nt]}(X_i^2 - 1)$$

with $d_N \sim N^{1-D}L(N)$ as $N \to \infty$.

Then the finite dimensional distributions of $Z_{N,2}(t)$ converge to the corresponding finite dimensional distributions of the Rosenblatt process $R_D(t)$.

By Theorem 3.1, we have the characteristic function of the Rosenblatt distribution $R_D(1)$ in a small neighborhood of the origin:

$$\exp(i\theta R_D(1)) = \exp\left\{\frac{1}{2}\sum_{n=2}^{\infty}\left[\frac{(2i\theta)^n}{n}\int_{[0,1]^n}\frac{1}{|x_2-x_1|^D|x_3-x_2|^D\dots|x_1-x_n|^D}d\mathbf{x}\right]\right\}, \tag{3.1}$$

The Rosenblatt process has the following representation as a Wiener-Itô integral:

$$R_D(t) = c(D)\int_{\mathbb{R}^2}\int_0^t (s-y_1)_+^{-\frac{1+D}{2}}(s-y_2)_+^{-\frac{1+D}{2}}dsdB(y_1)dB(y_2). \tag{3.2}$$

The constant $c(D)$ is chosen such that $\mathbb{E}\{R_D(1)^2\} = 1$. From (3.2), we see that the Rosenblatt process lives in the second order Wiener chaos and is a self-similar process with stationary increments. The Hurst index $H = 1 - D$.

3.2 Stochastic integral with respect to Rosenblatt process

The representation (3.2) is not very convenient to define stochastic integrals. Recall that for fraction Brownian motion, we have:

$$B_t^H = \int_0^t K^H(t,s)dB(s) \tag{3.3}$$

with $(B_t, t \in [0,T])$ a standard Brownian motion and

$$K^H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{2}{2}} u^{H-\frac{1}{2}} du \tag{3.4}$$

where $t > s$ and

$$c_H = \left(\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})} \right)^{\frac{1}{2}}. \tag{3.5}$$

There are similar results for Rosenblatt process:

Proposition 3.2. *Let K be the kernel in (3.4) and $(R_D(t))_{t\in[0,T]}$ a Rosenblatt process with Hurst index $H = 1-D$. Then it holds that*

$$R_D(t) = d(H) \int_0^t \int_0^t \left[\int_{y_1 \vee y_2}^t \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) du \right] dB(y_1) dB(y_2) \tag{3.6}$$

where $(B_t, t \in [0,T])$ is a standard Brownian motion, $H' = \frac{H+1}{2}$, and

$$d(H) = \frac{1}{H+1} \left(\frac{H}{2(2H-1)} \right)^{-\frac{1}{2}}. \tag{3.7}$$

By the representation (3.6), the stochastic integral with respect to Rosenblatt process can be defined as follows. We first rewrite

$$R_D(t) = \int_0^T \int_0^T I(1_{[0,t]})(y_1, y_2) dB(y_1) dB(y_2) \tag{3.8}$$

where the operator I is defined on the set of functions $f : [0,T] \to \mathbb{R}$, takes values in the set of functions $g : [0,T]^2 \to \mathbb{R}^2$ and it is given by

$$I(f)(y_1, y_2) = d(H) \int_{y_1 \vee y_2}^T f(u) \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) du. \tag{3.9}$$

If f is an element of the set $\mathcal{E}$ of step functions on $[0,T]$ of the form

$$f = \sum_{i=0}^{n-1} a_i 1_{(t_i, t_{i+1}]}, \quad 0 = t_0 < t_1 < \ldots < t_n = T \tag{3.10}$$

it is natural to define its stochastic integral with respect to $R_D(t)$ as

$$\int_0^T f(t)dR_D(t) = \sum_{i=0}^{n-1} a_i(R_D(t_{i+1}) - R_D(t_i)) = \int_0^T \int_0^T I(f)(y_1, y_2)dB(y_1)dB(y_2). \quad (3.11)$$

Let $\mathcal{H}$ be the set of deterministic functions f such that

$$||f||_{\mathcal{H}}^2 = H(2H-1)\int_0^T \int_0^T f(u)f(v)|u-v|^{2H-2}dudv < \infty. \quad (3.12)$$

It can be shown that

$$||f||_{\mathcal{H}}^2 = 2\int_0^T \int_0^T I(f)(y_1, y_2)^2 dy_1 dy_2 = 2\mathbb{E}\{(\int_0^T f(t)dR_D(t))^2\}. \quad (3.13)$$

Therefore the mapping

$$f \to \int_0^T f(t)dR_D(t) \quad (3.14)$$

defines an isometry from $\mathcal{E}$ to $L^2(\Omega)$ and it can be extended by continuity to an isometry from $\mathcal{H}$ to $L^2(\Omega)$ because $\mathcal{E}$ is dense in $\mathcal{H}$. We call this extension the Wiener integral of $f \in \mathcal{H}$ with respect to $R_D(t)$.

4 Analysis of oscillatory integrals

From (2.3) and (2.4), we can see that the rescaled corrector $\frac{u_\varepsilon - \bar{u}}{\varepsilon^\alpha}$ contains oscillatory integrals of the form:

$$\int_{\mathbb{R}} \frac{1}{\varepsilon^\alpha} q(\frac{x}{\varepsilon})h(x)dx \quad (4.1)$$

for some compactly supported $h(x)$. The main goal of section is to prove the following results:

Proposition 4.1. *Assume that $h(x)$ is compactly supported in $[0, \infty)$ and continuous, then*

$$\int_{\mathbb{R}} \frac{1}{\varepsilon^\alpha}(g^2(\frac{x}{\varepsilon}) - 1)h(x)dx \xrightarrow[\varepsilon \to 0]{\text{distribution}} \kappa_g \int_{\mathbb{R}} h(x)dR_D(x) \quad (4.2)$$

where $R_D(x)$ is the Rosenblatt process with $D = \alpha$.

Proof. Since $g(x) = \xi_{[x+U]}$, the LHS of (4.2) can be written as

$$\int_{\mathbb{R}} \frac{1}{\varepsilon^\alpha}[g^2(\frac{x}{\varepsilon}) - 1]h(x)dx = \sum_{k=-\infty}^{\infty} A_{k,\varepsilon}(\xi_k^2 - 1) \quad (4.3)$$

where

$$A_{k,\varepsilon} = \int_{\varepsilon(k-U)}^{\varepsilon(k+1-U)} \frac{h(x)}{\varepsilon^\alpha}dx. \quad (4.4)$$

Since $h(x)$ is compactly supported, the sum contains finitely many terms. We compute the conditional characteristic function of (4.3) as follows:

$$\begin{aligned} c_\varepsilon(\theta) =& \mathbb{E}\left\{\exp\left(i\theta\int_{\mathbb{R}}\frac{1}{\varepsilon^\alpha}(g(\frac{x}{\varepsilon})^2-1)h(x)dx\right) \mid U\right\} \\ =& \mathbb{E}\left\{\exp\left(i\theta\sum_{k=-\infty}^{\infty}A_{k,\varepsilon}(\xi_k^2-1)\right) \mid U\right\} \end{aligned} \tag{4.5}$$

Freeze U, then $A_{k,\varepsilon}$ are constants. If we assume $k=m,\ldots,n$ and $n-m+1=N$, then we have

$$\begin{aligned} c_\varepsilon(\theta) &= \frac{1}{(2\pi)^{\frac{N}{2}}|\Sigma_N|^{\frac{1}{2}}}\int_{\mathbb{R}^N}\exp\left(i\theta\sum_{k=m}^{n}A_{k,\varepsilon}(x_k^2-1)\right)\exp\left(-\frac{1}{2}\mathbf{x}'\Sigma_N^{-1}\mathbf{x}\right)d\mathbf{x} \\ &= |\Sigma_N|^{-\frac{1}{2}}||\Sigma_N^{-1}-2A_N(\varepsilon,\theta)|^{-\frac{1}{2}}\exp(-\mathrm{Tr}(A_N(\varepsilon,\theta))) \\ &= |I_N-2\Sigma_N A_N(\varepsilon,\theta)|^{-\frac{1}{2}}\exp(-\mathrm{Tr}(A_N(\varepsilon,\theta))) \end{aligned} \tag{4.6}$$

where Σ_N is the covariance matrix of $(\xi_m,\ldots,\xi_n)$, and $A_N(\varepsilon,\theta)$ is the $N\times N$ diagonal matrix where the diagonals are $i\theta A_{k,\varepsilon}, k=m,\ldots,n$.

Let $\lambda_{k,\varepsilon}(\theta), k=1,\ldots,N$ be the eigenvalues of $\Sigma_N A_N(\varepsilon,\theta)$, we claim that there exists $\delta>0$, such that if $|\theta|<\delta$, we have

$$\begin{aligned} c_\varepsilon(\theta) &= \exp\left(-\sum_{k=1}^{N}\lambda_{k,\varepsilon}(\theta)\right)\prod_{k=1}^{N}(1-2\lambda_{k,\varepsilon}(\theta))^{-\frac{1}{2}} \\ &= \exp\left(-\sum_{k=1}^{N}(\lambda_{k,\varepsilon}(\theta)+\frac{1}{2}\ln(1-2\lambda_{k,\varepsilon}(\theta)))\right) = \exp\left(\frac{1}{2}\sum_{n=2}^{\infty}\frac{2^n}{n}\sum_{k=1}^{N}\lambda_{k,\varepsilon}(\theta)^n\right). \end{aligned} \tag{4.7}$$

To see this, we only need to show that when $|\theta|<\delta$, then for every N, we have

$$\sum_{n=2}^{\infty}\frac{2^n}{n}\sum_{k=1}^{N}|\lambda_{k,\varepsilon}(\theta)|^n<\infty. \tag{4.8}$$

Actually, $|\lambda_{k,\varepsilon}(\theta)|\le M|\theta|\varepsilon^{1-\alpha}\sqrt{\sum_{i,j=1}^{N}|r(i-j)|^2}$ for some constant M. Since $N\varepsilon$ converges as $\varepsilon\to 0$, and by the result in [11], $\sum_{i,j=1}^{N}|r(i-j)|^2\sim O(N^{2-2\alpha})$, the claim is proved.

Next, we show that as $\varepsilon\to 0$, $\sum_{k=1}^{N}\lambda_{k,\varepsilon}(\theta)^n$ converges for each $n\ge 2$.

Since $\lambda_{k,\varepsilon}(\theta)$ are the eigenvalues of $\Sigma_N A_N(\varepsilon,\theta)$, we have

$$\sum_{k=1}^{N}\lambda_{k,\varepsilon}(\theta)^n=\mathrm{Tr}((\Sigma_N A_N(\varepsilon,\theta))^n). \tag{4.9}$$

If we denote $(\Sigma_N A_N(\varepsilon,\theta))_{ij}=\rho_{ij}$, we can write the RHS of (4.9) as follows:

$$\mathrm{Tr}((\Sigma_N A_N(\varepsilon,\theta))^n)=\sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\ldots\sum_{i_n=1}^{N}\rho_{i_1i_2}\rho_{i_2i_3}\ldots\rho_{i_{n-1}i_n}\rho_{i_ni_1}. \tag{4.10}$$

It is straightforward to check that

$$\rho_{kj} = r(|j-k|) i\theta A_{j+m-1,\varepsilon}. \tag{4.11}$$

By stationarity, we have

$$\begin{aligned} &\mathrm{Tr}((\Sigma_N A_N(\varepsilon,\theta))^n) \\ =&(i\theta)^n \sum_{i_1,\ldots,i_n=m,\ldots,n} A_{i_1,\varepsilon}A_{i_2,\varepsilon}\ldots A_{i_n,\varepsilon} r(|i_2-i_1|)r(|i_3-i_2|)\ldots r(|i_1-i_n|). \end{aligned} \tag{4.12}$$

By (4.4), we have

$$A_{k,\varepsilon} = \frac{1}{N^{1-\alpha}}(N\varepsilon)^{1-\alpha}\frac{1}{\varepsilon}\int_{\varepsilon(k-U)}^{\varepsilon(k+1-U)} h(x)dx := \frac{1}{N^{1-\alpha}}(N\varepsilon)^{1-\alpha}B_{k,N} \tag{4.13}$$

where

$$B_{k,N} = \frac{1}{\varepsilon}\int_{\varepsilon(k-U)}^{\varepsilon(k+1-U)} h(x)dx, \quad k=m,\ldots,n \tag{4.14}$$

is an approximation to $h(x)$. Therefore,

$$\begin{aligned} &\mathrm{Tr}((\Sigma_N A_N(\varepsilon,\theta))^n) \\ =&(i\theta)^n (N\varepsilon)^{n(1-\alpha)} \sum_{i_1,\ldots,i_n=m,\ldots,n} \frac{1}{N^n} B_{i_1,N}\ldots B_{i_n,N} r(|i_2-i_1|)N^\alpha \ldots r(|i_1-i_n|)N^\alpha. \end{aligned} \tag{4.15}$$

By assumption, $r(k) \sim \kappa_g |k|^{-\alpha}$ so that there exists some constant M independent of n and ε such that

$$|\mathrm{Tr}((\Sigma_N A_N(\varepsilon,\theta))^n)| \le |\theta|^n M^n \tag{4.16}$$

and

$$\mathrm{Tr}((\Sigma_N A_N(\varepsilon,\theta))^n) \to (i\theta\kappa_g)^n \int_{\mathbb{R}^n} \frac{h(x_1)h(x_2)\ldots h(x_n)}{|x_2-x_1|^\alpha |x_3-x_2|^\alpha \ldots |x_1-x_n|^\alpha} dx_1\ldots dx_n \tag{4.17}$$

as $\varepsilon \to 0$, where we have used the fact that $h(x)$ is continuous. Therefore, if $|\theta| < \delta$, we have

$$c_\varepsilon(\theta) \to \exp\left(\frac{1}{2}\sum_{n=2}^{\infty} \frac{(2i\theta\kappa_g)^n C_n}{n}\right) \tag{4.18}$$

where

$$C_n = \int_{\mathbb{R}^n} \frac{h(x_1)h(x_2)\ldots h(x_n)}{|x_2-x_1|^\alpha |x_3-x_2|^\alpha \ldots |x_1-x_n|^\alpha} dx_1\ldots dx_n. \tag{4.19}$$

We have shown that the conditional characteristic function $c_\varepsilon(\theta)$ converges in a small neighborhood of the origin, and we verify that

$$c_0(z) = \exp\left(\frac{1}{2}\sum_{n=2}^{\infty} \frac{(2iz\kappa_g)^n C_n}{n}\right) \tag{4.20}$$

is analytic when $|z| < \delta$. Therefore, $c_0(z)$ agrees with a unique characteristic function for all real values of z. We still denote the characteristic function as $c_0(z)$. Take expectation of $c_\varepsilon(\theta)$ in (4.5), by the Dominated Convergence Theorem, we have

$$\mathbb{E}\left\{\exp\left(i\theta\int_{\mathbb{R}}\frac{1}{\varepsilon^\alpha}(g(\frac{x}{\varepsilon})^2-1)h(x)dx\right)\right\}\to c_0(\theta). \tag{4.21}$$

Therefore, the random variable $\int_{\mathbb{R}}\frac{1}{\varepsilon^\alpha}(g(\frac{x}{\varepsilon})^2-1)h(x)dx$ converges in distribution and we claim that the limit is $\kappa_g\int_{\mathbb{R}}h(x)dR_D(x)$. To see this, we only have to prove that the characteristic function of $\kappa_g\int_{\mathbb{R}}h(x)dR_D(x)$ agrees with $c_0(\theta)$ when $|\theta|<\delta$.

Let $h^\varepsilon(x)=\sum_{i=1}^N a_{\varepsilon,i}1_{(t^\varepsilon_{i-1},t^\varepsilon_i]}(x), 0=t^\varepsilon_0<t^\varepsilon_1<\ldots<t^\varepsilon_N=T$ be an approximation to $h(x)$ in the sense that

$$||h^\varepsilon(x)-h(x)||_{\mathcal{H}}\to 0 \tag{4.22}$$

as $\varepsilon\to 0$. So $\int_{\mathbb{R}}h^\varepsilon(x)dR_D(x)=\sum_{i=1}^N a_{\varepsilon,i}(R_D(t^\varepsilon_i)-R_D(t^\varepsilon_{i-1}))$. We claim that

$$\exp\left(i\theta\int_{\mathbb{R}}h^\varepsilon(x)dR_D(x)\right)=\exp\left(\frac{1}{2}\sum_{n=2}^\infty\frac{(2i\theta)^n}{n}\int_{\mathbb{R}^n}\frac{h^\varepsilon(x_1)h^\varepsilon(x_2)\ldots h^\varepsilon(x_n)}{|x_2-x_1|^\alpha|x_3-x_2|^\alpha\ldots|x_1-x_n|^\alpha}d\mathbf{x}\right) \tag{4.23}$$

when θ is sufficiently small.

To see this, we consider

$$\begin{aligned}&\int_{\mathbb{R}^n}\frac{h^\varepsilon(x_1)h^\varepsilon(x_2)\ldots h^\varepsilon(x_n)}{|x_2-x_1|^\alpha|x_3-x_2|^\alpha\ldots|x_1-x_n|^\alpha}d\mathbf{x}\\ &=\sum_{i_1,\ldots,i_n=1}^N\left[\left(\prod_{k=1}^n a_{\varepsilon,i_k}\right)\int_{t^\varepsilon_{i_1-1}}^{t^\varepsilon_{i_1}}\cdots\int_{t^\varepsilon_{i_n-1}}^{t^\varepsilon_{i_n}}\frac{1}{|x_2-x_1|^\alpha|x_3-x_2|^\alpha\ldots|x_1-x_n|^\alpha}d\mathbf{x}\right]\end{aligned} \tag{4.24}$$

If we define $\theta_{\varepsilon,i}=a_{\varepsilon,i}-a_{\varepsilon,i+1}$ for $i=1,\ldots,N-1$ and $\theta_{\varepsilon,N}=a_{\varepsilon,N}$, then we have $a_{\varepsilon,i}=\sum_{k=i}^N\theta_{\varepsilon,k}$ and (4.24) can be rewritten as

$$\begin{aligned}&\int_{\mathbb{R}^n}\frac{h^\varepsilon(x_1)h^\varepsilon(x_2)\ldots h^\varepsilon(x_n)}{|x_2-x_1|^\alpha|x_3-x_2|^\alpha\ldots|x_1-x_n|^\alpha}d\mathbf{x}\\ &=\sum_{j_1,\ldots,j_n=1}^N\left[\left(\prod_{k=1}^n\theta_{\varepsilon,j_k}\right)\int_0^{t^\varepsilon_{j_1}}\cdots\int_0^{t^\varepsilon_{j_n}}\frac{1}{|x_2-x_1|^\alpha|x_3-x_2|^\alpha\ldots|x_1-x_n|^\alpha}d\mathbf{x}\right].\end{aligned} \tag{4.25}$$

On the other hand,

$$\int_{\mathbb{R}}h^\varepsilon(x)dR_D(x)=\sum_{k=1}^N\theta_{\varepsilon,k}R_D(t^\varepsilon_k). \tag{4.26}$$

By the results in [11], the characteristic function of $\int_{\mathbb{R}}h^\varepsilon(x)dR_D(x)$ in a small neighborhood of the origin is

$$\begin{aligned}&\exp\left(i\theta\int_{\mathbb{R}}h^\varepsilon(x)dR_D(x)\right)\\ &=\exp\left\{\frac{1}{2}\sum_{n=2}^\infty\left[\frac{(2i\theta)^n}{n}\sum_{j_1,\ldots,j_n=1}^N\left(\prod_{k=1}^n\theta_{\varepsilon,j_k}\right)\int_0^{t^\varepsilon_{j_1}}\cdots\int_0^{t^\varepsilon_{j_n}}\frac{1}{|x_2-x_1|^\alpha|x_3-x_2|^\alpha\ldots|x_1-x_n|^\alpha}d\mathbf{x}\right]\right\}\end{aligned} \tag{4.27}$$

Therefore, (4.23) is proved.

As $\varepsilon \to 0$,

$$\begin{aligned}&\exp\left(\frac{1}{2}\sum_{n=2}^{\infty}\frac{(2i\theta)^n}{n}\int_{\mathbb{R}^n}\frac{h^\varepsilon(x_1)h^\varepsilon(x_2)\dots h^\varepsilon(x_n)}{|x_2-x_1|^\alpha|x_3-x_2|^\alpha\dots|x_1-x_n|^\alpha}d\mathbf{x}\right)\\ \to&\exp\left(\frac{1}{2}\sum_{n=2}^{\infty}\frac{(2i\theta)^n}{n}\int_{\mathbb{R}^n}\frac{h(x_1)h(x_2)\dots h(x_n)}{|x_2-x_1|^\alpha|x_3-x_2|^\alpha\dots|x_1-x_n|^\alpha}d\mathbf{x}\right).\end{aligned} \tag{4.28}$$

Since $\int_{\mathbb{R}} h^\varepsilon(x)dR_D(x) \to \int_{\mathbb{R}} h(x)dR_D(x)$ in $L^2(\Omega)$ by the definition of stochastic integral with respect to Rosenblatt process, we have

$$\exp\left(i\theta\int_{\mathbb{R}}h(x)dR_D(x)\right)=\exp\left(\frac{1}{2}\sum_{n=2}^{\infty}\frac{(2i\theta)^n}{n}\int_{\mathbb{R}^n}\frac{h(x_1)h(x_2)\dots h(x_n)}{|x_2-x_1|^\alpha|x_3-x_2|^\alpha\dots|x_1-x_n|^\alpha}d\mathbf{x}\right) \tag{4.29}$$

in a small neighborhood of the origin. This completes our proof. □

Proposition 4.2. *Under the same assumption as in Proposition 4.1, we have*

$$\int_{\mathbb{R}}\frac{1}{\varepsilon^\alpha}q(\frac{x}{\varepsilon})h(x)dx \xrightarrow[\varepsilon\to 0]{\text{distribution}} \frac{V_2\kappa_g}{2}\int_{\mathbb{R}}h(x)dR_D(x). \tag{4.30}$$

Proof. By Hermite expansion,

$$q(\frac{x}{\varepsilon})=\Phi(g(\frac{x}{\varepsilon}))=\sum_{n=2}^{\infty}\frac{V_n}{n!}H_n(g(\frac{x}{\varepsilon})). \tag{4.31}$$

We claim that

$$\int_{\mathbb{R}}\frac{1}{\varepsilon^\alpha}(q(\frac{x}{\varepsilon})-\frac{V_2}{2}(g(\frac{x}{\varepsilon})^2-1))h(x)dx\to 0 \tag{4.32}$$

in probability.

Actually, we have

$$\mathbb{E}\left\{\left(\int_{\mathbb{R}}\frac{1}{\varepsilon^\alpha}(q(\frac{x}{\varepsilon})-\frac{V_2}{2}(g(\frac{x}{\varepsilon})^2-1))h(x)dx\right)^2\right\}=\sum_{n=3}^{\infty}\int_{\mathbb{R}^2}\frac{1}{\varepsilon^{2\alpha}}\frac{V_n^2}{n!}R_g(\frac{x-y}{\varepsilon})^n h(x)h(y)dxdy \tag{4.33}$$

and

$$\begin{aligned}&\frac{1}{\varepsilon^{2\alpha}}\left|\int_{\mathbb{R}^2}R_g(\frac{x-y}{\varepsilon})^n h(x)h(y)dxdy\right|\\ \leq&\frac{M}{\varepsilon^{2\alpha}}\int_{|x-y|<M\varepsilon}|h(x)h(y)|dxdy+\frac{M}{\varepsilon^{2\alpha}}\int_{|x-y|>M\varepsilon}\frac{\varepsilon^{n\alpha}}{|x-y|^{n\alpha}}|h(x)h(y)|dxdy\end{aligned} \tag{4.34}$$

for some constant M. Since $\alpha\in(0,\frac{1}{2})$ and $n\geq 3$, we show that the RHS of (4.34) is uniformly bounded in n and converges to 0 as $\varepsilon\to 0$.

Because $\sum_{n=3}^{\infty} \frac{V_n^2}{n!} < \infty$, by the Dominated Convergence Theorem, the LHS of (4.33) converges to 0 as $\varepsilon \to 0$. By Proposition 4.1,

$$\int_{\mathbb{R}} \frac{1}{\varepsilon^\alpha} \frac{V_2}{2} (g(\frac{x}{\varepsilon})^2 - 1) h(x) dx \to \frac{V_2 \kappa_g}{2} \int_{\mathbb{R}} h(x) dR_D(x) \tag{4.35}$$

which completes the proof. □

Remark 4.3. Although we assume that $h(x)$ is continuous in Propositions 4.1 and 4.2, we see from the above proof that $h(x)$ can be allowed to have finitely many jump discontinuities, and we will use this fact later.

5 Proof of the main theorem

Recalling (2.3) and (2.4), we have

$$u_\varepsilon(x) - \bar{u}(x) = -\int_0^x q(\frac{y}{\varepsilon}) F(y) dy + (c_\varepsilon - c^*) \frac{x}{a^*} + c^* \int_0^x q(\frac{y}{\varepsilon}) dy + r_\varepsilon(x) \tag{5.1}$$

where

$$r_\varepsilon(x) = (c_\varepsilon - c^*) \int_0^x q(\frac{y}{\varepsilon}) dy \tag{5.2}$$

and

$$c_\varepsilon - c^* = a^* \int_0^1 (F(y) - \int_0^1 F(z) dz - a^* b) q(\frac{y}{\varepsilon}) dy + \rho_\varepsilon \tag{5.3}$$

with ρ_ε the remainder term.

Define

$$U_\varepsilon(x) = -\int_0^x q(\frac{y}{\varepsilon}) F(y) dy + (c_\varepsilon - c^* - \rho_\varepsilon) \frac{x}{a^*} + c^* \int_0^x q(\frac{y}{\varepsilon}) dy \tag{5.4}$$

so that

$$u_\varepsilon(x) - \bar{u}(x) = U_\varepsilon(x) + r_\varepsilon(x) + \rho_\varepsilon \frac{x}{a^*} \tag{5.5}$$

The proof of Theorem 2.2 contains two steps. First, we prove the weak convergence of $\frac{1}{\varepsilon^\alpha} U_\varepsilon(x)$ as a process in $\mathcal{C}([0,1])$. Then we control the remainder term $r_\varepsilon(x) + \rho_\varepsilon \frac{x}{a^*}$. We use the notation $a \lesssim b$ when there exists a constant M such that $a \leq Mb$.

5.1 Weak convergence in $\mathcal{C}([0,1])$

Rewrite

$$\begin{aligned} U_\varepsilon(x) = & \int_{\mathbb{R}} (c^* 1_{[0,x]}(y) - F(y) 1_{[0,x]}(y)) q(\frac{y}{\varepsilon}) dy \\ & + \left(\int_0^1 (F(y) - \int_0^1 F(z) dz - a^* b) q(\frac{y}{\varepsilon}) dy \right) x. \end{aligned} \tag{5.6}$$

Define $F(x,y) = c^* 1_{[0,x]}(y) - F(y) 1_{[0,x]}(y) + x(F(y) - \int_0^1 F(z) dz - a^* b) 1_{[0,1]}(y)$ so that

$$\frac{1}{\varepsilon^\alpha} U_\varepsilon(x) = \frac{1}{\varepsilon^\alpha} \int_{\mathbb{R}} F(x,y) q(\frac{y}{\varepsilon}) dy. \tag{5.7}$$

Lemma 5.1. *Let*

$$\bar{U}(x) = \frac{V_2 \kappa_g}{2} \int_{\mathbb{R}} F(x,y) dR_D(y). \tag{5.8}$$

Then

$$\frac{1}{\varepsilon^{\alpha}} U_{\varepsilon}(x) \xrightarrow[\varepsilon \to 0]{\text{distribution}} \bar{U}(x) \tag{5.9}$$

in $\mathcal{C}([0,1])$.

Proof. We first prove the weak convergence of finite dimensional distributions and then prove tightness.

$\forall x_1, x_2, \ldots, x_n \in [0,1]$ and $c_1, c_2, \ldots, c_n \in \mathbb{R}$, consider

$$\sum_{i=1}^{n} c_i \frac{1}{\varepsilon^{\alpha}} U_{\varepsilon}(x_i) = \frac{1}{\varepsilon^{\alpha}} \int_{\mathbb{R}} \sum_{i=1}^{n} c_i F(x_i, y) q(\frac{y}{\varepsilon}) dy. \tag{5.10}$$

We see that $\sum_{i=1}^{n} c_i F(x_i, y)$ is compactly supported and has only finitely many discontinuities. Then by Proposition 4.2 we have

$$\sum_{i=1}^{n} c_i \frac{1}{\varepsilon^{\alpha}} U_{\varepsilon}(x_i) \xrightarrow[\varepsilon \to 0]{\text{distribution}} \sum_{i=1}^{n} c_i \bar{U}(x_i). \tag{5.11}$$

Therefore, we have proved the weak convergence of the finite dimensional distributions. To prove tightness, we apply the Kolmogorov criteria [5]. Note that $U_{\varepsilon}(0) = 0$, so we only need to show that there exist $\delta, \beta, C > 0$ such that

$$\mathbb{E}\{|\frac{1}{\varepsilon^{\alpha}} U_{\varepsilon}(x) - \frac{1}{\varepsilon^{\alpha}} U_{\varepsilon}(y)|^{\beta}\} \leq C|x-y|^{1+\delta}. \tag{5.12}$$

Define $F_1(y) = c^* - F(y)$ and $F_2(y) = F(y) - \int_0^1 F(z)dz - a^* b$. Then for $0 \leq y < x \leq 1$, we have

$$\frac{1}{\varepsilon^{\alpha}}(U_{\varepsilon}(x) - U_{\varepsilon}(y)) = \frac{1}{\varepsilon^{\alpha}} \int_y^x F_1(z) q(\frac{z}{\varepsilon}) dz + \frac{1}{\varepsilon^{\alpha}}(x-y) \int_0^1 F_2(z) q(\frac{z}{\varepsilon}) dz. \tag{5.13}$$

So

$$\begin{aligned} \mathbb{E}\{|\frac{1}{\varepsilon^{\alpha}} U_{\varepsilon}(x) - \frac{1}{\varepsilon^{\alpha}} U_{\varepsilon}(y)|^2\} \leq & \frac{2}{\varepsilon^{2\alpha}} \int_{[y,x]^2} F_1(z_1) F_1(z_2) R(\frac{z_1 - z_2}{\varepsilon}) dz_1 dz_2 \\ & + \frac{2}{\varepsilon^{2\alpha}} (x-y)^2 \int_{[0,1]^2} F_2(z_1) F_2(z_2) R(\frac{z_1 - z_2}{\varepsilon}) dz \\ := & (I) + (II). \end{aligned} \tag{5.14}$$

F_2 is bounded, $\alpha \in (0, \frac{1}{2})$ and by Lemma 2.1, we have

$$(II) \lesssim (x-y)^2. \tag{5.15}$$

For (I), we distinguish the cases $|y-x| < \varepsilon$ and $|y-x| \geq \varepsilon$.

If $|y-x|<\varepsilon$, since F_1 and R are both bounded, we have $(I)\lesssim|x-y|^{2-2\alpha}$.

If $|y-x|\ge\varepsilon$, by Lemma 2.1, we have

$$
\begin{aligned}
(I) &\lesssim \int_{[y,x]^2}\frac{1}{|z_1-z_2|^{2\alpha}}dz_1dz_2 \\
&\lesssim |x-y|\int_0^{x-y}\frac{1}{t^{2\alpha}}dt \ \lesssim \ |x-y|^{2-2\alpha}.
\end{aligned}
\tag{5.16}
$$

Choose $\delta=1-2\alpha$. We have

$$
\mathbb{E}\{|\frac{1}{\varepsilon^\alpha}U_\varepsilon(x)-\frac{1}{\varepsilon^\alpha}U_\varepsilon(y)|^2\}\le C|x-y|^{1+\delta} \tag{5.17}
$$

for some constant C. The proof is completed. □

5.2 The remainder term

To analyze the remainder term, we first write

$$
c_\varepsilon-c^*=\frac{\int_0^1 F(y)q(\frac{y}{\varepsilon})dy}{\int_0^1\frac{1}{a_\varepsilon(y)}dy}+\left(b+\frac{1}{a^*}\int_0^1F(y)dy\right)\left(\frac{1}{\int_0^1\frac{1}{a_\varepsilon(y)}dy}-\frac{1}{\frac{1}{a^*}}\right), \tag{5.18}
$$

which gives

$$
\rho_\varepsilon=\frac{a^*}{\int_0^1\frac{1}{a(\frac{y}{\varepsilon})}dy}\left[(a^*b+\int_0^1F(y)dy)\left(\int_0^1q(\frac{y}{\varepsilon})dy\right)^2-\int_0^1F(y)q(\frac{y}{\varepsilon})dy\int_0^1q(\frac{y}{\varepsilon})dy\right]. \tag{5.19}
$$

We have the following lemma:

Lemma 5.2.

$$
\mathbb{E}\{|\rho_\varepsilon|\}+\sup_{x\in[0,1]}\mathbb{E}\{|r_\varepsilon(x)|\}\le M\varepsilon^{2\alpha} \tag{5.20}
$$

for some constant M. Furthermore, we have

$$
\frac{1}{\varepsilon^\alpha}(\rho_\varepsilon\frac{x}{a^*}+r_\varepsilon(x))\xrightarrow[\varepsilon\to0]{\text{probability}}0 \tag{5.21}
$$

in $\mathcal{C}([0,1])$.

Proof.

$$
\mathbb{E}\left\{\left(\int_0^1q(\frac{y}{\varepsilon})dy\right)^2\right\}=\int_{[0,1]^2}R(\frac{y-z}{\varepsilon})dydz. \tag{5.22}
$$

By lemma 2.1, $R(\frac{y-z}{\varepsilon})\sim\kappa\frac{\varepsilon^{2\alpha}}{|y-z|^{2\alpha}}$, so we have

$$
\mathbb{E}\left\{\left(\int_0^1q(\frac{y}{\varepsilon})dy\right)^2\right\}\lesssim\varepsilon^{2\alpha}. \tag{5.23}
$$

By the Cauchy-Schwartz inequality, we can show in the same way that

$$\mathbb{E}\left\{|\int_0^1 F(y)q(\frac{y}{\varepsilon})dy\int_0^1 q(\frac{y}{\varepsilon})dy|\right\}\lesssim\varepsilon^{2\alpha}. \tag{5.24}$$

Since $\int_0^1\frac{1}{a(\frac{y}{\varepsilon})}dy$ is bounded from below, we have $\mathbb{E}\{|\rho_\varepsilon|\}\lesssim\varepsilon^{2\alpha}$.

For $r_\varepsilon(x)=(c_\varepsilon-c^*)\int_0^x q(\frac{y}{\varepsilon})dy$, we write it in two parts:

$$\begin{aligned} r_\varepsilon(x)=&\left(a^*\int_0^1(F(y)-\int_0^1 F(z)dz-a^*b)q(\frac{y}{\varepsilon})dy\right)\int_0^x q(\frac{y}{\varepsilon})dy+\rho_\varepsilon\int_0^x q(\frac{y}{\varepsilon})dy\\ =&r_\varepsilon^1(x)+r_\varepsilon^2(x). \end{aligned} \tag{5.25}$$

By Cauchy-Schwartz, $\mathbb{E}\{|r_\varepsilon^1(x)|\}\lesssim\varepsilon^{2\alpha}$ and the constant does not depend on x. $q(y)$ is bounded and since $\mathbb{E}\{|\rho_\varepsilon|\}\lesssim\varepsilon^{2\alpha}$, we also have $\sup_{x\in[0,1]}\mathbb{E}\{|r_\varepsilon^2(x)|\}\lesssim\varepsilon^{2\alpha}$. Thus, we have proved (5.20), and we have

$$\sup_{x\in[0,1]}\frac{1}{\varepsilon^\alpha}\mathbb{E}\{|\rho_\varepsilon\frac{x}{a^*}+r_\varepsilon(x)|\}\to 0. \tag{5.26}$$

So we have the weak convergence of finite dimensional distribution. Now we prove tightness.

We have $r_\varepsilon(0)=0$, and

$$\begin{aligned} &r_\varepsilon(x_1)-r_\varepsilon(x_2)\\ =&\left(a^*\int_0^1(F(y)-\int_0^1 F(z)dz-a^*b)q(\frac{y}{\varepsilon})dy\right)\int_{x_2}^{x_1} q(\frac{y}{\varepsilon})dy+\rho_\varepsilon\int_{x_2}^{x_1} q(\frac{y}{\varepsilon})dy. \end{aligned} \tag{5.27}$$

Following the proof of (5.14), we have

$$\mathbb{E}\{|r_\varepsilon(x_1)-r_\varepsilon(x_2)|^2\}\lesssim|x_1-x_2|^{2-2\alpha}. \tag{5.28}$$

Therefore,

$$\mathbb{E}\{|\rho_\varepsilon\frac{x_1}{a^*}-\rho_\varepsilon\frac{x_2}{a^*}+r_\varepsilon(x_1)-r_\varepsilon(x_2)|^2\}\le C|x_1-x_2|^{2-2\alpha} \tag{5.29}$$

for some constant C.

Thus $\varepsilon^{-\alpha}(\rho_\varepsilon\frac{x}{a^*}+r_\varepsilon(x))$ converges in distribution to 0 as $\varepsilon\to 0$, so it converges in probability to 0, which completes the proof. □

Recall that

$$\frac{u_\varepsilon(x)-\bar{u}(x)}{\varepsilon^\alpha}=\frac{1}{\varepsilon^\alpha}U_\varepsilon(x)+\frac{1}{\varepsilon^\alpha}(r_\varepsilon(x)+\rho_\varepsilon\frac{x}{a^*}). \tag{5.30}$$

We only need to combine lemma 5.1 and 5.2 to complete the proof of Theorem 2.2.

6 Conclusions and further discussion

We considered the homogenization and corrector (random fluctuation) theory of a one dimensional elliptic equation with highly oscillatory coefficients. For a certain class of random coefficients with long range correlations, we were able to show that the properly rescaled corrector converges in distribution in the space of continuous function to a stochastic integral with respect to the Rosenblatt process. Moreover, the corrector's amplitude is of order ε^α and $\alpha \in (0, 1/2)$ such that $R(x) \sim \kappa|x|^{-2\alpha}$. Therefore, the longer the range of the correlations, the larger is the amplitude of the corrector.

The appearance of the Rosenblatt process is due to the fact that the Hermite rank of Φ is 2. It is natural to ask what would happen if the Hermite rank of Φ was greater than 2. In [7, 12], the non-central limit theorems for functionals of Gaussian fields of arbitrary Hermite rank was proved. When the Hermite rank is greater than 2, the limit is the so-called Hermite process. In that case, we expect the properly rescaled corrector to converge in distribution to some stochastic integral with respect to the Hermite process although we have not carried out the calculations in detail.

It would also be interesting to generalize proposition 4.1 and 4.2 to the case of elliptic equations in higher dimensions, at least in the setting considered in [2]. To do this, we would have to first find the counterpart of the Rosenblatt process in higher dimensions, which is a nontrivial problem.

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