Convergence to SPDEs in Stratonovich form

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Abstract

We consider the perturbation of parabolic operators of the form $\partial_t + P(x, D)$ by large-amplitude highly oscillatory spatially dependent potentials modeled as Gaussian random fields. The amplitude of the potential is chosen so that the solution to the random equation is affected by the randomness at the leading order. We show that, when the dimension is smaller than the order of the elliptic pseudo-differential operator $P(x, D)$, the perturbed parabolic equation admits a solution given by a Duhamel expansion. Moreover, as the correlation length of the potential vanishes, we show that the latter solution converges in distribution to the solution of a stochastic parabolic equation with multiplicative noise that should be interpreted in the Stratonovich sense. The theory of mild solutions for such stochastic partial differential equations is developed.

The behavior described above should be contrasted to the case of dimensions larger than or equal to the order of the elliptic pseudo-differential operator $P(x, D)$. In the latter case, the solution to the random equation converges strongly to the solution of a homogenized (deterministic) parabolic equation as is shown in [2]. A stochastic limit is obtained only for sufficiently small space dimensions in this class of parabolic problems.

keywords: Partial differential equations with random coefficients, Stochastic partial differential equations, Gaussian potential, iterated Stratonovich integral, Wiener-Itô chaos expansion

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1 Introduction

We consider the parabolic equation

$$\begin{aligned}&\frac{\partial u_\varepsilon}{\partial t} + P(x, D)u_\varepsilon - \frac{1}{\varepsilon^{\frac{d}{2}}} q\big(\frac{x}{\varepsilon}\big) u_\varepsilon = 0 \\ &u_\varepsilon(0, x) = u_0(x),\end{aligned} \tag{1}$$

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where $P(x, D)$ is an elliptic pseudo-differential operator with principal symbol of order $\mathfrak{m} > d$ and $x \in \mathbb{R}^d$. The initial condition $u_0(x)$ is assumed to belong to $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. We assume that $q(x)$ is a mean zero, Gaussian, stationary field defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with integrable correlation function $R(x) = \mathbb{E}\{q(0)q(x)\}$.

The main objective of this paper is to construct a solution to the above equation in $L^2(\Omega \times \mathbb{R}^d)$ uniformly in time on bounded intervals (see Theorem 3 below) and to show that the solution converges in distribution as $\varepsilon \to 0$ to the unique mild solution of the following stochastic partial differential equation (SPDE)

$$\begin{aligned} &\frac{\partial u}{\partial t} + P(x, D)u - \sigma u \circ \dot{W} = 0 \\ &u(0, x) = u_0(x), \end{aligned} \tag{2}$$

where $\dot{W}$ denotes spatial white noise, $\circ$ denotes the Stratonovich product and sigma is defined as

$$\sigma^2 := (2\pi)^d \hat{R}(0) = \int_{\mathbb{R}^d} \mathbb{E}\{q(0)q(x)\}dx. \tag{3}$$

We denote by $G(t, x; y)$ the Green's function associated to the above unperturbed operator. In other words, $G(t, x; y)$ is the distribution kernel of the operator $e^{-tP(x,D)}$. Our main assumptions on the unperturbed problem are that $G(t, x; y) = G(t, y; x)$ is continuous and satisfies the following regularity conditions:

$$\sup_{t,y} \int_{\mathbb{R}^d} |G(t, x; y)|dx + \sup_{t,y} t^\alpha \int_{\mathbb{R}^d} |G(t, x; y)|^2 dx + \sup_{t,x,y} t^\alpha |G(t, x; y)| < \infty. \tag{4}$$

Here, we have defined $\alpha := \frac{d}{\mathfrak{m}}$. Note that $0 < \alpha < 1$. Note that the L^2 bound is a consequence of the L^1 and L^∞ bounds. Such regularity assumptions may be verified e.g. for parabolic equations with $\mathfrak{m} = 2$ and $d = 1$ or more generally for equations with $\mathfrak{m} = 2\mathfrak{n}$ an even number and $d < \mathfrak{m}$. The convergence of the random solution to the solution of the SPDE is obtained under the additional continuity constraint

$$\sup_{s \in (0,T), \zeta} s^\gamma \int_{\mathbb{R}^d} |G(s, x, \zeta) - G(s, x + y, \zeta)|dx \to 0 \quad \text{as } y \to 0 \quad \text{for} \quad \gamma = 2\Big(1 - \frac{d}{\mathfrak{m}}\Big). \tag{5}$$

Such a constraint may also be verified for Green's functions of parabolic equations with $\mathfrak{m} = 2\mathfrak{n}$ and $d < \mathfrak{m}$ as well as for operators of the form $P(x, D) = (-\Delta)^{\frac{\mathfrak{m}}{2}}$; see lemma 4.1 below.

We look for mild solutions of (2), which we recast as

$$u(t, x) = e^{-tP(D)} u_0(x) + \int_0^t \int_{\mathbb{R}^d} G(t - s, x; y) u(s, y) \circ \sigma dW(y) ds. \tag{6}$$

Here, dW is the standard Wiener measure on $\mathbb{R}^d$ and $\circ$ means that the integral is defined as a (anticipative) Stratonovich integral. In section 2, we define the Stratonovich integral for an appropriate class of random variables and construct a solution to the above equation in $L^2(\Omega \times \mathbb{R}^d)$ uniformly in time on bounded intervals by the method of Duhamel expansion; see Theorem 1 below. In section 3, we show that the solution to the above equation is unique in an adapted functional setting. The convergence of the solution $u_\varepsilon(t)$ to its limit $u(t)$ is addressed in section 4; see Theorem 4 below.

The analysis of stochastic partial differential equations of the form (2) with $\mathfrak{m} = 2$ and with the Stratonovich product replaced by an Itô (Skorohod) product or a Wick product and the white noise in space replaced by a white noise in space time is well developed; we refer the reader to e.g. [6, 9, 12, 15, 16, 22]. The case of space white noise with Itô product is analyzed in e.g. [10]. One of the salient features obtained in these references is that solutions to stochastic equations of the form (2) are found to be square-integrable for sufficiently small spatial dimensions d and to be elements in larger distributional spaces for larger spatial dimensions; see in particular [6] for sharp criteria on the existence of locally mean square random processes solution to stochastic equations.

The theory presented in this paper shows that the solution to (2) may indeed be seen as the $\varepsilon \to 0$ limit of solutions to a parabolic equation (1) with highly oscillatory coefficient when the spatial dimension is sufficiently small. In larger spatial dimensions, the behavior observed in [2] is different. The solution to (1) with a properly scaled potential (of amplitude proportional to $\varepsilon^{-\frac{\mathfrak{m}}{2}}$ for $\mathfrak{m} < d$) converges to the deterministic solution of a homogenized equation, at least for sufficiently small times. The solution to a stochastic model of the form (2) with multiplicative noise interpreted either as an Itô product or a Stratonovich product, when it exists, no longer represents the asymptotic behavior of the solution to an equation of the form (1) with highly oscillatory random coefficients. The justification for the stochastic models of the form (2) is then more difficult.

The analysis of equations with highly oscillatory random coefficients of the form (1) has also been performed in other similar contexts. We refer the reader to [18] for a recent analysis of the case $\mathfrak{m} = 2$ and $d = 1$ with much more general potentials than the Gaussian potentials considered in this paper. When the potential has smaller amplitude, then the limiting solution as $\varepsilon \to 0$ is given by the unperturbed solution of the parabolic equation where q has been set to 0. The analysis of the random fluctuations beyond the unperturbed solution were addressed in e.g. [1, 8]. The equation (1) may also be seen as a continuous version of the parabolic Anderson problem; see e.g. [5].

2 Stratonovich integrals and Duhamel solutions

The analysis of (6) requires that we define the multi-parameter Stratonovich integral used in the construction of a solution to the SPDE. The construction of Stratonovich integrals and their relationships to Itô integrals is well-studied. We refer the reader to e.g. [7, 11, 13, 17, 20]. The construction that we use below closely follows the functional setting presented in [14]. The convergence of processes to multiple Stratonovich integrals may be found in e.g. [3, 4].

Let $f(x_1, \dots, x_n)$ be a function of n variables in $\mathbb{R}^d$. We want to define the iterated Stratonovich integral $\mathcal{I}_n(f)$. Let us first assume that f separates as a product of n functions defined on $\mathbb{R}^d$, i.e., $f(x_1, \dots, x_n) = \prod_{k=1}^n f_k(x_k)$. Then we define

$$\mathcal{I}_n\big(\prod_{k=1}^n f_k(x_k)\big) = \prod_{k=1}^n \mathcal{I}_1(f_k(x_k)), \tag{7}$$

where $\mathcal{I}_1(f) = \int_{\mathbb{R}^d} f(x)dW(x)$ is the usual multi-parameter Wiener integral. It then remains to extend this definition of the integral to more general functions $f(x)$.

We define the symmetrized function

$$f_{\mathfrak{s}}(x_1, \dots, x_n) = \frac{1}{n!} \sum_{\mathfrak{s} \in \mathfrak{S}_n} f(x_{\mathfrak{s}(1)}, \dots, x_{\mathfrak{s}(n)}), \tag{8}$$

where the sum is taken over the $n!$ permutations of the variables $x_1, \dots, x_n$. We then define $\mathcal{I}_n(f) = \mathcal{I}_n(f_{\mathfrak{s}})$ and thus now consider functions that are symmetric in their arguments.

For the rest of the paper, we write Stratonovich integrals using the notation dW rather than $\circ\, dW$. For the Itô convention of integration, we use the notation δW. Let f and g be two functions of n variables. We formally define the inner product

$$\begin{aligned} \langle f, g \rangle_n &= \mathbb{E}\Big\{ \int_{\mathbb{R}^{nd}} f(x) dW(x_1) \dots dW(x_n) \int_{\mathbb{R}^{nd}} g(x) dW(x_1) \dots dW(x_n) \Big\} \\ &= \int_{\mathbb{R}^{2nd}} f(x) g(x') \mathbb{E}\big\{ dW(x_1) \dots dW(x_n) dW(x_{n+1}) \dots dW(x_{2n}) \big\}, \end{aligned} \tag{9}$$

since the latter has to hold for functions defined as in (7). Here, $x' = (x_{n+1}, \dots, x_{2n})$. We need to expand the moment of order $2n$ of Gaussian random variables. The moment is defined as follows:

$$\mathbb{E}\big\{ \prod_{k=1}^{2n} dW(x_k) \big\} = \sum_{\mathfrak{p} \in \mathfrak{P}} \prod_{k \in A_0(\mathfrak{p})} \delta(x_k - x_{l(k)}) dx_k dx_{l(k)}. \tag{10}$$

Here, $\mathfrak{p}$ runs over all possible pairings of $2n$ variables. There are

$$\mathrm{card}(\mathfrak{P}) = \mathrm{c_n} = \frac{(\mathrm{2n}-1)!}{(\mathrm{n}-1)! 2^{\mathrm{n}-1}} = \frac{(\mathrm{2n})!}{\mathrm{n}! 2^{\mathrm{n}}} = (\mathrm{2n}-1)!! \tag{11}$$

such pairings. Each pairing is defined by a map $l = l(\mathfrak{p})$ constructed as follows. The domain of definition of l is the subset $A_0 = A_0(\mathfrak{p})$ of $\{1, \dots, 2n\}$ and the image of l is $B_0 = B_0(\mathfrak{p}) = l(A_0)$ defined as the complement of A_0 in $\{1, \dots, 2n\}$. The cardinality of A_0 and B_0 is thus n and there are c_n choices of the function l such that $l(k) \geq k+1$. The formula (10) thus generalizes the case $n = 1$, where $\mathbb{E}\{dW(x)dW(y)\} = \delta(x-y)dxdy$.

We extend by density the iterated Stratonovich integral defined in (7) to the Banach space $\mathcal{B}_n$ of functions f that are bounded for the norm

$$\|f\|_n = \Big(\sum_{\mathfrak{p} \in \mathfrak{P}} \int_{\mathbb{R}^{2nd}} |f \otimes f|(x_1, \dots, x_{2n}) \prod_{k \in A_0(\mathfrak{p})} \delta(x_k - x_{l(k)}) dx_k dx_{l(k)} \Big)^{\frac{1}{2}}. \tag{12}$$

The above Banach space may be constructed as the completion of smooth functions with compact support for the above norm [19]. Since the sum of product of functions of one $d-$dimensional variable are dense in the space of continuous functions, they are dense in the above Banach space and the Stratonovich integral is thus defined for such integrands $f(x)$. A more explicit expression may be obtained for the above norm for functions $f(x)$ that are symmetric in their arguments. Since we do not use the explicit expression in this paper, we shall not derive it explicitly. We note however that

$$\|f\|_n^2 = \mathbb{E}\{\mathcal{I}_{n+n}(|f \otimes f|)\}, \tag{13}$$

since $\mathcal{I}_n(f)\mathcal{I}_n(f) = \mathcal{I}_{2n}(f \otimes f)$.

Note that the above space is a Banach subspace of the Hilbert space of square integrable functions since the L^2 norm of f appears for the pairing $\prod_k \delta(x_k - x_{k+n})$. Note also that the above space is dense in $L^2(\mathbb{R}^{nd})$ for its natural norm. Indeed, let f be a square integrable function. We can construct a sequence of functions f^k that vanish on a set of measure k^{-1} in the vicinity of the sets of Lebesgue measure 0 where the distributions $\delta(x_k - x_l)$, $1 \le k, l \le n$, are supported and equal to f outside of this set. For such functions, we verify that $\|f^k\|_n$ is the $L^2(\mathbb{R}^{nd})$ norm of f^k. Moreover, f^k converges to f as $k \to \infty$ as an application of the dominated Lebesgue convergence theorem so that $\mathcal{B}_n$ is dense in $L^2(\mathbb{R}^{nd})$. Note finally that the above expression still defines a norm for functions that are not necessarily symmetric in their arguments. This norm applied to non-symmetric functions is not optimal as far as the definition of iterated Stratonovich integrals is concerned since many cancellations may happen by symmetrization (8). However, the above norm is sufficient in the construction of a Duhamel expansion solution to the SPDE.

Duhamel solution. Let us define formally the integral

$$\mathcal{H}u(t,x) = \sigma \int_0^t \int_{\mathbb{R}^d} G(t-s,x;y)u(s,y)dW(y)ds, \tag{14}$$

where we recall that dW means an integral in the Stratonovich sense. The Duhamel solution is defined formally as

$$u(t,x) = \sum_{n=0}^{\infty} u_n(t,x), \quad u_{n+1}(t,x) = \mathcal{H}u_n(t,x), \quad u_0(t,x) = e^{-tP(x,D)}[u_0(x)], \tag{15}$$

where $u_0(x)$ is the initial conditions of the stochastic equation, which we assume is integrable. The above solution is thus defined formally as a sum of iterated Stratonovich integrals $u_n(t,x) = \mathcal{I}_n(f_n(t,x,\cdot))$.

The main result of this section is the following.

Theorem 1 *Let $u(t,x)$ be the function defined in* (15)*. The iterated integrals $u_n(t,x) = \mathcal{I}_n(f_n(t,x,\cdot))$ are defined in $L^2(\mathbb{R}^d;\mathcal{B}_n)$ uniformly in time $t \in (0,T)$ for all $T > 0$ and $n \ge 1$. When the initial condition $u_0(x) \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, then $u(t,x)$ is a mild solution to the SPDE in $L^2(\mathbb{R}^d \times \Omega)$ uniformly in time $t \in (0,T)$ for all $T > 0$. When $u_0(x) \in L^1(\mathbb{R}^d)$, then the deterministic component $u_0(t,x)$ in $u(t,x)$ satisfies $t^{\frac{d}{2\mathrm{m}}} u_0(t,x) \in L^2(\mathbb{R}^d)$ uniformly in time.*

Proof. The L^2 norm of $u(t,x)$ is defined by

$$\begin{aligned}\int_{\mathbb{R}^d} \mathbb{E}\{u^2(t,x)\}dx &= \sum_{n,m\ge 0} \int_{\mathbb{R}^d} \mathbb{E}\{\mathcal{I}_{n+m}(f_n(t,x,\cdot) \otimes f_m(t,x,\cdot))\}dx \\ &\le \sum_{n,m\ge 0} \int_{\mathbb{R}^d} \mathbb{E}\{\mathcal{I}_{n+m}(|f_n(t,x,\cdot) \otimes f_m(t,x,\cdot)|)\}dx.\end{aligned}$$

We now prove that the latter is bounded uniformly in time on compact intervals. The proof shows that $f_n(t,\cdot)$ is also uniformly bounded in $L^2(\mathbb{R}^d;\mathcal{B}_n)$ so that the iterated integrals $u_n(t,x)$ are indeed well defined.

Note that $n+m=2\bar{n}$ for otherwise the above integral vanishes. Then, using the notation $t_0=s_0=t$, we have

$$
\begin{aligned}
I_{n,m}(t) = & \int_{\mathbb{R}^d} \mathbb{E}\{\mathcal{I}_{n+m}(|f_n(t,x)\otimes f_m(t,x)|)\}dx = \\
& \int_{\mathbb{R}^d} \prod_{k=0}^{n-1} \int_0^{t_k} \int_{\mathbb{R}^{dn}} \prod_{k=0}^{n-1} |G|(t_k-t_{k+1},x_k;x_{k+1})\Big| \int_{\mathbb{R}^d} G(t_n,x_n;\xi)u_0(\xi)d\xi\Big| \prod_{k=1}^{n} dt_k \\
& \prod_{l=0}^{m-1} \int_0^{s_l} \int_{\mathbb{R}^{dm}} \prod_{l=0}^{m-1} |G|(s_l-s_{l+1},y_l;y_{l+1})\Big| \int_{\mathbb{R}^d} G(s_m,y_m;\zeta)u_0(\zeta)d\zeta\Big| \prod_{l=1}^{m} ds_l \\
& \delta(x_0-x)\delta(y_0-x)\sigma^{n+m}\mathbb{E}\{\prod_{k=1}^{n} dW(x_k)\prod_{l=1}^{m} dW(y_l)\}dx.
\end{aligned}
$$

Using the fact that $2ab \le a^2+b^2$ with a and b the Green's functions involving x and the fact that $\tau^{\frac{d}{\mathfrak{m}}}\int G^2(\tau,x,y)dx$ is uniformly bounded, we bound the integral in x by a constant. Let us define $\phi(s)=|t-s|^{-\frac{d}{\mathfrak{m}}}$. As a consequence, we obtain that

$$
\begin{aligned}
I_{n,m}(t) \lesssim & \int_0^t \phi(t_1) \prod_{k=1}^{n-1} \int_0^{t_k} \int_{\mathbb{R}^{d(n-1)}} \prod_{k=1}^{n-1} |G|(t_k-t_{k+1},x_k;x_{k+1})\Big| \int_{\mathbb{R}^d} G(t_n,x_n;\xi)u_0(\xi)d\xi\Big| \prod_{k=1}^{n} dt_k \\
& \int_0^t \prod_{l=1}^{m-1} \int_0^{s_l} \int_{\mathbb{R}^{d(m-1)}} \prod_{l=1}^{m-1} |G|(s_l-s_{l+1},y_l;y_{l+1})\Big| \int_{\mathbb{R}^d} G(s_m,y_m;\zeta)u_0(\zeta)d\zeta\Big| \prod_{l=1}^{m} ds_l \\
& \sigma^{n+m}\mathbb{E}\{\prod_{k=1}^{n} dW(x_k)\prod_{l=1}^{m} dW(y_l)\}.
\end{aligned}
$$

Here $a \lesssim b$ means that $a \le Cb$ for some constant $C>0$. In the above term $\phi(t_1)$ should be replaced by $\phi(t_1)+\phi(s_1)$. The second contribution involving $\phi(s_1)$ is treated in the same manner as that involving $\phi(t_1)$, which we now analyze.

Let us re-label $x_{n+l}=y_l$ and $t_{n+l}=s_l$ for $1\le l\le m$. We also define $\mathbf{x}=(x_0,\ldots,x_{n+m+1})$. Then we find that

$$
\begin{aligned}
I_{n,m}(t) &\le \sigma^{n+m}\int_{\mathbb{R}^{2\bar{n}d}} H_{n,m}(t,\mathbf{x})\,\mathbb{E}\Big\{\prod_{k=1}^{2\bar{n}} dW(x_k)\Big\}, \\
H_{n,m}(t,\mathbf{x}) &= \int_0^t \phi(t_1)\prod_{k=1}^{n-1}\int_0^{t_k}\int_0^t \prod_{l=1}^{m-1}\int_0^{t_{n+1+l}} \prod_{k=1,k\neq n}^{n+m-1} |G|(t_k-t_{k+1},x_k;x_{k+1}) \\
&\qquad \Big|\int_{\mathbb{R}^d} G(t_n,x_n;\xi)u_0(\xi)d\xi\Big|\Big|\int_{\mathbb{R}^d} G(t_{n+m},x_{n+m};\zeta)u_0(\zeta)d\zeta\Big|\prod_{k=1}^{n+m} dt_k.
\end{aligned}
$$

We now recall the pairings introduced in (10) and replace n by $\bar{n}$ there. Let us introduce the notation

$$
y_k = \begin{cases} x_{k+1} & k\neq n,n+m \\ \xi & k=n \\ \zeta & k=n+m \end{cases} \qquad \tau_k = \begin{cases} t_{k+1} & k\neq n,n+m \\ 0 & k=n,n+m, \end{cases}
$$

so that $H_{n,m}(t,\mathbf{x})$ is bounded by

$$\int_{\mathbb{R}^{2d}}\int_0^t \phi(t_1)\prod_{k=1}^{n-1}\int_0^{t_k}\int_0^t\prod_{l=1}^{m-1}\int_0^{t_{n+l}}\prod_{k=1}^{2\bar{n}}|G|(t_k-\tau_k,x_k;y_k)|u_0(\xi)|\,|u_0(\zeta)|\,d\xi d\zeta\prod_{k=1}^{n+m}dt_k.$$

Now, we have for each pairing $\mathfrak{p}\in\mathfrak{P}$,

$$\prod_{k=1}^{2\bar{n}}|G|(t_k-\tau_k,x_k;y_k)=\prod_{k\in A_0}|G|(t_k-\tau_k,x_k;y_k)|G|(t_{l(k)}-\tau_{l(k)},x_{l(k)};y_{l(k)}),$$

and as a consequence, using the delta functions appearing in (10),

$$\begin{aligned}I_{n,m}(t)\leq \sigma^{n+m}&\sum_{\mathfrak{p}\in\mathfrak{P}}\int_{\mathbb{R}^{2d}}\int_0^t\phi(t_1)\prod_{k=1}^{n-1}\int_0^{t_k}\int_0^t\prod_{l=1}^{m-1}\int_0^{t_{n+l}}|u_0(\xi)|\,|u_0(\zeta)|\\ &\int_{\mathbb{R}^{\bar{n}d}}\prod_{k\in A_0}\Big(|G|(t_k-\tau_k,x_k;y_k)|G|(t_{l(k)}-\tau_{l(k)},x_k;y_{l(k)})dx_k\Big)\,d\xi d\zeta\prod_{k=1}^{n+m}dt_k \qquad (16)\\ &\leq\sum_{\mathfrak{p}\in\mathfrak{P}}\int_0^t\phi(t_1)\prod_{k=1}^{n-1}\int_0^{t_k}\int_0^t\prod_{l=1}^{m-1}\int_0^{t_{n+l}}\prod_{k\in A_0}\frac{C}{(t_{\mathfrak{k}(k)}-\tau_{\mathfrak{k}(k)})^\alpha}\,\|u_0\|_{L^1}^2\prod_{k=1}^{n+m}dt_k,\end{aligned}$$

for some positive constant C in which we absorb σ^2. On the second line above, the $y_{l(k)}$ are evaluated at $x_{l(k)}=x_k$. The function $k\mapsto\mathfrak{k}(k)$ for $k\in A_0$ is at the moment an arbitrary function such that $\mathfrak{k}(k)=k$ or $\mathfrak{k}(k)=l(k)$. The last line is obtained iteratively in increasing values of k in A_0 by using that one of the Green's function is integrable in x_k uniformly in the other variables and that the other Green's function is bounded independent of the spatial variables by a constant times the time variable to the power $-\alpha$, where we recall that $\alpha=\frac{d}{\mathfrak{m}}$. We have used here assumption (4). It then remains to integrate in the variables ξ and ζ and we use the initial condition $u_0(x)$ for this.

Let us now choose the map $\mathfrak{k}(k)$. It is constructed as follows. When both k and $l(k)$ belong to $\{1,\ldots,n\}$ or both belong to $\{n+1,n+m\}$, then we set $\mathfrak{k}(k)=k$. When $k\in\{1,\ldots,n\}$ and $l(k)\in\{n+1,n+m\}$ (i.e., when there is a crossing from the n first variables to the m last variables), then we choose $\mathfrak{k}(k)=k$ for half of these crossings and $\mathfrak{k}(k)=l(k)$ for the other half. When the number of crossings is odd, the last crossing is chosen with $\mathfrak{k}(k)=k$.

Let us define $A_0^1=\mathfrak{k}(A_0)\cap\{0,\ldots,n\}$ and $A_0^2=\mathfrak{k}(A_0)\backslash A_0^1$. Let $n_0=n_0(\mathfrak{p})$ be the number of elements in A_0^1 and $m_0=m_0(\mathfrak{p})$ be the number of elements in A_0^2 such that $n_0+m_0=\bar{n}$. Let $p=p(\mathfrak{p})$ be the number of crossings in $\mathfrak{p}$. Then, by construction of m, we have

$$n_0=\frac{n-p}{2}+\Big[\frac{p+1}{2}\Big],\qquad m_0=\frac{m-p}{2}+\Big[\frac{p}{2}\Big], \qquad (17)$$

where $[\frac{p+1}{2}]=\frac{p+1}{2}$ if p is odd and $\frac{p}{2}$ if p is even, with $[\frac{p+1}{2}]+[\frac{p}{2}]=p$. Thus, n_0 is bounded by $\frac{n+1}{2}$ and m_0 by $\frac{m}{2}$.

We thus obtain that

$$I_{n,m}(t) \le C^{\bar n}\|u_0\|_{L^1}^2 \sum_{\mathfrak{p}\in\mathfrak{P}} \Big[\prod_{k=0}^{n-1}\int_0^{t_k} \phi(t_1) \prod_{k\in A_0^1} \frac{1}{(t_k-t_{k+1})^\alpha} \prod_{k=1}^n dt_k\Big] \Big[\prod_{l=0}^{m-1}\int_0^{s_l} \prod_{n+l\in A_0^2} \frac{1}{(s_l-s_{l+1})^\alpha}\prod_{l=1}^m ds_l\Big],$$

with the convention that $t_0=s_0=t$, $t_{n+1}=0$ and $s_{m+1}=0$. It remains to estimate the time integrals, which are very small, and sum over a very large number of them. It turns out that these integrals admit explicit expressions. The construction of the mapping $\mathfrak{k}(k)$ ensures that the number of singular terms of the form $\tau^{-\alpha}$ is not too large in the integrals over the t and the s variables.

Let α_k for $0\le k\le n$ be defined such that $\alpha_0=\alpha$, $\alpha_k=\alpha$ for $k\in A_0^1$ and $\alpha_k=0$ otherwise. Still with the convention that $t_{n+1}=0$, we thus want to estimate

$$I_n = I_n(\mathfrak{p}) = \prod_{k=0}^{n-1}\int_0^{t_k}\prod_{k=0}^n \frac{1}{(t_k-t_{k+1})^{\alpha_k}}\prod_{k=1}^n dt_k. \tag{18}$$

The integrals are calculated as follows. Let us consider the last integral:

$$\int_0^{t_{n-1}} \frac{1}{(t_{n-1}-t_n)^{\alpha_{n-1}}}\frac{1}{t_n^{\alpha_n}}dt_n = t_{n-1}^{1-\beta_{n-1}}\int_0^1 \frac{1}{(1-u)^{\alpha_{n-1}}u^{\alpha_n}}du,$$

where we define $\beta_n=\alpha_n$ and $\beta_m=\beta_{m+1}+\alpha_m$ for $0\le m\le n-1$. The latter integral is thus given by

$$t_{n-1}^{1-\beta_{n-1}}B(1-\beta_n, 1-\alpha_{n-1}),$$

where $B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is the Beta function and $\Gamma(x)$ the Gamma function equal to $(x-1)!$ for $x\in\mathbb{N}^*$. The integration in t_{n-2} then yields

$$\int_0^{t_{n-2}} \frac{t_{n-1}^{1-\beta_{n-1}}}{(t_{n-2}-t_{n-1})^{\alpha_{n-2}}}dt_{n-2} = t_{n-2}^{2-\beta_{n-2}}B(2-\beta_{n-1}, 1-\alpha_{n-2}).$$

By induction, we thus obtain that

$$I_n = t_0^{n-\beta_0}\prod_{k=0}^{n-1}B(n-k-\beta_{k+1}, 1-\alpha_k) = t_0^{n-\beta_0}\prod_{k=0}^{n-1}\frac{\Gamma(n-k-\beta_{k+1})\Gamma(1-\alpha_k)}{\Gamma(n-k+1-\beta_{k+1}-\alpha_k)}. \tag{19}$$

Since $\beta_{k+1}+\alpha_k=\beta_k$, we obtain by telescopic cancellations that

$$I_n = t_0^{n-\beta_0}\frac{\Gamma(1-\beta_n)}{\Gamma(n+1-\beta_0)}\prod_{k=0}^{n-1}\Gamma(1-\alpha_k).$$

Then with our explicit choices for the coefficients α_k above, we find that $\beta_0=(n_0+1)\alpha$ so that

$$I_n = t_0^{n-(n_0+1)\alpha}\frac{\Gamma(1-\alpha_n)\Gamma^{n_0}(1-\alpha)}{\Gamma(n+1-(n_0+1)\alpha)}.$$

For a fixed $\mathfrak{p}$, we see that the contribution of the time integrals in $I_{n,m}(t)$ is bounded by a constant (since $\Gamma(1-\alpha)$ is bounded as $\alpha < 1$) times

$$\frac{t_0^{n+m-\alpha\bar{n}-\alpha}\Gamma^{\bar{n}}(1-\alpha)}{\Gamma(n+1-(n_0+1)\alpha)\Gamma(m+1-m_0\alpha)} \leq \frac{t_0^{n+m-\alpha\bar{n}-\alpha}\Gamma^{\bar{n}}(1-\alpha)}{\Gamma((n+1)(1-\frac{\alpha}{2})-\alpha)\Gamma((m+1)(1-\frac{\alpha}{2}))}$$

based on the values of n_0 and m_0. Using Stirling's formula $\Gamma(z) \sim (\frac{2\pi}{z})^{\frac{1}{2}}(\frac{z}{e})^z$ so that $\Gamma(z)$ is bounded from below by $(\frac{z}{C})^z$ for $C < e$, we find that the latter term is bounded by

$$\frac{t_0^{n+m-\alpha\bar{n}-\alpha}C^{n+m}}{n^{n(1-\frac{\alpha}{2})}m^{m(1-\frac{\alpha}{2})}}, \tag{20}$$

for some positive constant C. The latter bound holds for each $\mathfrak{p} \in \mathfrak{P}$. Using the Stirling formula again, we observe that the number of graphs in $\mathfrak{P}$ is bounded by $(\frac{2\bar{n}}{e})^{\bar{n}}$. As a consequence, we have

$$I_{m,n} \leq t_0^{n+m-\alpha\bar{n}-\alpha}\Big(\frac{2\bar{n}}{e}\Big)^{\bar{n}}\frac{C^nC^m}{n^{n(1-\frac{\alpha}{2})}m^{m(1-\frac{\alpha}{2})}} \tag{21}$$

Using the concavity of the log function, we have

$$n^nm^m \geq \Big(\frac{n^2+m^2}{n+m}\Big)^{n+m} \geq \Big(\frac{n+m}{2}\Big)^{n+m},$$

so that

$$\bar{n}^{\bar{n}} \leq C^nC^mn^{\frac{n}{2}}m^{\frac{m}{2}}.$$

As a consequence, we have

$$I_{n,m} \leq J_{n,m}(t) := t_0^{(n+m)(1-\frac{\alpha}{2})-\alpha}C^nC^m\frac{1}{n^{\frac{n}{2}(1-\alpha)}m^{\frac{m}{2}(1-\alpha)}}. \tag{22}$$

The bound with $n = m$ shows that for $n \geq 1$, $u_n(t,x)$ belongs to $L^2(\mathbb{R}^d;\mathcal{B}_n)$ uniformly in time on compact intervals since $2(1-\alpha) > 0$. Now the deterministic component $u_0(t,x)$ corresponding to $n = m = 0$ is in $L^2(\mathbb{R}^d)$ uniformly in time when $u_0(x) \in L^2(\mathbb{R}^d)$ while $t^{\frac{\alpha}{2}}u_0(t,x)$ is in $L^2(\mathbb{R}^d)$ uniformly in time when $u_0(x) \in L^1(\mathbb{R}^d)$. Upon summing the above bound over n and m, we indeed deduce that $u(t,x)$ belongs to $L^2(\Omega\times\mathbb{R}^d)$ uniformly in time on compact intervals when $u_0 \in L^2(\mathbb{R}^d)$.

The above uniform convergence shows that $\mathcal{H}u(t,x)$ is well defined in $L^2(\Omega\times\mathbb{R}^d)$ uniformly in time. Moreover, we verify that $\mathcal{H}u(t,x) = \sum_{n\geq 1}u_n(t,x) = u(t,x)-u_0(t,x)$. This shows that $u(t,x)$ is a mild solution of the stochastic partial differential equation and concludes the proof of the theorem. ◻

3 Uniqueness of the SPDE solution

Let us assume that two solutions exist in a linear vector space $\mathfrak{M}$. Then their difference, which we call u, solves the equation

$$u = \mathcal{H}u = \mathcal{H}^pu,$$

for all $p \geq 0$. The space $\mathfrak{M}$ is defined so that $\mathcal{H}^p u$ is well-defined and is constructed as follows.

We construct $u \in \mathfrak{M}$ as a sum of iterated Stratonovich integrals

$$u(t,x) = \sum_{n \geq 0} \mathcal{I}_n(f_n(t,x,\cdot)).$$

Because the iterated Stratonovich integrals do not form an orthogonal basis of random variables in $L^2(\Omega)$, the above sum is formal and needs to be defined carefully. We need to ensure that the sum converges in an appropriate sense and that $\mathfrak{M}$ is closed under the application of $\mathcal{H}$.

One way to do so is to construct $u(t,x)$ using the classical Wiener-Itô chaos expansion

$$u(t,x) = \sum_{m \geq 0} I_m(g_m(t,x,\cdot)),$$

where I_m is the iterated Itô integral, and to show that the above series is well defined. We then also impose that the chaos expansion of $\mathcal{H}^p u$ is also well-defined.

We first need a calculus to change variables from a definition in terms of iterated Stratonovich integrals to one in terms of iterated Itô integrals. This is done by using the Hu-Meyer formulas. We re-derive this expression as follows. We denote by δW an Itô integral and by dW a Stratonovich integral. We project Stratonovich integrals onto the orthogonal basis of Itô integrals as follows

$$\mathbb{E}\{\mathcal{I}_n(f_n) I_m(\phi_m)\} = \mathbb{E}\{I_m(g_m) I_m(\phi_m)\} = m! \int_{\mathbb{R}^{md}} g_m \phi_m dx,$$

where ϕ_m is a test function. We find that $\mathbb{E}\{\mathcal{I}_n(f_n) I_m(\phi_m)\}$ is equal to

$$\int_{\mathbb{R}^{(n+m)d}} f_n(x_1,\ldots,x_n)\phi_m(y_1,\ldots,y_m)\mathbb{E}\{dW(x_1)\ldots dW(x_n)\delta W(y_1)\ldots\delta W(y_m)\}.$$

The moment of product of Gaussian variables is handled as in (10) with the exception that $\mathbb{E}\{\delta W(y_k)\delta W(y_l)\} = 0$ for $k \neq l$ by renormalization of the Itô-Skorohod integral. The functions f_n and ϕ_m are symmetric in their arguments (i.e., invariant by permutation of their variables). We observe that the variables y need be paired with m variables x. There are $\binom{n}{m}$ ways of pairing the y variables. There remain $n - m = 2k$ variables that need be paired, for a possible number of pairings equal to

$$\frac{(2k-1)!}{(k-1)!2^{k-1}}.$$

The above term is thus given by

$$\binom{m+2k}{m} \frac{(2k-1)!}{(k-1)!2^{k-1}} \int \Big(\int f_{m+2k}(y_1,\ldots y_m, x_1, x_1, \ldots x_k, x_k) \prod_{l=1}^k dx_l \Big) \phi_m(y_1,\ldots y_m) \prod_{p=1}^m dy_p.$$

This shows that g_m is given by

$$g_m(x_1,\ldots,x_m) = \frac{(m+2k)!}{m!k!2^k} \int f_{m+2k}(x_1,\ldots,x_m, y_1^{\otimes 2},\ldots,y_k^{\otimes 2}) \prod_{p=1}^k dy_k.$$

Here $y^{\otimes 2} \equiv (y, y)$. The coefficients g_m are therefore obtained by integrating $n - m$ factors pairwise in the coefficients f_n. This allows us to write the iterated Stratonovich integral as a sum of iterated Itô integrals as follows:

$$\mathcal{I}_n(f_n) = \sum_{k=0}^{[\frac{n}{2}]} \frac{n!}{(n-2k)!k!2^k} I_{n-2k}\Big(\int_{\mathbb{R}^{kd}} f_n(x_{n-2k}, y^{\otimes 2}) dy \Big).$$

This is the Hu-Meyer formula. More interesting for us is the reverse change of coordinates. Let us define formally

$$f = \sum_{n\geq 0} \mathcal{I}_n(f_n) = \sum_{m\geq 0} I_m(g_m).$$

Then we find that

$$g_m(x) = \sum_{k\geq 0} \frac{(m+2k)!}{m!k!2^k} \int_{\mathbb{R}^{kd}} f_{m+2k}(x, y^{\otimes 2}) dy.$$

The square integrability of the coefficients g_m is a necessary condition for the random variables f to be square integrable, and more generally, to be in larger spaces of distributions [9]. The above formula provides the type of constraints we need to impose on the traces of the coefficients f_n. For square integrable variables, we consider the normed vector space $\mathfrak{M}_f$ of random variables

$$f = \sum_{n\geq 0} \mathcal{I}_n(f_n)$$

where the coefficients $\{f_n\}$ are bounded for the norm

$$\|f\|_{\mathfrak{M}_f} = \Big(\sum_{m\geq 0} m! \int \Big(\sum_{k\geq 0} \frac{(m+2k)!}{m!k!2^k} \int_{\mathbb{R}^{kd}} |f_{m+2k}|(x, y^{\otimes 2}) dy \Big)^2 dx \Big)^{\frac{1}{2}} < \infty.$$

Note that the above defines a norm as the triangle inequality is clearly satisfied and for $k = 0$, we find that the L^2 norm of each f_m has to vanish, so that $f_m \equiv 0$ for all m when the norm vanishes. Note also that $\mathfrak{M}_f$ is a dense subset of $L^2(\Omega)$ as any square integrable function g_m may be approximated by a function f_m^k, which vanishes in a set of Lebesgue measure at most k^{-1} in the vicinity of the measure 0 set of diagonals given by the support of the distributions $\delta(x_k - x_l)$. For such functions, we verify that $f_m^k = g_m^k$ so that the Itô and Stratonovich iterated integrals agree. We also have that g_m^k converges to g_m by density. Since every square integrable random variable may be approximated by a finite number of terms in the chaos expansion, this concludes our proof that $\mathfrak{M}_f$ is dense in $L^2(\Omega)$ equipped with its natural metric.

Let us now move to the analysis of the stochastic integral $\mathcal{H}$. It turns out that $\mathfrak{M}_f$ is not stable under $\mathcal{H}$ nor is it in any natural generalization of $\mathfrak{M}_f$. Let us define

$$u(t, x) = \sum_{n\geq 0} \mathcal{I}_n(f_n(t, x, \cdot)), \qquad \mathcal{H}u(t, x) = \sum_{n\geq 0} \mathcal{I}_n((\mathcal{H}f)_n(t, x, \cdot))$$

We then observe that

$$\mathcal{H}f_{n+1}(t,x,y) = \sigma\mathfrak{s}\Big[\int_0^t G(t-s,x,y_1)f_n(s,y)ds\Big],$$

where $\mathfrak{s}$ is the symmetrization with respect to the $d(n+1)-$dimensional y variables. Let us consider $\mathcal{H}^2 f_{n+2}$, which depends only on f_n. Let $c_{m,k} = \frac{(m+2k)!}{m!k!2^k}$ the coefficient that appears in the definition of g_m. Then, for $\mathcal{H}^2 f_{n+2}$ relative to f_n, the coefficients indexed by k are essentially replaced by coefficients indexed by $k+1$. Since $c_{m,k+1}$ is not bounded by a multiple of $c_{m,k}$ uniformly, the integral operator $\mathcal{H}^2$ cannot be bounded in $\mathfrak{M}_f$. The reason why solutions to the stochastic equation may still be found is because the integrations in time after n iterations of the integral $\mathcal{H}$ provide a factor inversely proportional to $n!$. This factor allows us to stabilize the growth in the traces that appears by going from $c_{m,k}$ to $c_{m,k+1}$. Uniqueness of the solution may thus only be obtained in a space where the factor $n!$ appears, at least implicitly.

A suitable functional space is constructed as follows. Let g_m be the chaos expansion coefficients associated to the coefficients $|f_n|$ and $g_{m,p}$ the chaos expansion coefficients associated to the coefficients $|\mathcal{H}^p f_n|$.

Then we impose that the coefficients $\{f_n\}$ be bounded for the norm

$$\sup_{m\geq 0}\sup_{p\geq 0}\sup_{t\in(0,T)}\Big(c_p\int g^2_{m,p}(t,x,y)dxdy\Big)^{\frac12} < \infty, \tag{23}$$

where c_p is an increasing series such that $c_p \to \infty$ as $p \to \infty$. Here T is a fixed (arbitrary) positive time. We denote by $\mathfrak{M} = \mathfrak{M}(T)$ the normed vector space of random fields $u(t,x)$ for which the decomposition in iterated Stratonovich integrals satisfies the above constraint.

We are now ready to state the main result of this section.

Theorem 2 *Let $T > 0$ be an arbitrary time and $u_0(x) \in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$. The solution constructed in Theorem 1 is the unique mild solution to the stochastic partial differential equation* (2) *in the space $\mathfrak{M} = \mathfrak{M}(T)$ for an appropriate sequence of terms $c_p \to \infty$ as $p \to \infty$.*

Proof. Let us first prove uniqueness in $\mathfrak{M}$. We have $u = \mathcal{H}^p u$ for all $p \geq 0$. This implies that $g_m(t,x,\cdot) = g_{m,p}(t,x\cdot)$. The latter converges to 0 in the L^2 sense as $p \to \infty$. This implies that $g_m(t,x,\cdot)$ uniformly vanishes for all m so that $u \equiv 0$.

Let now $u(t,x)$ be given by the following Duhamel expansion

$$u(t,x) = \sum_{n\geq 0} u_n(t,x), \quad u_{n+1}(t,x) = \mathcal{H}u_n(t,x) = \mathcal{H}^{n+1}u_0(t,x), \quad u_0(t,x) = e^{-tP}u_0(x).$$

We thus verify that

$$\mathcal{H}^k u(t,x) = \sum_{n\geq k} u_n(t,x).$$

The proof of construction of the Duhamel solution shows that

$$\int \mathbb{E}\{(\mathcal{H}^k u)^2(t,x)\}dx \leq \mathbb{E}\sum_{n,m\geq k}\int \mathcal{I}_{n+m}\big(|f_n|\otimes|f_m|\big)(t,x)dx := \sum_{m\geq 0} m!\int g^2_{m,k}(t,x,y)dxdy$$

converges to 0 and is bounded by a constant we call $c_k^{-1}\|u_0\|^2_{L^1(\mathbb{R}^d)}$ for $k \geq 1$. Here, c_k may be chosen independently of $u_0 \in L^1(\mathbb{R}^d)$. For such a sequence of terms $c_k \to \infty$ as $k \to \infty$, (23) is clearly satisfied. This shows that u belongs to $\mathfrak{M}$ so constructed. □

The same theory holds when the supremum in m is replaced by a sum with weight $m!$ so that $\mathfrak{M}$ becomes a subspace of L^2. In some sense, the subspace created above is the smallest we can consider that is stable under application of $\mathcal{H}$. When $u_0(x) \in L^1(\mathbb{R}^d)$ not necessarily in $L^2(\mathbb{R}^d)$, then the deterministic component $u_0(t,x)$ is not square integrable uniformly in time. The space $\mathfrak{M}$ may then be replaced by a different space where c_p in (23) is replaced by $t^{\frac{\alpha}{2}} c_p$.

4 Convergence result

Let us now come back to the solution of the equation with random coefficients (1). The theory of existence for such an equation is very similar to that for the stochastic limit. We define formally the integral

$$\mathcal{H}_\varepsilon u(t,x) = \int_0^t \int_{\mathbb{R}^d} G(t-s,x;y) u(s,y) q_\varepsilon(y) dy ds, \tag{24}$$

where we have defined $q_\varepsilon(y) = \varepsilon^{-\frac{d}{2}} q(\frac{y}{\varepsilon})$. The Duhamel solution is defined formally as

$$u_\varepsilon(t,x) = \sum_{n=0}^\infty u_{n,\varepsilon}(t,x), \quad u_{n+1,\varepsilon}(t,x) = \mathcal{H}_\varepsilon u_{n,\varepsilon}(t,x), \quad u_0(t,x) = e^{-tP(x,D)}[u_0(x)], \tag{25}$$

where u_0 is the initial condition of the stochastic equation, which we assume is integrable and square integrable. We have the first result:

Theorem 3 *The function $u_\varepsilon(t,x)$ defined in (25) solves*

$$u_\varepsilon(t,x) = \mathcal{H}_\varepsilon u_\varepsilon(t,x) + e^{-tP(x,D)}[u_0(x)], \tag{26}$$

and is in $L^2(\mathbb{R}^d \times \Omega)$ uniformly in time $t \in (0,T)$ for all $T > 0$.

Proof. The proof goes along the same lines as that of Theorem 1. The L^2 norm of $u_\varepsilon(t,x)$ is defined by

$$\begin{aligned} \int_{\mathbb{R}^d} \mathbb{E}\{u_\varepsilon^2(t,x)\} dx &= \sum_{n,m\geq 0} \int_{\mathbb{R}^d} \mathbb{E}\{u_{n,\varepsilon}(t,x) u_{m,\varepsilon}(t,x)\} dx \\ &\leq \sum_{n,m\geq 0} \int_{\mathbb{R}^d} \mathbb{E}\{|u_{n,\varepsilon}|(t,x)|u_{m,\varepsilon}|(t,x)\} dx \leq I_{m,n,\varepsilon}(t), \end{aligned}$$

where

$$\begin{aligned} I_{n,m,\varepsilon}(t) = \int_{\mathbb{R}^d} \prod_{k=0}^{n-1} \int_0^{t_k} \int_{\mathbb{R}^{dn}} \prod_{k=0}^{n-1} |G|(t_k - t_{k+1}, x_k; x_{k+1}) \Big| \int_{\mathbb{R}^d} G(t_n, x_n; \xi) u_0(\xi) d\xi \Big| \prod_{k=1}^n dt_k \\ \prod_{l=0}^{m-1} \int_0^{s_l} \int_{\mathbb{R}^{dm}} \prod_{l=0}^{m-1} |G|(s_l - s_{l+1}, y_l; y_{l+1}) \Big| \int_{\mathbb{R}^d} G(s_m, y_m; \zeta) u_0(\zeta) d\zeta \Big| \prod_{l=1}^m ds_l \\ \delta(x_0 - x)\delta(y_0 - x) \mathbb{E}\{\prod_{k=1}^n q_\varepsilon(x_k) dx_k \prod_{l=1}^m q_\varepsilon(y_l) dy_l\} dx. \end{aligned}$$

Following the proof of Theorem 1, we obtain

$$I_{n,m,\varepsilon}(t) \le \int_{\mathbb{R}^{2\bar{n}d}} H_{n,m}(t,\mathbf{x})\,\mathbb{E}\Big\{\prod_{k=1}^{2\bar{n}} q_\varepsilon(x_k)dx_k\Big\}.$$

The statement (10) now becomes

$$\mathbb{E}\Big\{\prod_{k=1}^{2\bar{n}} q_\varepsilon(x_k)dx_k\Big\} = \sum_{\mathfrak{p}\in\mathfrak{P}} \prod_{k\in A_0(\mathfrak{p})} \varepsilon^{-d} R\big(\frac{x_k - x_{l(k)}}{\varepsilon}\big) dx_k dx_{l(k)}, \tag{27}$$

where we recall that $R(x) = \mathbb{E}\{q(0)q(x)\}$ is the correlation function of the Gaussian field q. As in the construction of the Duhamel solution for (2), this yields

$$I_{n,m,\varepsilon}(t) \le \sum_{\mathfrak{p}\in\mathfrak{P}} \int_{\mathbb{R}^{2d}} \int_0^t \phi(t_1) \prod_{k=1}^{n-1}\int_0^{t_k} \int_0^t \prod_{l=1}^{m-1}\int_0^{t_{n+l}} |u_0(\xi)|\,|u_0(\zeta)| \int_{\mathbb{R}^{\bar{n}d}} \prod_{k\in A_0}$$
$$\Big(|G|(t_k-\tau_k, x_k; y_k)|G|(t_{l(k)}-\tau_{l(k)}, x_{l(k)}; y_{l(k)})\varepsilon^{-d}\Big|R\big(\frac{x_k - x_{l(k)}}{\varepsilon}\big)\Big| dx_k dx_{l(k)}\Big)\, d\xi d\zeta \prod_{k=1}^{n+m} dt_k.$$

For each $k \in A_0$ considered iteratively with increasing order, the term between parentheses is bounded by the L^1 norm of the Green's function integrated in $x_{\mathfrak{k}(k)}$ times the integral of the correlation function in the variable $x_{\mathfrak{k}'(k)}$, with $(\mathfrak{k}(k), \mathfrak{k}'(k)) = (k, l(k))$, which gives a σ^2 contribution, thanks to the definition (3), times the L^∞ norm of the Green's function in the variable $x_{\mathfrak{k}'(k)}$. Using (4) and the integrability of the correlation function $R(x)$, this shows that

$$I_{n,m,\varepsilon}(t) \le \sum_{\mathfrak{p}\in\mathfrak{P}} \int_0^t \phi(t_1) \prod_{k=1}^{n-1}\int_0^{t_k} \int_0^t \prod_{l=1}^{m-1}\int_0^{t_{n+l}} \prod_{k\in A_0} \frac{C}{(t_{\mathfrak{k}(k)} - \tau_{\mathfrak{k}(k)})^\alpha} \|u_0\|_{L^1}^2 \prod_{k=1}^{n+m} dt_k,$$

as in the proof of Theorem 1. The rest of the proof is therefore as in Theorem 1 and shows that each $u_{n,\varepsilon}(t,x)$ is well defined in $L^2(\mathbb{R}^d\times\Omega)$ uniformly in time and that the series defining $u(t,x)$ converges uniformly in the same sense. □

Mollification and convergence result. We now have defined a sequence of solutions $u_\varepsilon(t,x)$ and a limiting solution $u(t,x)$. When q_ε and the white noise W used in the construction of $u(t,x)$ are independent, then the best we can hope for is that u_ε converges in distribution to u. The convergence is in fact much stronger by constructing $q_\varepsilon dx$ as a mollifier of dW. Let $\hat{R}(\xi)$ be the power spectrum of q, which is defined as the Fourier transform of $R(x)$. By Bochner's theorem, the power spectrum is non-negative and we may define $\hat{\rho}(\xi) = (\hat{R}(\xi))^{\frac{1}{2}}$. Let $\rho(x)$ be the inverse Fourier transform of $\hat{\rho}$. We may then define

$$\tilde{q}(x) = \int_{\mathbb{R}^d} \rho(x-y)dW(y), \tag{28}$$

and obtain a stationary Gaussian process $\tilde{q}(x)$. This process is mean-zero and its correlation function is given by

$$\tilde{R}(x) = \int_{\mathbb{R}^d} \rho(x-y)\rho(y)dy = R(x),$$

by inverse Fourier transform of a product. As a consequence, $q(x)$ and $\tilde{q}(x)$ have the same law since they are mean zero and their correlation functions agree. The corresponding Duhamel solutions u_ε and $\tilde{u}_\varepsilon$ also have the same law by inspection. It thus obviously remains to understand the limiting law of $\tilde{u}_\varepsilon$ to obtain that of u_ε. It turns out that $\tilde{u}_\varepsilon$ may be interpreted as a mollifier of $u(t,x)$, the solution constructed in Theorem 1, and as such converges strongly to its limit.

In addition to the assumptions on the Green's function in (4) and (5), we also assume that $\rho(x) \in L^1(\mathbb{R}^d)$. Then we have

Theorem 4 *Let $u_\varepsilon(t,x)$ be the solution constructed in Theorem 3 and $u(t,x)$ the solution constructed in Theorem 1. Then we have that $u_\varepsilon(t,x)$ converges in distribution to $u(t,x)$ as $\varepsilon \to 0$. More precisely, let $\tilde{u}_\varepsilon(t,x)$ be the Duhamel solution corresponding to the random potential $\tilde{q}$ in (28). Then we have that*

$$\|\tilde{u}_\varepsilon(t) - u(t)\|_{L^2(\mathbb{R}^d\times\Omega)} \to 0, \qquad \varepsilon \to 0, \tag{29}$$

uniformly in time over compact intervals.

Proof. Let us drop the upper ˜ to simplify notation. We have

$$\begin{aligned}\delta I_\varepsilon(t) &= \int_{\mathbb{R}^d} \mathbb{E}\{(u(t)-u_\varepsilon(t))^2\}dx = \sum_{n,m} \delta I_{\varepsilon,n,m}(t)\\ \delta I_{\varepsilon,n,m}(t) &= \int_{\mathbb{R}^d} \mathbb{E}\{(u_n(t)-u_{n,\varepsilon}(t))(u_m(t)-u_{m,\varepsilon}(t))\}dx.\end{aligned}$$

Following the proofs of Theorems 1 and 3, we observe that

$$\begin{aligned}&\delta I_{\varepsilon,n,m}(t) = \\ &\quad\int_{\mathbb{R}^d} \prod_{k=0}^{n-1}\int_0^{t_k}\int_{\mathbb{R}^{dn}}\prod_{k=0}^{n-1} G(t_k-t_{k+1},x_k;x_{k+1})\int_{\mathbb{R}^d} G(t_n,x_n;\xi)u_0(\xi)d\xi\prod_{k=1}^n dt_k\\ &\quad\prod_{l=0}^{m-1}\int_0^{s_l}\int_{\mathbb{R}^{dm}}\prod_{l=0}^{m-1} G(s_l-s_{l+1},y_l;y_{l+1})\int_{\mathbb{R}^d} G(s_m,y_m;\zeta)u_0(\zeta)d\zeta\prod_{l=1}^m ds_l\delta(x_0-x)\\ &\quad\delta(y_0-x)\mathbb{E}\Big\{\Big(\prod_{k=1}^n \sigma dW(x_k)-\prod_{k=1}^n q_\varepsilon(x_k)dx_k\Big)\Big(\prod_{l=1}^m \sigma dW(y_k)-\prod_{l=1}^m q_\varepsilon(y_l)dy_l\Big)\Big\}dx.\end{aligned}$$

Here, we have again that $t_0 = s_0 = t$. The integration in x is handled as in the proof of Theorem 1 so that (with $\phi(t_1)+\phi(s_1)$ replaced by $\phi(t_1)$ as above)

$$\begin{aligned}&|\delta I_{\varepsilon,n,m}(t)| \lesssim \\ &\quad\Big|\int_0^t \phi(t_1)\prod_{k=1}^{n-1}\int_0^{t_k}\int_{\mathbb{R}^{d(n-1)}}\prod_{k=1}^{n-1} G(t_k-t_{k+1},x_k;x_{k+1})\int_{\mathbb{R}^d} G(t_n,x_n;\xi)u_0(\xi)d\xi\prod_{k=1}^n dt_k\\ &\quad\int_0^t\prod_{l=1}^{m-1}\int_0^{s_l}\int_{\mathbb{R}^{d(m-1)}}\prod_{l=1}^{m-1} G(s_l-s_{l+1},y_l;y_{l+1})\int_{\mathbb{R}^d} G(s_m,y_m;\zeta)u_0(\zeta)d\zeta\prod_{l=1}^m ds_l\\ &\quad\mathbb{E}\Big\{\Big(\prod_{k=1}^n \sigma dW(x_k)-\prod_{k=1}^n q_\varepsilon(x_k)dx_k\Big)\Big(\prod_{l=1}^m \sigma dW(y_k)-\prod_{l=1}^m q_\varepsilon(y_l)dy_l\Big)\Big\}\Big|.\end{aligned}$$

The main difference with respect to previous proofs is that we cannot bound the Green's functions by their absolute values just yet. The moment of Gaussian variables is handled as follows. We recast it as

$$\Big(\prod_{k=1}^{n}\sigma dW(x_k)-\prod_{k=1}^{n}q_\varepsilon(x_k)dx_k\Big)\prod_{l=1}^{m}q_\varepsilon(y_l)dy_l$$

plus a second contribution that is handled similarly. We denote by $\delta I^1_{\varepsilon,n,m}(t)$ the corresponding contribution in $\delta I_{\varepsilon,n,m}(t)$ and by $\delta I^2_{\varepsilon,n,m}(t)=\delta I_{\varepsilon,n,m}(t)-\delta I^1_{\varepsilon,n,m}(t)$. The above contribution is recast as

$$\sum_{q=1}^{n}\prod_{p=1}^{q-1}\sigma dW(x_k)\Big(\sigma dW(x_q)-q_\varepsilon(x_q)dx_q\Big)\prod_{p=q+1}^{n+m}q_\varepsilon(x_p)dx_p, \tag{30}$$

where we have defined $x_{n+l}=y_l$ for $1\le l\le m$. We have therefore n (or more precisely $n\wedge m$ by decomposing the product over m variables when $m<n$) terms of the form

$$\mathbb{E}\Big\{\prod_{p=1}^{q-1}\sigma dW(x_k)\Big(\sigma dW(x_q)-q_\varepsilon(x_q)dx_q\Big)\prod_{p=q+1}^{n+m}q_\varepsilon(x_p)dx_p\Big\}:=\mathbb{E}\Big\{\prod_{k=1}^{2\bar n}a_{k,\varepsilon}(dx_k)\Big\},$$

where each measure $a_{k,\varepsilon}(dx_k)$ is Gaussian. Then, (10) is replaced in this context by

$$\begin{aligned}\mathbb{E}\Big\{\prod_{k=1}^{2\bar n}a_{k,\varepsilon}(dx_k)\Big\} &= \sum_{\mathfrak{p}\in\mathfrak{P}}\prod_{k\in A_0(\mathfrak{p})}\mathbb{E}\big\{a_{k,\varepsilon}(dx_k)a_{l(k),\varepsilon}(dx_{l(k)})\big\}\\ &:= \sum_{\mathfrak{p}\in\mathfrak{P}}\prod_{k\in A_0(\mathfrak{p})}h_{\varepsilon,k}(x_k-x_{l(k)})dx_kdx_{l(k)}.\end{aligned} \tag{31}$$

The functions $h_{\varepsilon,k}(x_k-x_{l(k)})$ come in five different forms according as

$$\begin{aligned}\mathbb{E}\{dW(x)dW(y)\} &= \sigma^2\delta(x-y)dxdy\\ \mathbb{E}\{dW(x)q_\varepsilon(y)dy\} &= \sigma\frac{1}{\varepsilon^d}\rho\Big(\frac{x-y}{\varepsilon}\Big)dxdy\\ \mathbb{E}\{q_\varepsilon(x)dxq_\varepsilon(y)dy\} &= \frac{1}{\varepsilon^d}R\Big(\frac{x-y}{\varepsilon}\Big)dxdy\\ \mathbb{E}\{(dW(x)-q_\varepsilon(x)dx)dW(y)\} &= \Big(\sigma^2\delta(x-y)-\sigma\frac{1}{\varepsilon^d}\rho\Big(\frac{x-y}{\varepsilon}\Big)\Big)dxdy\\ \mathbb{E}\{(dW(x)-q_\varepsilon(x)dx)q_\varepsilon(y)dy\} &= \Big(\sigma\frac{1}{\varepsilon^d}\rho\Big(\frac{x-y}{\varepsilon}\Big)-\frac{1}{\varepsilon^d}R\Big(\frac{x-y}{\varepsilon}\Big)\Big)dxdy.\end{aligned} \tag{32}$$

At this point, we have obtained that

$$\begin{aligned}|\delta I^1_{\varepsilon,n,m}(t)| \lesssim \sum_{\mathfrak{p}\in\mathfrak{P}}\\ \Big|\int_0^t\phi(t_1)\prod_{k=1}^{n-1}\int_0^{t_k}\int_{\mathbb{R}^{d(n-1)}}\prod_{k=1}^{n-1}G(t_k-t_{k+1},x_k;x_{k+1})\int_{\mathbb{R}^d}G(t_n,x_n;\xi)u_0(\xi)d\xi\prod_{k=1}^{n}dt_k\\ \int_0^t\prod_{l=1}^{m-1}\int_0^{s_l}\int_{\mathbb{R}^{d(m-1)}}\prod_{l=1}^{m-1}G(s_l-s_{l+1},y_l;y_{l+1})\int_{\mathbb{R}^d}G(s_m,y_m;\zeta)u_0(\zeta)d\zeta\prod_{l=1}^{m}ds_l\\ \prod_{k\in A_0(\mathfrak{p})}h_{\varepsilon,k}(x_k-x_{l(k)})dx_kdx_{l(k)}\Big|.\end{aligned}$$

Using the notation as in the proof of Theorem 1, we obtain that

$$
\begin{aligned}
|\delta I^1_{\varepsilon,n,m}(t)| &\lesssim \sum_{\mathfrak{p}\in\mathfrak{P}} \Big| \int_{\mathbb{R}^{2d}} \int_0^t \phi(t_1) \prod_{k=1}^{n-1} \int_0^{t_k} \int_0^t \prod_{l=1}^{m-1} \int_0^{t_{n+l}} u_0(\xi)u_0(\zeta) \int_{\mathbb{R}^{\bar{n}d}} \prod_{k\in A_0(\mathfrak{p})} \\
&\Big(G(t_k-\tau_k, x_k; y_k) G(t_{l(k)}-\tau_{l(k)}, x_{l(k)}; y_{l(k)}) h_{\varepsilon,k}(x_k - x_{l(k)}) dx_k dx_{l(k)} \Big) d\xi d\zeta \prod_{k=1}^{n+m} dt_k \Big| \\
&\lesssim \sum_{\mathfrak{p}\in\mathfrak{P}} \int_{\mathbb{R}^{2d}} \int_0^t \phi(t_1) \prod_{k=1}^{n-1} \int_0^{t_k} \int_0^t \prod_{l=1}^{m-1} \int_0^{t_{n+l}} |u_0(\xi)||u_0(\zeta)| \Big| \int_{\mathbb{R}^{\bar{n}d}} \prod_{k\in A_0(\mathfrak{p})} \\
&\Big(G(t_k-\tau_k, x_k; y_k) G(t_{l(k)}-\tau_{l(k)}, x_{l(k)}; y_{l(k)}) h_{\varepsilon,k}(x_k - x_{l(k)}) dx_k dx_{l(k)} \Big) \Big| d\xi d\zeta \prod_{k=1}^{n+m} dt_k.
\end{aligned}
$$

It remains to handle the multiple integral between absolute values. For $k \in A_0(\mathfrak{p})$ for which $h_{\varepsilon,k}$ is of the form given in the last two lines of (32), we observe that the corresponding term between parentheses in the above expression is of the form

$$
\begin{aligned}
& \int_{\mathbb{R}^{2d}} G(s,x;\zeta) G(\tau,y;\xi) h_\varepsilon(x-y) dx dy \\
=& \int_{\mathbb{R}^d} G(s,x,\zeta) \Big(\int_{\mathbb{R}^d} \frac{1}{\varepsilon^d} g\Big(\frac{x-y}{\varepsilon}\Big) \big(G(\tau,x;\xi) - G(\tau,y;\xi)\big) dy \Big) dx \qquad (33)\\
=& \int_{\mathbb{R}^d} G(s,x,\zeta) \Big(\int_{\mathbb{R}^d} g(y) \big(G(\tau,x;\xi) - G(\tau,x+\varepsilon y;\xi)\big) dy \Big) dx,
\end{aligned}
$$

where the function $g(x)$ is given by either

$$
g(x) = \pm\sigma\rho(x) \qquad \text{or} \qquad g(x) = \pm(R(x) - \sigma\rho(x)). \qquad (34)
$$

This is because ρ averages to σ while R averages to σ^2. Let k_0 be the index for which h_{ε,k_0} is in the form of a difference as above. This yields, with $g = g[k_0]$ as above,

$$
\begin{aligned}
|\delta I^1_{\varepsilon,n,m}(t)| &\lesssim \sum_{\mathfrak{p}\in\mathfrak{P}} \int_{\mathbb{R}^{2d}} \int_0^t \phi(t_1) \prod_{k=1}^{n-1} \int_0^{t_k} \int_0^t \prod_{l=1}^{m-1} \int_0^{t_{n+l}} |u_0(\xi)||u_0(\zeta)| \int_{\mathbb{R}^{\bar{n}d}} \prod_{k_0\neq k\in A_0(\mathfrak{p})} \\
&\Big| G(t_k-\tau_k, x_k; y_k) G(t_{l(k)}-\tau_{l(k)}, x_{l(k)}; y_{l(k)}) h_{\varepsilon,k}(x_k - x_{l(k)}) \Big| dx_k dx_{l(k)} \\
&|G(t_{\mathfrak{k}(k_0)} - \tau_{\mathfrak{k}(k_0)}, x_{\mathfrak{k}(k_0)}; y_{\mathfrak{k}(k_0)})||g(x_{\mathfrak{k}'(k_0)})| d\xi d\zeta \prod_{k-1}^{n+m} dt_k \\
&\Big| \big(G(\cdot,\cdot,\cdot) - G(\cdot, \cdot + \varepsilon x_{\mathfrak{k}'(k_0)}, \cdot)\big)(t_{\mathfrak{k}'(k_0)} - \tau_{\mathfrak{k}'(k_0)}, x_{\mathfrak{k}(k_0)}; y_{\mathfrak{k}'(k_0)}) \Big| dx_{k_0} dx_{l(k_0)}.
\end{aligned}
$$

The above term is now handled as in the proof of Theorem 1. For $k \neq k_0$, the bounds are obtained as before because ρ and R are integrable functions by hypothesis. The Green's function $|G(t_{\mathfrak{k}(k_0)} - \tau_{\mathfrak{k}(k_0)}, x_{\mathfrak{k}(k_0)}; y_{\mathfrak{k}(k_0)})|$ is bounded by a constant times $|t_{\mathfrak{k}(k_0)} - \tau_{\mathfrak{k}(k_0)}|^{-\alpha}$. The integration $dx_{k_0} dx_{l(k_0)} = dx_{\mathfrak{k}(k_0)} dx_{\mathfrak{k}'(k_0)}$ then yields a contribution bounded by $|t_{\mathfrak{k}'(k_0)} - \tau_{\mathfrak{k}'(k_0)}|^{-\gamma}$ times

$$
M_\varepsilon = \sup_{\tau\in(0,T),\xi\in\mathbb{R}^d} \tau^\gamma \int_{\mathbb{R}^{2d}} |g(y)| \big| G(\tau,x;\xi) - G(\tau,x+\varepsilon y;\xi) \big| dx dy. \qquad (35)
$$

The presence of the factor γ is sufficient to ensure that M_ε converges to 0 as $\varepsilon \to 0$. As a consequence, as in the derivation of (16), we obtain that

$$|\delta I_{\varepsilon,n,m}(t)| \lesssim \sum_{\mathfrak{p}\in\mathfrak{P}} \int_0^t \prod_{k=1}^{n-1} \int_0^{t_k} \int_0^t \prod_{l=1}^{m-1} \int_0^{t_{n+l}} \frac{2nM_\varepsilon \phi(t_1)}{|t_{\mathfrak{k}'(k_0)} - \tau_{\mathfrak{k}'(k_0)}|^\gamma} \prod_{k\in A_0} \frac{C}{\big(t_{\mathfrak{k}(k)} - \tau_{\mathfrak{k}(k)}\big)^{\frac{d}{\mathfrak{m}}}} \prod_{k=1}^{n+m} dt_k.$$

The factor $2n$ comes from twice the summed contributions in (30). The presence of the factor γ increases the time integrals as follows. Assume that $k_0 \leq n$ for concreteness; the case $k_0 \geq n+1$ is handled similarly. Then β_0 in the proof of Theorem 1 should be replaced by $\beta_0 + \gamma$. This does not significantly modify the analysis of the Γ functions and the contribution of each graph is still bounded by a term of the form (20). The behavior in time, however, is modified by the presence of the contribution γ and we find that

$$|\delta I_{\varepsilon,n,m}(t)| \leq CnM_\varepsilon t_0^{(n+m)(1-\frac{\alpha}{2})-\alpha-\gamma} C^n C^m \frac{1}{n^{\frac{n}{2}(1-\alpha)} m^{\frac{m}{2}(1-\alpha)}}.$$

The above bound is of interest for $n+m \geq 2$ since the case $n=m=0$ corresponds to the ballistic component $u_0(t,x)$, which is the same for $u_\varepsilon(t,x)$ and $u(t,x)$ so that $\delta I_{\varepsilon,0,0} = 0$. By choosing $\gamma = 2(1-\alpha) > 0$, we observe that $2(1-\frac{\alpha}{2}) - \alpha - \gamma \geq 0$ for $n+m \geq 2$ so that $|\delta I_{\varepsilon,n,m}(t)|$ is bounded uniformly in time. The new factor n may be absorbed into C^n so that after summation over n and m, we get

$$\|u_\varepsilon(t) - u(t)\|^2_{L^2(\mathbb{R}^d\times\Omega)} \leq CM_\varepsilon. \tag{36}$$

By assumption (5), the integrand in (35) converges point-wise to 0 and an application of the dominated Lebesgue convergence theorem shows that $M_\varepsilon \to 0$. This concludes the proof of the convergence result. □

A continuity lemma. We conclude this paper by showing that the constraints (4) and (5) imposed on the Green's functions of the unperturbed problem throughout the paper are satisfied for a natural class of parabolic operators.

Lemma 4.1 *Let $G(t,x)$ be defined as the Fourier transform of $e^{-t|\xi|^{\mathfrak{m}}}$, i.e., the Green's function of the operator $\partial_t + (-\Delta)^{\frac{\mathfrak{m}}{2}}$ for $\mathfrak{m} > d$. Then the conditions in (4) and (5) are satisfied. Moreover, when $\mathfrak{m}$ is an even number, then M_ε in (35) satisfies the bound*

$$M_\varepsilon \lesssim \varepsilon^\beta, \qquad \beta = 2(\mathfrak{m}-d) \wedge 1.$$

Proof. By scaling invariance, we find that $G(t,x) = t^{-\frac{d}{\mathfrak{m}}} G(1, t^{-\frac{1}{\mathfrak{m}}} x)$. Since $|\xi|^p e^{-|\xi|^{\mathfrak{m}}}$ is integrable for all p, we obtain that $G(1,x)$ belongs to $C^\infty(\mathbb{R}^d)$. Since $G(1,x)$ is bounded, then so is $t^{\frac{d}{\mathfrak{m}}} G(t,x)$ uniformly in t and x.

By the above scaling, $G(t,x)$ belongs to $L^1(\mathbb{R}^d)$ uniformly in time if and only if $G(1,x)$ does. When $\mathfrak{m}$ in an even integer, then $e^{-|\xi|^{\mathfrak{m}}}$ belongs to $\mathcal{S}(\mathbb{R}^d)$, the space of Schwartz functions, so that $G(1,x) \in \mathcal{S}(\mathbb{R}^d)$ as well. It is therefore integrable and has an integrable gradient. When $\mathfrak{m}$ is not an even integer, we have

$$e^{-|\xi|^{\mathfrak{m}}} - 1 = \sum_{k=1}^\infty \frac{(-1)^k}{k!} |\xi|^{k\mathfrak{m}}.$$

The Fourier transform of the homogeneous function $|\xi|^{k\mathfrak{m}}$ is given by [21]

$$c(k)|x|^{-k\mathfrak{m}-d}, \qquad c(k) = C_d 2^{k\mathfrak{m}} \frac{\Gamma(\frac{1}{2}(k\mathfrak{m}+d))}{\Gamma(-\frac{1}{2}k\mathfrak{m})},$$

where C_d is a normalization constant independent of k. The Fourier transform of $e^{-|\xi|^{\mathfrak{m}}}$ may then be written as a constant times $|x|^{-(d+\mathfrak{m})}$ plus a smoother contribution that converges faster to 0 (for instance because it belongs to some $H^s(\mathbb{R}^d)$ with $s > \frac{d}{2}+k$ sufficiently large so that k derivatives of this contribution are integrable). It is therefore integrable for $\mathfrak{m} > 0$. The L^2 bound follows from the L^1 and L^∞ bounds.

We obtain by scaling invariance and from the definition of $G(t,x)$ that

$$t^\gamma \int_{\mathbb{R}^d} |G(t,x) - G(t,x+\varepsilon y)| dx = t^\gamma \int_{\mathbb{R}^d} |G(1,x) - G(1,x+t^{-\frac{1}{\mathfrak{m}}}\varepsilon y)| dx.$$

The above derivation shows that the gradient of G is also integrable for $\mathfrak{m} > 1$ so we may bound the above quantity by $t^\gamma(1 \wedge t^{-\frac{1}{\mathfrak{m}}}\varepsilon|y|)$. Now,

$$\sup_{t<T}(t^\gamma \wedge t^{\gamma-\frac{1}{\mathfrak{m}}}\varepsilon|y|) \lesssim (\varepsilon|y|)^{\gamma\mathfrak{m}} \vee \varepsilon|y| = (\varepsilon|y|)^{2(\mathfrak{m}-d)} \vee \varepsilon|y|,$$

according as $\gamma\mathfrak{m} < 1$ or $\gamma\mathfrak{m} \geq 1$. We thus obtain (5) by sending $\varepsilon y \to 0$. When $g(y)$ is sufficiently regular, then we obtain the more precise bound

$$M_\varepsilon \lesssim \varepsilon^{2(\mathfrak{m}-d)} \int_{\mathbb{R}^d} |g(y)||y|^{2(\mathfrak{m}-d)} dy \vee \varepsilon \int_{\mathbb{R}^d} |g(y)||y| dy,$$

provided that the latter integrals are well-defined. □

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