

Homogenization with large spatial random potential

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Abstract

We consider the homogenization of parabolic equations with large spatially-dependent potentials modeled as Gaussian random fields. We derive the homogenized equations in the limit of vanishing correlation length of the random potential. We characterize the leading effect in the random fluctuations and show that their spatial moments converge in law to Gaussian random variables. Both results hold for sufficiently small times and in sufficiently large spatial dimensions $d \geq \mathfrak{m}$, where \mathfrak{m} is the order of the spatial pseudo-differential operator in the parabolic equation. In dimension $d < \mathfrak{m}$, the solution to the parabolic equation is shown to converge to the (non-deterministic) solution of a stochastic equation in [2]. The results are then extended to cover the case of long range random potentials, which generate larger, but still asymptotically Gaussian, random fluctuations.

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1 Introduction

Let $\mathfrak{m} > 0$ and $P(D)$ the pseudo-differential operator with symbol $\hat{p}(\xi) = |\xi|^{\mathfrak{m}}$. We consider the following evolution equation in dimension $d \geq \mathfrak{m}$:

$$\begin{aligned} \left(\frac{\partial}{\partial t} + P(D) - \frac{1}{\varepsilon^\alpha} q\left(\frac{x}{\varepsilon}\right) \right) u_\varepsilon(t, x) &= 0, & x \in \mathbb{R}^d, \quad t > 0, \\ u_\varepsilon(0, x) &= u_0(x), & x \in \mathbb{R}^d. \end{aligned} \tag{1}$$

Here, $u_0 \in L^2(\mathbb{R}^d)$ and $q(x)$ is a mean zero stationary (real valued) Gaussian process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that $q(x)$ has bounded and integrable correlation function $R(x) = \mathbb{E}\{q(y)q(x+y)\}$, where \mathbb{E} is the mathematical expectation associated with \mathbb{P} , and bounded, continuous in the vicinity of 0, and integrable power

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spectrum $(2\pi)^d \hat{R}(\xi) = \int_{\mathbb{R}^d} e^{-i\xi \cdot x} R(x) dx$ in the sense that $\int_{\mathbb{R}^d \setminus B(0,1)} \hat{R}(\xi) |\xi|^{-m} d\xi < \infty$. The size of the potential is constructed so that the limiting solution as $\varepsilon \rightarrow 0$ is different from the unperturbed solution obtained by setting $q = 0$. The appropriate size of the potential is given by

$$\varepsilon^\alpha = \begin{cases} \varepsilon^{\frac{m}{2}} |\ln \varepsilon|^{\frac{1}{2}} & d = m, \\ \varepsilon^{\frac{m}{2}} & d > m. \end{cases} \quad (2)$$

Therefore, $\alpha = \frac{m}{2}$ except in dimension $d = m$ where a logarithmic correction appears. We still use the convenient notation ε^α in that case with $\alpha = \alpha(\varepsilon) = \frac{m}{2} + \frac{1}{2} \frac{\ln |\ln \varepsilon|}{|\ln \varepsilon|}$.

The above equation may be seen as a continuous version of the parabolic Anderson model, which is extensively studied as a model (typically for $m = 2$) where localization and intermittency can develop; see e.g. [4]. This paper concerns the regime where u_ε is well approximated by the solution to a deterministic (homogenized) equation.

The potential is bounded \mathbb{P} -a.s. on bounded domains but is unbounded \mathbb{P} -a.s. on \mathbb{R}^d . It is therefore unclear that (1) admits solutions a priori. By using a method based on the Duhamel expansion, we obtain that for a sufficiently small time $T > 0$, the parabolic equation (1) indeed admits a weak solution $u_\varepsilon(t, \cdot) \in L^2(\Omega \times \mathbb{R}^d)$ uniformly in time $t \in (0, T)$ and $0 < \varepsilon < \varepsilon_0$.

Homogenization theory. As $\varepsilon \rightarrow 0$, we show that the solution $u_\varepsilon(t)$ to (1) converges strongly in $L^2(\Omega \times \mathbb{R}^d)$ uniformly in $t \in (0, T)$ to its limit $u(t)$ solution of the following homogenized evolution equation

$$\begin{aligned} \left(\frac{\partial}{\partial t} + P(D) - \rho \right) u(t, x) &= 0, & x \in \mathbb{R}^d, \quad t > 0, \\ u(0, x) &= u_0(x), & x \in \mathbb{R}^d, \end{aligned} \quad (3)$$

where the effective (non-negative) potential is given by

$$\rho = \begin{cases} c_d \hat{R}(0) & d = m, \\ \int_{\mathbb{R}^d} \frac{\hat{R}(\xi)}{|\xi|^m} d\xi & d > m. \end{cases} \quad (4)$$

Here, c_d is the volume of the unit sphere S^{d-1} . We denote by \mathcal{G}_t^ρ the propagator for the above equation, which to $u_0(x)$ associates $\mathcal{G}_t^\rho u_0(x) = u(t, x)$ solution of (3).

We assume that the non-negative (by Bochner's theorem) power spectrum $\hat{R}(\xi)$ is bounded by $f(|\xi|)$, where $f(r)$ is a positive, bounded, radially symmetric, and integrable function $f(r) \leq \tau_f r^{-n}$ for some $0 \leq n < d - m$ in dimension $d > m$ and $n = 0$ when $d \leq m$, with $\rho_f := c_d \int_0^\infty f(r) r^{d-m-1} dr \vee \tau_f < \infty$. Here, $a \vee b = \max(a, b)$.

Theorem 1 *Let $T > 0$ such that $4T\rho_f < 1$. Then there exists a solution to (1) $u_\varepsilon(t) \in L^2(\Omega \times \mathbb{R}^d)$ uniformly in $0 < \varepsilon < \varepsilon_0$ for all $t \in [0, T]$. Moreover, let us assume that $\hat{R}(\xi)$ is of class $\mathcal{C}^\gamma(\mathbb{R}^d)$ for some $0 \leq \gamma \leq 2$ and let $u(t, x)$ be the unique solution in $L^2(\mathbb{R}^d)$ to (3). Then, we have the convergence results*

$$\begin{aligned} \|(u_\varepsilon - u)(t)\|_{L^2(\Omega \times \mathbb{R}^d)} &\lesssim \varepsilon^{\frac{\beta}{2}} \|u_0\|_{L^2(\mathbb{R}^d)}, \\ \|(u_\varepsilon - u)(t)\|_{L^2(\mathbb{R}^d)} &\lesssim \varepsilon^{\gamma \wedge \beta} \|u_0\|_{L^2(\mathbb{R}^d)}, \end{aligned} \quad (5)$$

where $a \lesssim b$ means $a \leq Cb$ for some $C > 0$, $a \wedge b = \min(a, b)$, where $\mathbf{u}_\varepsilon(t, \cdot)$ is a deterministic function in $L^2(\mathbb{R}^d)$ uniformly in time, and where we have defined

$$\varepsilon^\beta = \begin{cases} |\ln \varepsilon|^{-1} & d = \mathbf{m}, \\ \varepsilon^{d-\mathbf{m}} & \mathbf{m} < d < 2\mathbf{m}, \\ \varepsilon^\mathbf{m} |\ln \varepsilon| & d = 2\mathbf{m}, \\ \varepsilon^\mathbf{m} & d > 2\mathbf{m}. \end{cases} \quad (6)$$

In (5), $\mathbf{u}_\varepsilon(t, x)$ is the Fourier transform of $\mathfrak{U}_\varepsilon(t, \xi) = e^{-i(|\xi|^\mathbf{m} - \rho_\varepsilon(\xi))\hat{u}_0(\xi)}$, where $\rho_\varepsilon = \int_{\mathbb{R}^d} \frac{\hat{R}(\xi_1 - \varepsilon\xi)}{\xi_1^\mathbf{m}} d\xi_1$ when $d > \mathbf{m}$ and $\rho_\varepsilon = c_d \hat{R}(\varepsilon\xi)$ when $d = \mathbf{m}$ and where the Fourier transform of the initial condition $u_0(x)$ is define as $\hat{u}_0(\xi) = \int_{\mathbb{R}^d} e^{-i\xi \cdot x} u_0(x) dx$.

Note that the effective potential $-\rho$ is non-positive. The above theorem thus shows an exponential growth in time of the low-frequency components of u_ε , at least for sufficiently small times. The method used below to prove the above theorem does not extend to arbitrary times $T < \infty$ fixed independent of ε .

Theory of random fluctuations. The error term $u_\varepsilon - u$ is dominated by deterministic components when $\varepsilon^{\gamma \wedge \beta} \gg \varepsilon^{\frac{d-2\alpha}{2}}$ and by random fluctuations when $\varepsilon^{\gamma \wedge \beta} \ll \varepsilon^{\frac{d-2\alpha}{2}}$. In both situations, the random fluctuations may be estimated as follows. We show that

$$u_{1,\varepsilon}(t, x) = \frac{1}{\varepsilon^{\frac{d-2\alpha}{2}}} (u_\varepsilon - \mathbb{E}\{u_\varepsilon\})(t, x), \quad (7)$$

converges weakly in space and in distribution to a Gaussian random variable. More precisely, we have

Theorem 2 *Let M be a test function such that its Fourier transform $\hat{M} \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Then we find that for all $t \in (0, T)$*

$$(u_{1,\varepsilon}(t, \cdot), M) \xrightarrow{\varepsilon \rightarrow 0} \int_{\mathbb{R}^d} \mathcal{M}_t(x) \sigma dW_x, \quad \mathcal{M}_t(x) = \int_0^t \mathcal{G}_s^\rho M(x) \mathcal{G}_{t-s}^\rho u_0(x) ds, \quad (8)$$

where convergence holds in the sense of distributions, dW_x is the standard multiparameter Wiener measure on \mathbb{R}^d and σ is the standard deviation defined by

$$\sigma^2 := (2\pi)^d \hat{R}(0) = \int_{\mathbb{R}^d} \mathbb{E}\{q(0)q(x)\} dx. \quad (9)$$

This shows that the fluctuations of the solution are asymptotically given by a Gaussian random variable, which is consistent with the central limit theorem. We may recast the convergence in (8) as $(u_{1,\varepsilon}(t, \cdot), M) \rightarrow (u_1(t, \cdot), M)$, where u_1 is the solution to the following stochastic partial differential equation with additive noise

$$\frac{\partial u_1}{\partial t} + P(D)u_1 - \rho u_1 = \sigma u \dot{W}, \quad (10)$$

with initial conditions $u_1(0, x) = 0$. Here, \dot{W} is spatial white noise. Let $G^\rho(t, x - y)$ be the Green's kernel of the limiting equation (3). Then, u_1 is given by

$$u_1(t, x) = \int_0^t \int_{\mathbb{R}^d} G^\rho(t - s, x - y) u(s, y) \sigma dW_y ds,$$

where the above is defined as a Wiener integral.

Transition to stochasticity. We observe a sharp transition in the behavior of u_ε at $d = \mathbf{m}$. For $d < \mathbf{m}$, the following holds. The size of the potential that generates an order $O(1)$ perturbation is now given by (see the last inequality in lemma 2.1)

$$\varepsilon^\alpha = \varepsilon^{\frac{d}{2}}.$$

Using the same methods as for the case $d \geq \mathbf{m}$, we may obtain that $u_\varepsilon(t)$ is uniformly bounded and thus converges weakly in $L^2(\Omega \times \mathbb{R}^d)$ for sufficiently small times to a function $u(t)$. The problem is addressed in [2], where it is shown that $u(t)$ is the solution to the stochastic partial differential equation in Stratonovich form

$$\frac{\partial u}{\partial t} + P(D)u + u \circ \sigma \frac{dW}{dx} = 0, \quad (11)$$

with $u(0, x) = u_0(x)$, σ defined as in (9), and $\frac{dW}{dx}$ d -parameter spatial white noise “density”. The above equation admits a unique solution that belongs to $L^2(\Omega \times \mathbb{R}^d)$ locally uniformly in time. Stochastic equations have also been analyzed in the case where $d \geq \mathbf{m}$ (i.e., $d \geq 2$ when $P(D) = -\Delta$), see [10, 13]. However, our results show that such solutions cannot be obtained as a limit in $L^2(\Omega \times \mathbb{R}^d)$ of solutions corresponding to vanishing correlation length so that their physical justification is more delicate. In the case $d = 1$ and $\mathbf{m} = 2$ with $q(x)$ a bounded potential, we refer the reader to [15] for more details on the above stochastic equation.

Random fluctuations and long range correlations. The above theorems 1 and 2 assume short range correlations for the random potential. Mathematically, this is modeled by an integrable correlation function, or equivalently a bounded value for $\hat{R}(0)$. Longer range correlations correspond to correlation functions $R(x)$ that decay like $|x|^{-(d-\mathbf{n})}$ as $|x| \rightarrow \infty$ for some $0 < \mathbf{n} < d$ and may be modeled by unbounded power spectra in the vicinity of the origin, for instance by assuming that

$$\hat{R}(\xi) = h(\xi)\hat{S}(\xi), \quad 0 < h(\lambda\xi) = |\lambda|^{-\mathbf{n}}h(\xi), \quad (12)$$

where $h(\xi)$ is thus a positive function homogeneous of degree $-\mathbf{n}$ and $\hat{S}(\xi)$ is bounded on $B(0, 1)$. We assume that $\hat{R}(\xi)$ is still bounded on $\mathbb{R}^d \setminus B(0, 1)$.

Provided that $d > \mathbf{m} + \mathbf{n}$ so that ρ defined in (4) is still bounded, the results of theorems 1 and 2 may then be extended to the case of long range fluctuations. The convergence properties stated in theorem 1 still hold with β replaced by $\beta - \mathbf{n}$. The random fluctuations are now asymptotically Gaussian processes of amplitude of order $\varepsilon^{\frac{d-\mathbf{m}-\mathbf{n}}{2}}$, which may conveniently be written as stochastic integrals with respect to some multiparameter fractional Brownian motion in place of the Wiener measure appearing in (8). More precisely, we have the following result:

Theorem 3 *Let us assume that $h(\xi) = |\xi|^{-\mathbf{n}}$ for $\mathbf{n} > 0$ and $\mathbf{m} + \mathbf{n} < d$. We also impose the following regularity on \hat{u}_0 :*

$$\int_{B(0,1)} |\hat{u}_0(\xi + \tau)|^2 h(\xi) d\xi \leq C, \quad \text{for all } \tau \in \mathbb{R}^d. \quad (13)$$

Then theorem 1 holds with β replaced by $\beta - \mathbf{n}$.

Let us define the random corrector

$$u_{1,\varepsilon}(t, x) = \frac{1}{\varepsilon^{\frac{d-m-n}{2}}} (u_\varepsilon - \mathbb{E}\{u_\varepsilon\})(t, x). \quad (14)$$

Then its spatial moments $(u_{1,\varepsilon}(t, x), M(x))$ converge in law to centered Gaussian random variables $\mathcal{N}(0, \Sigma_M(t))$ with variance given by

$$\Sigma_M(t) = (2\pi)^d \hat{S}(0) \int_{\mathbb{R}^{2d}} \mathcal{M}_t(x) \varphi(x-y) \mathcal{M}_t(y) dx dy. \quad (15)$$

Results for more general homogeneous functions $h(\xi)$ are described in section 3.3. The above theorem shows that $u_{1,\varepsilon}$ converges weakly and in distribution to u_1 solution of the following SPDE with additive noise:

$$\frac{\partial u_1}{\partial t} + P(D)u_1 - \rho u_1 = \sigma u \dot{W}^H, \quad (16)$$

where \dot{W}^H is the centered Gaussian field with covariance function

$$\mathbb{E}\{\dot{W}^H(x)\dot{W}^H(x+y)\} = \frac{c_n}{|y|^{d-n}}, \quad c_n = \frac{\Gamma(\frac{d-n}{2})}{(2^n \pi^{\frac{d}{2}} \Gamma(\frac{n}{2}))}.$$

Up to multiplication by a normalization constant, \dot{W}^H may be regarded as a multiparameter fractional white noise with Hurst index defined by $2H = 1 + \frac{n}{d}$. The above SPDE thus generalizes (10) to the case of long range correlations.

Outline. The rest of the paper is structured as follows. Section 2 recasts (1) as an infinite Duhamel series of integrals in the Fourier domain. The cross-correlations of the terms appearing in the series are analyzed by calculating moments of Gaussian variables and estimating the contributions of graphs similar to those introduced in [6, 12]. These estimates allow us to construct a solution to (1) in $L^2(\Omega \times \mathbb{R}^d)$ uniformly in time for sufficiently small times $t \in (0, T)$. The maximal time T of validity of the theory depends on the power spectrum $\hat{R}(\xi)$. The estimates on the graphs are then used in section 3 to characterize the limit and the leading random fluctuations of the solution $u_\varepsilon(t, x)$. The extension of the results to long range correlations is presented in section 3.3. A roadmap of the proof of the main theorems is presented in section 2.3.

The analysis of (1) and of similar operators has been performed for smaller potentials than those given in (2) in e.g. [1, 7] when u_ε converges strongly to the solution of the unperturbed equation (with $q \equiv 0$). The results presented in this paper may thus be seen as generalizations to the case of sufficiently strong potentials so that the unperturbed solution is no longer a good approximation of u_ε . The analysis presented below is based on simple estimates for the Feynman diagrams corresponding to Gaussian random potentials and does not extend to other potentials such as Poisson point potentials, let alone potentials satisfying some mild mixing conditions. Extension to other potentials would require more sophisticated estimates of the graphs than those presented here or a different functional setting than the $L^2(\Omega \times \mathbb{R}^d)$ setting considered here. For related estimates on the graphs appearing in Duhamel expansion, we refer the reader to e.g. [5, 6, 12].

2 Duhamel expansion and existence theory

Since $q(x)$ is a stationary mean zero Gaussian random field, it admits the following spectral representation

$$q(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\xi \cdot x} \hat{q}(\xi) d\xi, \quad (17)$$

where $\hat{q}(\xi)d\xi$ is the complex spectral process such that

$$\mathbb{E} \left\{ \int_{\mathbb{R}^d} f(\xi) \hat{q}(\xi) d\xi \overline{\int_{\mathbb{R}^d} g(\xi) \hat{q}(\xi) d\xi} \right\} = \int_{\mathbb{R}^d} f(\xi) \bar{g}(\xi) (2\pi)^d \hat{R}(\xi) d\xi,$$

for all f and g in $L^2(\mathbb{R}^d; \hat{R}(\xi)d\xi)$ with the power spectrum and correlation function of q respectively defined by

$$0 \leq (2\pi)^d \hat{R}(\xi) = \int_{\mathbb{R}^d} e^{-i\xi \cdot x} R(x) dx, \quad R(x) = \mathbb{E}\{q(y)q(x+y)\}. \quad (18)$$

Note that $\mathbb{E}\{\hat{q}(\xi)\hat{q}(\zeta)\} = \hat{R}(\xi)\delta(\xi + \zeta)$ and $\mathbb{E}\{\hat{q}(\xi)\overline{\hat{q}(\zeta)}\} = \hat{R}(\xi)\delta(\xi - \zeta)$.

2.1 Duhamel expansion

Let us introduce $\hat{q}_\varepsilon(\xi) = \varepsilon^{d-\alpha} \hat{q}(\varepsilon\xi)$, the Fourier transform of $\varepsilon^{-\alpha} q(\frac{x}{\varepsilon})$. We may now recast the parabolic equation (1) as

$$\left(\frac{\partial}{\partial t} + \xi^m\right) \hat{u}_\varepsilon = \hat{q}_\varepsilon * \hat{u}_\varepsilon, \quad (19)$$

with $\hat{u}_\varepsilon(0, \xi) = \hat{u}_0(\xi)$, where

$$\hat{q}_\varepsilon * \hat{u}_\varepsilon(t, \xi) = \int_{\mathbb{R}^d} \hat{u}_\varepsilon(t, \xi - \zeta) \hat{Q}_\varepsilon(d\zeta) \equiv \int_{\mathbb{R}^d} \hat{u}_\varepsilon(t, \xi - \zeta) \hat{q}_\varepsilon(\zeta) d\zeta.$$

Here and below, we use the notation $\xi^m = |\xi|^m$. After integration in time, the above equation becomes

$$\hat{u}_\varepsilon(t, \xi) = e^{-t\xi^m} \hat{u}_0(\xi) + \int_0^t e^{-s\xi^m} \int_{\mathbb{R}^d} \hat{q}_\varepsilon(\xi - \xi_1) \hat{u}_\varepsilon(t-s, \xi_1) d\xi_1 ds. \quad (20)$$

This allows us to write the formal Duhamel expansion

$$\hat{u}_\varepsilon(t, \xi) = \sum_{n \in \mathbb{N}} \hat{u}_{n,\varepsilon}(t, \xi), \quad (21)$$

$$\hat{u}_{n,\varepsilon}(t, \xi) = \int_{\mathbb{R}^{nd}} \prod_{k=0}^{n-1} \int_0^{t_k(\mathbf{s})} e^{-\xi_k^m s_k} e^{-(t - \sum_{k=0}^{n-1} s_k) \xi_n^m} \prod_{k=0}^{n-1} \hat{q}_\varepsilon(\xi_k - \xi_{k+1}) \hat{u}_0(\xi_n) ds d\xi. \quad (22)$$

Here, we have introduced the following notation:

$$\mathbf{s} = (s_0, \dots, s_{n-1}), \quad t_k(\mathbf{s}) = t - s_0 - \dots - s_{k-1}, \quad t_0(\mathbf{s}) = t, \quad ds = \prod_{k=0}^{n-1} ds_k, \quad d\xi = \prod_{k=1}^n d\xi_k.$$

We now show that for sufficiently small times, the expansion (21) converges (uniformly for all ε sufficiently small) in the $L^2(\Omega \times \mathbb{R}^d)$ sense. Moreover, the L^2 norm of $u_\varepsilon(t)$ is bounded by the $L^2(\mathbb{R}^d)$ norm of \hat{u}_0 , which gives us an a priori estimate for the solution. The convergence results are based on the analysis of the following moments

$$U_\varepsilon^{n,m}(t, \xi, \zeta) = \mathbb{E}\{\hat{u}_{\varepsilon,n}(t, \xi) \overline{\hat{u}_{\varepsilon,m}(t, \zeta)}\}, \quad (23)$$

which, thanks to (22), are given by

$$\int_{\mathbb{R}^{d(n+m)}} \prod_{k=0}^{n-1} \int_0^{t_k(\mathbf{s})} \prod_{l=0}^{m-1} \int_0^{t_l(\boldsymbol{\tau})} e^{-s_k \xi_k^m} e^{-(t - \sum_{k=0}^{n-1} s_k) \xi_n^m} e^{-\tau_l \zeta_l^m} e^{-(t - \sum_{l=0}^{m-1} \tau_l) \zeta_m^m} \\ \mathbb{E}\left\{ \prod_{k=0}^{n-1} \prod_{l=0}^{m-1} \hat{q}_\varepsilon(\xi_k - \xi_{k+1}) \bar{\hat{q}}_\varepsilon(\zeta_l - \zeta_{l+1}) \right\} \hat{u}_0(\xi_n) \bar{\hat{u}}_0(\zeta_m) ds d\boldsymbol{\tau} d\xi d\boldsymbol{\zeta}.$$

Let us introduce the notation $s_n(\mathbf{s}) = t_n(\mathbf{s}) = t - \sum_{k=0}^{n-1} s_k$ and $\tau_m(\boldsymbol{\tau}) = t_m(\boldsymbol{\tau}) = t - \sum_{l=0}^{m-1} \tau_l$. We also define $\xi_{n+k+1} = \zeta_{m-k}$ and $s_{n+k+1} = \tau_{m-k}$ for $0 \leq k \leq m$. Since q_ε is real-valued, we find that

$$U_\varepsilon^{n,m}(t, \xi_0, \xi_{n+m+1}) = \int \prod_{k=0}^{n+m+1} e^{-s_k \xi_k^m} \mathbb{E}\left\{ \prod_{k=0, k \neq n}^{n+m} \hat{q}_\varepsilon(\xi_k - \xi_{k+1}) \right\} \hat{u}_0(\xi_n) \bar{\hat{u}}_0(\xi_{n+1}) ds d\xi,$$

where the domain of integration in the s and ξ variables is inherited from the previous expression. Note that no integration is performed in the variables $s_n(\mathbf{s})$ and $s_{n+1}(\boldsymbol{\tau})$. The integral may be recast as

$$\int \prod_{k=0}^{n+m+1} e^{-s_k \xi_k^2} \mathbb{E}\left\{ \prod_{k=0, k \neq n}^{n+m} \hat{q}_\varepsilon(\xi_k - \xi_{k+1}) \right\} \hat{u}_0(\xi_n) \bar{\hat{u}}_0(\xi_{n+1}) \delta(t - \sum_{k=0}^n s_k) \delta(t - \sum_{k=n+1}^{n+m+1} s_k) ds d\xi,$$

where the integrals in all the s_k variables for $0 \leq k \leq n+m+1$ are performed over $(0, \infty)$. The δ functions ensure that the integration is equivalent to the one presented above. The latter form is used in the proof of lemma 2.3 below.

We need to introduce additional notation. The moments of $\hat{u}_{\varepsilon,n}$ are defined as

$$U_\varepsilon^n(t, \xi) = \mathbb{E}\{\hat{u}_{\varepsilon,n}(t, \xi)\}. \quad (24)$$

We also introduce the following covariance function

$$V_\varepsilon^{n,m}(t, \xi, \zeta) = \text{cov}(\hat{u}_{\varepsilon,n}(t, \xi), \hat{u}_{\varepsilon,m}(t, \zeta)) = U_\varepsilon^{n,m}(t, \xi, \zeta) - U_\varepsilon^n(t, \xi) \overline{U_\varepsilon^m(t, \zeta)}. \quad (25)$$

These terms allow us to analyze the convergence properties of the solution $\hat{u}_\varepsilon(t, \xi)$. Let $\hat{M}(\xi)$ be a smooth (integrable and square integrable is sufficient) test function on \mathbb{R}^d . We introduce the two random variables

$$I_\varepsilon(t) = \int_{\mathbb{R}^d} |\hat{u}_\varepsilon(t, \xi)|^2 d\xi \quad (26)$$

$$X_\varepsilon(t) = \int_{\mathbb{R}^d} \hat{u}_\varepsilon(t, \xi) \overline{\hat{M}(\xi)} d\xi. \quad (27)$$

2.2 Summation over graphs

We now need to estimate moments of the Gaussian process \hat{q}_ε . The expectation in $U_\varepsilon^{n,m}$ vanishes unless there is $\bar{n} \in \mathbb{N}$ such that $n + m = 2\bar{n}$ is even. The expectation of a product of Gaussian variables has an explicit structure written as a sum over all possible products of pairs of indices of the form $\xi_k - \xi_{k+1}$. The moments are thus given as a sum of products of the expectation of *pairs* of terms $\hat{q}_\varepsilon(\xi_k - \xi_{k+1})$, where the sum runs over all possible pairings. We define the **pair** (ξ_k, ξ_l) , $1 \leq k < l$, as the contribution in the product given by

$$\mathbb{E}\{\hat{q}_\varepsilon(\xi_{k-1} - \xi_k)\hat{q}_\varepsilon(\xi_{l-1} - \xi_l)\} = \varepsilon^{d-2\alpha} \hat{R}(\varepsilon(\xi_k - \xi_{k-1}))\delta(\xi_k - \xi_{k-1} + \xi_l - \xi_{l-1}).$$

We have used here the fact that $\hat{R}(-\xi) = \hat{R}(\xi)$.

The number of pairings in a product of $n + m = 2\bar{n}$ terms (i.e., the number of allocations of the set $\{1, \dots, 2\bar{n}\}$ into \bar{n} unordered pairs) is equal to

$$\frac{(2\bar{n} - 1)!}{2^{\bar{n}-1}(\bar{n} - 1)!} = \frac{(2\bar{n})!}{\bar{n}!2^{\bar{n}}} = (2\bar{n} - 1)!!.$$

There is consequently a very large number of terms appearing in $U_\varepsilon^{n,m}(t, \xi_0, \xi_{n+m+1})$. In each instance of the pairings, we have \bar{n} terms k and \bar{n} terms $l \equiv l(k)$. Note that $l(k) \geq k + 1$. We denote by **simple pairs** the pairs such that $l(k) = k + 1$, which thus involve a delta function of the form $\delta(\xi_{k+1} - \xi_{k-1})$.

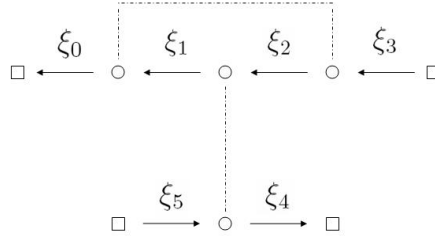


Figure 1: Graph with $n = 3$ and $m = 1$ corresponding to the pairs (ξ_1, ξ_3) and (ξ_2, ξ_5) and the delta functions $\delta(\xi_1 - \xi_0 + \xi_3 - \xi_2)$ and $\delta(\xi_2 - \xi_1 + \xi_5 - \xi_4)$.

The collection of pairs $(\xi_k, \xi_{l(k)})$ for \bar{n} values of k and \bar{n} values of $l(k)$ constitutes a graph $\mathbf{g} \in \mathfrak{G}$ constructed as follows; see Fig.1 and [6]. The upper part of the graph with n bullets represents $\hat{u}_{\varepsilon,n}$ while the lower part with m bullets represents $\hat{u}_{\varepsilon,m}$. The two squares on the left of the graph represent the variables ξ_0 and ξ_{n+m+1} in $U_\varepsilon^{n,m}(t, \xi_0, \xi_{n+m+1})$ while the squares on the right represent $\hat{u}_0(\xi_n)$ and $\hat{u}_0(\xi_{n+1})$. The dotted pairing lines represent the pairs of the graph \mathbf{g} . Here, \mathfrak{G} denotes the collection of all possible $|\mathfrak{G}| = \frac{(2\bar{n}-1)!}{2^{\bar{n}-1}(\bar{n}-1)!}$ graphs that can be constructed for a given \bar{n} .

We denote by $A_0 = A_0(\mathbf{g})$ the collection of the \bar{n} values of k and by $B_0 = B_0(\mathbf{g})$ the collection of the \bar{n} values of $l(k)$. We then find that

$$\mathbb{E}\left\{\prod_{k=1, k \neq n+1}^{n+m+1} \hat{q}_\varepsilon(\xi_{k-1} - \xi_k)\right\} = \sum_{\mathbf{g} \in \mathfrak{G}} \prod_{k \in A_0(\mathbf{g})} \varepsilon^{d-2\alpha} \hat{R}(\varepsilon(\xi_k - \xi_{k-1}))\delta(\xi_k - \xi_{k-1} + \xi_{l(k)} - \xi_{l(k)-1}).$$

This provides us with an explicit expression for $U_\varepsilon^{n,m}(t, \xi_0, \xi_{n+m+1})$ as a summation over all possible graphs generated by moments of Gaussian random variables. We need to introduce several classes of graphs.

We say that the graph has a **crossing** if there is a $k \leq n$ such that $l(k) \geq n+2$. We denote by $\mathfrak{G}_c \subset \mathfrak{G}$ the set of graphs with at least one crossing and by $\mathfrak{G}_{nc} = \mathfrak{G} \setminus \mathfrak{G}_c$ the **non-crossing** graphs. We observe that $V_\varepsilon^{n,m}(t, \xi_0, \xi_{n+m+1})$ is the sum over the crossing graphs and that $U_\varepsilon^n(t, \xi_0) \overline{U_\varepsilon^m(t, \xi_{n+m+1})}$ is the sum over the non-crossing graphs in $U_\varepsilon^{n,m}(t, \xi_0, \xi_{n+m+1})$.

The unique graph \mathfrak{g}_s with only simple pairs is called the **simple graph** and we define $\mathfrak{G}_{ns} = \mathfrak{G} \setminus \mathfrak{g}_s$. We denote by \mathfrak{G}_{cs} the **crossing simple graphs** with only simple pairs except for exactly one crossing. The complement of \mathfrak{G}_{cs} in the crossing graphs is denoted by $\mathfrak{G}_{cns} = \mathfrak{G}_c \setminus \mathfrak{G}_{cs}$.

As we shall see, only the simple graph \mathfrak{g}_s contributes an $O(1)$ term in the limit $\varepsilon \rightarrow 0$ and only the graphs in \mathfrak{G}_{cs} contribute to the leading order $O(\varepsilon^{\frac{1}{2}(d-2\alpha)})$ in the fluctuations of \hat{u}_ε .

The graphs are defined similarly in the calculation of $U_\varepsilon^n(t, \xi_0)$ in (24) for $n = 2\bar{n}$ and $m = 0$, except that crossing graphs have no meaning in such a context. A summation over $k \in A_0(\mathfrak{g})$ of all the arguments $\xi_k - \xi_{k-1} + \xi_{l(k)} - \xi_{l(k)-1}$ of the δ functions shows that the last delta function may be replaced without modifying the integral in $U_\varepsilon^n(t, \xi_0)$ by $\delta(\xi_0 - \xi_n)$.

This allows us to summarize the above calculations as follows:

$$U_\varepsilon^{n,m}(t, \xi_0, \xi_{n+m+1}) = \int \prod_{k=0}^{n+m+1} e^{-s_k \xi_k^m} \hat{u}_0(\xi_n) \tilde{u}_0(\xi_{n+1}) \sum_{\mathfrak{g} \in \mathfrak{G}} \prod_{k \in A_0(\mathfrak{g})} \varepsilon^{d-2\alpha} \hat{R}(\varepsilon(\xi_k - \xi_{k-1})) \delta(\xi_k - \xi_{k-1} + \xi_{l(k)} - \xi_{l(k)-1}) ds d\xi. \quad (28)$$

Similarly,

$$U_\varepsilon^n(t, \xi_0) = \hat{u}_0(\xi_0) \int \prod_{k=0}^n e^{-s_k \xi_k^m} \sum_{\mathfrak{g} \in \mathfrak{G}} \prod_{k \in A_0(\mathfrak{g})} \varepsilon^{d-2\alpha} \hat{R}(\varepsilon(\xi_k - \xi_{k-1})) \delta(\xi_k - \xi_{k-1} + \xi_{l(k)} - \xi_{l(k)-1}) ds d\xi. \quad (29)$$

2.3 Roadmap for the proof of the main theorems

We are now ready to sketch the main steps in the proof of theorems 1, 2, and 3. Lemma 2.1 below presents the main result allowing us to estimate each of the terms in the products in (28) and (29). These estimates are followed in section 2.5 by an analysis of the crossing graphs, which are shown to be negligible in the limit $\varepsilon \rightarrow 0$. Non-crossing graphs are also shown in section 2.6 to be negligible in the limit $\varepsilon \rightarrow 0$ except for the simple graph \mathfrak{g} . As a consequence, the random solution is well-approximated by a still ε -dependent deterministic solution; see (63) below. The convergence of the latter deterministic solution to its limit as $\varepsilon \rightarrow 0$ is handled in section 3.1, which concludes the proof of theorem 1. The theory of fluctuations is considered in section 3.2. There, it is proved that the single crossing graphs are the ones that contribute most to the

random fluctuations. These graphs are shown to involve those terms in the expansion (22) that are linear in the potential \hat{q}_ε , i.e., the terms that have a Gaussian distribution; see (76) below. Upon passing to the limit in the latter expression, we obtain theorem 2. Relatively minor modifications in the analysis of the Gaussian fluctuations allow us to obtain theorem 3 in the presence of potentials with long range correlations.

2.4 Preliminary lemmas in the analysis of the graphs

The products appearing in (28) and (29) above are analyzed by means of estimates that we state and prove in the following two lemmas.

Lemma 2.1 *Let us assume that \hat{R} is bounded by a smooth radially symmetric, decreasing function $f(r)$. We also assume that $f(r) \leq \tau_f r^{-n}$ for some $0 \leq n < d - m$ in dimension $d > m$ and $n = 0$ when $d \leq m$. Then we obtain the following estimates. For $d > m$, we have*

$$\int \frac{1}{|\xi_k|^m} \hat{R}(\xi_k - y) d\xi_k \leq \rho_f := c_d \int_0^\infty \frac{1}{|\xi|^m} f(|\xi|) |\xi|^{d-1} d|\xi| \vee \tau_f,$$

uniformly in $y \in \mathbb{R}^d$, where $c_d = |S^{d-1}|$ and $a \vee b = \max(a, b)$. Moreover,

$$\int \frac{1}{|\xi_k|^m} \hat{R}(\xi_k - y) \left(\frac{\varepsilon^m}{|\xi_k - z|^m} \wedge t \right) d\xi_k \leq C \rho_f \begin{cases} \varepsilon^{m-n} & d > 2m - n \\ \varepsilon^{m-n} |\ln \varepsilon| & d = 2m - n \\ \varepsilon^{d-m-n} & m < d < 2m - n, \end{cases}$$

for C a constant independent of y and z . For $d = m$, we define $\rho_f = c_d f(0)$ and have

$$\int \left(\frac{\varepsilon^m}{|\xi_k - z|^m} \wedge t \right)^l \hat{R}(\xi_k - y) d\xi_k \lesssim \rho_f \begin{cases} \varepsilon^m |\ln \varepsilon| & l = 1 \\ \varepsilon^m & l = 2. \end{cases}$$

For $d < m$, we have

$$\int \left(\frac{\varepsilon^m}{|\xi_k - z|^m} \wedge t \right)^l \hat{R}(\xi_k - y) d\xi_k \lesssim \varepsilon^d, \quad l \geq 1.$$

Proof. Once \hat{R} is bounded above by a decreasing, radially symmetric, function $f(r)$, the above integrals are maximal when $y = z = 0$ thanks to lemma 2.2 below since $|\xi|^{-m}$ and $(\varepsilon^m |\xi|^{-m} \wedge t)$ are radially symmetric and decreasing. The first bound is then obvious and defines ρ_f . The second bound is obvious in dimension $d > 2m$ since $|\xi_k|^{-2m}$ is integrable.

All the bounds in the lemma are thus obtained from a bound for

$$\int_0^\infty \left(\frac{\varepsilon^m}{r^m} \wedge t \right)^l r^{d-1} f(r) dr.$$

We obtain that the above integral restricted to $r \in (1, \infty)$ is bounded by a constant times $\varepsilon^{ml} \rho_f$ for $d \geq m$ and by a constant times ε^{ml} for $d < m$. It thus remains to bound the integral on $r \in (0, 1)$, which is equal to

$$\int_0^{\varepsilon t^{-\frac{1}{m}}} t^l r^{d-1} f(r) dr + \int_{\varepsilon t^{-\frac{1}{m}}}^1 \frac{\varepsilon^{lm}}{r^{lm}} r^{d-1} f(r) dr.$$

Replacing $f(r)$ by $\tau_f r^{-n}$, we find that the first integral is bounded by a constant times ε^{d-n} and the second integral by a constant times $\varepsilon^{d-n} \vee \varepsilon^{lm}$ when $d - n - lm \neq 0$ and $\varepsilon^{2m} |\ln \varepsilon|$ when $d = 2m - n$. It remains to divide through by ε^m when $l = 2$ to obtain the desired results. \square

Lemma 2.2 *Let f , g , and h be non negative, bounded, integrable, and radially symmetric functions on \mathbb{R}^d that are decreasing as a function of radius. Then the integral*

$$I_{\zeta, \tau} = \int_{\mathbb{R}^d} f(\xi - \zeta) g(\xi - \tau) h(\xi) d\xi, \quad (30)$$

which is well defined, is maximal at $\zeta = \tau = 0$.

Proof. In a first step, we rotate ζ to align it with τ . The first claim is that the integral cannot increase while doing so. Then we send ζ and τ to 0. The second claim is that the integral again does not increase.

We assume that the functions f , g , and h are smooth and obtain the result in the general case by density. We choose a system of coordinates so that $\tau = |\tau|e_1$, where (e_1, \dots, e_d) is an orthonormal basis of \mathbb{R}^d , and $\zeta = |\zeta|\hat{\theta}$ with $\hat{\theta} = (\cos \theta, \sin \theta, 0, \dots, 0)$. Without loss of generality, we may assume that $\theta \in (0, \pi)$. Then $I_{\zeta, \tau}$ may be recast as I_θ and we find that

$$I_\theta = \int_0^\infty |\xi|^{d-1} h(|\xi|) J_\theta(|\xi|) d|\xi|,$$

where we denote $h(|\xi|) \equiv h(\xi)$ with the same convention for f and g and define

$$J_\theta(|\xi|) = \int_{S^{d-1}} f(|\xi|\psi - \zeta) g(|\xi|\psi - \tau) d\psi.$$

It is sufficient to show that $\partial_\theta J_\theta \leq 0$. We find

$$\partial_\theta J_\theta = \int_{S^{d-1}} -\hat{\theta}^\perp \cdot \nabla f(|\xi|\psi - \zeta) g(|\xi|\psi - \tau) d\psi,$$

with $\hat{\theta}^\perp = (-\sin \theta, \cos \theta, 0, \dots, 0)$. We decompose the sphere as $\psi = (\psi \cdot \hat{\theta}, \tilde{\psi})$ and find, for some positive weight $w(\mu)$ that

$$\partial_\theta J_\theta = \int_{-1}^1 d(\psi \cdot \hat{\theta}) (-f')(|\xi|\psi - \zeta|) w(\psi \cdot \hat{\theta}) \int_{S^{d-2}} (\hat{\theta}^\perp \cdot \tilde{\psi}) g(|\xi|\psi - \tau) d\tilde{\psi}.$$

We now observe that

$$\begin{aligned} & \int_{S^{d-2}} (\hat{\theta}^\perp \cdot \tilde{\psi}) g(|\xi|\psi - \tau) d\tilde{\psi} \\ &= \int_{\hat{\theta}^\perp \cdot \tilde{\psi} > 0} (\hat{\theta}^\perp \cdot \tilde{\psi}) (g(|\xi|(\hat{\theta} \cdot \psi \hat{\theta} + \tilde{\psi}) - \tau) - g(|\xi|(\hat{\theta} \cdot \psi \hat{\theta} - \tilde{\psi}) - \tau)) d\tilde{\psi} \leq 0, \end{aligned}$$

as $|\xi|(\hat{\theta} \cdot \psi \hat{\theta} + \tilde{\psi}) - \tau \leq |\xi|(\hat{\theta} \cdot \psi \hat{\theta} - \tilde{\psi}) - \tau$ by construction. Indeed, we find that $|\xi|(\hat{\theta} \cdot \psi \hat{\theta} \pm \tilde{\psi}) - \tau|^2 - |\xi|^2 - |\tau|^2 + 2|\tau||\xi|\hat{\theta} \cdot \psi \hat{\theta} \cdot \tau = \pm 2|\tau||\xi|\tilde{\psi} \cdot \tau = \pm 2|\tau||\xi|\hat{\theta}^\perp \cdot \tau$ whereas $\hat{\theta}^\perp \cdot \tau = -\sin \theta |\tau| < 0$ by construction. This shows that $|\xi|(\hat{\theta} \cdot \psi \hat{\theta} + \tilde{\psi})$ is closer to τ

than $|\xi|(\hat{\theta} \cdot \psi \hat{\theta} - \tilde{\psi})$ is, and since $g(r)$ is decreasing, that $\partial_{\theta} J_{\theta} \leq 0$. This concludes the proof of the first claim.

If $\beta = 0$ or $\tau = 0$, we set $b = 0$ below. Otherwise, we may assume without loss of generality that $\tau = -b\zeta$ for some $b > 1$. We still define $\zeta = |\zeta|\hat{\theta}$. We now define the integral $I_a = I_{a\zeta, b\zeta}$, $0 \leq a \leq 1$, and compute

$$\partial_a I_a = \int_{\mathbb{R}^d} -\zeta \cdot \nabla f(\xi - a\zeta) g(\xi + b\zeta) h(\xi) d\xi = \int_{\mathbb{R}^d} -\zeta \cdot \nabla f(\xi) g(\xi + (b-a)\zeta) h(\xi + a\zeta) d\xi.$$

Define $l(\xi, \zeta) = g(\xi + (b-a)\zeta) h(\xi + a\zeta)$. Then because f is radially symmetric, we have

$$\partial_a I_a = \int_0^\infty m(|\xi|) |\xi|^{d-1} d|\xi|, \quad m(|\xi|) = -f'(|\xi|) \int_{S^{d-1}} \hat{\theta} \cdot \psi l(|\xi|\psi, \zeta) d\psi.$$

We recast

$$m(|\xi|) = -f'(|\xi|) \int_{\hat{\theta} \cdot \psi > 0} (\hat{\theta} \cdot \psi) (l(|\xi|\psi, \zeta) - l(-|\xi|\psi, \zeta)) d\psi \leq 0,$$

since $|\xi|\psi + \gamma\zeta| \geq | -|\xi|\psi + \gamma\zeta|$ by construction for all $\gamma > 0$ and thus for $\gamma = a$ and $\gamma = b - a$. This shows that $\partial_a I_a \leq 0$ and concludes the proof of the second claim. \square

2.5 Analysis of crossing graphs

We now analyze the influence of the crossing graphs on $I_\varepsilon(t)$ and $X_\varepsilon(t)$ defined in (26) and (27), respectively, for sufficiently small times. We obtain from (25) and (28) that

$$\begin{aligned} V_\varepsilon^{n,m}(t, \xi_0, \xi_{n+m+1}) &= \sum_{\mathfrak{g} \in \mathfrak{G}_c} \int \prod_{k=0}^{n+m+1} e^{-s_k \xi_k^m} \hat{u}_0(\xi_n) \bar{\hat{u}}_0(\xi_{n+1}) \\ &\quad \prod_{k \in A_0(\mathfrak{g})} \varepsilon^{d-2\alpha} \hat{R}(\varepsilon(\xi_k - \xi_{k-1})) \delta(\xi_k - \xi_{k-1} + \xi_{l(k)} - \xi_{l(k)-1}) ds d\xi, \end{aligned} \quad (31)$$

involves the summation over the crossing graphs \mathfrak{G}_c . Let us consider a graph $\mathfrak{g} \in \mathfrak{G}_c$ with M crossing pairs, $M \geq 1$. Crossing pairs are defined by $k \leq n$ and $l(k) \geq n+2$.

Denote by $(\xi_{q_m}, \xi_{l(q_m)})$, $1 \leq m \leq M$ the crossing pairs and define $Q = \max_m \{q_m\}$. By summing the arguments inside the delta functions for all $k \leq n$, we observe that the last of these delta functions may be replaced when $Q < n$ by

$$\delta(\xi_0 - \xi_n + \sum_{m=1}^M \xi_{q_m} - \xi_{q_m-1}) = \delta(\xi_Q - \xi_{Q-1} + \xi_0 - \xi_n + \sum_{m=1}^{M-1} \xi_{q_m} - \xi_{q_m-1}). \quad (32)$$

The above delta function will be used to integrate in the variable ξ_Q when $Q < n$. Similarly, by summing over all pairs with $k \geq n+2$, we obtain that when $l(q) < n+m+1$, the last of these delta functions may be replaced by

$$\delta(\xi_{n+1} - \xi_{n+m+1} + \sum_{m=1}^M \xi_{l(q_m)} - \xi_{l(q_m)-1}).$$

Using the last two independent δ functions, we thus obtain that the product of delta functions in (31) involves in particular the following constraint:

$$\delta(\xi_{n+m+1} - \xi_{n+1} + \xi_n - \xi_0). \quad (33)$$

The analysis of the contributions of the crossing graphs is slightly different for the energy in (26) and for the spatial moments in (27). We start with the energy.

Analysis of the crossing terms in $I_\varepsilon(t)$. We evaluate $|V_\varepsilon^{n,m}(t, \xi_0, \xi_0)|$ in (31) at $\xi_{n+m+1} = \xi_0$ and integrate in the ξ_0 variable over \mathbb{R}^d . Let us define $A' = A_0 \setminus \{Q\}$. For each $k \in A' \cup \{0\}$, we perform the change of variables $\xi_k \rightarrow \frac{\xi_k}{\varepsilon}$. We then define

$$\xi_k^\varepsilon = \begin{cases} \xi_k & k \notin A' \cup \{0\} \\ \frac{\xi_k}{\varepsilon} & k \in A' \cup \{0\}. \end{cases} \quad (34)$$

Note that $\xi_n = \xi_{n+1}$ since $\xi_{n+m+1} = \xi_0$. This allows us to obtain that

$$\begin{aligned} \int_{\mathbb{R}^d} |V_\varepsilon^{n,m}(t, \xi_0, \xi_0)| d\xi_0 &\leq \sum_{\mathfrak{g} \in \mathfrak{G}_c} \int e^{-(s_0 + s_{n+m+1})\varepsilon - m \xi_0^m} \prod_{k=1}^{n+m} e^{-s_k (\xi_k^\varepsilon)^m} |\hat{u}_0(\xi_n)|^2 \\ &\prod_{k \in A'(\mathfrak{g})} \varepsilon^{-2\alpha} \hat{R}(\xi_k - \varepsilon \xi_{k-1}^\varepsilon) \delta\left(\frac{\xi_k}{\varepsilon} - \xi_{k-1}^\varepsilon + \xi_{l(k)} - \xi_{l(k)-1}^\varepsilon\right) \\ &\varepsilon^{-2\alpha} \hat{R}(\xi_0 - \varepsilon \xi_n^\varepsilon + \sum_{m=1}^{M-1} \xi_{q_m} - \varepsilon \xi_{q_m-1}^\varepsilon) \delta(\xi_Q - \xi_{Q-1} + \xi_{l(Q)} - \xi_{l(Q)-1}) ds d\boldsymbol{\xi}. \end{aligned} \quad (35)$$

Here $d\boldsymbol{\xi}$ also includes the integration in the variable ξ_0 . The estimates for $V_\varepsilon^{n,m}$ here and in subsequent sections rely on integrating selected time variables. All estimates are performed as the following lemma indicates.

Lemma 2.3 *Let $t > 0$ given and consider an integral of the form*

$$I_{n-1} = \prod_{k=0}^{n-1} \int_0^{t_k(\mathfrak{s})} \left(\prod_{k=0}^{n-1} f_k(s_k) \right) \prod_{k=0}^{n-1} ds_k, \quad (36)$$

where $0 \leq f_k(\mathfrak{s}) \leq 1$ for $0 \leq k \leq n$ and assume that $\int_0^t f_{n-1}(s_{n-1}) ds_{n-1} \leq h \wedge t$. Then

$$I_{n-2} \leq (h \wedge t) I_{n-1}. \quad (37)$$

Moreover, let \mathfrak{s} be a permutation of the indices $0 \leq k \leq n-1$. Define $I_{n-1}^\mathfrak{s}$ as I_{n-1} with f_k replaced by $f_{\mathfrak{s}(k)}$. Then $I_{n-1}^\mathfrak{s} = I_{n-1}$.

Using the above result with the permutation leaving all indices fixed except $\mathfrak{s}(n-1) = K$ and $\mathfrak{s}(K) = n-1$ for some $0 \leq K \leq n-2$ allows us to estimate I_{n-1} by integrating in the K th variable.

Proof. The derivation of (37) is immediate. We also calculate

$$\begin{aligned} I_{n-1} &= \int_{\mathbb{R}_+^{n+1}} \left(\prod_{k=0}^{n-1} f_k(s_k) \right) \delta\left(t - \sum_{k=0}^n s_k\right) \prod_{k=0}^n ds_k \\ &= \int_{\mathbb{R}_+^{n+1}} \left(\prod_{k=0}^{n-1} f_{\mathfrak{s}(k)}(s_{\mathfrak{s}(k)}) \right) \delta\left(t - \sum_{k=0}^n s_{\mathfrak{s}(k)}\right) \prod_{k=0}^n ds_k \\ &= \int_{\mathbb{R}_+^{n+1}} \left(\prod_{k=0}^{n-1} f_{\mathfrak{s}(k)}(s_k) \right) \delta\left(t - \sum_{k=0}^n s_k\right) \prod_{k=0}^n ds_k = I_{n-1}^\mathfrak{s}. \end{aligned}$$

□

Note that $e^{-s_n(\mathbf{s})(\xi_\varepsilon^n)^m}$ and $e^{-s_{n+1}(\mathbf{s})(\xi_\varepsilon^{n+1})^m}$ are bounded by 1. We now estimate the integrals in the variables s_0 , s_{n+m+1} , and s_k for $k \in A'$ in (35). Note that $n+1$ cannot belong to A' and that n does not belong to A' either since either $n = Q$ (last crossing) or $n \in B_0$ is a receiving end of the pairing line $k \rightarrow l(k)$. Each integral is bounded by:

$$\int_0^{\tau \wedge t} e^{-s\varepsilon^{-m}\xi^m} ds \leq \frac{\varepsilon^m}{\xi^m} \wedge t. \quad (38)$$

The remaining exponential terms $e^{-s_k(\xi_k^\varepsilon)^m}$ are bounded by 1. Using lemma 2.3, this allows us to obtain that

$$\begin{aligned} \int_{\mathbb{R}^d} |V_\varepsilon^{n,m}(t, \xi_0, \xi_0)| d\xi_0 &\leq \sum_{\mathfrak{g} \in \mathfrak{G}_c} \left(\int d\tilde{\mathbf{s}} \right) \int |\hat{u}_0(\xi_n)|^2 \\ &\prod_{k \in A'(\mathfrak{g})} \varepsilon^{-2\alpha} \left(\frac{\varepsilon^m}{\xi_k^m} \wedge t \right) \hat{R}(\xi_k - \varepsilon \xi_{k-1}^\varepsilon) \delta\left(\frac{\xi_k}{\varepsilon} - \xi_{k-1}^\varepsilon + \xi_{l(k)} - \xi_{l(k)-1}^\varepsilon\right) \\ &\varepsilon^{-2\alpha} \left(\frac{\varepsilon^m}{\xi_0^m} \wedge t \right)^2 \hat{R}(\xi_0 - \varepsilon \xi_n^\varepsilon + \sum_{m=1}^{M-1} (\xi_{q_m} - \varepsilon \xi_{q_m-1}^\varepsilon)) \delta(\xi_Q - \xi_{Q-1} + \xi_{l(Q)} - \xi_{l(Q)-1}) d\boldsymbol{\xi}. \end{aligned}$$

Here, $d\tilde{\mathbf{s}}$ corresponds to the integration in the remaining time variables s_k for $k \notin A' \cup \{0\} \cup \{n+m+1\}$. There are $2\bar{n} - (\bar{n} + 1) = \bar{n} - 1$ such variables. Note the square on the last line, which comes from time integration in both variables s_0 and s_{n+m+1} .

We now estimate the above product. When $Q < n$, let us define k_0 such that $n = l(k_0)$. Define k_1 such that $l(k_1) = n+m+1$. Assume first that $Q < n$ and $l(Q) < n+m+1$. For each $k \in A'(\mathfrak{g}) \setminus (k_0 \cup k_1)$, we use lemma 2.1 to find the estimate

$$\int \varepsilon^{-2\alpha} \left(\frac{\varepsilon^m}{\xi_k^m} \wedge t \right) \hat{R}(\xi_k - \varepsilon \xi_{k-1}^\varepsilon) \delta\left(\frac{\xi_k}{\varepsilon} - \xi_{k-1}^\varepsilon + \xi_{l(k)} - \xi_{l(k)-1}^\varepsilon\right) d\xi_k d\xi_{l(k)} \leq \rho_f. \quad (39)$$

The integration in the $\xi_{l(Q)}$ variable is estimated by using the above delta function. The delta function for $k = k_0 \in A'(\mathfrak{g})$ may be written thanks to (32) in the form $\delta(\xi_Q - \mu)$, where μ is a linear combination of the variables ξ_j with $j \neq Q$ and is thus used to integrate in the variable ξ_Q . The term $\hat{R}(\xi_{k_0} - \varepsilon \xi_{k_0-1}^\varepsilon)$ is used to integrate in the variable ξ_{k_0} . The integral in ξ_0 is estimated using lemma 2.1 by

$$\int \varepsilon^{-2\alpha} \left(\frac{\varepsilon^m}{\xi_0^m} \wedge t \right)^2 \hat{R}(\xi_0 - \varepsilon \xi_n^\varepsilon + \sum_{m=1}^{M-1} (\xi_{q_m} - \varepsilon \xi_{q_m-1}^\varepsilon)) d\xi_0 \leq C \rho_f \varepsilon^\beta.$$

The delta function $\delta(\xi_{k_1} - \xi_{k_1-1} - \xi_{n+m+1} - \xi_{n+m})$ is seen to be equivalent to $\delta(\xi_n - \xi_{n+1})$ thanks to (33), which handles the integration in the variable ξ_{n+1} . When $Q = n$ or $l(Q) = n+m+1$, then a simplified version of what we just described provides the same estimates.

It remains to use the initial conditions to integrate in the last variable ξ_n and obtain the bound

$$\int_{\mathbb{R}^d} |V_\varepsilon^{n,m}(t, \xi_0, \xi_0)| d\xi_0 \leq C \sum_{\mathfrak{g} \in \mathfrak{G}_c} \left(\int d\tilde{\mathbf{s}} \right) \rho_f^{\bar{n}-1} \|\hat{u}_0\|^2 \rho_f \varepsilon^\beta = C \sum_{\mathfrak{g} \in \mathfrak{G}_c} \left(\int d\tilde{\mathbf{s}} \right) \rho_f^{\bar{n}} \varepsilon^\beta \|\hat{u}_0\|^2. \quad (40)$$

Using Stirling's formula, we find that $|\mathfrak{G}_c| < \frac{(2\bar{n}-1)!}{2^{\bar{n}-1}(\bar{n}-1)!}$ is bounded by $(\frac{2\bar{n}}{e})^{\bar{n}}$. It remains to evaluate the integrals in time. We verify that

$$\prod_{k=0}^{n-1} \int_0^{t_k(\mathbf{s})} ds_0 \cdots ds_{n-1} = \frac{t^n}{n!}, \quad t_k(\mathbf{s}) = t - s_0 - \dots - s_{k-1}. \quad (41)$$

Let $\bar{p} = \bar{p}(\mathbf{g})$ be the number of s_k for $k \leq n$ in $\tilde{\mathbf{s}}$ and $\bar{q} = \bar{q}(\mathbf{g})$ be the number of s_k for $k \geq n+1$ in $\tilde{\mathbf{s}}$, with $\bar{p} + \bar{q} = \bar{n} - 1$. Using (41), we thus find that

$$\left(\int d\tilde{\mathbf{s}} \right) = \frac{t^{\bar{p}} t^{\bar{q}}}{\bar{p}! \bar{q}!} = \frac{t^{\bar{n}-1}}{(\bar{n}-1)!} \binom{\bar{n}-1}{\bar{p}} \leq t^{\bar{n}-1} \left(\frac{\bar{n}-1}{2e} \right)^{-\bar{n}+1} \leq t^{\bar{n}-1} \bar{n} \left(\frac{\bar{n}}{2e} \right)^{-\bar{n}}$$

using Stirling's formula. This shows that

$$\rho_f^{\bar{n}} \sum_{\mathbf{g} \in \mathfrak{G}_c} \left(\int d\tilde{\mathbf{s}} \right) \leq \frac{\bar{n}}{T} (4\rho_f T)^{\bar{n}}, \quad (42)$$

uniformly for $t \in (0, T)$. We thus need to choose T sufficiently small so that $4\rho_f T < 1$. Then, for \mathfrak{r} such that $4\rho_f T < \mathfrak{r}^2 < 1$, we find that

$$\int |V_\varepsilon^{n,m}(t, \xi_0, \xi_0)| d\xi_0 \leq C \mathfrak{r}^{n+m} \varepsilon^\beta \|\hat{u}_0\|^2, \quad (43)$$

for some positive constant C . It remains to sum over n and m to obtain that

$$|\mathbb{E}\{I_\varepsilon(t)\} - \int_{\mathbb{R}^d} \mathbb{E}\{\hat{u}_\varepsilon(t, \xi)\}^2 d\xi| \leq \frac{C}{(1-\mathfrak{r})^2} \varepsilon^\beta \|\hat{u}_0\|^2. \quad (44)$$

We shall analyze the non-crossing terms generating $|\mathbb{E}\{\hat{u}_\varepsilon(t, \xi)\}^2$ shortly. Before doing so, we analyze the influence of the crossing terms on X_ε . We can verify that the error term ε^β in (44) is optimal, for instance by looking at the contribution of the graph with $n = m = 1$.

Analysis of the crossing terms in X_ε . It turns out that the contribution of the crossing terms is smaller for the moment X_ε than it is for the energy I_ε . More precisely, we show that the smallest contribution to the variance of X_ε is of order $\varepsilon^{d-2\alpha}$ for graphs in \mathfrak{G}_{cs} and of order $\varepsilon^{d-2\alpha+\beta}$ for the other crossing graphs.

We come back to (31) and this time perform the change of variables $\xi_k \rightarrow \frac{\xi_k}{\varepsilon}$ for $k \in A'$ only. We re-define

$$\xi_k^\varepsilon = \begin{cases} \xi_k & k \notin A' \\ \frac{\xi_k}{\varepsilon} & k \in A', \end{cases} \quad (45)$$

and find that

$$\begin{aligned} V_\varepsilon^{n,m}(t, \xi_0, \xi_{n+m+1}) &= \sum_{\mathbf{g} \in \mathfrak{G}_c} \int \prod_{k=0}^{n+m+1} e^{-s_k(\xi_k^\varepsilon)^m} \hat{u}_0(\xi_n) \bar{\hat{u}}_0(\xi_{n+1}) \\ &\quad \prod_{k \in A'(\mathbf{g})} \varepsilon^{-2\alpha} \hat{R}(\xi_k - \varepsilon \xi_{k-1}^\varepsilon) \delta\left(\frac{\xi_k}{\varepsilon} - \xi_{k-1}^\varepsilon + \xi_{l(k)} - \xi_{l(k)-1}^\varepsilon\right) \\ &\quad \varepsilon^{d-2\alpha} \hat{R}(\varepsilon(\xi_Q - \xi_{Q-1}^\varepsilon)) \delta(\xi_Q - \xi_{Q-1} + \xi_{l(Q)} - \xi_{l(Q)-1}) ds d\xi. \end{aligned} \quad (46)$$

Note that neither n nor $n + m + 1$ belong to $A'(\mathfrak{g})$. For each $k \in A'(\mathfrak{g})$, we integrate in s_k and obtain using (38) that

$$\begin{aligned} |V_\varepsilon^{n,m}(t, \xi_0, \xi_{n+m+1})| &\leq \sum_{\mathfrak{g} \in \mathfrak{G}_c} \int \prod_{k \notin A'(\mathfrak{g})} e^{-s_k \xi_k^m} |\hat{u}_0(\xi_n) \bar{\hat{u}}_0(\xi_{n+1})| \\ &\prod_{k \in A'(\mathfrak{g})} \varepsilon^{-2\alpha} \left(\frac{\varepsilon^m}{\xi_k^m} \wedge t \right) \hat{R}(\xi_k - \varepsilon \xi_{k-1}^\varepsilon) \delta\left(\frac{\xi_k}{\varepsilon} - \xi_{k-1}^\varepsilon + \xi_{l(k)} - \xi_{l(k)-1}^\varepsilon\right) \\ &\varepsilon^{d-2\alpha} \hat{R}(\varepsilon(\xi_Q - \xi_{Q-1}^\varepsilon)) \delta(\xi_Q - \xi_{Q-1} + \xi_{l(Q)} - \xi_{l(Q)-1}) d\tilde{s} d\xi. \end{aligned} \quad (47)$$

By assumption on $\hat{R}(\xi)$, we know the existence of a constant \hat{R}_∞ such that

$$\varepsilon^{d-2\alpha} \hat{R}(\varepsilon(\xi_Q - \xi_{Q-1}^\varepsilon)) \leq \varepsilon^{d-2\alpha} \hat{R}_\infty. \quad (48)$$

This is where the factor $\varepsilon^{d-2\alpha}$ arises. We need however to ensure that the integral in ξ_Q is well-defined. We have two possible scenarios: either $Q = n$ or $n \in B_0$. When $Q = n$, the integration in ξ_Q is an integration in ξ_n for which we use $\hat{u}_0(\xi_n)$. When $n \in B_0$, we thus have $n = l(k_0)$ for some k_0 and we replace the delta function involving ξ_n by a delta function involving ξ_Q given by (32). In either scenario, we can integrate in the variable ξ_Q without using the term $\hat{R}(\varepsilon(\xi_Q - \xi_{Q-1}))$.

All the other variables are handled as in the analysis preceding (40) except that we non longer have that $\xi_n = \xi_{n+1}$. Rather, we use the inequality

$$|\hat{u}_0(\xi_n) \bar{\hat{u}}_0(\xi_{n+1})| \leq \frac{1}{2} \left(|\hat{u}_0(\xi_n)|^2 + |\hat{u}_0(\xi_n - \xi_0 + \xi_{n+m+1})|^2 \right), \quad (49)$$

to obtain the bound

$$|V_\varepsilon^{n,m}(t, \xi_0, \xi_{n+m+1})| \leq \varepsilon^{d-2\alpha} \hat{R}_\infty \sum_{\mathfrak{g} \in \mathfrak{G}_c} \left(\int d\tilde{s} \right) \rho_f^{\bar{n}-1} \|\hat{u}_0\|^2. \quad (50)$$

The bound is uniform in ξ_0 and ξ_{n+m+1} . Using (42) and (43), we obtain

$$|V_\varepsilon^{n,m}(t, \xi_0, \xi_{n+m+1})| \leq \varepsilon^{d-2\alpha} \mathfrak{r}^{n+m} \|\hat{u}_0\|^2. \quad (51)$$

After summation in $n, m \in \mathbb{N}$, we thus find that

$$\mathbb{E}\{(X_\varepsilon - \mathbb{E}\{X_\varepsilon\})^2\} \leq \frac{C}{(1 - \mathfrak{r})^2} \varepsilon^{d-2\alpha} \|\hat{u}_0\|^2 \|\hat{M}\|_1^2. \quad (52)$$

Similarly, by setting $\xi_{n+m+1} = \xi_0$, we find that

$$\left| \mathbb{E}\left\{ \int_{\mathbb{R}^d} |\hat{u}_\varepsilon|^2(t, \xi) \varphi(\xi) d\xi \right\} - \int_{\mathbb{R}^d} |\mathbb{E}\{\hat{u}_\varepsilon(t, \xi)\}|^2 \varphi(\xi) d\xi \right| \leq \frac{C}{(1 - \mathfrak{r})^2} \varepsilon^{d-2\alpha} \|\hat{u}_0\|^2 \|\varphi\|_1, \quad (53)$$

for any test function $\varphi \in L^1(\mathbb{R}^d)$. This local energy estimate is to be compared with the global estimate obtained in (44).

Analysis of the leading crossing terms in X_ε . The preceding estimate on X_ε may be refined as only the crossing graphs in \mathfrak{G}_{cs} have contributions of order $\varepsilon^{d-2\alpha}$. We return to the bound (47) and obtain that

$$\begin{aligned}
|V_\varepsilon^{n,m}(t, \xi_0, \xi_{n+m+1})| &\leq \varepsilon^{d-2\alpha} \hat{R}_\infty \sum_{\mathfrak{g} \in \mathfrak{G}_c} \int \prod_{k \notin A'(\mathfrak{g})} e^{-s_k(\xi_k)^m} |\hat{u}_0(\xi_n) \bar{\hat{u}}_0(\xi_{n+1})| \\
&\prod_{k \in A'(\mathfrak{g})} \varepsilon^{-2\alpha} \left(\frac{\varepsilon^m}{\xi_k^m} \wedge t \right) \hat{R}(\xi_k - \varepsilon \xi_{k-1}^\varepsilon) \delta\left(\frac{\xi_k}{\varepsilon} - \xi_{k-1}^\varepsilon + \xi_{l(k)} - \xi_{l(k)-1}^\varepsilon\right) \\
&\delta(\xi_Q - \xi_{Q-1} + \xi_{l(Q)} - \xi_{l(Q)-1}) d\tilde{s} d\xi.
\end{aligned} \tag{54}$$

The $\bar{n} + 2$ variables in time left are $s_0, s_Q, s_{l(Q)}$, and the $\bar{n} - 1$ variables $s_{l(A'(\mathfrak{g}))}$.

Let $\mathfrak{g} \in \mathfrak{G}_c$. Let us assume that for some $k = \kappa$ such that $(\xi_k, \xi_{l(k)})$ is not a crossing pair, we have $l(k) - 1 > k$, i.e., $\mathfrak{g} \in \mathfrak{G}_{ncs}$. The non-crossing pairs are not affected by the possible change of a delta function involving ξ_n to a delta function involving ξ_Q . We may then integrate in the variable $s_{l(k)}$ and obtain the bound for the integral

$$\begin{aligned}
&\varepsilon^{d-2\alpha} \hat{R}_\infty \int d\tilde{s} d\xi |\hat{u}_0(\xi_n) \bar{\hat{u}}_0(\xi_{n+1})| \delta(\xi_Q - \xi_{Q-1} + \xi_{l(Q)} - \xi_{l(Q)-1}) \\
&\times \prod_{k \in A'(\mathfrak{g}) \setminus \kappa} \varepsilon^{-2\alpha} \left(\frac{\varepsilon^m}{\xi_k^m} \wedge t \right) \hat{R}(\xi_k - \varepsilon \xi_{k-1}^\varepsilon) \delta\left(\frac{\xi_k}{\varepsilon} - \xi_{k-1}^\varepsilon + \xi_{l(k)} - \xi_{l(k)-1}^\varepsilon\right) \\
&\times \left(\frac{\varepsilon^m}{\xi_\kappa^m} \wedge t \right) \left(\frac{\varepsilon^m}{|\xi_\kappa - \varepsilon \xi_{\kappa-1}^\varepsilon - \varepsilon \xi_{l(\kappa)-1}^\varepsilon|^m} \wedge t \right) \hat{R}(\xi_\kappa - \varepsilon \xi_{\kappa-1}^\varepsilon) \delta\left(\frac{\xi_\kappa}{\varepsilon} - \xi_{\kappa-1}^\varepsilon + \xi_{l(\kappa)} - \xi_{l(\kappa)-1}^\varepsilon\right) \\
&\leq C \varepsilon^\beta \varepsilon^{d-2\alpha} \left(\int d\tilde{s} \right) \hat{R}_\infty \rho_f^{\bar{n}-1} \|\hat{u}_0\|^2,
\end{aligned}$$

thanks to lemma 2.1. The summation over all graphs in \mathfrak{G}_{ncs} of any quantity derived from $V_\varepsilon^{n,m}(t, \xi_0, \xi_{n+m+1})$ is therefore ε^β smaller than the corresponding sum over all graphs in \mathfrak{G}_c . We thus see that any non-crossing pair has to be of the form $l(k) - 1 = k$, i.e., a simple pair, in order for the graph to correspond to a contribution of order $\varepsilon^{d-2\alpha}$.

Let us consider the graphs composed of crossings and simple pairs. We may delete the simple pairs from the graph since they contribute integrals of order $O(1)$ thanks to lemma 2.1 and assume that the graph is composed of crossings only, thus with $n = m$ and $Q = n$ after deletion of the simple pairs. Let us consider $k < n$ with $l(k) \geq n + 1$ so that the delta function

$$\delta\left(\frac{\xi_k}{\varepsilon} - \xi_{k-1}^\varepsilon + \xi_{l(k)} - \xi_{l(k)-1}^\varepsilon\right)$$

is present in the integral defining $V_\varepsilon^{n,m}$. We find for the same reason as above that the contribution of the corresponding graph is of order $\varepsilon^{d-2\alpha} \varepsilon^\beta$ by integration in the variable $s_{l(k)}$. As a consequence, the only graph composed exclusively of crossing pairs that generates a contribution of order $\varepsilon^{d-2\alpha}$ is the graph with $n = m = 1$. This concludes our proof that the contribution of order $\varepsilon^{d-2\alpha}$ in $V_\varepsilon^{n,m}$ is given by the nm graphs in \mathfrak{G}_{cs} when both n and m are odd numbers (otherwise, \mathfrak{G}_{cs} is empty). All other graphs in \mathfrak{G}_c provide a contribution of order ε^β smaller than what we obtained in (51). In other

words, let us define

$$V_{\varepsilon,s}^{n,m}(t, \xi_0, \xi_{n+m+1}) = \sum_{\mathfrak{g} \in \mathfrak{G}_{cs}} \int \prod_{k=0}^{n+m+1} e^{-s_k \xi_k^m} \hat{u}_0(\xi_n) \tilde{u}_0(\xi_{n+1}) \prod_{k \in A_0(\mathfrak{g})} \varepsilon^{d-2\alpha} \hat{R}(\varepsilon(\xi_k - \xi_{k-1})) \delta(\xi_k - \xi_{k-1} + \xi_{l(k)} - \xi_{l(k)-1}) ds d\boldsymbol{\xi}. \quad (55)$$

We have found that

$$|V_{\varepsilon}^{n,m}(t, \xi_0, \xi_{n+m+1}) - V_{\varepsilon,s}^{n,m}(t, \xi_0, \xi_{n+m+1})| \lesssim \varepsilon^{d-2\alpha+\beta} \mathbf{r}^{n+m} \|\hat{u}_0\|^2. \quad (56)$$

2.6 Analysis of non-crossing graphs

We now apply the estimates obtained in the preceding section to the analysis of the moments $U_{\varepsilon}^n(t)$ defined in (24) and given more explicitly in (29). Our objective is to show that only the simple graph \mathfrak{g} contributes a term of order $O(1)$ in (29) whereas all other graphs in \mathfrak{G}_{ns} contribute (summable in n) terms of order $O(\varepsilon^\beta)$. Note that $n = 2\bar{n}$, for otherwise, $U_{\varepsilon}^n(t) = 0$. We recall that the simple graph is defined by $l(k) = k + 1$. We thus define the simple graph contribution as

$$\begin{aligned} U_{\varepsilon,s}^n(t, \xi_0) &= \mathcal{U}_{\varepsilon}^n(t, \xi_0) \hat{u}_0(\xi_0) \\ \mathcal{U}_{\varepsilon}^n(t, \xi_0) &= \int \prod_{k=0}^n e^{-s_k \xi_k^m} \prod_{k=0}^{\bar{n}-1} \varepsilon^{d-2\alpha} \hat{R}(\varepsilon(\xi_{2k+1} - \xi_{2k})) \delta(\xi_{2(k+1)} - \xi_{2k}) ds d\boldsymbol{\xi}, \end{aligned} \quad (57)$$

and

$$U_{\varepsilon,s}(t, \xi_0) = \sum_{n \in \mathbb{N}} U_{\varepsilon,s}^n(t, \xi_0) := \mathcal{U}_{\varepsilon}(t, \xi_0) \hat{u}_0(\xi_0). \quad (58)$$

For all $k \in A_0$, we perform the change of variables $\xi_k \rightarrow \frac{\xi_k}{\varepsilon}$ and (re-)define as before

$$\xi_k^{\varepsilon} = \begin{cases} \xi_k & k \notin A_0 \\ \frac{\xi_k}{\varepsilon} & k \in A_0. \end{cases} \quad (59)$$

This gives

$$\begin{aligned} U_{\varepsilon}^n(t, \xi_0) &= \hat{u}_0(\xi_0) \sum_{\mathfrak{g} \in \mathfrak{G}} \int \prod_{k=0}^n e^{-s_k (\xi_k^{\varepsilon})^m} \\ &\quad \prod_{k \in A_0(\mathfrak{g})} \varepsilon^{-2\alpha} \hat{R}(\xi_k - \varepsilon \xi_{k-1}^{\varepsilon}) \delta\left(\frac{\xi_k}{\varepsilon} - \xi_{k-1}^{\varepsilon} + \xi_{l(k)} - \xi_{l(k)-1}^{\varepsilon}\right) ds d\boldsymbol{\xi}. \end{aligned} \quad (60)$$

Assuming that $l(k) - 1 > k$ for one of the pairings $k = \kappa$, we obtain as in the analysis leading to (56) the following bound for the corresponding graph:

$$\begin{aligned} &|\hat{u}_0(\xi_0)| \int \prod_{k \in A_0(\mathfrak{g}) \setminus \kappa} \varepsilon^{-2\alpha} \left(\frac{\varepsilon^m}{\xi_k^m} \wedge t\right) \hat{R}(\xi_k - \varepsilon \xi_{k-1}^{\varepsilon}) \delta\left(\frac{\xi_k}{\varepsilon} - \xi_{k-1}^{\varepsilon} + \xi_{l(k)} - \xi_{l(k)-1}^{\varepsilon}\right) \\ &\quad \times \varepsilon^{-2\alpha} \left(\frac{\varepsilon^m}{\xi_k^m} \wedge t\right) \left(\frac{\varepsilon^m}{|\xi_k - \varepsilon \xi_{k-1}^{\varepsilon} - \varepsilon \xi_{l(k)-1}^{\varepsilon}|^m} \wedge t\right) \\ &\quad \times \hat{R}(\xi_{\kappa} - \varepsilon \xi_{\kappa-1}^{\varepsilon}) \delta\left(\frac{\xi_{\kappa}}{\varepsilon} - \xi_{\kappa-1}^{\varepsilon} + \xi_{l(\kappa)} - \xi_{l(\kappa)-1}^{\varepsilon}\right) \quad d\bar{s} d\boldsymbol{\xi} \\ &\leq \varepsilon^{\beta} \left(\int d\bar{s}\right) \rho_f^{\bar{n}} |\hat{u}_0(\xi_0)|. \end{aligned}$$

This shows that

$$|U_\varepsilon^n(t, \xi_0) - U_{\varepsilon, s}^n(t, \xi_0)| \leq |\hat{u}_0(\xi_0)| \varepsilon^\beta \mathbf{r}^n, \quad (61)$$

so that

$$|\mathbb{E}\{\hat{u}_\varepsilon\}(t, \xi) - U_{\varepsilon, s}(t, \xi)| \lesssim \frac{1}{1 - \mathbf{r}} \varepsilon^\beta |\hat{u}_0(\xi)|, \quad (62)$$

at least for sufficiently small times $t \in (0, T)$ such that $4\rho_f T < 1$. It remains to analyze the limit of $U_{\varepsilon, s}(t, \xi)$ to obtain the limiting behavior of X_ε and $I_{\varepsilon, \varphi}$. This analysis is carried out in the next section. Another application of lemma 2.1 shows that $U_{\varepsilon, s}(t, \xi)$ is square integrable and that its $L^2(\mathbb{R}^d)$ norm is bounded by $\|\hat{u}_0\|$. In other words, we have constructed a weak solution $\hat{u}_\varepsilon(t) \in L^2(\Omega \times \mathbb{R}^d)$ to (19) since the series (21) converges uniformly in $L^2(\Omega \times \mathbb{R}^d)$ for sufficiently small times $t \in (0, T)$ such that $4\rho_f T < 1$.

Collecting the results obtained in (44) and (62), we have shown that

$$\|(\hat{u}_\varepsilon - U_{\varepsilon, s})(t)\|_{L^2(\Omega \times \mathbb{R}^d)} \lesssim \varepsilon^{\frac{\beta}{2}} \|\hat{u}_0\|_{L^2(\mathbb{R}^d)}, \quad (63)$$

where $U_{\varepsilon, s}$ is the deterministic term given in (58). The analysis of $U_{\varepsilon, s}$ and X_ε is carried out in the following section.

3 Homogenized limit and Gaussian fluctuations

In this section, we conclude the proof of theorems 1 and 2.

3.1 Homogenization theory for u_ε

We come back to the analysis of $U_{\varepsilon, s}(t, \xi)$ defined in (57). Since only the simple graph is retained in the definition of mean field solution $U_{\varepsilon, s}(t, \xi)$, the equation it satisfies may be obtained from that for \hat{u}_ε by simply assuming the mean field approximation $\mathbb{E}\{\hat{q}_\varepsilon \hat{q}_\varepsilon \hat{u}_\varepsilon\} \sim \mathbb{E}\{\hat{q}_\varepsilon \hat{q}_\varepsilon\} \mathbb{E}\{\hat{u}_\varepsilon\}$ since the Duhamel expansions then agree. As a consequence, we find that $U_{\varepsilon, s}$ is the solution to the following integral equation

$$\begin{aligned} U_{\varepsilon, s}(t, \xi) &= e^{-t\xi^{\mathbf{m}}} \hat{u}_0(\xi) \\ &\quad + \int_0^t e^{-\xi^{\mathbf{m}} s} \int_0^{t-s} e^{-\xi_1^{\mathbf{m}} s_1} \int \varepsilon^{d-2\alpha} \hat{R}(\varepsilon(\xi_1 - \xi)) U_{\varepsilon, s}(t-s-s_1, \xi) d\xi_1 ds ds_1 \\ &= e^{-t\xi^{\mathbf{m}}} \hat{u}_0(\xi) + \int_0^t \int_0^v e^{-\xi^{\mathbf{m}}(v-s_1)} e^{-\xi_1^{\mathbf{m}} s_1} \varepsilon^{d-2\alpha} \int \hat{R}(\varepsilon(\xi_1 - \xi)) U_{\varepsilon, s}(t-v, \xi) d\xi_1 ds_1 dv \\ &= e^{-t\xi^{\mathbf{m}}} \hat{u}_0(\xi) + \varepsilon^{\mathbf{m}-2\alpha} \int_0^t \int_0^{\frac{v}{\varepsilon^{\mathbf{m}}}} e^{-\xi^{\mathbf{m}}(v-\varepsilon^{\mathbf{m}} s_1)} e^{-\xi_1^{\mathbf{m}} s_1} \int \hat{R}(\xi_1 - \varepsilon\xi) d\xi_1 ds_1 U_{\varepsilon, s}(t-v, \xi) dv \\ &:= e^{-t\xi^{\mathbf{m}}} \hat{u}_0(\xi) + A_\varepsilon U_{\varepsilon, s}(t, \xi). \end{aligned} \quad (64)$$

The last integral results from the change of variables $\varepsilon\xi_1 \rightarrow \xi_1$ and $s_1\varepsilon^{-\mathbf{m}} \rightarrow s_1$. It remains to analyze the convergence properties of the solution to the latter integral equation. Note that ξ acts as a parameter in that equation. Let us decompose

$$A_\varepsilon U(t, \xi) = \rho_\varepsilon \int_0^t e^{-\xi^{\mathbf{m}} v} U(t-v, \xi) dv + E_\varepsilon U(t, \xi), \quad (65)$$

with $\rho_\varepsilon = \int_{\mathbb{R}^d} \frac{\hat{R}(\xi_1 - \varepsilon\xi)}{\xi_1^{\mathbf{m}}} d\xi_1$ when $d > \mathbf{m}$ and $\rho_\varepsilon = c_d \hat{R}(\varepsilon\xi)$ when $d = \mathbf{m}$. Then we have

Lemma 3.1 *Let $\xi \in \mathbb{R}^d$ and $f(r)$ as in lemma 2.1. Then the operator E_ε defined above in (65) is bounded in the Banach space of continuous functions on $(0, T)$. Moreover, we have*

$$\|E_\varepsilon\|_{\mathcal{L}(C(0,T))} \lesssim \varepsilon^{\beta-\mathbf{n}}. \quad (66)$$

Proof. We start with the case $d > \mathbf{m}$ so that $\varepsilon^{\mathbf{m}-2\alpha} = 1$. Note that \mathbf{n} in lemma 2.1 is defined such that $d > \mathbf{m} - \mathbf{n}$ as well. With $B_\varepsilon = A_\varepsilon - E_\varepsilon$ in (65), we find that

$$B_\varepsilon U_{\varepsilon,s}(t, \xi) = \int_0^t e^{-\xi^\mathbf{m} v} \int_0^\infty \int e^{-\xi_1^\mathbf{m} s_1} \hat{R}(\xi_1 - \varepsilon \xi) d\xi_1 ds_1 U_{\varepsilon,s}(t-v, \xi) dv.$$

The remainder E_ε is then given by

$$\begin{aligned} E_\varepsilon U_{\varepsilon,s}(t, \xi) &= \int_0^t \int_0^{\frac{v}{\varepsilon^\mathbf{m}}} \int e^{-\xi^\mathbf{m} v} (e^{\varepsilon^\mathbf{m} \xi^\mathbf{m} s_1} - 1) e^{-\xi_1^\mathbf{m} s_1} \hat{R}(\xi_1 - \varepsilon \xi) d\xi_1 ds_1 U_{\varepsilon,s}(t-v, \xi) dv \\ &\quad - \int_0^t \int_{\frac{v}{\varepsilon^\mathbf{m}}}^\infty \int e^{-\xi^\mathbf{m} v} e^{-\xi_1^\mathbf{m} s_1} \hat{R}(\xi_1 - \varepsilon \xi) d\xi_1 ds_1 U_{\varepsilon,s}(t-v, \xi) dv. \end{aligned}$$

The continuity of $E_\varepsilon U_{\varepsilon,s}(t, \xi)$ in time is clear when $U_{\varepsilon,s}(t, \xi)$ is continuous in time. Without loss of generality, we assume that $U_{\varepsilon,s}(\cdot, \xi)$ is bounded by 1 in the uniform norm. We decompose the integral in the s_1 variable in the first term of the definition of E_ε into two integrals on $0 \leq s_1 \leq \frac{v}{2\varepsilon^\mathbf{m}}$ and $\frac{v}{2\varepsilon^\mathbf{m}} \leq s_1 \leq \frac{v}{\varepsilon^\mathbf{m}}$. Because $e^{-\xi^\mathbf{m} v} (e^{\varepsilon^\mathbf{m} \xi^\mathbf{m} s_1} - 1) \leq 1$, the second integral is estimated as

$$\begin{aligned} &\int_0^t \int_{\frac{v}{2\varepsilon^\mathbf{m}}}^{\frac{v}{\varepsilon^\mathbf{m}}} \int e^{-\xi^\mathbf{m} v} (e^{\varepsilon^\mathbf{m} \xi^\mathbf{m} s_1} - 1) e^{-\xi_1^\mathbf{m} s_1} \hat{R}(\xi_1 - \varepsilon \xi) d\xi_1 ds_1 dv \\ &\leq \int_0^t \int \frac{1}{\xi_1^\mathbf{m}} e^{-\xi_1^\mathbf{m} \frac{v}{2\varepsilon^\mathbf{m}}} \hat{R}(\xi_1 - \varepsilon \xi) d\xi_1 dv \leq \int \frac{2}{\xi_1^\mathbf{m}} \left(\frac{\varepsilon^\mathbf{m}}{\xi_1^\mathbf{m}} \wedge t \right) \hat{R}(\xi_1 - \varepsilon \xi) d\xi_1 \lesssim \varepsilon^{\beta-\mathbf{n}} \rho_f, \end{aligned}$$

thanks to lemma 2.1. The above bound is uniform in ξ . The last integral defining E_ε on the interval $s_1 \geq \frac{v}{\varepsilon^\mathbf{m}}$ is treated in the exact same way and also provides a contribution of order $O(\varepsilon^{\beta-\mathbf{n}})$.

The final contribution involves the integration over the interval $0 \leq s_1 \leq \frac{v}{2\varepsilon^\mathbf{m}}$. Using $e^{-\xi^\mathbf{m} v} (e^{\varepsilon^\mathbf{m} \xi^\mathbf{m} s_1} - 1) \leq \varepsilon^\mathbf{m} \xi^\mathbf{m} s_1 e^{-\frac{\xi^\mathbf{m} v}{2}}$ on that interval, it is bounded by

$$\begin{aligned} I_3 &:= \int_0^t \int_0^{\frac{v}{2\varepsilon^\mathbf{m}}} \int_{\mathbb{R}^d} \varepsilon^\mathbf{m} \xi^\mathbf{m} s_1 e^{-\frac{\xi^\mathbf{m} v}{2}} e^{-\xi_1^\mathbf{m} s_1} \hat{R}(\xi_1 - \varepsilon \xi) d\xi_1 ds_1 dv \\ &\leq \frac{2\varepsilon^\mathbf{m} \xi^\mathbf{m}}{\xi^\mathbf{m}} (1 - e^{-\frac{\xi^\mathbf{m} t}{2}}) \int_0^{\frac{t}{2\varepsilon^\mathbf{m}}} s_1 \int_{\mathbb{R}^d} e^{-\xi_1^\mathbf{m} s_1} \hat{R}(\xi_1 - \varepsilon \xi) d\xi_1 ds_1, \end{aligned}$$

by switching the variables $0 \leq s \leq \frac{v}{2\varepsilon^\mathbf{m}} \leq \frac{t}{2\varepsilon^\mathbf{m}}$. Using lemma 2.2, we may replace $\hat{R}(\xi_1 - \varepsilon \xi)$ by $\hat{R}(\xi_1)$ in the above expression. This shows that

$$I_3 \leq 2\varepsilon^\mathbf{m} \int_{\mathbb{R}^d} \int_0^{\frac{t}{2\varepsilon^\mathbf{m}}} s_1 e^{-\xi_1^\mathbf{m} s_1} ds_1 \hat{R}(\xi_1) d\xi_1.$$

We observe that

$$\int_0^\tau s_1 e^{-\xi_1^\mathbf{m} s_1} ds_1 \lesssim \frac{1}{\xi_1^{2\mathbf{m}}} \wedge \tau^2,$$

so that

$$I_3 \lesssim \varepsilon^m \int_0^\infty f(r) r^{d-1} (r^{-2m} \wedge \tau^2) dr, \quad \tau = \frac{t}{2\varepsilon^m} \vee 1.$$

The integral over $(1, \infty)$ is bounded by $\varepsilon^m \rho_f$. Using the assumption that $f(r) \lesssim r^{-n}$, we obtain that the integral over $(0, 1)$ is bounded by a constant times

$$\int_0^{\tau^{-\frac{1}{m}}} r^{d-1-n} dr + \int_{\tau^{-\frac{1}{m}}}^1 r^{d-1-n-2m} dr \lesssim \tau^{2-\frac{d-n}{m}} \vee 1,$$

when $d - n - 2m \neq 0$ and $|\ln \tau|$ when $d = n + 2m$. Since τ is bounded by a constant times ε^{-m} , this shows that I_3 is bounded by ε^{d-m-n} when $d - n - 2m \neq 0$ and $\varepsilon^d |\ln \varepsilon|$ when $d = n + 2m$. This concludes the proof when $d > m - n$.

We now consider the proof when $d = m$ with $n = 0$. Then, $\varepsilon^{m-2\alpha} = \frac{1}{|\ln \varepsilon|}$. The leading term is given by $U_{\varepsilon,s}$, which solves the integral equation:

$$\begin{aligned} U_{\varepsilon,s}(t, \xi) &= e^{-t\xi^m} \hat{u}_0(\xi) \\ &+ \int_0^t e^{-\xi^m s} \int_0^{t-s} e^{-\xi_1^m s_1} \int \frac{1}{|\ln \varepsilon|} \hat{R}(\varepsilon(\xi_1 - \xi)) U_{\varepsilon,s}(t-s-s_1, \xi) d\xi_1 ds ds_1 \\ &= e^{-t\xi^m} \hat{u}_0(\xi) + \frac{1}{|\ln \varepsilon|} \int_0^t \int_0^{\frac{v}{\varepsilon^m}} e^{-\xi^m (v-\varepsilon^m s_1)} e^{-\xi_1^m s_1} \hat{R}(\xi_1 - \varepsilon\xi) d\xi_1 ds_1 U_{\varepsilon,s}(t-v, \xi) dv \\ &= e^{-t\xi^m} \hat{u}_0(\xi) + A_\varepsilon U_{\varepsilon,s}(t, \xi), \quad A_\varepsilon = B_\varepsilon + E_\varepsilon. \end{aligned} \tag{67}$$

Here we have defined

$$B_\varepsilon U(t, \xi) = \rho_\varepsilon \int_0^t e^{-\xi^m s} U(t-s, \xi) ds, \quad \rho_\varepsilon = c_d \hat{R}(\varepsilon\xi),$$

and E_ε is the remainder. As in the case $d > m$, a contribution to $|\ln \varepsilon| E_\varepsilon$ comes from

$$\int_0^t \int_0^{\frac{v}{\varepsilon^m}} \int e^{-\xi^m v} (e^{\varepsilon^m \xi^m s_1} - 1) e^{-\xi_1^m s_1} \hat{R}(\xi_1 - \varepsilon\xi) d\xi_1 ds_1 U_{\varepsilon,s}(t-v, \xi) dv.$$

We again decompose the integral in s_1 into $0 \leq s_1 \leq \frac{v}{2\varepsilon^m}$ and $\frac{v}{2\varepsilon^m} \leq s_1 \leq \frac{v}{\varepsilon^m}$. We have

$$\begin{aligned} &\int_0^t \int_0^{\frac{v}{\varepsilon^m}} \int e^{-\xi^m v} (e^{\varepsilon^m \xi^m s_1} - 1) e^{-\xi_1^m s_1} \hat{R}(\xi_1 - \varepsilon\xi) d\xi_1 ds_1 dv \\ &\leq \int \frac{2}{\varepsilon^m} \left(\frac{\varepsilon^m}{\xi_1^m} \wedge t \right)^2 \hat{R}(\xi_1 - \varepsilon\xi) d\xi_1 \lesssim \rho_f, \end{aligned}$$

according to lemma 2.1. Also,

$$\int_0^t \int_0^{\frac{v}{2\varepsilon^m}} \int e^{-\xi^m v} (e^{\varepsilon^m \xi^m s_1} - 1) e^{-\xi_1^m s_1} \hat{R}(\xi_1 - \varepsilon\xi) d\xi_1 ds_1 dv \lesssim \varepsilon^m \left(\frac{t}{2\varepsilon^m} \vee 1 \right)$$

according to the calculations performed above on I_3 , which is uniformly bounded, and thus provides a $|\ln \varepsilon|^{-1}$ contribution to E_ε .

We are thus left with the analysis of

$$U(t, \xi) \mapsto \int_0^t e^{-\xi^m v} \left(\frac{1}{|\ln \varepsilon|} \int \frac{1 - e^{-\frac{\xi_1^m v}{\varepsilon^m}}}{\xi_1^m} \hat{R}(\xi_1 - \varepsilon\xi) d\xi_1 - \rho_\varepsilon \right) U(t-v, \xi) dv,$$

as an operator in $\mathcal{L}(\mathcal{C}(0, T))$ for ξ fixed. Define $\hat{R}_\varepsilon(\xi_1) = \hat{R}(\xi_1 - \varepsilon\xi)$. The integral in ξ_1 may be recast as

$$\int_0^\infty \frac{1 - e^{-\frac{r^m v}{\varepsilon^m}}}{r} \left(\int_{S^{d-1}} \hat{R}_\varepsilon(r\theta) d\mu(\theta) \right) dr.$$

We observe that the integral on $(1, \infty)$ is bounded by $\|\hat{R}\|_1$. Assuming that \hat{R} is of class $\mathcal{C}^{0,\gamma}(\mathbb{R}^d)$ for $\gamma > 0$, we write $\hat{R}_\varepsilon(\xi_1) = \hat{R}_\varepsilon(0) + (\hat{R}_\varepsilon(\xi_1) - \hat{R}_\varepsilon(0))$. The second contribution generates a term proportional to r^γ in the integral and thus is bounded independent of ε . It remains to estimate

$$c_d \hat{R}_\varepsilon(0) \int_0^1 \frac{1 - e^{-\frac{r^m v}{\varepsilon^m}}}{r} dr = c_d \hat{R}_\varepsilon(0) \int_0^{\frac{1}{\varepsilon}} \frac{1 - e^{-r^m}}{r} dr.$$

The latter integral restricted to $(0, 1)$ is bounded. On $r \geq 1$, e^{-r^m}/r is uniformly integrable so that

$$c_d \hat{R}_\varepsilon(0) \int_0^1 \frac{1 - e^{-\frac{r^m v}{\varepsilon^m}}}{r} dr = c_d \hat{R}(\varepsilon\xi) |\ln \varepsilon| + O(1).$$

This shows that E_ε is of order $\frac{1}{|\ln \varepsilon|} = \varepsilon^\beta$ as an operator on $\mathcal{C}(0, T)$ and concludes the proof of the lemma. \square

Note that A_ε may be written as

$$A_\varepsilon U(t, \xi) = \int_0^t \varphi_\varepsilon(s, \xi) U(t - s\xi) ds,$$

where $\varphi_\varepsilon(s, \xi)$ is uniformly bounded in s, ξ , and ε by a constant φ_∞ . The equation

$$(I - A_\varepsilon)U(t, \xi) = S(t, \xi),$$

admits a unique (by Gronwall's lemma) solution given by the Duhamel expansion and bounded by

$$|U(t, \xi)| \leq \|S\|_\infty e^{t\varphi_\infty}.$$

As in the proof of lemma 3.1, let us define $B_\varepsilon = A_\varepsilon - E_\varepsilon$. We verify that $\mathfrak{U}_\varepsilon(t, \xi)$, the solution to

$$(I - B_\varepsilon)\mathfrak{U}_\varepsilon = e^{-t\xi^m} \hat{u}_0(\xi),$$

is given by

$$\mathfrak{U}_\varepsilon(t, \xi) = e^{-t(\xi^m - \rho_\varepsilon(\xi))} \hat{u}_0(\xi). \quad (68)$$

The solution may thus grow exponentially in time for low frequencies. The error $V_\varepsilon(t, \xi) = (U_{\varepsilon,s}(t, \xi) - \mathfrak{U}_\varepsilon(t, \xi))$ is a solution to

$$(I - A_\varepsilon)V_\varepsilon = E_\varepsilon \mathfrak{U}_\varepsilon(t, \xi),$$

so that over bounded intervals in time (with a constant growing exponentially with time but independent of ξ), we find that

$$|U_{\varepsilon,s}(t, \xi) - \mathfrak{U}_\varepsilon(t, \xi)| = |V_\varepsilon(t, \xi)| \lesssim \varepsilon^\beta |\hat{u}_0(\xi)|. \quad (69)$$

The above inequality combined with (63) yields the first estimate in (5).

Up to an order $O(\varepsilon^\beta |\hat{u}_0(\xi)|)$, we have thus obtained that $\mathbb{E}\{\hat{u}_\varepsilon(t, \xi)\}$ is given by

$$e^{-t(\xi^{\mathbf{m}} - \rho_\varepsilon(\xi))} \hat{u}_0(\xi),$$

which in the physical domain gives rise to a possibly non-local equation. It remains to analyze the limit of the above term, and thus the error $\rho_\varepsilon(\xi) - \rho$, which depends on the regularity of $\hat{R}(\xi)$. For $\hat{R}(\xi)$ of class $\mathcal{C}^2(\mathbb{R}^d)$, we find that

$$|e^{-t(\xi^{\mathbf{m}} - \rho_\varepsilon(\xi))} - e^{-t(\xi^{\mathbf{m}} - \rho)}| \leq t e^{Ct} e^{-\xi^{\mathbf{m}} t} |\rho_\varepsilon(\xi) - \rho| \lesssim e^{Ct} e^{-\xi^{\mathbf{m}} t} \varepsilon^2 t \xi^2.$$

The reason for the second order accuracy is that $\hat{R}(-\xi) = \hat{R}(\xi)$ and $\nabla \hat{R}(0) = 0$ so that first-order terms in the Taylor expansion vanish. For $\hat{R}(\xi)$ of class $\mathcal{C}^\gamma(\mathbb{R}^d)$ with $0 < \gamma < 2$, we obtain by interpolation that

$$|e^{-t(\xi^{\mathbf{m}} - \rho_\varepsilon(\xi))} - e^{-t(\xi^{\mathbf{m}} - \rho)}| \lesssim e^{Ct} e^{-\xi^{\mathbf{m}} t} \varepsilon^\gamma t \xi^\gamma.$$

When $\mathbf{m} \geq \gamma$, the above term is bounded by $O(\varepsilon^\gamma)$ uniformly in ξ and uniformly in time on bounded intervals. When $\mathbf{m} \leq \gamma$, the above term is bounded by $O(\varepsilon^{\mathbf{m}})$ uniformly in ξ and uniformly in time on bounded intervals. This concludes the proof of theorem 1. In terms of the propagators defined in (57), we may recast the above result as

$$|\mathcal{U}_\varepsilon(t, \xi) - \mathcal{U}(t, \xi)| \lesssim \varepsilon^{\gamma \wedge \beta} \quad \mathcal{U}(t, \xi) = e^{-(\xi^{\mathbf{m}} - \rho)t}, \quad (70)$$

where the bound is uniform in time for $t \in (0, T)$ and uniform in $\xi \in \mathbb{R}^d$. From this, we deduce the second inequality in (5), which concludes the proof of theorem 1.

3.2 Fluctuation theory for u_ε

We now address the proof of theorem 2. The first term in the decomposition of $\hat{u}_{n,\varepsilon}$ defined in (22) is its mean $\mathbb{E}\{\hat{u}_{n,\varepsilon}\}$, which was analyzed in the preceding section. The second contribution corresponds to the graphs \mathfrak{G}_{cs} in the analysis of the correlation function and is constructed as follows. Let $n = 2p + 1$, $p \in \mathbb{N}$. We introduce the corrector $\hat{u}_{n,\varepsilon}^c$ given by

$$\begin{aligned} \hat{u}_{n,\varepsilon}^c(t, \xi_0) &= \int \prod_{k=0}^n e^{-s_k \xi_k^{\mathbf{m}}} \sum_{q=0}^p \left[\prod_{r=1}^q \mathbb{E}\{\hat{q}_\varepsilon(\xi_{2(r-1)} - \xi_{2r-1}) \hat{q}_\varepsilon(\xi_{2r-1} - \xi_{2r})\} \right] \\ &\quad \hat{q}_\varepsilon(\xi_{2q} - \xi_{2q+1}) \left[\prod_{r=q+1}^p \mathbb{E}\{\hat{q}_\varepsilon(\xi_{2r-1} - \xi_{2r}) \hat{q}_\varepsilon(\xi_{2r} - \xi_{2r+1})\} \right] \hat{u}_0(\xi_n) ds d\xi. \end{aligned} \quad (71)$$

In other words, all the random terms are averaged as simple pairs except for one term. There are $p + 1$ such graphs. We define

$$\hat{u}_\varepsilon^c(t, \xi) = \sum_{n \geq 1} \hat{u}_{n,\varepsilon}^c(t, \xi). \quad (72)$$

We verify that

$$V_{\varepsilon,s}^{n,m}(t, \xi_0, \xi_{n+m+1}) := \mathbb{E}\{\hat{u}_{n,\varepsilon}^c(t, \xi_0) \bar{\hat{u}}_{n,\varepsilon}^c(t, \xi_{n+m+1})\}$$

is equal to the sum in $V_\varepsilon^{n,m}(t, \xi_0, \xi_{n+m+1})$ only over the graphs in \mathfrak{G}_{cs} . Indeed, the above correlation involves all the graphs composed of simple pairs with a single crossing.

Now let us define the variable

$$Y_\varepsilon = (\hat{u}_\varepsilon - \hat{u}_\varepsilon^c - \mathbb{E}\{\hat{u}_\varepsilon\}, \hat{M}). \quad (73)$$

Summing over $n, m \in \mathbb{N}$ the inequality in (56) as we did to obtain (52), we have demonstrated that

$$\mathbb{E}\{Y_\varepsilon^2\} \lesssim \varepsilon^{d-2\alpha+\beta} \|\hat{u}_0\|^2 \|\hat{M}\|_1^2, \quad (74)$$

for sufficiently small times. The leading term in the random fluctuations of u_ε is thus given by u_ε^c . It remains to analyze the convergence properties of

$$Z_\varepsilon(t) = \frac{1}{\varepsilon^{\frac{d-2\alpha}{2}}} (\hat{u}_\varepsilon^c, \hat{M}). \quad (75)$$

We thus come back to the analysis of \hat{u}_ε^c and observe that for $n = 2p + 1$,

$$\begin{aligned} \hat{u}_{n,\varepsilon}^c(t, \xi_0) &= \int \left(\prod_{k=0}^n e^{-s_k \xi_k^m} \right) \sum_{q=0}^p \left[\prod_{r=1}^q \varepsilon^{d-2\alpha} \hat{R}(\varepsilon(\xi_{2r-1} - \xi_0)) \delta(\xi_{2r} - \xi_0) \right] \\ &\quad \hat{q}_\varepsilon(\xi_0 - \xi_n) \left[\prod_{r=q+1}^p \varepsilon^{d-2\alpha} \hat{R}(\varepsilon(\xi_{2r} - \xi_n)) \delta(\xi_{2r-1} - \xi_n) \right] \hat{u}_0(\xi_n) ds d\xi. \end{aligned}$$

Using the propagator defined in (57), we verify that

$$\begin{aligned} \hat{u}_{n,\varepsilon}^c(t, \xi_0) &= \sum_{q=0}^p \int \prod_{k=0}^{2q-1} e^{-s_k \xi_k^m} \left[\prod_{r=1}^q \varepsilon^{d-2\alpha} \hat{R}(\varepsilon(\xi_{2r-1} - \xi_0)) \delta(\xi_{2r} - \xi_0) \right] \\ &\quad \hat{q}_\varepsilon(\xi_0 - \xi_n) \mathcal{U}_\varepsilon^{n-2q-1}(t_{2q+1}, \xi_n) \hat{u}_0(\xi_n) ds d\tilde{\xi} \\ &= \sum_{q=0}^p \int_0^t \int_{\mathbb{R}^d} \mathcal{U}_\varepsilon^{2q}(t - t_{2q+1}, \xi_0) \hat{q}_\varepsilon(\xi_0 - \xi_n) \mathcal{U}_\varepsilon^{n-2q-1}(t_{2q+1}, \xi_n) \hat{u}_0(\xi_n) dt_{2q+1} d\xi_n \\ &= \sum_{q=0}^p \int_0^t \int_{\mathbb{R}^d} \mathcal{U}_\varepsilon^{2q}(t - s, \xi_0) \hat{q}_\varepsilon(\xi_0 - \xi_1) \mathcal{U}_\varepsilon^{n-2q-1}(s, \xi_1) \hat{u}_0(\xi_1) ds d\xi_1. \end{aligned}$$

Upon summing over n , we obtain

$$\hat{u}_\varepsilon^c(t, \xi) = \int_0^t \int_{\mathbb{R}^d} \mathcal{U}_\varepsilon(t - s, \xi) \hat{q}_\varepsilon(\xi - \xi_1) \mathcal{U}_\varepsilon(s, \xi_1) \hat{u}_0(\xi_1) ds d\xi_1. \quad (76)$$

We can use the error on the propagator obtained in (70) to show that the leading order of \hat{u}_ε^c is not modified by replacing \mathcal{U}_ε by \mathcal{U} . In other words, replacing \mathcal{U}_ε by \mathcal{U} modifies Z_ε in (75) by a term of order $O(\varepsilon^{\frac{1}{2}(\beta \wedge \gamma)})$ in $L^2(\Omega \times \mathbb{R}^d)$, which thus goes to 0 in law.

Note that $\hat{u}_\varepsilon^c(t, \xi)$ is a mean zero Gaussian random variable. It is therefore sufficient to analyze the convergence of its variance in order to capture the convergent random variable for each t and ξ . The same is true for the random variable Z_ε . Up to a lower-order term, which does not modify the final convergence, we thus have that

$$(\hat{u}_\varepsilon^c, \hat{M}) = \int \int_0^t \bar{\mathcal{U}}_{\hat{M}}(t - s, \xi) \hat{q}_\varepsilon(\xi_1) \mathcal{U}_{\hat{u}_0}(s, \xi - \xi_1) ds d\xi d\xi_1.$$

We have defined $\mathcal{U}_f(t, \xi) = \mathcal{U}(t, \xi)f(\xi)$ for a function $f(\xi)$. As a consequence, we find that, still up a vanishing contribution,

$$\begin{aligned} \mathbb{E}\{|Z_\varepsilon|^2\} &= \int \int_0^t \int_0^t \bar{\mathcal{U}}_{\hat{M}}(t-s, \xi) \mathcal{U}_{\hat{M}}(t-\tau, \zeta) \hat{R}(\varepsilon\xi_1) \delta(\xi_1 - \zeta_1) \\ &\quad \times \mathcal{U}_{\hat{u}_0}(s, \xi - \xi_1) \bar{\mathcal{U}}_{\hat{u}_0}(\tau, \zeta - \zeta_1) d[s\tau\zeta_1\xi_1]. \end{aligned}$$

Here and below, we use the notation $d[x_1 \dots x_n] \equiv dx_1 \dots dx_n$. By the dominated Lebesgue convergence theorem, we obtain in the limit

$$\mathbb{E}\{|Z|^2\} := \hat{R}(0) \int \left| \int \int_0^t \mathcal{U}_{\hat{M}}(t-s, \xi) \mathcal{U}_{\hat{u}_0}(s, \xi - \xi_1) d\xi ds \right|^2 d\xi_1.$$

Here, Z is defined as a mean zero Gaussian random variable with the above variance. Let us define $\mathcal{G}_t^\rho f(x)$, the solution at time t of (3) with $f(x)$ as initial conditions, which is also the inverse Fourier transform of $\mathcal{U}_f(t, \xi)$. We then recognize in $\int \int_0^t \mathcal{U}_{\hat{M}}(t-s, \xi) \mathcal{U}_{\hat{u}_0}(s, \xi - \xi_1) d\xi_1 ds$ the Fourier transform of $\mathcal{M}_t(x)$ defined in (8) so that by an application of the Plancherel identity, we find that

$$\mathbb{E}\{Z^2\} = (2\pi)^d \hat{R}(0) \int_{\mathbb{R}^d} \left(\int_0^t \mathcal{G}_{t-s}^\rho M(x) \mathcal{G}_s^\rho u_0(x) ds \right)^2 dx = (2\pi)^d \hat{R}(0) \int_{\mathbb{R}^d} \mathcal{M}_t^2(x) dx. \quad (77)$$

This shows that $Z(t)$ is indeed the Gaussian random variable written on the right hand side in (8) by an application of the Itô isometry formula. This concludes the proof of theorem 2.

3.3 Long range correlations and correctors

We now consider long range correlations described by power spectra defined in (12). We present a proof of theorem 3 and consider possible generalizations.

Proof. The proof of theorem 1 relies on three estimates: those of lemma 2.1 and lemma 3.1 and the uniform bound in (48) for \hat{R} . Lemmas 2.1 and 3.1 were written to account for power spectra bounded by $|\xi|^{-n}$ in the vicinity of the origin. It thus remains to replace (48) by

$$\varepsilon^{d-m} \hat{R}(\varepsilon(\xi_Q - \xi_{Q-1}^\varepsilon)) \leq \varepsilon^{d-m-n} h(\xi_Q - \xi_{Q-1}^\varepsilon) \hat{S}_\infty,$$

when $|\xi_Q - \xi_{Q-1}^\varepsilon| \leq 1$ while we still use (48) otherwise. We have defined \hat{S}_∞ as the supremum of $\hat{S}(\xi)$ in $B(0, 1)$. It now remains to show that the integration with respect to ξ_Q in (47) is still well-defined. Note that either $Q = n$ or $\xi_Q - \xi_{Q-1}^\varepsilon$ may be written as $\xi_n - \zeta$ for some $\zeta \in \mathbb{R}^d$ thanks to (32). Upon using (49), we thus observe that in all cases, the integration with respect to ξ_Q in (47) is well-defined and bounded uniformly provided that (13) is satisfied uniformly in τ . Using the Hölder inequality, we verify that (13) holds e.g. when $\hat{u}_0(\cdot - \tau) \in L^q(B(0, 1))$ uniformly in τ for $q > \frac{2d}{d-n}$. This concludes the proof of the first part of the theorem.

Let us now define

$$\tilde{Z}_\varepsilon(t) = \frac{1}{\varepsilon^{\frac{d-m-n}{2}}} (\hat{u}_\varepsilon^c, \hat{M}) = \varepsilon^{\frac{n}{2}} Z_\varepsilon(t).$$

We verify as for the derivation of $\mathbb{E}\{Z_\varepsilon^2\}$ that

$$\begin{aligned} \mathbb{E}\{\tilde{Z}_\varepsilon^2\} &= \int_0^t \int_0^t \int_0^t \mathcal{U}_{\hat{M}}(t-s, \xi) \hat{\mathcal{U}}_{\hat{M}}(t-\tau, \zeta) \hat{S}(\varepsilon \xi_1) h(\xi_1) \delta(\xi_1 - \zeta_1) \\ &\quad \times \mathcal{U}_{\hat{u}_0}(s, \xi - \xi_1) \hat{\mathcal{U}}_{\hat{u}_0}(\tau, \zeta - \zeta_1) d[s\tau\zeta\zeta_1\xi\xi_1]. \end{aligned}$$

The dominated Lebesgue convergence theorem yields in the limit $\varepsilon \rightarrow 0$

$$\begin{aligned} \mathbb{E}\{\tilde{Z}^2\} &:= \hat{S}(0) \int \left| \int_0^t \int \mathcal{U}_{\hat{M}}(t-s, \xi) \mathcal{U}_{\hat{u}_0}(s, \xi - \xi_1) h^{\frac{1}{2}}(\xi_1) d\xi_1 ds \right|^2 d\xi \\ &= \hat{S}(0) \int |\hat{\mathcal{M}}_t(\xi)|^2 h(\xi) d\xi, \end{aligned}$$

where \mathcal{M}_t is defined in (8). An application of the inverse Fourier transform yields (15). \square

Note that (15) generalizes (77), where $\varphi(x) = \delta(x)$, to functions $\mathcal{M}_t(x) \in L_\varphi^2(\mathbb{R}^d)$ with inner product

$$(f, g)_\varphi = \int_{\mathbb{R}^{2d}} f(x)g(y)\varphi(x-y)dx dy. \quad (78)$$

For $h(\xi) = |\xi|^{-n}$, we find that $\varphi(x) = c_n|x|^{n-d}$, with $c_n = \Gamma(\frac{d-n}{2})/(2^n\pi^{\frac{d}{2}}\Gamma(\frac{n}{2}))$ a normalizing constant.

Following e.g. [8, 11], we may then define a stochastic integral with fractional Brownian motion

$$Z = \int_{\mathbb{R}^d} \mathcal{M}_t(x) dW^H(x), \quad (79)$$

where W^H is fractional Brownian motion defined such that

$$\mathbb{E}\{\dot{W}^H(x)\dot{W}^H(x+y)\} = \frac{c_n}{|y|^{d-n}}, \quad c_n = \frac{\Gamma(\frac{d-n}{2})}{2^n\pi^{\frac{d}{2}}\Gamma(\frac{n}{2})}.$$

We then verify that $\mathbb{E}\{Z^2\} = \Sigma_M$ so that the random variable Z is indeed given by the above formula (79). When $\mathbf{n} = 0$, we retrieve the value for the Hurst parameter $H = \frac{1}{2}$ so that $W^H = W$, the standard multiparameter Brownian motion.

In the analysis of stochastic equations [9, 14], the multiparameter fractional Brownian motion is often defined as the centered Gaussian field B^H with Hurst index $H = (H_1, \dots, H_d)$, $\frac{1}{2} < H_i < 1$, and covariance

$$\mathbb{E}\{B^H(x)B^H(y)\} = \frac{1}{2^d} \prod_{i=1}^d (|x_i|^{2H_i} + |y_i|^{2H_i} - |x_i - y_i|^{2H_i}).$$

With this definition, we then find the correlation for the fractional white noise

$$\mathbb{E}\{\dot{B}^H(x)\dot{B}^H(y)\} = \varphi_H(x-y) := \prod_{i=1}^d H_i(2H_i - 1)|x_i - y_i|^{2H_i-2}.$$

The above is then defined as the Fourier transform of

$$h_H(\xi) = \prod_{i=1}^d \frac{H_i(2H_i - 1)}{c_i} |\xi_i|^{-n_i}, \quad \sum_{i=1}^d \mathbf{n}_i = \mathbf{n}, \quad 2H_i = 1 + \frac{\mathbf{n}_i}{d}, \quad c_i = \frac{\Gamma(1 - H_i)}{2^{n_i} \pi^{\frac{1}{2}} \Gamma(H_i - \frac{1}{2})}.$$

The results of theorem 1 and 3 may also be extended to this framework by slightly modifying the proofs in lemmas 2.1 and 3.1.

Note that homogenization theory is valid as soon as $d > \mathbf{m} + \mathbf{n}$. When $d < \mathbf{m} + \mathbf{n}$, generalizations of the work in [2] considered elsewhere show that the limit for u_ε is the solution in $L^2(\Omega \times \mathbb{R}^d)$ to a stochastic differential equation of the form (11) with white noise replaced by fractional white noise.

The stochastic representation in (79) is not necessary since $\Sigma_M(t)$ fully characterizes the random variable Z . However, the representation emphasizes the following conclusion. Let Z_1^H and Z_2^H be the limiting random variables corresponding to two moments with weights $M_1(x)$ and $M_2(x)$ and a given Hurst parameter H . When $H = \frac{1}{2}$, we deduce directly from (79) that $\mathbb{E}\{Z_1^{\frac{1}{2}}Z_2^{\frac{1}{2}}\} = 0$ when $M_1(x)M_2(x) = 0$, i.e., when the supports of the moments are disjoint. This is not the case when $H \neq \frac{1}{2}$ as fractional Brownian motion does not have independent increments. Rather, we find that $\mathbb{E}\{Z_1^H Z_2^H\}$ is given by $(\mathcal{M}_{t,1}, \mathcal{M}_{t,2})_\varphi$, where the inner product is defined in (78) and $\mathcal{M}_{t,k}$ is defined in (8) with M replaced by M_k , $k = 1, 2$. Similar results were obtained in the context of the one-dimensional homogenization with long-range diffusion coefficients [3].

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