Wave field correlations in weakly mismatched random media

Guillaume Bal * Leonid Ryzhik †

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Abstract

This paper concerns the derivation of a Fokker-Planck equation for the correlation of two high frequency wave fields propagating in two different random media. The mismatch between the random media need be small, on the order of the wavelength, and their correlation length need be large relative to the wavelength. The loss of correlation caused by the mismatch in the random media is quantified. The derivation is based on a random Liouville equation to model high frequency correlations and on the method of characteristics to characterize mixing in the random Liouville equation. Applications of such correlation loss include the monitoring in time of random media and the analysis of time reversed waves in changing heterogeneous domains.

1 Introduction

The energy density of high frequency waves propagating in highly heterogeneous media can be modeled by a Fokker-Planck equation in the phase space, i.e., the space of positions and momenta. We refer to e.g. [2] for a mathematical derivation of such a macroscopic model for the wave energy density. The main assumption on the heterogeneous medium is that it is a random medium with a very large correlation length relative to the typical wavelength in the system. The Fokker-Planck equation may be seen as a highly-peaked-forward-scattering approximation to the radiative transfer equations, which are also used in the modeling of the energy density of waves in heterogeneous media when correlation length and wavelength are comparable; see e.g. [1, 5, 8].

Such macroscopic models for waves in random media can more generally be used to quantify the correlation function of two wave fields propagating in possibly two different media [1]. This has application in the temporal monitoring of the statistical properties of random media as well as in the analysis of the refocusing of time reversed waves [4, 3, 7]. This paper analyzes the effect of changes in the random media on the two-field correlation. In the Fokker-Planck regime, the two-field correlation decays as the two media separate and we present a quantitative estimate of such a decay.

Although the results in this paper generalize to fairly large classes of waves such as e.g., acoustic waves as in [2]. electromagnetic waves, and elastic waves (see [8]) we restrict ourselves to the case of a scalar Schrödinger equation to simplify. This models the effect of heterogeneities on a single particle represented by a quantum wave function. The changes in the heterogeneous medium considered here are sufficiently small so that the correlation of the two fields, one propagating in the unperturbed medium and the other one propagationg in the perturbed medium, still satisfies a random Liouville equation in the high frequency limit. The techniques used in [2] are then generalized to this new

*Department of Applied Physics and Applied Mathematics, Columbia University, New York NY, 10027; gb2030@columbia.edu

†Department of Mathematics, University of Chicago, Chicago IL, 60637; ryzhik@math.uchicago.edu

random Liouville equation. The limiting equation, of Fokker-Planck type, is obtained in the vanishing limit of the correlation length of the heterogeneous medium. Its derivation is based on the mixing properties of the bicharacteristics of a random Hamiltonian, as in [2], and on the mixing properties of a highly oscillatory functional of such bicharacteristics, which is the main new result obtained in this paper.

The rest of the paper is organized as follows. Section 2 presents the random Schrödinger equations and the limiting random Liouville equation for the Wigner transform of the two wave fields, which is the Fourier transform in the offset variable of the correlation function of the two fields. It next reviews in subsection sub:review the results obtained in [2] while adapting them to the case of a random Schrödinger equation. Finally in subsection 2.2 it derives the generalized Fokker-Planck equations (see (2.23) below) by formal analysis. The latter result is obtained rigorously in sections 3 and 4. The derivation of the Fokker-Planck equation is deduced from the limiting law (as the correlation length of the random medium goes to zero) of the bicharacteristics of the random Hamiltonian and of functionals of such bicharacteristics. The limiting law is presented in Theorem 2.1 below. The proof of Theorem 2.1 is postponed to section 4.

2 The random momenta Liouville equation

We consider the evolution of the correlation function of two solutions of the Schrödinger equation with a light mismatch between the random potentials but with the same initial data. The function ψ_ε satisfies

$$i\varepsilon\frac{\partial\psi_\varepsilon}{\partial t}+\frac{\varepsilon^2}{2}\Delta\psi_\varepsilon-V_\delta(x)\psi_\varepsilon=0 \tag{2.1}$$

and the function ϕ_ε satisfies

$$i\varepsilon\frac{\partial\phi_\varepsilon}{\partial t}+\frac{\varepsilon^2}{2}\Delta\phi_\varepsilon-[V_\delta(x)+\varepsilon S_\delta(x)]\phi_\varepsilon=0. \tag{2.2}$$

Both of the functions ϕ_ε and ψ_ε satisfy initially

$$\psi_\varepsilon(0,x)=\phi_\varepsilon(0,x)=\phi_0^\varepsilon(x). \tag{2.3}$$

The family ϕ_ε^0 is ε-oscillatory and compact at infinity. The random potentials V_δ and S_δ vary on a scale δ that is much larger than the wave length ε of the initial data but is much smaller than the overall propagation distance that is of the order $O(1)$: $\varepsilon\ll\delta\ll 1$. To keep a non-trivial correlation of ψ_ε and ϕ_ε the mismatch of the potentials has to be weak – hence the coefficient ε in front of S_δ. We will see that in order to produce an order one contribution we will have eventually to take $S_\delta(x)=\delta^{-1/2}S(x/\delta)$ making the overall strength of the mismatch be of the order $O(\varepsilon/\sqrt{\delta})$.

In order to study the correlation of ψ_ε and ϕ_ε we introduce the cross Wigner transform as

$$W_\varepsilon(t,x,k)=\int e^{ik\cdot y}\psi_\varepsilon\left(t,x-\frac{\varepsilon y}{2}\right)\bar\phi_\varepsilon\left(t,x+\frac{\varepsilon y}{2}\right)\frac{dy}{(2\pi)^d}. \tag{2.4}$$

The distribution $W_\varepsilon(t,x,k)$ does not have to be real if $\phi_\varepsilon\neq\psi_\varepsilon$. Its phase measures the decoherence of the functions ϕ_ε and ψ_ε.

In order to obtain an equation for W_ε we differentiate the Wigner transform with respect to time:

$$\begin{aligned}&\frac{\partial W_\varepsilon}{\partial t}+k\cdot\nabla_x W_\varepsilon\\ &=\frac{1}{i\varepsilon}\int e^{ik\cdot y}\left[V_\delta\left(x-\frac{\varepsilon y}{2}\right)-V_\delta\left(x+\frac{\varepsilon y}{2}\right)-\varepsilon S_\delta\left(x+\frac{\varepsilon y}{2}\right)\right]\psi_\varepsilon\left(x-\frac{\varepsilon y}{2}\right)\bar\phi_\varepsilon\left(x+\frac{\varepsilon y}{2}\right)\frac{dy}{(2\pi)^d}.\end{aligned}$$

Passing to the limit $\varepsilon \to 0$ we obtain an equation for the distribution $W_\delta(t,x,k)$, the weak limit of W_ε as $\varepsilon \to 0$:

$$\frac{\partial W_\delta}{\partial t} + k \cdot \nabla_x W_\delta - \nabla V_\delta(x) \cdot \nabla_k W_\delta = iS_\delta(x) W_\delta. \tag{2.5}$$

In order to obtain a non-trivial limit of $W_\delta(t,x,k)$ as the correlation length $\delta \to 0$ we choose the random potential $V_\delta(x)$ and the mismatch $S_\delta(x)$ to be of the form

$$V_\delta(x) = \sqrt{\delta} V\left(\frac{x}{\delta}\right), \quad S_\delta(x) = \frac{1}{\sqrt{\delta}} S\left(\frac{x}{\delta}\right).$$

Then (2.5) becomes

$$\begin{aligned} &\frac{\partial W_\delta}{\partial t} + k \cdot \nabla_x W_\delta - \frac{1}{\sqrt{\delta}} \nabla V\left(\frac{x}{\delta}\right) \cdot \nabla_k W_\delta = \frac{i}{\sqrt{\delta}} S\left(\frac{x}{\delta}\right) W_\delta \\ &W(0,x,k) = W_0(x,k). \end{aligned} \tag{2.6}$$

The initial data $W_0(x,k)$ is simply the limit Wigner measure of the family ϕ_ε^0. Equation (2.6) is the starting point of our analysis. We note that if we consider the initial data for the Schrödinger equation as a mixture of states then the error bound for the approximation for the Wigner transform W_ε by the solution of (2.6) may be estimated and the sequential limits $\varepsilon \to 0$ first, and $\delta \to 0$ second may be replaced by a joint limit $(\varepsilon, \delta) \to 0$; see [1, 6] for details. This may be done in a certain region of the (ε, δ)-plane that ensures that the scale separation $\varepsilon \ll \delta \ll 1$ is kept under control. We will not pursue this avenue in this paper in order to avoid unnecessary technical complications.

2.1 A review of the case in the absence of a mismatch

We first recall the known results of [3] (see also [4] for a longer time scale analysis) when there is no potential mismatch, that is, when $S = 0$ and $d \geq 3$; we restrict our attention in this paper also to that case though a generalization using the results of [2] and [5] is possible. Then solution of (2.6) is given explicitly in terms of the random characteristics $(X^\delta(t), K^\delta(t))$: define

$$\frac{dX^\delta(t)}{dt} = -K^\delta(t), \quad \frac{dK^\delta(t)}{dt} = \frac{1}{\sqrt{\delta}} \nabla V\left(\frac{X^\delta(t)}{\delta}\right), \quad X^\delta(0) = x, \ K^\delta(0) = k. \tag{2.7}$$

We have $W_\delta(t,x,k) = W_0(X^\delta(t), K^\delta(t))$. The characteristic trajectories satisfy a limit theorem; the process $K^\delta(t)$ converges to a Brownian motion $K(t)$ on the sphere $\{|k| = |k(0)|\}$ and $X^\delta(t)$ converges to

$$X(t) = x + \int_0^t K(s) ds.$$

Before formulating the limit theorem we describe first the necessary assumptions on the random potentials V and S; we will not, of course, need the assumptions on S in the present section but we will need them later on and it is convenient to put them here.

Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space, and let $\mathbb{E}$ denote the expectation with respect to $\mathbb{P}$. Let us denote the joint process $F(x,\omega) = (V(x,\omega), S(x,\omega))$ and assume that the random field F is stationary in the first variable. This means that for any shift $x \in \mathbb{R}^d$ and a collection of points $x_1, \ldots, x_n \in \mathbb{R}^d$ the laws of $(F(x_1 + x), \ldots, F(x_n + x))$ and $(F(x_1), \ldots, F(x_n))$ are identical. In addition, we assume that $\mathbb{E}\{S(x)\} = \mathbb{E}\{V(x)\} = 0$ for all $x \in \mathbb{R}^d$, the realizations of $V(x)$ and $S(x)$ are $\mathbb{P}$ a.s. C^2-smooth in x and they satisfy

$$M := \max_{|\alpha| \leq 2} \operatorname*{ess-sup}_{(x,\omega) \in \mathbb{R}^d \times \Omega} |\partial_x^\alpha F(x,\omega)| < +\infty. \tag{2.8}$$

We suppose further that the random field $F(x,\omega)$ is strongly mixing in the uniform sense. More precisely, for any $R>0$ we let $\mathcal{C}_R^i$ and $\mathcal{C}_R^e$ be the σ-algebras generated by random variables $F(x)$ for all $x\in\mathbb{B}_R$ and $x\in\mathbb{B}_R^c$ respectively. The uniform mixing coefficient between the σ-algebras is

$$\phi(\rho) := \sup[\,|\mathbb{P}(B)-\mathbb{P}(B|A)|:\ R>0,\ A\in\mathcal{C}_R^i,\ B\in\mathcal{C}_{R+\rho}^e\,], \tag{2.9}$$

for all $\rho>0$. We suppose that $\phi(\rho)$ decays faster than any power: for each $p>0$

$$h_p := \sup_{\rho\ge 0}\rho^p\phi(\rho) < +\infty. \tag{2.10}$$

The two-point spatial correlation 2×2 tensor of the random field F is denoted by $R(y)$ and has components

$$R^{VV}(y)=\mathbb{E}[V(0)V(y)],\ R^{VS}(y)=\mathbb{E}\{V(0)S(y)\},\ R^{SV}(y)=\mathbb{E}\{S(0)V(y)\},\ R^{SS}(y)=\mathbb{E}\{S(0)S(y)\}.$$

Note that (2.10) implies that for each $p>0$

$$h_p := \sum_{|\alpha|\le 4}\sup_{y\in\mathbb{R}^d}(1+|y|^2)^{p/2}|\partial_y^\alpha R(y)| < +\infty. \tag{2.11}$$

We also assume that the correlation tensor $R(y)$ is of the C^∞-class and that

$$\hat{R}^{VV}(k) \text{ does not vanish identically on any hyperplane } H_p=\{k:\ (k\cdot p)=0\}. \tag{2.12}$$

Here $\hat{R}^{VV}(k)=\int R^{VV}(x)\exp(-ik\cdot x)dx$ is the power spectrum of V.

Let us define the diffusion matrix D_{mn} by

$$D_{mn}(k) = -\frac{1}{2}\int_{-\infty}^{\infty}\frac{\partial^2 R^{VV}(ks)}{\partial x_n\partial x_m}ds = -\frac{1}{2|k|}\int_{-\infty}^{\infty}\frac{\partial^2 R^{VV}(s\hat{k})}{\partial x_n\partial x_m}ds,\qquad m,n=1,\dots,d,\quad \hat{k}=k/|k|. \tag{2.13}$$

Then we have the following result.

Theorem 2.1 *[3] Let W^δ be the solution of (2.6) with the initial data $W_0(x,k)$ supported in a compact set away from $k=0$: $supp\{W_0\}\subset\mathcal{S}\times A(M)$ with $A(M)=\{M^{-1}\le|k|\le M\}$ for some $M>0$ and a compact set $\mathcal{S}\subset\mathbb{R}^d$. Let the function $\bar{\phi}$ satisfy*

$$\frac{\partial\bar{\phi}}{\partial t}+k\cdot\nabla_x\bar{\phi} = \frac{\partial}{\partial k_m}\left(D_{mn}(k)\frac{\partial\bar{\phi}}{\partial k_n}\right) \tag{2.14}$$
$$\bar{\phi}(0,x,k)=W_0(x,k).$$

Then, there exist two constants $C(T)$ and $\alpha_0>0$ such that

$$\sup_{(t,x,k)\in[0,T]\times K}\left|\mathbb{E}W^\delta(t,x,k)-\bar{\phi}(t,x,k)\right| \le C(T)(1+\|W_0\|_{C^4})\delta^{\alpha_0} \tag{2.15}$$

for all compact sets $K\subset\mathcal{A}(M)=\mathbb{R}^d\times A(M)$.

Note that

$$D_{nm}(k)\hat{k}_m = -\frac{1}{2|k|}\int_{-\infty}^{\infty}\frac{\partial^2 R^{VV}(s\hat{k})}{\partial x_n\partial x_m}\hat{k}_m ds = -\frac{1}{2|k|}\int_{-\infty}^{\infty}\frac{d}{ds}\left(\frac{\partial R^{VV}(sk)}{\partial x_n}\right)ds = 0$$

and thus the K-process generated by (2.14) is indeed a diffusion process on a sphere $k=$ const, or, equivalently, equations (2.14) for different values of $|k|$ are decoupled. It is easy to check that assumption (2.12) implies that the matrix $D(k)$ has rank $d-1$ for each $k\in\mathbb{R}^d\{0\}$. It can be also shown that then equation (2.14) is hypoelliptic on the manifold $\mathbb{R}^d\times\{|k|=k_0\}$ for each $k_0>0$.

2.2 A formal analysis of the momenta Liouville equation

We first present a non-rigorous formal multiple scales analysis of (2.6), which provides a short and relatively quick way to the correct limit. We introduce a multiple scales expansion

$$W_\delta = W(t,x,k) + \sqrt{\delta} W_1(t,x,y,k) + \delta W_2(t,x,y,k) + \ldots, \quad y = x/\delta$$

and insert it into (2.6). As usual we make an additional assumption that the leading order term $W(t,x,k)$ is deterministic and does not depend on the fast scale variable y. In the leading order we obtain

$$k \cdot \nabla_y W_1 + \theta W_1 = \nabla V(y) \cdot \nabla_k W + iS(y) W.$$

Here $\theta > 0$ is an auxiliary regularizing parameter that we will send to zero at the end. Define the correctors χ_j and η as mean-zero solutions of

$$k \cdot \nabla_y \chi_j + \theta \chi_j = \frac{\partial V}{\partial y_j}$$

and

$$k \cdot \nabla_y \eta + \theta \eta = S(y).$$

They are given explicitly by

$$\chi_j(y,k) = \int_0^\infty \frac{\partial V(y - sk)}{\partial y_j} e^{-\theta s} ds \tag{2.16}$$

and

$$\eta(y,k) = \int_0^\infty e^{-\theta s} S(y - sk) ds. \tag{2.17}$$

The function W_1 is given in terms of the correctors as

$$W_1(t,x,y,k) = \sum_{j=1}^d \chi_j(y,k) \frac{\partial W(t,x,k)}{\partial k_j} + i\eta(y,k) W(t,x,k).$$

The equation for W_2 is

$$\frac{\partial W}{\partial t} + k \cdot \nabla_x W + k \cdot \nabla_y W_2 = \nabla V(y) \cdot \nabla_k W_1 + iS(y) W_1.$$

Averaging under the assumption that $\mathbb{E}\{k \cdot \nabla_y W_2\} = 0$ we obtain the following closed equation for the leading order term W:

$$\frac{\partial W}{\partial t} + k \cdot \nabla_x W = \mathbb{E}\left\{\nabla V(y) \cdot \nabla_k W_1 + iS(y) W_1\right\} = J_I + J_{II}. \tag{2.18}$$

The two terms on the right side are computed using the explicit expressions (2.16) and (2.17) for the correctors. The first term may be split as

$$J_I = \mathbb{E}\left\{\nabla V(y) \cdot \nabla_k W_1\right\} = J_I^1 + J_I^2$$

with

$$\begin{aligned} J_I^1 &= \mathbb{E}\left\{\frac{\partial V}{\partial y_j}(y) \frac{\partial}{\partial k_j}\left[\chi_m(y,k) \frac{\partial W(t,x,k)}{\partial k_m}\right]\right\} \\ &= \frac{\partial}{\partial k_j}\left[\mathbb{E}\left\{\frac{\partial V}{\partial y_j}(y) \int_0^\infty \frac{\partial V(y-sk)}{\partial y_m} e^{-\theta s} ds\right\} \frac{\partial W(t,x,k)}{\partial k_m}\right] = \frac{\partial}{\partial k_j}\left(D_{jm}(k) \frac{\partial W(t,x,k)}{\partial k_m}\right) \end{aligned}$$

where the diffusion matrix D_{jm} is given by (2.13). The term J_{I2} is

$$J_I^2 = \mathbb{E}\left\{\frac{\partial V}{\partial y_j}(y)\frac{\partial}{\partial k_j}\left[i\eta(y,k)W(t,x,k)\right]\right\} = i\frac{\partial}{\partial k_j}\left[\mathbb{E}\left\{\frac{\partial V}{\partial y_j}(y)\int_0^\infty S(y-sk)e^{-\theta s}ds\right\}W(t,x,k)\right]$$
$$= i\frac{\partial}{\partial k_j}\left(E_j'(k)W(t,x,k)\right)$$

with the drift

$$E_j'(k) = \int_0^\infty \frac{\partial R^{SV}(sk)}{\partial x_j}ds.$$

Now we look at the second term in the right side of (2.18)

$$J_{II} = \mathbb{E}\left\{iS(y)W_1\right\} = J_{II}^1 + J_{II}^2 \tag{2.19}$$

with

$$J_{II}^1 = \mathbb{E}\left\{iS(y)\chi_m(y,k)\frac{\partial W(t,x,k)}{\partial k_m}\right\} = i\mathbb{E}\left\{S(y)\int_0^\infty \frac{\partial V(y-sk)}{\partial y_m}e^{-\theta s}ds\right\}\frac{\partial W(t,x,k)}{\partial k_m}$$
$$= iE_m''(k)\frac{\partial W(t,x,k)}{\partial k_m}$$

with

$$E_m'' = -\int_0^\infty \frac{\partial R^{VS}(sk)}{\partial x_j}ds = \int_0^\infty \frac{\partial R^{SV}(-sk)}{\partial x_j}ds = \int_{-\infty}^0 \frac{\partial R^{SV}(sk)}{\partial x_j}ds.$$

Note that

$$J_I^2 + J_{II}^1 = i\frac{\partial}{\partial k_j}\left(E_j'(k)W(t,x,k)\right) + iF_m(k)\frac{\partial W(t,x,k)}{\partial k_m}$$
$$= i(E_j' + E_j'')\frac{\partial W(t,x,k)}{\partial k_j} + i(\nabla_k\cdot E')W(t,x,k) = E_j\frac{\partial W(t,x,k)}{\partial k_j} + FW(t,x,k)$$

with

$$E_j = E_j' + E_j'' = \int_{-\infty}^\infty \frac{\partial R^{SV}(sk)}{\partial x_j}ds \tag{2.20}$$

and

$$F = \nabla_k\cdot E' = \int_0^\infty s\Delta R_{SV}(sk)ds. \tag{2.21}$$

The last term in (2.19) is

$$II_2 = \mathbb{E}\left\{iS(\mathbf{z})\eta(\mathbf{z},k)W(t,x,k)\right\} = -\kappa(k)W(t,x,k)$$

with the absorption coefficient

$$\kappa(k) = \int_0^\infty R^{SS}(sk)ds. \tag{2.22}$$

Putting together all the terms above we get the equation for W:

$$\frac{\partial W}{\partial t} + k\cdot\nabla_x W = iE(k)\cdot\nabla_k W + iF(k)W + \frac{\partial}{\partial k_m}\left(D_{mn}(k)\frac{\partial W}{\partial k_n}\right) - \kappa(k)W. \tag{2.23}$$

If S and V are independent then $F = E = 0$ and this simplifies to

$$\frac{\partial W}{\partial t} + k\cdot\nabla_x W = \frac{\partial}{\partial k_m}\left(D_{mn}(k)\frac{\partial W}{\partial k_n}\right) - \kappa(k)W. \tag{2.24}$$

In the next section we present a rigorous derivation of the limit equation (2.23).

3 The Liouville equation and the phase diffusion

3.1 An example: the decorrelated case

The purpose of this section is to obtain the limit equation (2.23) as the limit of (2.6). Recall that solution of (2.6)

$$\frac{\partial W_\delta}{\partial t} + k \cdot \nabla_x W_\delta - \frac{1}{\sqrt{\delta}} \nabla V \left(\frac{x}{\delta}\right) \cdot \nabla_k W_\delta = \frac{i}{\sqrt{\delta}} S \left(\frac{x}{\delta}\right) W_\delta \tag{3.1}$$

may be obtained by the method of characteristics. Along the trajectories we have

$$\frac{dX^\delta}{dt} = -K^\delta, \quad \frac{dK^\delta}{dt} = \frac{1}{\sqrt{\delta}} \nabla V \left(\frac{X^\delta}{\delta}\right), \quad \frac{dZ^\delta}{dt} = \frac{1}{\sqrt{\delta}} S \left(\frac{X^\delta}{\delta}\right) \tag{3.2}$$

with the initial data

$$X^\delta(0) = x, \quad K^\delta(0) = k, \quad Z^\delta(0) = 0.$$

Then the solution of (3.1) is given by

$$W^\delta(t, x, k) = e^{iZ^\delta(t)} W_0(X^\delta(t), K^\delta(t)).$$

The results described in the previous section tell us that K^δ converges to a Brownian motion $K(t)$ on the sphere, with the diffusion matrix $D_{mn}(k)$, and X^δ converges to its time integral:

$$K^\delta(t) \to K(t), \quad X^\delta(t) \to X(t) = x + \int_0^t K(s) ds.$$

We will show below that Z^δ converges to a Brownian motion with the diffusion coefficient $\kappa(k)$ given by (2.22). In the simplest case when V and S are uncorrelated the Kolmogorov equation for the process $(X(t), K(t), Z(t))$ is

$$\frac{\partial f}{\partial t} + k \cdot \nabla_x f = \frac{\partial}{\partial k_m} \left(D_{mn}(k) \frac{\partial f}{\partial k_n} \right) + \kappa(k) \frac{\partial^2 f}{\partial z^2}.$$

Recall that

$$\overline{W}^\delta(t, x, k) := \mathbb{E}\{W^\delta(t, x, k)\} = \mathbb{E}_{x,k,z=0} \left(e^{iZ^\delta(t)} W_0(X^\delta(t), K^\delta(t)) \right).$$

Therefore, in the limit $\delta \to 0$, the function $\overline{W}^\delta(t, x, k)$ converges to

$$W(t, x, k) = g(t, x, k, z = 0).$$

Here the function g satisfies the Kolmogorov equation

$$\frac{\partial g}{\partial t} + k \cdot \nabla_x g = \frac{\partial}{\partial k_m} \left(D_{mn}(k) \frac{\partial g}{\partial k_n} \right) + \kappa(k) \frac{\partial^2 g}{\partial z^2} \tag{3.3}$$

$$g(0, x, k, z) = e^{iz} W_0(x, k). \tag{3.4}$$

It may be written as $g(t, x, k, z) = e^{iz} q(t, x, k)$, where the function q satisfies

$$\frac{\partial q}{\partial t} + k \cdot \nabla_x q = \frac{\partial}{\partial k_m} \left(D_{mn}(k) \frac{\partial q}{\partial k_n} \right) - \kappa(k) q \tag{3.5}$$

$$q(0, x, k) = W_0(x, k).$$

We see that actually $W(t, x, k) = g(t, x, k, z = 0) = q(t, x, k)$. Note that (3.5) is nothing but (2.24); this relates the approach of the present section to the formal result of the previous section. However, the Kolmogorov equation (3.3) provides the description of the whole limit process $Z(t)$ while (2.24) is just one of its reductions.

3.2 The main result

We will prove the following theorem.

Theorem 3.1 *The joint process $(X^\delta(t), K^\delta(t), Z^\delta(t))$ converges in law in the limit $\delta \to 0$ to the diffusion process $(X(t), K(t), Z(t))$ with the joint generator*

$$\mathcal{L}\phi = D_{mn}(k)\frac{\partial^2\phi}{\partial k_m \partial k_n} + [D_m(k) + D_m(-k)]\frac{\partial^2\phi}{\partial k_m \partial z} + D(k)\frac{\partial^2\phi}{\partial z^2} + E_m(k)\frac{\partial\phi}{\partial k_m} + E(k)\frac{\partial\phi}{\partial z} - k\cdot\nabla_x\phi \tag{3.6}$$

with the coefficients

$$D_{mn}(k) = -\int_0^\infty R_{mn}^{VV}(sk))\,ds, \qquad D(k) = \int_0^\infty R^{SS}(sk)ds, \tag{3.7}$$

$$D_m(k) = \int_0^\infty R_m^{SV}(sk)\,ds, \qquad E_m(k) = -\int_0^\infty s\Delta R_m^{VV}(sk)\,ds, \tag{3.8}$$

and

$$E(k) = \int_0^\infty s\Delta R^{SV}(sk)\,ds. \tag{3.9}$$

The generator may be written slightly more compactly as

$$\mathcal{L}\phi = \frac{\partial}{\partial k_n}\left(D_{mn}(k)\frac{\partial\phi}{\partial k_m}\right) + \frac{\partial}{\partial k_m}\left(D_m(k)\frac{\partial\phi}{\partial z}\right) + \frac{\partial}{\partial z}\left(D_m(-k)\frac{\partial\phi}{\partial k_m}\right) + D(k)\frac{\partial^2\phi}{\partial z^2} - k\cdot\nabla_x\phi. \tag{3.10}$$

3.3 A formal computation of the limit

We first present a formal computation, which is similar to the derivation in section 2.2 and leads to the generator (3.6). We start with the Liouville equation including the phase variable

$$\frac{\partial\phi}{\partial t} + k\cdot\nabla_x\phi - \frac{1}{\sqrt{\delta}}\nabla V\left(\frac{x}{\delta}\right)\cdot\nabla_k\phi - \frac{1}{\sqrt{\delta}}S\left(\frac{x}{\delta}\right)\frac{\partial\phi}{\partial z} = 0 \tag{3.11}$$

and consider an asymptotic expansion of the form

$$\phi(t,x,k,z) = \bar\phi(t,x,z,k) + \sqrt{\delta}\phi_1(t,x,y,z,k) + \delta\phi_2(t,x,y,z,k) + \ldots, \quad y = x/\delta.$$

In the leading order we get:

$$k\cdot\nabla_y\phi_1 + \theta\phi_1 = \nabla V(y)\cdot\nabla_k\bar\phi + S(y)\frac{\partial\bar\phi}{\partial z}.$$

As before $\theta > 0$ is a regularizing parameter that we will send to zero later. The leading order term $\bar\phi(t,x,k)$ is assumed to be deterministic and independent of the fast variable y. Using the correctors χ_j and η given by (2.16) and (2.17), respectively, we obtain an expression for ϕ_1 as

$$\phi_1(t,x,y,k,z) = \chi_j(y,k)\frac{\partial\bar\phi}{\partial k_j} + \eta(y,k)\frac{\partial\bar\phi}{\partial z}.$$

The equation for $\bar\phi$ is obtained in the same way as before, as a formal solvability condition for ϕ_2; it reads

$$\frac{\partial\bar\phi}{\partial t} + k\cdot\nabla_x\bar\phi = \mathbb{E}\left\{\nabla V(y)\cdot\nabla_k\phi_1 + S(y)\frac{\partial\phi_1}{\partial z}\right\}. \tag{3.12}$$

The first term on the right is

$$\mathbb{E}\left\{V\left(y\right)\cdot\nabla_k\phi_1\right\}=\frac{\partial}{\partial k_j}\mathbb{E}\left\{V_j(y)\left[\int_0^\infty V_m(y-sk)e^{-\theta s}ds\frac{\partial\bar{\phi}}{\partial k_m}+\int_0^\infty S(y-sk)e^{-\theta s}ds\frac{\partial\bar{\phi}}{\partial z}\right]\right\}$$
$$\to\frac{\partial}{\partial k_j}\left(D_{mj}(k)\frac{\partial\bar{\phi}}{\partial k_m}\right)+\frac{\partial}{\partial k_j}\left(\int_0^\infty R_j^{SV}(sk)ds\frac{\partial\bar{\phi}}{\partial z}\right) \tag{3.13}$$

in the limit $\theta\to 0$. The second term in the right side of (3.12) is

$$\mathbb{E}\left\{S\left(y\right)\frac{\partial\phi_1}{\partial z}\right\}=\frac{\partial}{\partial z}\mathbb{E}\left\{S(y)\left[\int_0^\infty V_m(y-sk)e^{-\theta s}ds\frac{\partial\bar{\phi}}{\partial k_m}+\int_0^\infty S(y-sk)e^{-\theta s}ds\frac{\partial\bar{\phi}}{\partial z}\right]\right\}$$
$$\to-\frac{\partial}{\partial z}\left(\int_0^\infty R_m^{VS}(sk)ds\frac{\partial\bar{\phi}}{\partial k_m}\right)+D\frac{\partial^2\bar{\phi}}{\partial z^2}=\frac{\partial}{\partial z}\left(\int_0^\infty R_m^{SV}(-sk)ds\frac{\partial\bar{\phi}}{\partial k_m}\right)+D\frac{\partial^2\bar{\phi}}{\partial z^2} \tag{3.14}$$

as $\theta\to 0$. Putting together (3.13) and (3.14) we obtain

$$\frac{\partial\bar{\phi}}{\partial t}+k\cdot\nabla_x\bar{\phi}=\frac{\partial}{\partial k_j}\left(D_{mj}(k)\frac{\partial\bar{\phi}}{\partial k_m}\right)+\frac{\partial}{\partial k_j}\left(D_j(k)\frac{\partial\bar{\phi}}{\partial z}\right)+\frac{\partial}{\partial z}\left(D_j(-k)\frac{\partial\bar{\phi}}{\partial k_j}\right)+D\frac{\partial^2\bar{\phi}}{\partial z^2}. \tag{3.15}$$

4 Proof of Theorem 3.1

Theorem 3.1 is a simple corollary of the following proposition. Let us first introduce some notation. Given a function $G\in C_b^3([0,+\infty)\times\mathbb{R}_*^{2d})$ and $t\geq 0$ let us introduce

$$N_t(G)=G(t,X(t),K(t),Z(t))-G(0,X(0),K(0),Z(0))-\int_0^t(\partial_\rho+\mathcal{L})G(\rho,X(\rho),K(\rho),Z(\rho))d\rho$$

with the operator $\mathcal{L}$ defined in (3.6). We also denote by $\mathbb{R}_*^d:=\mathbb{R}^d\setminus\{0\}$ and $\mathbb{R}_*^{2d}:=\mathbb{R}^d\times\mathbb{R}_*^d$ to avoid the singular point $k=0$. For any non-negative integers p,q,r and a function $G(t,x,k)$ that has p, q and r derivatives in the respective variables we define

$$\|G\|_{p,q,r,s}:=\sum\sup_{(t,x,k,z)}|\partial_t^\alpha\partial_x^\beta\partial_k^\gamma\partial_z^\eta G(t,x,k,z)|. \tag{4.1}$$

The summation range covers all integers $0\leq\alpha\leq p$, $0\leq\eta\leq s$ and all integer valued multi-indices $|\beta|\leq q$ and $|\gamma|\leq r$. We denote by $C_b^{p,q,r,s}([0,+\infty)\times\mathbb{R}_*^{2d}\times\mathbb{R})$ the space of all functions G with $\|G\|_{p,q,r,s}<+\infty$. We shall also consider spaces of bounded and a suitable number of times continuously differentiable functions $C_b^{p,q}(\mathbb{R}_*^{2d})$ and $C_b^p(\mathbb{R}_*^d)$ with the respective norms $\|\cdot\|_{p,q}$ and $\|\cdot\|_p$.

We let $\mathcal{C}:=C([0,+\infty);\mathbb{R}^d\times\mathbb{R}_*^d\times\mathbb{R})$ be the set of continuous paths of $(X(t),K(t),Z(t))$. For any $u\leq v$ denote by $\mathcal{M}_u^v$ the σ-algebra of subsets of $\mathcal{C}$ generated by $(X(t),K(t),Z(t))$, $t\in[u,v]$. We write $\mathcal{M}^v:=\mathcal{M}_0^v$ and $\mathcal{M}$ for the σ algebra of Borel subsets of $\mathcal{C}$. It coincides with the smallest σ-algebra that contains all $\mathcal{M}^t$, $t\geq 0$. We define $\mathcal{C}(T,M)$ as the set of paths $\pi\in\mathcal{C}$ so that both $(2M)^{-1}\leq|K(t)|\leq 2M$, and

$$X(t)-X(u)+\int_u^t K(s)ds=0,\ \text{ for all }\ 0\leq u<t\leq T.$$

We denote by $\mathbb{P}_{x,k,z}^\delta$ the probability measure on $\mathcal{C}(T,M)$ induced by the trajectories of (3.2) and by $\mathbb{E}_{x,k,z}^\delta$ the corresponding expectation.

Proposition 4.1 *Suppose that* $(x,k) \in \mathcal{A}(M) = \mathbb{R}^d \times \{M^{-1} \le |k| \le M\}$ *for some* $M > 0$ *and that a test function* $\zeta \in C_b((\mathbb{R}_*^{2d})^n)$ *is non-negative. Let* $\gamma_0 \in (0,1/2)$ *and let* $0 \le t_1 < \cdots < t_n \le T_* \le t < v \le T$. *Assume further that* $v - t \ge \delta^{\gamma_0}$. *Then, there exist constants* $\gamma_1 > 0$, $C(T)$ *such that for any function* $G \in C^3([T_*,T] \times \mathbb{R}_*^{2d} \times \mathbb{R})$ *we have*

$$\left| \mathbb{E}^{\delta}_{x,k,z} \left\{ [N_v(G) - N_t(G)]\, \tilde{\zeta} \right\} \right| \le C(T)\delta^{\gamma_1}(v-t)\|G\|_{1,1,3}\mathbb{E}^{\delta}_{x,k,z}\tilde{\zeta}. \tag{4.2}$$

Here $\tilde{\zeta}(\pi) := \zeta(X(t_1), K(t_1), Z(t_1), \ldots, X(t_n), K(t_n), Z(t_n))$, *and* $\pi = (X(t), K(t), Z(t))$ *is any continuous path. The choice of the constants* γ_1, C *does not depend on* (x,k,z), $\delta \in (0,1]$, ζ, *the times* $t_1, \ldots, t_n, T_*, T, v, t$, *or the test function* G.

Proof of Theorem 3.1. Theorem 3.1 is a simple consequence of Proposition 4.1. Let $\phi_0(x,k,z)$ be a test function and let the function $\bar{\phi}(t,x,k,z)$ solve the initial value problem

$$\begin{aligned} \frac{\partial \bar{\phi}}{\partial t} &= \mathcal{L}\bar{\phi} \\ \bar{\phi}(0,x,k,z) &= \phi_0(x,k,z). \end{aligned} \tag{4.3}$$

We apply Proposition 4.1 with $G(t,x,k,z) = \bar{\phi}(u-t,x,k,z)$, $t = \delta^{\gamma}$ and $u > \delta^{\gamma}$ with $1/2 < \gamma < 1$ and take $\tilde{\zeta} = 1$. It follows from (4.2) that

$$\left| \mathbb{E}^{\delta}_{x,k,z} \left[\phi_0(X(u),K(u),Z(u)) - \bar{\phi}(u-\delta^{\gamma}, X(\delta^{\gamma}), K(\delta^{\gamma}), Z(\delta^{\gamma})) \right] \right| \le C\|G\|_{1,1,3}\delta^{\gamma_1}. \tag{4.4}$$

Using the fact that $\bar{\phi}$ is a smooth function and $1/2 < \gamma < 1$ we conclude that

$$\left| \mathbb{E}^{\delta}_{x,k,z} \left[\phi_0(X^{\delta}(u), K^{\delta}(u), Z^{\delta}(u)) - \bar{\phi}(u,x,k,z) \right] \right| \le C\|G\|_{1,1,3}\delta^{\gamma_1}. \tag{4.5}$$

The conclusion of Theorem 3.1 now follows. □

4.1 The proof of Proposition 4.1

The proof of Proposition 4.1 is technical and uses some of the ideas of [1, 3, 4, 5]. However, the present situation is much simpler than in the aforementioned papers as we already know from Theorem 2.1 that the process (X^{δ}, K^{δ}) converges to a process with the generator

$$\tilde{\mathcal{L}} = \frac{\partial}{\partial k_n}\left(D_{nm}(k) \frac{\partial}{\partial k_m} \right) - k \cdot \nabla_x.$$

This means that the law of the process X^{δ} will be close to that of the limit $X(t)$; and in particular $X^{\delta}(t)$ does not approach a narrow tube around its past trajectory with a probability very close to one. This is the main difference with [1, 3, 4, 5] where this potential return was a major obstacle in the proof.

The strategy of the proof is as follows. We will need to deal with objects of, say, the form

$$\mathbb{E}\left\{ G(X^{\delta}(s), K^{\delta}(s), Z^{\delta}(s)) S\left(\frac{X^{\delta}(s)}{\delta}\right) V\left(\frac{X^{\delta}(s')}{\delta}\right) \right\} \tag{4.6}$$

with $s < s'$ but the difference $s' - s$ small. Then we will consider a slight pullback in time

$$\sigma(s) = s - \delta^{1-\gamma}, \tag{4.7}$$

with a sufficiently small $\gamma > 0$ and a linearization

$$L(\sigma, s) = X^\delta(\sigma) - (s - \sigma)K^\delta(\sigma). \tag{4.8}$$

The characteristic equations (3.2) allow us to estimate the difference between (4.6) and

$$\mathbb{E}\left\{G(X^\delta(\sigma), K^\delta(\sigma), Z^\delta(\sigma))S\left(\frac{L(\sigma, s)}{\delta}\right)V\left(\frac{L(\sigma, s')}{\delta}\right)\right\}, \tag{4.9}$$

and show that it is small. However, the latter expectation approximately splits:

$$\begin{aligned}\mathbb{E}\left\{G(X^\delta(\sigma), K^\delta(\sigma), Z^\delta(\sigma))S\left(\frac{L(\sigma, s)}{\delta}\right)V\left(\frac{L(\sigma, s')}{\delta}\right)\right\} \approx \\ \mathbb{E}\left\{G(X^\delta(\sigma), K^\delta(\sigma), Z^\delta(\sigma))\right\} R^{SV}\left(\frac{L(\sigma, s') - L(\sigma, s)}{\delta}\right).\end{aligned} \tag{4.10}$$

The reason for the expectation splitting is that the argument of the function G in (4.9) depends only on the potential in a tube close to the trajectory $X^\delta(t)$ until the the time $t = \sigma$, while $L(\sigma, s)$ and $L(\sigma, s')$ are at distance much larger than δ from this tube with a probability close to one. Hence, the values of $G(X^\delta(\sigma), K^\delta(\sigma), Z^\delta(\sigma))$ and, say, of $V\left(\frac{L(\sigma, s')}{\delta}\right)$ are nearly independent and expectation (4.9) splits. This is formalized by the following mixing lemma.

For any $t \geq 0$ we denote by $\mathcal{F}_t$ the σ-algebra generated by $(X^\delta(s), K^\delta(s), Z^\delta(s))$, $s \leq t$. Here we suppress, for the sake of abbreviation, writing the initial data in the notation of the trajectory. We assume that $X_1, X_2 : (\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d^2})^2 \to \mathbb{R}$ are certain continuous functions, Z is a random variable and g_1, g_2 are $\mathbb{R}^d$-valued random vectors. We suppose further that Z, g_1, g_2, are $\mathcal{F}_t$-measurable, while $\tilde{X}_1, \tilde{X}_2$ are random fields of the form

$$\tilde{X}_i(x) = X_i\left(F(x), \nabla_x F(x), \nabla_x^2 F(x)\right),$$

where, as before we denote for brevity $F = (V, S)$. We also let

$$U(\theta_1, \theta_2) := \mathbb{E}\left[\tilde{X}_1(\theta_1)\tilde{X}_2(\theta_2)\right], \quad \theta_1, \theta_2 \in \mathbb{R}^d \tag{4.11}$$

and recall that $\phi(r)$ is the mixing coefficient defined in (2.9). The following mixing lemma is formalizing the expectation splitting and can be proved in the same way as Lemmas 5.2 and 5.3 of [1].

Lemma 4.2 *(i) Assume that $r, t \geq 0$ and*

$$\inf_{u \leq t}\left|g_i - \frac{X^\delta(u)}{\delta}\right| \geq \frac{r}{\delta}, \tag{4.12}$$

$\mathbb{P}$-a.s. on the set $Z \neq 0$ for $i = 1, 2$. Then we have

$$\left|\mathbb{E}\left[\tilde{X}_1(g_1)\tilde{X}_2(g_2)Z\right] - \mathbb{E}\left[U(g_1, g_2)Z\right]\right| \leq 2\phi\left(\frac{r}{2\delta}\right)\|X_1\|_{L^\infty}\|X_2\|_{L^\infty}\|Z\|_{L^1(\Omega)}. \tag{4.13}$$

(ii) Let $\mathbb{E}X_1(0) = 0$ and assume that g_2 satisfies (4.12),

$$\inf_{u \leq t}\left|g_1 - \frac{X^\delta(u)}{\delta}\right| \geq \frac{r + r_1}{\delta} \tag{4.14}$$

and $|g_1 - g_2| \geq r_1\delta^{-1}$ for some $r_1 \geq 0$, $\mathbb{P}$-a.s. on the event $Z \neq 0$. Then, we have

$$\left|\mathbb{E}\left[\tilde{X}_1(g_1)\tilde{X}_2(g_2)\,Z\right] - \mathbb{E}\left[U(g_1, g_2)Z\right]\right| \leq C\phi^{1/2}\left(\frac{r}{2\delta}\right)\phi^{1/2}\left(\frac{r_1}{2\delta}\right)\|X_1\|_{L^\infty}\|X_2\|_{L^\infty}\|Z\|_{L^1(\Omega)} \tag{4.15}$$

for some absolute constant $C > 0$. Here the function U is given by (4.11).

We proceed now with the proof of Proposition 4.1. Let $G(k,z)$ be a sufficiently smooth function. We will establish the approximate martingale property (4.2) for G. It suffices to consider functions of the form $G(z,k) = g(z)r(k)$. The characteristic equations

$$Z^\delta(t) = z + \frac{1}{\sqrt{\delta}} \int_0^t S\left(\frac{X^\delta(s)}{\delta}\right) ds,$$

and

$$K_j^\delta(t) = k + \frac{1}{\sqrt{\delta}} \int_0^t V_j\left(\frac{X^\delta(s)}{\delta}\right) ds,$$

imply that we have

$$\begin{aligned} &G(K^\delta(u), Z^\delta(u)) - G(K^\delta(t), Z^\delta(t)) \\ &= \frac{1}{\sqrt{\delta}} \int_t^u \left[g'(Z^\delta(s)) S\left(\frac{X^\delta(s)}{\delta}\right) r(K^\delta(s)) + g(Z^\delta(s)) r_j(K^\delta(s)) V_j\left(\frac{X^\delta(s)}{\delta}\right)\right] ds. \end{aligned} \tag{4.16}$$

Here and below, we use the notation V_j for V_{x_j} and similarly R_k for R_{x_k} to simplify. In order to be able to make a backward step σ as in (4.7) we split the above integral as

$$G(K^\delta(u), Z^\delta(u)) - G(K^\delta(t), Z^\delta(t)) = A + B \tag{4.17}$$

with

$$A = \frac{1}{\sqrt{\delta}} \int_t^{t+\delta^{1-\gamma}} \left[g'(Z^\delta(s)) r(K^\delta(s)) S\left(\frac{X^\delta(s)}{\delta}\right) + g(Z^\delta(s)) r_j(K^\delta(s)) V_j\left(\frac{X^\delta(s)}{\delta}\right)\right] ds$$

and

$$B = \frac{1}{\sqrt{\delta}} \int_{t+\delta^{1-\gamma}}^u \left[g'(Z^\delta(s)) r(K^\delta(s)) S\left(\frac{X^\delta(s)}{\delta}\right) + g(Z^\delta(s)) r_j(K^\delta(s)) V_j\left(\frac{X^\delta(s)}{\delta}\right)\right] ds = B_1 + B_2.$$

The first term is small:

$$|A| \le C\delta^{1/2-\gamma} \|G\|_{1,1}$$

provided that $\gamma < 1/2$. The term B will be analyzed with the aforementioned ideas of a backward step and linearization, and using the mixing lemma. The terms B_1 and B_2 may be written as

$$\begin{aligned} B_1 = \frac{1}{\sqrt{\delta}} \int_{t+\delta^{1-\gamma}}^u &\left\{ g'(Z^\delta(\sigma(s))) r(K^\delta(\sigma(s)) + [g'(Z^\delta(s)) r(K^\delta(s)) - g'(Z^\delta(\sigma(s))) r(K^\delta(\sigma(s))] \right\} \\ &\times S\left(\frac{X^\delta(s)}{\delta}\right) ds = I + II \end{aligned} \tag{4.18}$$

and

$$\begin{aligned} B_2 = \frac{1}{\sqrt{\delta}} \int_{t+\delta^{1-\gamma}}^u &\left\{ g(Z^\delta(\sigma(s))) r_j(K^\delta(\sigma(s)) + [g(Z^\delta(s)) r_j(K^\delta(s)) - g(Z^\delta(\sigma(s))) r_j(K^\delta(\sigma(s))] \right\} \\ &\times V_j\left(\frac{X^\delta(s)}{\delta}\right) ds = III + IV \end{aligned} \tag{4.19}$$

with

$$I = \frac{1}{\sqrt{\delta}} \int_{t+\delta^{1-\gamma}}^u g'(Z^\delta(\sigma(s))) r(K^\delta(\sigma(s))) S\left(\frac{X^\delta(s)}{\delta}\right) ds \tag{4.20}$$

and

$$II = \frac{1}{\sqrt{\delta}} \int_{t+\delta^{1-\gamma}}^{u} [g'(Z^\delta(s))r(K^\delta(s)) - g'(Z^\delta(\sigma(s)))r(K^\delta(\sigma(s))]S\left(\frac{X^\delta(s)}{\delta}\right) ds \tag{4.21}$$

while

$$III = \frac{1}{\sqrt{\delta}} \int_{t+\delta^{1-\gamma}}^{u} g(Z^\delta(\sigma(s)))r_j(K^\delta(\sigma(s))V_j\left(\frac{X^\delta(s)}{\delta}\right) ds \tag{4.22}$$

and

$$IV = \frac{1}{\sqrt{\delta}} \int_{t+\delta^{1-\gamma}}^{u} [g(Z^\delta(s))r_j(K^\delta(s)) - g(Z^\delta(\sigma(s)))r_j(K^\delta(\sigma(s))]V_j\left(\frac{X^\delta(s)}{\delta}\right) ds. \tag{4.23}$$

We summarize the contributions of each of the terms above in the following lemma.

Lemma 4.3 *The terms I, II, III and IV satisfy the following estimates:*

$$\left|\mathbb{E}\left\{\left[I - \int_t^u \int_0^\infty \theta \Delta R^{SV}\left(\theta K^\delta(s)\right) d\theta g'(Z^\delta(s))r(K^\delta(s))ds\right]\tilde{\zeta}\right\}\right| \tag{4.24}$$
$$\le C\delta^{\gamma_1}\|G\|_{2,2}\left[\|\zeta\|_\infty + \mathbb{E}\left\{\tilde{\zeta}\right\}\right](u-t),$$

$$\left|\mathbb{E}\left[\left\{II - \int_t^u \int_0^\infty g''(Z^\delta(s))r(K^\delta(s))R_{SS}(\theta K^\delta(s))d\theta ds\right.\right.\right. \tag{4.25}$$
$$\left.\left.\left. - \int_t^u \int_0^\infty g'(Z^\delta(s))r_j(K^\delta(s))R_j^{SV}(\theta K^\delta(s))d\theta ds\right\}\tilde{\zeta}\right]\right| \le C\delta^{\gamma_2}\|G\|_3\left[\|\zeta\|_\infty + E\left\{\tilde{\zeta}\right\}\right](u-t),$$

$$\left|\mathbb{E}\left\{III\tilde{\zeta}\right\} + \int_t^u \int_0^\infty \theta g(Z^\delta(s))r_j(K^\delta(s))\Delta R_j^{VV}\left(\theta K^\delta(s)\right) d\theta ds\right| \tag{4.26}$$
$$\le C\delta^\alpha\|G\|_1\left(\|\zeta\|_\infty + \mathbb{E}[\tilde{\zeta}]\right)(u-t)$$

$$\left|\mathbb{E}\{IV\tilde{\zeta}\} - \int_t^u \int_0^\infty g'(Z^\delta(s))r_j(K^\delta(s))R_j^{SV}\left(-\theta K^\delta(s)\right) d\theta ds\right. \tag{4.27}$$
$$\left. + \int_t^u \int_0^\infty g(Z^\delta(s))r_{jm}(K^\delta(s))R_{mj}^{VV}\left(\theta K^\delta(s)\right) d\theta ds\tilde{\zeta}\right| \le C\delta^\alpha\|G\|_2\left(\|\tilde{\zeta}\|_\infty + E\{\tilde{\zeta}\}\right)(u-t).$$

It remains now to prove Lemma 4.3 as the four individual contributions above combine to the operator $\mathcal{L}$ in (3.6).

4.2 The estimate for I

We first recall the linear approximation (4.8) and define the interpolation

$$R(v,\sigma,s) = (1-v)L(\sigma,s) + vX^\delta(s).$$

This allows us to write a linear approximation for S as

$$S\left(\frac{X^\delta(s)}{\delta}\right) = S\left(\frac{R(1,\sigma,s)}{\delta}\right) = S\left(\frac{R(0,\sigma,s)}{\delta}\right) + \frac{1}{\delta}\int_0^1 S_i\left(\frac{R(v,\sigma,s)}{\delta}\right)(X_i^\delta(s) - L_i(\sigma,s))dv$$
$$= S\left(\frac{L(\sigma,s)}{\delta}\right) + \frac{1}{\delta}\int_0^1 S_i\left(\frac{R(v,\sigma,s)}{\delta}\right)(X_i^\delta(s) - L_i(\sigma,s))dv.$$

Now we split I as

$$I = J_1 + J_2 \tag{4.28}$$

according to the above, with

$$J_1 = \frac{1}{\sqrt{\delta}} \int_{t+\delta^{1-\gamma}}^{u} g'(Z^\delta(\sigma(s)))r(K^\delta(\sigma(s))S\left(\frac{L(\sigma,s)}{\delta}\right) ds \tag{4.29}$$

and

$$J_2 = \frac{1}{\delta^{3/2}} \int_{t+\delta^{1-\gamma}}^{u} \int_0^1 g'(Z^\delta(\sigma(s)))r(K^\delta(\sigma(s)))S_i\left(\frac{R(v,\sigma,s)}{\delta}\right)(X_i^\delta(s) - L_i(\sigma,s))dvds. \tag{4.30}$$

The term J_1 is ready for an application of the mixing lemma: the arguments of the function G (that is, of g and r), and of the field S are separated by a distance of the order $O(\delta^{1-\gamma})$ that is much larger than δ, with a probability close to one. In order to make this statement precise we introduce a stopping time τ_δ that ensures that until τ_δ the trajectory $X^\delta(t)$ "goes forward" and does not come back to its past.

Let $0 < \varepsilon_1 < \varepsilon_2 < 1/2$, $\varepsilon_3 \in (0, 1/2 - \varepsilon_2)$, $\varepsilon_4 \in (1/2, 1 - \varepsilon_1 - \varepsilon_2)$ be small positive constants and set

$$N = [\delta^{-\varepsilon_1}], \quad p = [\delta^{-\varepsilon_2}], \quad q = p\,[\delta^{-\varepsilon_3}], \quad N_1 = Np\,[\delta^{-\varepsilon_4}]. \tag{4.31}$$

The requirement is that ε_i, $i \in \{1,2,3\}$ should be sufficiently small and ε_4 is bigger than $1/2$, less than one and can be made as close to one as we would need it. It is important that $\varepsilon_1 < \varepsilon_2$ so that $N \ll p$ when $\delta \ll 1$. We introduce the following $(\mathcal{M}^t)_{t\geq 0}$ stopping times. Let $t_k^{(p)} := kp^{-1}$ be a mesh of times, and $\pi \in \mathcal{C}$ be a path. We define the "violent turn" stopping time

$$V_\delta(\pi) := \inf\left[t \geq 0 : \text{ for some } k \geq 0 \text{ we have } t \in \left[t_k^{(p)}, t_{k+1}^{(p)}\right) \text{ and} \right. \tag{4.32}$$
$$\left. \hat{K}(t_{k-1}^{(p)}) \cdot \hat{K}(t) \leq 1 - \frac{1}{N}, \text{ or } \hat{K}\left(t_k^{(p)} - \frac{1}{N_1}\right) \cdot \hat{K}(t) \leq 1 - \frac{1}{N}\right],$$

where by convention we set $\hat{K}(-1/p) := \hat{K}(0)$. Note that with the above choice of ε_4 we have $\hat{K}\left(t_k^{(p)} - 1/N_1\right) \cdot \hat{K}(t_k^{(p)}) > 1 - 1/N$, provided that $\delta \in (0, \delta_0]$ and δ_0 is sufficiently small. The stopping time V_δ is triggered when the trajectory performs a sudden turn; this is undesirable as the trajectory may then return back to the region it has already visited and create correlations with the past.

For each $t \geq 0$, we denote by $\mathfrak{X}_t(\pi) := \bigcup_{0\leq s\leq t} X(s;\pi)$ the trace of the spatial component of the path π up to time t, and by $\mathfrak{X}_t(q;\pi) := [x : \text{dist }(x, \mathfrak{X}_t(\pi)) \leq 1/q]$ a tubular region around the path. We introduce the stopping time

$$U_\delta(\pi) := \inf\left[t \geq 0 : \exists\, k \geq 1 \text{ and } t \in [t_k^{(p)}, t_{k+1}^{(p)}) \text{ for which } X(t) \in \mathfrak{X}_{t_{k-1}^{(p)}}(q)\right]. \tag{4.33}$$

It is associated with the return of the X component of the trajectory to the tube around its past; this is again an undesirable way to create correlations with the past. Finally, we set the stopping time

$$\tau_\delta(\pi) := V_\delta(\pi) \wedge U_\delta(\pi). \tag{4.34}$$

Lemma 4.4 *[4] The probability of the event $[\,\tau_\delta < T\,]$ for a fixed $T > 0$ goes to zero, as $\delta \to 0$: there exists $\alpha_0 > 0$ so that*

$$\mathbb{P}\{[\,\tau_\delta < T\,]\} \leq C(T)\delta^{\alpha_0}. \tag{4.35}$$

We apply part (i) of Lemma 4.2 to $E\{J_1\tilde{\zeta}\}$ with $\tilde{X}_1(x) = S(x)$ and $\tilde{X}_2 = 1$,

$$\mathcal{Z} = g'(Z^\delta(\sigma))r(K^\delta(\sigma(s)))\mathbf{1}[\tau_\delta > T]\tilde{\zeta},$$

and $g_1 = L(\sigma, s)/\delta$. Note that g_1 and Z are both $\mathcal{F}_\sigma$ measurable. It follows from the definition of the stopping time τ_δ that when $\mathcal{Z} \neq 0$ then the linearization also stays away from the past trajectory: for all $0 \leq \rho \leq \sigma$ we have

$$\left|L(\sigma, s) - X^\delta(\rho)\right| \geq C\delta^{1-\gamma} \tag{4.36}$$

and hence

$$\inf_{0\leq\rho\leq\sigma(s)}\left|g_1 - \frac{X^\delta(\rho)}{\delta}\right| \geq \frac{r}{\delta}$$

with $r = C\delta^{1-\gamma}$. We also decompose J_1 according to whether the stopping time has occurred or not before time T:

$$\begin{aligned} J_1 &= \frac{1}{\sqrt{\delta}}\int_{t+\delta^{1-\gamma}}^u g'(Z^\delta(\sigma(s)))r(K^\delta(\sigma(s)))\mathbf{1}[\tau_\delta > T]S\left(\frac{L(\sigma,s)}{\delta}\right)ds \\ &+ \frac{1}{\sqrt{\delta}}\int_{t+\delta^{1-\gamma}}^u g'(Z^\delta(\sigma(s)))r(K^\delta(\sigma(s)))(1 - \mathbf{1}[\tau_\delta > T])S\left(\frac{L(\sigma,s)}{\delta}\right)ds = J_{11} + J_{12}. \end{aligned}$$

However, (4.35) implies that

$$\mathbb{E}\left\{\left|J_{12}\tilde{\zeta}\right|\right\} \leq C\delta^{\alpha_0}\|G\|_{1,1}(u-t) \tag{4.37}$$

so we have to deal only with J_{11}. Using the mixing lemma as above, with the point separation as in (4.36), and the fact that $\mathbb{E}[S(x)] = 0$ (whence $U = 0$ in (4.13)) we estimate

$$\left|\mathbb{E}\left(J_{11}\tilde{\zeta}\right)\right| \leq \frac{C}{\sqrt{\delta}}\phi\left(C\delta^{-\gamma}\right)(u-t)\|g\|_1\|r\|_0\mathbb{E}[\tilde{\zeta}] \leq C\delta\|G\|_{1,1}\mathbb{E}[\tilde{\zeta}](u-t). \tag{4.38}$$

Here $\phi(r)$ is the mixing coefficient that decays faster than any power of r; see (2.10). We conclude that

$$\left|\mathbb{E}\left(J_1\tilde{\zeta}\right)\right| \leq C\delta\|G\|_{1,1}\mathbb{E}[\tilde{\zeta}](u-t) + C\delta^{\alpha_0}\|G\|_{1,1}(u-t) \tag{4.39}$$

so the term J_1 produces only a small contribution.

Now we estimate the second term J_2 in (4.28); it is given explicitly by (4.30). We split it further by using the next order expansion

$$S_i\left(\frac{R(v,\sigma,s)}{\delta}\right) = S_i\left(\frac{L(\sigma,s)}{\delta}\right) + \frac{1}{\delta}\int_0^v S_{ij}\left(\frac{R(\theta,\sigma,s)}{\delta}\right)(X_j^\delta(s) - L_j(\sigma,s))d\theta.$$

This leads to the corresponding expression $J_2 = J_{21} + J_{22}$ with

$$J_{21} = \frac{1}{\delta^{3/2}}\int_{t+\delta^{1-\gamma}}^u g'(Z^\delta(\sigma(s)))r(K^\delta(\sigma(s)))S_i\left(\frac{L(\sigma,s)}{\delta}\right)(X_i^\delta(s) - L_i(\sigma,s))ds$$

and

$$\begin{aligned} J_{22} = \frac{1}{\delta^{5/2}}\int_{t+\delta^{1-\gamma}}^u\int_0^1\int_0^v g'(Z^\delta(\sigma(s)))r(K^\delta(\sigma(s)))S_{ij}\left(\frac{R(\theta,\sigma,s)}{\delta}\right) \\ \times (X_i^\delta(s) - L_i(\sigma,s))(X_j^\delta(s) - L_j(\sigma,s))d\theta dv ds. \end{aligned}$$

Note that the characteristic equations and the definition (4.7) of the time $\sigma(s)$ imply that

$$|L(\sigma,s)-X^\delta(s)|\le C\delta^{2-2\gamma-1/2}=C\delta^{3/2-2\gamma}. \tag{4.40}$$

It follows that

$$\left|\mathbb{E}\left\{J_{22}\tilde\zeta\right\}\right|\le C\|G\|_{1,1}\mathbb{E}\{\tilde\zeta\}\delta^{-5/2}\delta^{3-4\gamma}(u-t)\le C\delta^{1/2-4\gamma}\|G\|_{1,1}\mathbb{E}\{\tilde\zeta\}(u-t), \tag{4.41}$$

which is small if $\gamma<1/8$. This is an important characteristic feature of the weak coupling limit; after several linearizations the remainder becomes deterministically small while the linearized terms may be controlled with the mixing lemma.

Next, we look at J_{21} that is the only potentially surviving (not small) in the limit $\delta\to 0$ contribution to I: to do so we write, using the evolution equation for X^δ and a further linearization for the function V_i:

$$\begin{aligned}
X_i^\delta(s)-L_i(\sigma,s)&=-\int_\sigma^s[K_i^\delta(v)-K_i^\delta(\sigma)]dv=-\frac{1}{\sqrt\delta}\int_\sigma^s\int_\sigma^v V_i\left(\frac{X^\delta(\rho)}{\delta}\right)d\rho dv\\
&=-\frac{1}{\sqrt\delta}\int_\sigma^s(s-\rho)V_i\left(\frac{X^\delta(\rho)}{\delta}\right)d\rho=-\frac{1}{\sqrt\delta}\int_\sigma^s(s-\rho)V_i\left(\frac{L(\sigma,\rho)}{\delta}\right)d\rho\\
&-\frac{1}{\sqrt\delta}\int_\sigma^s(s-\rho)\left[V_i\left(\frac{X^\delta(\rho)}{\delta}\right)-V_i\left(\frac{L(\sigma,\rho)}{\delta}\right)\right]d\rho=-\frac{1}{\sqrt\delta}\int_\sigma^s(s-\rho)V_i\left(\frac{L(\sigma,\rho)}{\delta}\right)d\rho\\
&-\frac{1}{\delta^{3/2}}\int_\sigma^s\int_0^1(s-\rho)V_{im}\left(\frac{R(v,\sigma,\rho)}{\delta}\right)\left(X_m^\delta(\rho)-L_m(\sigma,\rho)\right)dvd\rho.
\end{aligned}\tag{4.42}$$

This produces a further split

$$J_{21}=J_{21}^1+J_{21}^2$$

with

$$J_{21}^1=-\frac{1}{\delta^2}\int_{t+\delta^{1-\gamma}}^u\int_\sigma^s(s-\rho)g'(Z^\delta(\sigma(s)))r(K^\delta(\sigma(s)))S_i\left(\frac{L(\sigma,s)}{\delta}\right)V_i\left(\frac{L(\sigma,\rho)}{\delta}\right)d\rho ds$$

and

$$\begin{aligned}
J_{21}^2=-\frac{1}{\delta^3}\int_{t+\delta^{1-\gamma}}^u\int_\sigma^s\int_0^1(s-\rho)g'(Z^\delta(\sigma(s)))r(K^\delta(\sigma(s)))S_i\left(\frac{L(\sigma,s)}{\delta}\right)V_{im}\left(\frac{R(v,\sigma,\rho)}{\delta}\right)\\
\times\left(X_m^\delta(\rho)-L_m(\sigma,\rho)\right)dvd\rho ds.
\end{aligned}$$

Note that we have linearized enough to achieve a deterministic estimate

$$|J_{21}^2|\le C\delta^{-3}\delta^{1-\gamma}\delta^{1-\gamma}\delta^{3/2-\gamma}\|G\|_1(u-t)=C\delta^{1/2-3\gamma}\|G\|_{1,1}(u-t) \tag{4.43}$$

which is small if $\gamma<1/6$. Hence, J_{21}^1 remains the only potentially contributing term in I: it is analyzed with the help of the mixing lemma. We take

$$g_1=\frac{L(\sigma,s)}{\delta},\quad g_2=\frac{L(\sigma,\rho)}{\delta},\quad X_1(x)=S_i(x),\quad X_2(x)=V_i(x),\quad r=\rho-\sigma,\quad r_1=s-\rho,$$

and

$$\mathcal{Z}=g'(Z^\delta(\sigma))r(K^\delta(\sigma(s)))\mathbf{1}[\tau_\delta>T]\tilde\zeta.$$

Let us denote

$$R_{ij}^{SV}(x) = -\mathbb{E}\left\{S_i(y)V_j(x+y)\right\}.$$

Observe that

$$R_{ij}^{SV}(x) = -\mathbb{E}\left\{S_i(y)V_j(x+y)\right\} = -\frac{\partial}{\partial x_j}\mathbb{E}\left\{S_i(y)V(x+y)\right\} = \frac{\partial}{\partial x_j}\mathbb{E}\left\{S(y)V_i(x+y)\right\} = \frac{\partial^2 R^{SV}(x)}{\partial x_i \partial x_j}.$$

The mixing lemma implies that

$$\left|\mathbb{E}\left[J_{21}^1\tilde{\zeta}\right] + \frac{1}{\delta^2}\int\limits_{t+\delta^{1-\gamma}}^{u}\int\limits_{\sigma}^{s}(s-\rho)\mathbb{E}\left\{g'(Z^\delta(\sigma(s)))r(K^\delta(\sigma(s)))\left(-R_{ii}^{SV}\left(\frac{L(\sigma,\rho)-L(\sigma,s)}{\delta}\right)\right)\right\}d\rho ds\right|$$
$$\leq \frac{C\|G\|_{1,1}\mathbb{E}(\tilde{\zeta})}{\delta^2}\int\limits_{t+\delta^{1-\gamma}}^{u}\int\limits_{\sigma}^{s}(s-\rho)\phi^{1/2}\left(\frac{\rho-\sigma}{\delta}\right)\phi^{1/2}\left(\frac{s-\rho}{\delta}\right)d\rho ds \leq C\delta\|G\|_{1,1}\mathbb{E}(\tilde{\zeta})(u-t)$$

since the mixing coefficient ϕ is rapidly decaying. It follows that

$$\left|\mathbb{E}\left[J_{21}^1\tilde{\zeta}\right] - \frac{1}{\delta^2}\int_{t+\delta^{1-\gamma}}^{u}\int_{\sigma}^{s}(s-\rho)\mathbb{E}\left\{g'(Z^\delta(\sigma(s)))r(K^\delta(\sigma(s)))\left(\Delta R^{SV}\left(\frac{(s-\rho)K^\delta(\sigma)}{\delta}\right)\right)\right\}d\rho ds\right|$$
$$\leq C\delta\|G\|_{1,1}\mathbb{E}(\tilde{\zeta})(u-t)$$

The integral above from σ to s may be massaged as

$$\frac{1}{\delta^2}\int_{\sigma}^{s}(s-\rho)\left(\Delta R^{SV}\left(\frac{(s-\rho)K^\delta(\sigma)}{\delta}\right)\right)d\rho = \int_0^{\delta^{-\gamma}}\theta\Delta R^{SV}\left(\theta K^\delta(\sigma)\right)d\theta.$$

Observe that

$$|K^\delta(s) - K^\delta(\sigma)| \leq C\delta^{1/2-\gamma}$$

and the function R^{SV} is smooth and rapidly decaying. This allows us to replace σ in the argument of R^{SV} by s. The same can be done with the functions g and r; we conclude that

$$\left|\mathbb{E}\left\{\left[J_{21}^1 - \int_t^u\int_0^\infty \theta\Delta R^{SV}\left(\theta K^\delta(s)\right)d\theta g'(Z^\delta(s))r(K^\delta(s))ds\right]\tilde{\zeta}\right\}\right| \leq C\delta^{\gamma_1}\|G\|_{2,2}\mathbb{E}\left\{\tilde{\zeta}\right\}(u-t) \tag{4.44}$$

with some $\gamma_1 > 0$. Therefore, putting all the work of this section together, see (4.39), (4.41) and (4.43), we also have

$$\left|\mathbb{E}\left\{\left[I - \int_t^u\int_0^\infty \theta\Delta R^{SV}\left(\theta K^\delta(s)\right)d\theta g'(Z^\delta(s))r(K^\delta(s))ds\right]\tilde{\zeta}\right\}\right| \leq C\delta^{\gamma_1}\|G\|_{2,2}\left[1+\mathbb{E}\left\{\tilde{\zeta}\right\}\right](u-t). \tag{4.45}$$

This proves the estimate (4.24) in Lemma 4.3.

4.3 Estimate for II

We now look at the term II given by (4.21) and split it further as

$$II = \frac{1}{\sqrt{\delta}} \int_{t+\delta^{1-\gamma}}^{u} [g'(Z^\delta(s))r(K^\delta(s)) - g'(Z^\delta(\sigma(s)))r(K^\delta(\sigma(s))]S\left(\frac{X^\delta(s)}{\delta}\right) ds$$
$$= \frac{1}{\delta} \int_{t+\delta^{1-\gamma}}^{u} \int_{\sigma(s)}^{s} g''(Z^\delta(\rho))r(K^\delta(\rho))S\left(\frac{X^\delta(\rho)}{\delta}\right) S\left(\frac{X^\delta(s)}{\delta}\right) d\rho ds$$
$$+ \frac{1}{\delta} \int_{t+\delta^{1-\gamma}}^{u} \int_{\sigma(s)}^{s} g'(Z^\delta(\rho))r_j(K^\delta(\rho))V_j\left(\frac{X^\delta(\rho)}{\delta}\right) S\left(\frac{X^\delta(s)}{\delta}\right) d\rho ds,$$

The estimation of II is very similar to that of I both in spirit and in mechanics but is even somewhat simpler since as all we have to justify is the replacement of the arguments of S and V_j by the corresponding linear approximation from the time $\sigma(s)$. This is done as in the previous section with the help of the mixing lemma and linearization and leads to

$$\left|\mathbb{E}\left\{II\tilde{\zeta}\right\} - \mathbb{E}\left\{II'\tilde{\zeta}\right\}\right| \le C\delta^{\gamma_2}\|G\|_{3,3}\left[1+\mathbb{E}\{\tilde{\zeta}\}\right](u-t) \qquad (4.46)$$

with

$$II' = \frac{1}{\delta} \int_{t+\delta^{1-\gamma}}^{u} \int_{\sigma(s)}^{s} g''(Z^\delta(\sigma))r(K^\delta(\sigma))S\left(\frac{L(\sigma,\rho)}{\delta}\right) S\left(\frac{L(\sigma,s)}{\delta}\right) d\rho ds$$
$$+ \frac{1}{\delta} \int_{t+\delta^{1-\gamma}}^{u} \int_{\sigma(s)}^{s} g'(Z^\delta(\sigma))r_j(K^\delta(\sigma))V_j\left(\frac{L(\sigma,\rho)}{\delta}\right) S\left(\frac{L(\sigma,s)}{\delta}\right) d\rho ds = II_1 + II_2.$$

The mixing lemma and rapid decay of the mixing coefficient imply that

$$\left|\mathbb{E}\left[\left\{II_1 - \frac{1}{\delta}\int_{t+\delta^{1-\gamma}}^{u}\int_{\sigma}^{s} g''(Z^\delta(\sigma))r(K^\delta(\sigma))R^{SS}\left(\frac{L(\sigma,s)-L(\sigma,\rho)}{\delta}\right) d\rho ds\right\}\tilde{\zeta}\right]\right| \le C\delta\|G\|_3 E\{\tilde{\zeta}\}(u-t)$$

The inner integral may be re-written as

$$\int_{\sigma}^{s} g''(Z^\delta(\sigma))r(K^\delta(\sigma))R_{SS}\left(\frac{L(\sigma,s)-L(\sigma,\rho)}{\delta}\right) d\rho$$
$$= \int_{\sigma}^{s} g''(Z^\delta(\sigma))r(K^\delta(\sigma))R_{SS}\left(-\frac{(s-\rho)K^\delta(\sigma)}{\delta}\right) d\rho$$
$$= \int_{0}^{\delta^{-\gamma}} g''(Z^\delta(\sigma))r(K^\delta(\sigma))R_{SS}(\theta K^\delta(\sigma))d\theta$$

Using the rapid decay of R_{SS}, smoothness of G and closeness of s and σ we obtain

$$\left|\mathbb{E}\left[\left\{II_1 - \int_t^u \int_0^\infty g''(Z^\delta(s))r(K^\delta(s))R_{SS}(\theta K^\delta(s))d\theta ds\right\}\tilde{\zeta}\right]\right| \le C\delta^{\gamma_2}\|G\|_3 E\left\{\tilde{\zeta}\right\}(u-t). \qquad (4.47)$$

Similarly, we have for II_2:

$$\left|\mathbb{E}\left[\left\{II_2+\frac{1}{\delta}\int\limits_{t+\delta^{1-\gamma}}^{u}\int\limits_{\sigma}^{s}g'(Z^\delta(\sigma))r_j(K^\delta(\sigma))R_j^{VS}\Big(\frac{L(\sigma,s)-L(\sigma,\rho)}{\delta}\Big)d\rho ds\right\}\tilde{\zeta}\right]\right|\le C\delta\|G\|_3E\{\tilde{\zeta}\}(u-t)$$

The inner integral may be re-written as

$$\begin{aligned}
&\int\limits_{\sigma}^{s}g'(Z^\delta(\sigma))r_j(K^\delta(\sigma))R_j^{VS}\left(\frac{L(\sigma,s)-L(\sigma,\rho)}{\delta}\right)d\rho\\
&=\int_{\sigma}^{s}g'(Z^\delta(\sigma))r_j(K^\delta(\sigma))R_j^{VS}\left(-\frac{(s-\rho)K^\delta(\sigma)}{\delta}\right)d\rho\\
&=\int_0^{\delta^{-\gamma}}g'(Z^\delta(\sigma))r_j(K^\delta(\sigma))R_j^{VS}(-\theta K^\delta(\sigma))d\theta=-\int_0^{\delta^{-\gamma}}g'(Z^\delta(\sigma))r_j(K^\delta(\sigma))R_j^{SV}(\theta K^\delta(\sigma))d\theta
\end{aligned}$$

Using the rapid decay of R_{SS}, smoothness of G and closeness of s and σ we obtain

$$\left|\mathbb{E}\left[\left\{II_2-\int_t^u\int_0^\infty g'(Z^\delta(s))r_j(K^\delta(s))R_j^{SV}(\theta K^\delta(s))d\theta ds\right\}\tilde{\zeta}\right]\right|\le C\delta^{\gamma_2}\|G\|_3E\left\{\tilde{\zeta}\right\}(u-t). \quad (4.48)$$

Putting (4.46), (4.47) and (4.47) together we obtain (4.25).

4.4 Estimate of III

We look at the third term in (4.19) given by (4.22)

$$III=\frac{1}{\sqrt{\delta}}\int_{t+\delta^{1-\gamma}}^{u}g(Z^\delta(\sigma(s)))r_j(K^\delta(\sigma(s))V_j\left(\frac{X^\delta(s)}{\delta}\right)ds. \quad (4.49)$$

This term is similar to I in that before using the mixing lemma we have to expand a little bit:

$$\begin{aligned}
V_j\left(\frac{X^\delta(s)}{\delta}\right)&=V_j\left(\frac{R(1,\sigma,s)}{\delta}\right)=V_j\left(\frac{R(0,\sigma,s)}{\delta}\right)+\int_0^1\frac{d}{dv}V_j\left(\frac{R(v,\sigma,s)}{\delta}\right)dv\\
&=V_j\left(\frac{L(\sigma,s)}{\delta}\right)+\frac{1}{\delta}\int_0^1V_{jk}\left(\frac{R(v,\sigma,s)}{\delta}\right)(X_k^\delta(s)-L_k(\sigma,s))dv.
\end{aligned}$$

Accordingly we split III as $III=III_1+III_2$ with

$$III_1=\frac{1}{\sqrt{\delta}}\int_{t+\delta^{1-\gamma}}^{u}g(Z^\delta(\sigma(s)))r_j(K^\delta(\sigma(s))V_j\left(\frac{L(\sigma,s)}{\delta}\right)ds \quad (4.50)$$

and

$$III_2=\frac{1}{\delta^{3/2}}\int_{t+\delta^{1-\gamma}}^{u}\int_0^1g(Z^\delta(\sigma(s)))r_j(K^\delta(\sigma(s))V_{jk}\left(\frac{R(v,\sigma,s)}{\delta}\right)(X_k^\delta(s)-L_k(\sigma,s))dvds. \quad (4.51)$$

The expectation of the first term on the event $[\tau_\delta>T]$ is small by the mixing lemma because the points $X^\delta(\sigma)$ and $L(\sigma,s)$ are at distance of order $\delta^{1-\gamma}$, $\mathbb{E}[V_j(x)]=0$, and the mixing coefficient is rapidly decaying. On the other hand, $\mathbb{P}[\tau_\delta<T]$ is small according to Lemma 4.4. We conclude that

$$\left|\mathbb{E}\left\{III_1\tilde{\zeta}\right\}\right|\le C\delta\|G\|_1\left(\|\zeta\|_\infty+\mathbb{E}[\tilde{\zeta}]\right)(u-t). \quad (4.52)$$

In order to estimate III_2 we write

$$V_{jk}\left(\frac{R(v,\sigma,s)}{\delta}\right) = V_{jk}\left(\frac{L(\sigma,s)}{\delta}\right) + \frac{1}{\delta}\int_0^v V_{jkm}\left(\frac{R(\theta,\sigma,s)}{\delta}\right)(X_m^\delta(s) - L_m(\sigma,s))d\theta,$$

which splits $III_2 = III_{21} + III_{22}$ (note that the v-variable integrates out in III_{21}):

$$III_{21} = \frac{1}{\delta^{3/2}}\int_{t+\delta^{1-\gamma}}^u g(Z^\delta(\sigma(s)))r_j(K^\delta(\sigma(s))V_{jk}\left(\frac{L(\sigma,s)}{\delta}\right)(X_k^\delta(s) - L_k(\sigma,s))ds, \tag{4.53}$$

and

$$\begin{aligned} III_{22} = \frac{1}{\delta^{5/2}}\int_{t+\delta^{1-\gamma}}^u \int_0^1\int_0^v g(Z^\delta(\sigma(s)))r_j(K^\delta(\sigma(s))V_{jkm}\left(\frac{R(\theta,\sigma,s)}{\delta}\right) \\ \times (X_m^\delta(s) - L_m(\sigma,s))(X_k^\delta(s) - L_k(\sigma,s))d\theta dv ds. \end{aligned} \tag{4.54}$$

We have linearized sufficiently to make III_{22} be deterministically small because of (4.40):

$$|III_{22}| \le C\delta^{-5/2}\delta^{3-2\gamma}\|G\|_{1,1}(u-t). \tag{4.55}$$

This leaves us with III_{21} to take care of. This we do with the help of (4.42)

$$\begin{aligned} X_i^\delta(s) - L_i(\sigma,s) &= -\int_\sigma^s [K(v) - K(\sigma)]dv = -\frac{1}{\sqrt{\delta}}\int_\sigma^s\int_\sigma^v V_j\left(\frac{X^\delta(\rho)}{\delta}\right)d\rho dv \\ &= -\frac{1}{\sqrt{\delta}}\int_\sigma^s (s-\rho)V_j\left(\frac{X^\delta(\rho)}{\delta}\right)d\rho = -\frac{1}{\sqrt{\delta}}\int_\sigma^s (s-\rho)V_j\left(\frac{L(\sigma,\rho)}{\delta}\right)d\rho \\ &-\frac{1}{\sqrt{\delta}}\int_\sigma^s (s-\rho)\left[V_j\left(\frac{X^\delta(\rho)}{\delta}\right) - V_j\left(\frac{L(\sigma,\rho)}{\delta}\right)\right]d\rho = -\frac{1}{\sqrt{\delta}}\int_\sigma^s (s-\rho)V_j\left(\frac{L(\sigma,\rho)}{\delta}\right)d\rho \\ &-\frac{1}{\delta^{3/2}}\int_\sigma^s\int_0^1 (s-\rho)V_{jm}\left(\frac{R(v,\sigma,\rho)}{\delta}\right)\left(X_m^\delta(\rho) - L_m(\sigma,\rho)\right)dv d\rho. \end{aligned}$$

that allows us to decompose $III_{21} = III_{21}^1 + III_{21}^2$ with

$$III_{21}^1 = -\frac{1}{\delta^2}\int_{t+\delta^{1-\gamma}}^u\int_\sigma^s (s-\rho)g(Z^\delta(\sigma(s)))r_j(K^\delta(\sigma(s))V_{jk}\left(\frac{L(\sigma,s)}{\delta}\right)V_j\left(\frac{L(\sigma,\rho)}{\delta}\right)d\rho ds, \tag{4.56}$$

and

$$\begin{aligned} III_{21}^2 = -\frac{1}{\delta^3}\int_{t+\delta^{1-\gamma}}^u\int_\sigma^s\int_0^1 (s-\rho)g(Z^\delta(\sigma(s)))r_j(K^\delta(\sigma(s))V_{jk}\left(\frac{L(\sigma,s)}{\delta}\right)V_{jm}\left(\frac{R(v,\sigma,\rho)}{\delta}\right) \\ \times\left(X_m^\delta(\rho) - L_m(\sigma,\rho)\right)dv d\rho ds. \end{aligned} \tag{4.57}$$

Again, III_{21}^2 is deterministically small because of (4.40):

$$|III_{21}^2| \le C\delta^{-3}\delta^{2-2\gamma}\delta^{3/2-\gamma}\|G\|_{1,1}(u-t) \le C\delta^{1/2-3\gamma}\|G\|_{1,1}(u-t).$$

Now, for III_{21}^1 we first compute

$$\begin{aligned} \mathbb{E}\{V_{jk}(x+y)V_j(y)\} &= \frac{\partial^2}{\partial x_j\partial x_k}\mathbb{E}\{V(x+y)V_j(y)\} = -\frac{\partial^2}{\partial x_j\partial x_k}\mathbb{E}\{V_j(x+y)V(y)\} \\ &= -\frac{\partial^3}{\partial x_j^2\partial x_k}\mathbb{E}\{V(x+y)V(y)\} = -\Delta R_k^{VV}(x) \end{aligned}$$

and use the mixing lemma to conclude that

$$\left|\mathbb{E}\left\{III_{21}^{1}\tilde{\zeta}\right\} - \frac{1}{\delta^2}\int_{t+\delta^{1-\gamma}}^{u}\int_{\sigma}^{s}(s-\rho)g(Z^\delta(\sigma(s)))r_j(K^\delta(\sigma(s))\Delta R_j^{VV}\left(\frac{L(\sigma,s)-L(\sigma,\rho)}{\delta}\right)d\rho ds\right|$$
$$\leq C\delta\|G\|_1\mathbb{E}[\tilde{\zeta}](u-t).$$

As before we change variables in the inner integrals above and replace the argument $\sigma(s)$ of smooth functions appearing above by s, as well replacing the limits of integration by their values in the limit $\delta\to 0$, to conclude that

$$\left|\mathbb{E}\left\{III_{21}^{1}\tilde{\zeta}\right\} - \int_t^u\int_0^\infty \theta g(Z^\delta(s))r_j(K^\delta(s))\Delta R_j^{VV}\left(-\theta K^\delta(s)\right)d\theta ds\right|$$
$$\leq C\delta\|G\|_{2,2}\mathbb{E}[\tilde{\zeta}](u-t).$$

Overall, the work of this section implies that

$$\left|\mathbb{E}\left\{III\tilde{\zeta}\right\} + \int_t^u\int_0^\infty \theta g(Z^\delta(s))r_j(K^\delta(s))\Delta R_j^{VV}\left(\theta K^\delta(s)\right)d\theta ds\right| \tag{4.58}$$
$$\leq C\delta^\alpha\|G\|_1\left(\|\zeta\|_\infty+\mathbb{E}[\tilde{\zeta}]\right)(u-t)$$

which is nothing but (4.26).

4.5 Estimate of IV

We are now down to the last term IV in (4.19) that is estimated as II with the help of the mixing lemma and no additional expansions:

$$IV = \frac{1}{\delta}\int\limits_{t+\delta^{1-\gamma}}^{u}\int\limits_{\sigma(s)}^{s} g'(Z^\delta(\rho))r_j(K^\delta(\rho))S\left(\frac{X^\delta(\rho)}{\delta}\right)V_j\left(\frac{X^\delta(s)}{\delta}\right)d\rho ds \tag{4.59}$$
$$+\frac{1}{\delta}\int\limits_{t+\delta^{1-\gamma}}^{u}\int\limits_{\sigma(s)}^{s} g(Z^\delta(\rho))r_{jm}(K^\delta(\rho))V_m\left(\frac{X^\delta(\rho)}{\delta}\right)V_j\left(\frac{X^\delta(s)}{\delta}\right)d\rho ds.$$

First, we observe using the mixing lemma and smoothness of G that (compare to (4.46))

$$\left|\mathbb{E}\left\{IV\tilde{\zeta}\right\} - \mathbb{E}\left\{IV'\tilde{\zeta}\right\}\right| \leq C\delta^{\gamma_2}\|G\|_3\left[\|\tilde{\zeta}\|_\infty + \mathbb{E}\{\tilde{\zeta}\}\right](u-t) \tag{4.60}$$

with

$$IV' = \frac{1}{\delta}\int\limits_{t+\delta^{1-\gamma}}^{u}\int\limits_{\sigma(s)}^{s} g'(Z^\delta(\sigma))r_j(K^\delta(\sigma))S\left(\frac{L(\sigma,\rho)}{\delta}\right)V_j\left(\frac{L(\sigma,s)}{\delta}\right)d\rho ds \tag{4.61}$$
$$+\frac{1}{\delta}\int\limits_{t+\delta^{1-\gamma}}^{u}\int\limits_{\sigma(s)}^{s} g(Z^\delta(\sigma))r_{jm}(K^\delta(\sigma))V_m\left(\frac{L(\sigma,\rho)}{\delta}\right)V_j\left(\frac{L(\sigma,s)}{\delta}\right)d\rho ds = IV_1 + IV_2.$$

Now, we have by the mixing lemma

$$\left|\mathbb{E}\{IV_1\tilde{\zeta}\} - \frac{1}{\delta}\int\limits_{t+\delta^{1-\gamma}}^{u}\int\limits_{\sigma(s)}^{s} g'(Z^\delta(\sigma))r_j(K^\delta(\sigma))R_j^{SV}\left(\frac{L(\sigma,s)-L(\sigma,\rho)}{\delta}\right)d\rho ds\tilde{\zeta}\right|$$
$$\leq C\delta^\alpha\|G\|_2\left(\|\tilde{\zeta}\|_\infty + E\{\tilde{\zeta}\}\right)(u-t)$$

and thus

$$\left|\mathbb{E}\{IV_1\tilde{\zeta}\} - \int_t^u \int_0^\infty g'(Z^\delta(s)) r_j(K^\delta(s)) R_j^{SV}\left(-\theta K^\delta(s)\right) d\theta ds \tilde{\zeta}\right|$$
$$\leq C\delta^\alpha \|G\|_2 \left(\|\tilde{\zeta}\|_\infty + E\{\tilde{\zeta}\}\right)(u-t). \tag{4.62}$$

Similarly, for the other contribution we have

$$\left|\mathbb{E}\{IV_2\tilde{\zeta}\} + \frac{1}{\delta}\int_{t+\delta^{1-\gamma}}^u \int_{\sigma(s)}^s g(Z^\delta(\sigma)) r_{jm}(K^\delta(\sigma)) R_{mj}^{VV}\left(\frac{L(\sigma,s)-L(\sigma,\rho)}{\delta}\right) d\rho ds \tilde{\zeta}\right|$$
$$\leq C\delta^\alpha \|G\|_2 \left(\|\tilde{\zeta}\|_\infty + E\{\tilde{\zeta}\}\right)(u-t)$$

and

$$\left|\mathbb{E}\{IV_2\tilde{\zeta}\} + \int_t^u \int_0^\infty g(Z^\delta(s)) r_{jm}(K^\delta(s)) R_{mj}^{VV}\left(\theta K^\delta(s)\right) d\theta ds \tilde{\zeta}\right|$$
$$\leq C\delta^\alpha \|G\|_2 \left(\|\tilde{\zeta}\|_\infty + E\{\tilde{\zeta}\}\right)(u-t). \tag{4.63}$$

Together, (4.62) and (4.63) imply (4.27). This finishes the proof of Lemma 4.3 and thus of Proposition 4.1 as well. □

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