Asymptotics of the phase of the solutions of the random Schrödinger equation

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Abstract

We consider solutions of the Schrödinger equation with a weak time-dependent random potential. It is shown that when the two-point correlation function of the potential is rapidly decaying then the Fourier transform $\hat{\zeta}_\varepsilon(t,\xi)$ of the appropriately scaled solution converges point-wise in ξ to a deterministic limit, exponentially decaying in time. On the other hand, when the two-point correlation function decays slowly, we show that the limit of $\hat{\zeta}_\varepsilon(t,\xi)$ has the form $\hat{\zeta}_0(\xi)\exp(iB_\kappa(t,\xi))$ where $B_\kappa(t,\xi)$ is a fractional Brownian motion.

1 Introduction and the main results

We consider solutions of the Schrödinger equation

$$\begin{aligned} & i\frac{\partial\phi}{\partial t}+\frac{1}{2}\Delta\phi-\gamma V(t,x)\phi=0,\quad x\in\mathbb{R}^d, \\ & \phi(0,x)=\phi_0(x), \end{aligned} \tag{1.1}$$

with a random potential $V(t,x)$ in the spatial dimension $d\geq 1$. Here $\gamma\ll 1$ is the small parameter that measures the relative strength of the (weak) random fluctuations. The long time behavior of the Wigner transform [13] of the solutions of (1.1) defined as

$$W(t,x,k)=\int e^{ik\cdot y}\phi(x-\frac{y}{2})\phi^*(t,x+\frac{y}{2})\frac{dy}{(2\pi)^d}$$

has been extensively studied in the past: it can be shown that a properly rescaled (to allow for long distance and large time propagation) limit of $\mathbb{E}(W(t,x,k))$ converges as $\gamma\to 0$ to the solution of the radiative transport equation [3, 4, 7, 8, 9, 14, 18, 24]

$$\bar{W}_t+k\cdot\nabla_x\bar{W}=\int\hat{R}(p-k,\frac{p^2-k^2}{2})(\bar{W}(t,x,p)-\bar{W}(t,x,k))dp. \tag{1.2}$$

This result holds under the assumption that $V(t,x)$ is a spatially and temporally homogeneous mean-zero random field with the two-point correlation function

$$R(t,x)=\mathbb{E}[V(s,y)V(t+s,x+y)],$$

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whose power spectrum

$$\hat{R}(\omega, k) = \int R(t,x)e^{-ik\cdot x - i\omega t}dxdt$$

appears in (1.2). In addition, it has been shown that the limit is often self-averaging, that is, given any test function $\eta(x,k) \in \mathcal{S}(\mathbb{R}^{2d})$, $\langle W, \eta\rangle \to \langle \bar{W}, \eta\rangle$ in probability [1, 2, 4, 5, 6, 20, 21]. However, this result does not hold strongly, that is, point-wise in x and k. Here we denoted

$$\langle W, \eta\rangle = \int W(x,k)\eta(x,k)dxdk.$$

On the other hand, surprisingly, the solution $\phi(t,x)$ of (1.1) itself seems to be much less studied – an obvious reason for this is that $\phi(t,x)$ becomes highly oscillatory after propagation on long distances while the Wigner transform is a macroscopic quantity. The goal of the present paper is to understand the behavior of $\phi(t,x)$ after propagation over long distances and also to study the effect of the slow spatial and temporal decay of the correlation function $R(t,x)$ on the behavior of solutions, long time limit and self-averaging properties.

We are interested in the long time, large propagation distances effect of the random inhomogeneities, so we consider temporal and spatial scales of the order $t \sim O(\varepsilon^{-1})$ and $x \sim O(\varepsilon^{-1})$ with $\varepsilon = \varepsilon(\gamma) \ll 1$ a small parameter depending on γ, to be determined later. Finding an appropriate length and time scale $O(\varepsilon^{-1})$, on which one observes a non-trivial behavior, as functional of $\gamma \ll 1$ is part of the problem. Let us recast (1.1) as an equation for the rescaled function $\phi_\varepsilon(t,x) = \phi(t/\varepsilon, x/\varepsilon)$:

$$\begin{aligned} &i\varepsilon\frac{\partial \phi_\varepsilon}{\partial t} + \frac{\varepsilon^2}{2}\Delta\phi_\varepsilon - \gamma V(\frac{t}{\varepsilon}, \frac{x}{\varepsilon})\phi_\varepsilon = 0, \\ &\phi_\varepsilon(0,x) = \phi_0(x/\varepsilon). \end{aligned} \tag{1.3}$$

In particular, we have $\hat{\phi}_\varepsilon(0,\xi) = \varepsilon^d\hat{\phi}_0(\varepsilon\xi)$. We assume that the spatial power spectrum has the form

$$\tilde{R}(t,k) = e^{-\mathfrak{g}(k)|t|}\hat{R}(k), \tag{1.4}$$

where $\hat{R}(k) \in L^1(\mathbb{R}^d)$, and

$$\tilde{R}(t,k) = \int e^{-ik\cdot x}R(t,x)dx.$$

The space-time power energy spectrum is then

$$\hat{R}(\omega,k) = \frac{2\mathfrak{g}(k)\hat{R}(k)}{\omega^2 + \mathfrak{g}^2(k)}. \tag{1.5}$$

Rapidly decaying correlations. The first result of this paper is the following theorem concerning the usually considered situation when the function $R(t,x)$ is rapidly decaying.

Theorem 1.1 *Assume that $V(t,x)$ is a spatially homogeneous mean-zero Gaussian and Markovian in time random field with the two-point correlation function $R(t,x)$ and the spatial power spectrum $\tilde{R}(t,k)$ of the form* (1.4) *with*

$$\int \frac{\hat{R}(p)dp}{\mathfrak{g}(p)} < +\infty. \tag{1.6}$$

Let $\varepsilon = \gamma^2$, and define

$$\hat{\zeta}_\varepsilon(t,\xi) = \frac{1}{\varepsilon^d}\hat{\phi}_\varepsilon(t,\xi/\varepsilon)e^{i|\xi|^2t/(2\varepsilon)}. \tag{1.7}$$

Then, for each $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^d$ fixed, $\hat{\zeta}_\varepsilon(t,\xi)$ converges in probability, as $\varepsilon \to 0$, to

$$\bar{\zeta}_0(t,\xi) = \hat{\phi}_0(\xi)e^{-\frac{1}{2}D_\xi t}, \tag{1.8}$$

where

$$D_\xi = 2\int \frac{\hat{R}(p)}{\mathfrak{g}(p) - i(\xi\cdot p - |p|^2/2)}\frac{dp}{(2\pi)^d} = 2\int \frac{\hat{R}(\xi - p)}{\mathfrak{g}(\xi - p) + i\big(\frac{|p|^2}{2} - \frac{|\xi|^2}{2}\big)}\frac{dp}{(2\pi)^d}. \tag{1.9}$$

This result agrees with the qualitative predictions of the kinetic theory: roughly speaking, the field $\hat{\zeta}(t,\xi)$ captures the ballistic part of the solution of the radiative transport equation (1.2) that decays exponentially in time. In the geometric optics regime (that is different from the weak coupling regime considered here) a result similar to Theorem 1.1 could be deduced using [2, 5]. We stress that, somewhat surprisingly, the phase has a deterministic limit after subtracting the fast phase contribution $e^{i|\xi|^2 t/\varepsilon}$. Moreover, the self-averaging limit holds point-wise in ξ: no averaging or integration against a test function is required. This is different from the previous results in the literature for the Wigner transform mentioned above where some averaging in phase space is always required.

Slowly decaying correlations. Suppose now that the spatial power spectrum has the form

$$\hat{R}(p) = \frac{a(p)}{|p|^{2\alpha+d-2}} \tag{1.10}$$

and the spectral gap is

$$\mathfrak{g}(p) = \mu|p|^{2\beta} \tag{1.11}$$

for some $\alpha < 1$, $0 \le \beta \le 1/2$, $\mu > 0$, and a compactly supported, non-negative, bounded measurable function $a(p)$. We assume that $a(p)$ is continuous at $p = 0$ and $a(0) > 0$. Observe that in order for (1.6) to hold we need to assume that $\alpha + \beta < 1$. Our second result concerns the case when the correlation function decays slowly so that $\alpha + \beta > 1$. This implies that $\alpha \in (\frac{1}{2}, 1)$. Let us first define the constants

$$K_1(\alpha,\beta,\mu) = \Omega_d \int_0^{+\infty} e^{-\mu\rho^{2\beta}}\frac{d\rho}{\rho^{2\alpha-1}}, \tag{1.12}$$

where Ω_d is the surface area of the unit sphere in $\mathbb{R}^d$, and

$$K_2(\xi;\alpha,\mu) = \int_0^{+\infty} e^{-\mu\rho}\frac{d\rho}{\rho^{2\alpha-1}}\int_{\mathbb{S}^{d-1}} e^{i|\xi|\rho\omega\cdot e_1}S(d\omega). \tag{1.13}$$

Theorem 1.2 *Assume that the two-point space-time correlation function $R(t,x)$ has the form (1.4) with $\hat{R}(p)$ and $\mathfrak{g}(p)$ as in (1.10) and (1.11), and that $\alpha + \beta > 1$, $1/2 < \alpha < 1$ and $\beta \le 1/2$. Let $\varepsilon = \gamma^{1/\kappa}$, with $\kappa = (\alpha + 2\beta - 1)/(2\beta) = 1 - \frac{1-\alpha}{2\beta}$. Then, for each $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^d$ fixed, $\hat{\zeta}_\varepsilon(t,\xi)$ converges in law, as $\varepsilon \to 0$ to the random variable*

$$\bar{\zeta}_0(t,\xi) = \hat{\phi}_0(\xi)e^{i\sqrt{D(\xi)}B_\kappa(t)}, \tag{1.14}$$

where $B_\kappa(t;\xi)$ is a standard scalar fractional Brownian motion and its variance D is given by

$$D = \frac{a(0)K_1(\alpha,\beta,\mu)}{2\kappa(2\pi)^d}. \tag{1.15}$$

when $\beta < 1/2$, and

$$D(\xi) = \frac{a(0)K_2(\xi;\alpha,\mu)}{2\alpha(2\pi)^d}, \tag{1.16}$$

when $\beta = 1/2$.

We note that there are several important differences between the rapidly decorrelating case considered in Theorem 1.1 and the slowly decorrelating case in Theorem 1.2. First of all, the time scale of $\hat{\zeta}_\varepsilon(t,\xi)$ now is not γ^{-2} but rather $\gamma^{-1/\kappa}$. In particular, it is no longer universal but rather depends on the parameters α and β when $\alpha+\beta>1$. On the other hand, if we fix the ratio ε of the overall propagation distance and the correlation length of the medium, then the strength of the heterogeneities $\gamma=\varepsilon^\kappa=\varepsilon^{1-(1-\alpha)/2\beta}$ that produces a non-trivial effect also decreases when α and β increase. This shows that weaker fluctuations generate a macroscopic effect in the presence of long range (in space and time) correlations. The main qualitative difference between the two regimes, however, is that the phase is no longer self-averaging: the limit is truly stochastic and has a self-similar behavior.

Let us also point out a difference between the evolution of the energy and the phase of the wave: while Theorems 1.1 and 1.2 show that the time scale on which the phase evolves depends very much on the nature of the correlations of the random medium this does not seem to be the case for the wave energy. Indeed, while the total scattering cross-section

$$\Sigma=\int\hat{R}(p-k,\frac{p^2-k^2}{2})dp$$

is infinite in the regime of slowly decaying correlations, with the parameters α and β as in Theorem 1.2, the transport equation (1.2) still makes sense because of the regularizing effect of the difference $W(t,x,p)-W(t,x,k)$ that appears in the right side of (1.2). Hence, we believe that even in this range of parameters wave energy evolves on the time scale $O(\gamma^{-2})$, as in the rapidly decorrelating case. Thus, the slow decay of correlations leads to time-separation of the energy and phase evolutions, a phenomenon we plan to address in detail elsewhere.

Let us mention that to the best of our knowledge the first study of wave propagation in random media with slowly decaying correlations was done in the one-dimensional case [12, 19] where it was shown that a pulse going through a random medium with long range correlations performs a fractional Brownian motion around its mean position, as opposed to the regular Brownian motion in the rapidly decorrelating case [11]. On the other hand, motion of particles in such random media leading to fractional Brownian limits was considered in [10, 16, 17]. The main contributions of the present paper are that, first, the full limit process of the wave field is identified (we are not aware of any such results for waves in any regime in dimensions higher than one), and, second, it is shown that slow decay of correlations may induce loss of self-averaging properties.

The paper is organized as follows: in Section 2 we consider the Duhamel expansion for (1.1) that is the basis for our considerations. We prove Theorem 1.1 in Section 3 and Theorem 1.2 in Section 4. In both proofs we first identify the limit of $\mathbb{E}(\hat{\zeta}_\varepsilon(t,\xi))$ by summing the Feynman diagrams for the Duhamel expansion – this is the same strategy used in [7, 8, 9] to obtain the kinetic limit in the rapidly decorrelating case. Here, however, the diagram estimation is simpler since the potential is time-dependent. On the other hand, the new aspect in the case of slowly decaying correlations is that all diagrams contribute to the limit and not only the ladder diagrams as in the rapidly decorrelating case. The next step in the proofs of Theorems 1.1 and 1.2 is to identify the limit of the higher moments of $\hat{\zeta}_\varepsilon(t,\xi)$. It suffices to consider the second moment in order to prove convergence in probability in Theorem 1.1, while all moments have to considered in the proof of Theorem 1.2 to identify the limit process since the limit is not deterministic.

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2 The Duhamel expansion

We may re-write (1.3) as an integral in time equation

$$\hat{\phi}_\varepsilon(t,\xi) = \hat{\phi}_0(\xi)e^{-i\varepsilon|\xi|^2 t/2} + \frac{\gamma}{i\varepsilon}\int_0^t \int \frac{\hat{V}(s_1/\varepsilon, dp_1)}{(2\pi)^d}\hat{\phi}_\varepsilon(s_1, \xi - \frac{p_1}{\varepsilon})e^{-i\varepsilon|\xi|^2(t-s_1)/2}ds_1.$$

Hence, the function $\hat{\zeta}_\varepsilon(t,\xi)$ given by (1.7) solves

$$\hat{\zeta}_\varepsilon(t,\xi) = \hat{\phi}_0(\xi) + \frac{\gamma}{i\varepsilon}\int_0^t \int \frac{\hat{V}(s_1/\varepsilon, dp_1)}{(2\pi)^d}\hat{\zeta}_\varepsilon(s_1, \xi - p_1)e^{i(|\xi|^2 - |\xi-p_1|^2)s_1/(2\varepsilon)}ds_1, \tag{2.1}$$

as $\hat{\zeta}(0,\xi) = \hat{\phi}_0(\xi)$. Iterating (2.1) leads to an infinite series expansion for $\hat{\zeta}_\varepsilon(t,\xi)$:

$$\hat{\zeta}_\varepsilon(t,\xi) = \sum_{n=0}^{\infty}\hat{\zeta}_n^\varepsilon(t,\xi), \tag{2.2}$$

with the individual terms of the form

$$\hat{\zeta}_n^\varepsilon(t,\xi) = \left[\frac{\gamma}{i\varepsilon(2\pi)^d}\right]^n \int\limits_{\Delta_n(t)} d\mathbf{s}^{(n)} \int \hat{V}(\frac{s_1}{\varepsilon}, dp_1)\ldots\hat{V}(\frac{s_n}{\varepsilon}, dp_n)\hat{\phi}_0(\xi - p_1 - \cdots - p_n)e^{iG_n(\mathbf{s}^{(n)},\mathbf{p}^{(n)})/\varepsilon}, \tag{2.3}$$

with the phase

$$G_n(\mathbf{s}^{(n)},\mathbf{p}^{(n)}) = \sum_{k=1}^{n}(|\xi - p_1 - \cdots - p_{k-1}|^2 - |\xi - p_1 - \cdots - p_k|^2)\frac{s_k}{2} = A_n(\mathbf{s}^{(n)},\mathbf{p}^{(n)}) - B_n(\mathbf{s}^{(n)},\mathbf{p}^{(n)}). \tag{2.4}$$

Here we use the notation $p_0 = 0$, $\mathbf{s}^{(n)} = (s_1,\ldots,s_n) \in \mathbb{R}^n$, $\mathbf{p}^{(n)} = (p_1,\ldots,p_n) \in \mathbb{R}^{nd}$, so that $d\mathbf{s}^{(n)} = ds_1 ds_2 \ldots ds_n$. We have also split the phase into

$$A_n(\mathbf{s}^{(n)},\mathbf{p}^{(n)}) = \sum_{m=1}^{n}(\xi\cdot p_m)s_m, \tag{2.5}$$
$$B_n(\mathbf{s}^{(n)},\mathbf{p}^{(n)}) = \sum_{m=1}^{n} s_m p_m \cdot \left(\sum_{j=1}^{m-1} p_j\right) + \frac{1}{2}\sum_{m=1}^{n} s_m|p_m|^2.$$

Finally, $\Delta_n(t)$ denotes the time simplex

$$\Delta_n(t) = \{(s_1, s_2, \ldots, s_n) : \quad 0 \le s_n \le s_{n-1} \le \cdots \le s_1 \le t\}.$$

The next proposition shows that the series (2.2) converges almost surely and, moreover, one can take the expectation term-wise for $\varepsilon > 0$ and $\gamma > 0$ fixed. This will allow us to work with term-wise estimates for each $\mathbb{E}(\hat{\zeta}_\varepsilon^n)$ separately in the proof of Theorems 1.1 and 1.2.

Proposition 2.1 *(i) The series* (2.2) *for the function* $\hat{\zeta}_\varepsilon(t,\xi)$ *converges almost surely for all values of* $\gamma, \varepsilon \in (0,1)$ *and* $\phi_0 \in C_c^\infty(\mathbb{R}^d)$. *(ii) Moreover, for each* $(t,\xi) \in \mathbb{R}^{1+d}$ *fixed, we have*

$$\mathbb{E}\hat{\zeta}_\varepsilon(t,\xi) = \sum_{n=0}^{\infty}\mathbb{E}\hat{\zeta}_n^\varepsilon(t,\xi). \tag{2.6}$$

Proof. We may assume without loss of generality that $\gamma = \varepsilon = 1$. Let $\theta_\rho(p) = (1 + |p|^2)^\rho$ for any $\rho \in \mathbb{R}$, and set $d_* = [d/2] + 1$. The right side of (2.3) can be rewritten as follows:

$$\hat{\zeta}_n^\varepsilon(t,\xi) = \frac{1}{(2\pi)^{nd}} \int\limits_{\Delta_n(t)} d\mathbf{s}^{(n)} \int \prod_{k=1}^n \left[\theta_{4d_*}(p_k)\hat{V}(s_k, dp_k)\right] \hat{\phi}_0(\xi - \sum_{j=1}^n p_j) e^{iG_n(\mathbf{s}^{(n)},\mathbf{p}^{(n)})} \prod_{k=1}^n \theta_{-4d_*}(p_k)$$

$$= \int\limits_{\Delta_n(t)} d\mathbf{s}^{(n)} \int \prod_{k=1}^n W(s_k, x_k) f_n(-x_1, \dots, -x_n) dx^{(n)}, \tag{2.7}$$

where $W(s,x) = (I - \Delta_x)^{4d_*} V(s,x)$, and

$$\hat{f}_n(p_1, \dots, p_n) = \hat{\phi}_0(\xi - \sum_{j=1}^n p_j) e^{iG_n(\mathbf{s}^{(n)},\mathbf{p}^{(n)})} \prod_{k=1}^n \theta_{-4d_*}(p_k)$$

while $dx^{(n)} = dx_1 \dots, dx_n$. We can further transform the utmost right side of (2.7) as follows

$$\hat{\zeta}_n^\varepsilon(t,\xi) = \int\limits_{\Delta_n(t)} d\mathbf{s}^{(n)} \int \prod_{k=1}^n \left[\theta_{-d_*}(x_k) W(s_k, x_k)\right] g_n(-x_1, \dots, -x_n) dx^{(n)}, \tag{2.8}$$

where

$$g_n(x_1, \dots, x_n) = \prod_{k=1}^n \left[\theta_{d_*}(x_k)\right] f_n(x_1, \dots, x_n) \tag{2.9}$$

$$= \frac{1}{(2\pi)^{nd}} \mathcal{F}\left\{(I - \Delta_{p_1})^{d_*} \dots (I - \Delta_{p_n})^{d_*} \left[\hat{f}(p_1, \dots, p_n)\right]\right\}(-x_1, \dots, -x_n).$$

The following lemma can be concluded directly from (2.9) and the choice of d_*.

Lemma 2.2 *There exists a constant $M > 0$ such that*

$$\|g_n\|_\infty \le M^n, \quad \forall\, n \ge 1.$$

We now recall Theorem 3.2 of [22].

Lemma 2.3 *Let $W(t,x)$ be a stationary, continuous trajectory Gaussian field $W(t,x)$ with a two-point correlation function $R_W(h,x) = \mathbb{E}(W(t,y)W(t+h, y+x))$. Assume that there exist $C > 0$ and $r > 0$ such that $|R_W(h,x) - R_W(0,0)| \le C(|h| + |x|)^r$ for all $|h| + |x| \le 1$. Then, for any $\gamma > 1$ there exists a positive random variable F such that*

$$\sup_{t \in [0,T]} |W(t,x;\omega)| \le F(\omega)(1 + \log^+ |x|)^{\gamma/2}, \quad \forall\, x \in \mathbb{R}^d. \tag{2.10}$$

In addition, there exists a constant $C > 0$ such that

$$\mathbb{P}[F \ge \lambda] \le Ce^{-\lambda^2/C}, \quad \forall\, \lambda > 0. \tag{2.11}$$

Combining the above two lemmas we can estimate the right hand side of (2.8) by

$$\frac{[CF(\omega)]^n}{n!} \left\{\int \theta_{-d_*}(x)(1 + \log^+ |x|)^{\gamma/2} dx\right\}^n, \tag{2.12}$$

which proves part (i) of Proposition 2.1. Part (ii) follows from estimate (2.12) and the tail estimates of random variable F given in (2.11). □

3 Proof of Theorem 1.1

We now prove Theorem 1.1, that is, we consider the case when the two-point correlation function decays sufficiently rapidly so that the phase obeys a deterministic limit. We shall assume in the course of the proof of Theorem 1.1 that $\gamma = \varepsilon^{1/2}$.

Outline of the proof

The proof is based on working with the Duhamel expansion (2.2) and, in particular, with the series (2.6) for $\mathbb{E}(\hat{\zeta}_\varepsilon(t,\xi))$. The first step in the proof is the following uniform bound for the individual terms of (2.6).

Proposition 3.1 *For all $T > 0$, $n \geq 0$ and all $\xi \in \mathbb{R}^d \setminus \{0\}$ there exists a constant $C(T)$ such that*

$$\sup_{t\in[0,T]} |\mathbb{E}\hat{\zeta}_n^\varepsilon(t,\xi)| \leq \frac{C^n(T;\xi)}{n!} \tag{3.1}$$

for all $\varepsilon \in (0,1]$.

As a consequence, we may interchange the limit $\varepsilon \downarrow 0$ and the summation in n.

Corollary 3.2 *We have*

$$\lim_{\varepsilon\downarrow 0} \mathbb{E}\hat{\zeta}_\varepsilon(t,\xi) = \sum_{n=0}^{\infty} \lim_{\varepsilon\downarrow 0} \mathbb{E}\hat{\zeta}_n^\varepsilon(t,\xi), \tag{3.2}$$

for all $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^d \setminus \{0\}$.

Next, we identify the limit of the individual terms in the right side of (3.2).

Proposition 3.3 *We have $\mathbb{E}\hat{\zeta}_n^\varepsilon(t,\xi) = 0$ when n is odd and*

$$\lim_{\varepsilon\downarrow 0} \mathbb{E}\hat{\zeta}_{2n}^\varepsilon(t,\xi) = \hat{\phi}_0(\xi)\frac{(-tD_\xi)^n}{2^n n!} \tag{3.3}$$

for all $t \in \mathbb{R}$, $n \in \mathbb{N}$ and $\xi \in \mathbb{R}^d \setminus \{0\}$.

This implies convergence of the expectation:

Corollary 3.4 *We have*

$$\lim_{\varepsilon\downarrow 0} \mathbb{E}\hat{\zeta}_\varepsilon(t,\xi) = \hat{\phi}_0(\xi)e^{-tD_\xi/2}, \tag{3.4}$$

for all $t \in \mathbb{R}$and $\xi \in \mathbb{R}^d \setminus \{0\}$.

The final step is to establish the following result, which implies, in particular, Theorem 1.1.

Proposition 3.5 *We have, for all $t \geq 0$ and $\xi \neq 0$:*

$$\lim_{\varepsilon\downarrow 0} \mathbb{E}[\hat{\zeta}_\varepsilon(t,\xi)]^2 = [\bar{\zeta}_0(t,\xi)]^2. \tag{3.5}$$

Proof of Proposition 3.1

Of course, only the case of even n-s requires a proof as the expectation vanishes for n odd. Note that

$$
\begin{aligned}
|\mathbb{E}\hat{\zeta}^{\varepsilon}_{2n}(t,\xi)| &= \left[\frac{1}{\varepsilon^{1/2}(2\pi)^d}\right]^{2n}\left|\int\limits_{\Delta_{2n}(t)} d\mathbf{s}^{(2n)}\int \mathbb{E}\left[\hat{V}(\frac{s_1}{\varepsilon},dp_1)\dots\hat{V}(\frac{s_{2n}}{\varepsilon},dp_{2n})\right]\hat{\phi}_0(\xi-p_1-\dots-p_{2n})\right.\\
&\left.\times e^{iG_n(\mathbf{s}^{(2n)},\mathbf{p}^{(2n)})/\varepsilon}\right| \le \frac{C^n\|\phi_0\|_\infty}{\varepsilon^n}\int\limits_{\Delta_{2n}(t)} d\mathbf{s}^{(2n)}\int\left|\mathbb{E}\left[\hat{V}(\frac{s_1}{\varepsilon},dp_1)\dots\hat{V}(\frac{s_{2n}}{\varepsilon},dp_{2n})\right]\right|\\
&= \frac{C^n\|\phi_0\|_\infty}{(2n)!\varepsilon^n}\int_0^t\dots\int_0^t d\mathbf{s}^{(2n)}\int\left|\mathbb{E}\left[\hat{V}(\frac{s_1}{\varepsilon},dp_1)\dots\hat{V}(\frac{s_{2n}}{\varepsilon},dp_{2n})\right]\right|. \qquad (3.6)
\end{aligned}
$$

The last step above uses the symmetry of the integrand in $s_1,\dots,s_{2n}$ that brings about the factorial in the dominator. Using the relation

$$
\mathbb{E}\left[\hat{V}(t,dp)\hat{V}(s,dq)\right] = (2\pi)^d e^{-\mathfrak{g}(p)|t-s|}\delta(p+q)\hat{R}(p)dpdq, \qquad (3.7)
$$

and the rules of computing $2n$-th joint moment of mean zero Gaussian random variables we conclude that the right hand side of (3.6) can be estimated by

$$
\frac{C^n\|\phi_0\|_\infty}{(2n)!\varepsilon^n}\sum_{\mathcal{F}}\int_0^t\dots\int_0^t d\mathbf{s}^{(2n)}\int d\mathbf{p}^{(2n)}\prod_{(k,l)\in\mathcal{F}} e^{-\mathfrak{g}(p_k)|s_k-s_l|/\varepsilon}\delta(p_k+p_l)\hat{R}(p_k), \qquad (3.8)
$$

where the summation extends over all Feynman diagrams formed over vertices $1,\dots,2n$. We recall that a Feynman diagram for the set $S=\{1,2,\dots,2n\}$ is a partition of S into n pairs of numbers (lr), such that each element of S appears in exactly one of the pairs. If a pair (lr) is present in a Feynman diagram $\mathcal{F}$ and $l<r$ we say that l is a left vertex and r is a right vertex.

Changing variables $s_k' := s_k/\varepsilon$ we obtain that expression (3.8) equals

$$
\begin{aligned}
&\frac{C^n\|\phi_0\|_\infty}{(2n)!}\sum_{\mathcal{F}}\int d\mathbf{p}^{(2n)}\prod_{(k,l)\in\mathcal{F}}\left[\varepsilon\int_0^{t/\varepsilon}\int_0^{t/\varepsilon} e^{-\mathfrak{g}(p_k)|s_k-s_l|}ds_k ds_l\right]\delta(p_k+p_l)\hat{R}(p_k) \qquad (3.9)\\
&\le \frac{C^n t^n\|\phi_0\|_\infty}{(2n)!}\sum_{\mathcal{F}}\int\prod_{(k,l)\in\mathcal{F}}\delta(p_k+p_l)\frac{\hat{R}(p_k)}{\mathfrak{g}(p_k)}d\mathbf{p}^{(2n)} = \frac{C^n t^n\|\hat{\phi}_0\|_\infty}{2^n n!}\left[\int\frac{\hat{R}(p)}{\mathfrak{g}(p)}dp\right]^n.
\end{aligned}
$$

In the last step above we used the fact that the total number of Feynman diagrams for a set of $2n$ elements is $(2n-1)!!$. Now, the conclusion of Proposition 3.1 follows. $\square$

The above argument actually shows the following.

Proposition 3.6 *There exists a constant C such that for all $n\ge 1$, $t>0$*

$$
\sum_{\mathcal{F}}\int\dots\int_{\Delta_{2n}(t)} d\mathbf{s}^{(2n)}\int d\mathbf{p}^{(2n)}\prod_{(k,l)\in\mathcal{F}} e^{-\mathfrak{g}(p_k)|s_k-s_l|/\varepsilon}\delta(p_k+p_l)\hat{R}(p_k)\le\frac{(Ct\varepsilon)^n}{n!}, \qquad (3.10)
$$

where the summation extends over all Feynman diagrams formed over $\{1,\dots,2n\}$.

Proof of Proposition 3.3

Let us introduce some terminology: the Feynman diagram $(1,2),\ldots,(2n-1,2n)$ shall be called *a ladder* diagram. For a given diagram $\mathcal{F}$ we let

$$\mathcal{I}_\varepsilon(t;\mathcal{F}) := \int\limits_{\Delta_{2n}(t)} d\mathbf{s}^{(2n)} \int d\mathbf{p}^{(2n)} \prod_{(k,l)\in\mathcal{F}} e^{-\mathfrak{g}(p_k)|s_k-s_l|/\varepsilon}\delta(p_k+p_l)\hat{R}(p_k). \tag{3.11}$$

As a conclusion of Proposition 3.6, we obtain, in particular, that

$$\mathcal{I}(\mathcal{F}) = \limsup_{\varepsilon\downarrow 0} \sup_{t\in[0,T]} \varepsilon^{-n}\mathcal{I}_\varepsilon(t;\mathcal{F}) < +\infty, \tag{3.12}$$

for any Feynman diagram $\mathcal{F}$. We will now show that $\mathcal{I}(\mathcal{F})=0$ for all non-ladder diagrams, and then identify the actual limit of $\varepsilon^{-n}\mathcal{I}_\varepsilon(\mathcal{F})$ for the ladder diagrams completing the proof of Proposition 3.3. We start with non-ladder diagrams.

Lemma 3.7 *Suppose that $\mathcal{F}$ is not a ladder diagram. Then,*

$$\lim_{\varepsilon\downarrow 0} \sup_{t\in[0,T]} \varepsilon^{-n}\mathcal{I}_\varepsilon(t;\mathcal{F}) = 0, \tag{3.13}$$

for any $T>0$.

Proof. This lemma shall be proved by induction on n – the number of edges of a Feynman diagram. First, we verify it for $n=2$. We have to consider then two diagrams $\mathcal{F}_1=\{(1,3),(2,4)\}$ and $\mathcal{F}_2=\{(1,4),(2,3)\}$. Start with the first one. Suppose that $\kappa\in(0,1)$ and consider the sets of the following times: $A_1=[|s_1-s_3|\geq\varepsilon^\kappa]$ and $A_2=[|s_2-s_4|\geq\varepsilon^\kappa]$, as well as $A_3=A_1^c\cup A_2^c$. Consider the expressions

$$I_i(\varepsilon) = \int\limits_{\Delta_4(t)\cap A_i} ds_1\ldots ds_4 \int dp_1dp_2 \exp\left\{-[\mathfrak{g}(p_1)(s_1-s_3)+\mathfrak{g}(p_2)(s_2-s_4)]/\varepsilon\right\}\hat{R}(p_1)\hat{R}(p_2),$$

for $i=1,2,3$, then

$$\mathcal{I}_\varepsilon(t;\mathcal{F}_1) \leq \sum_{i=1}^3 I_i(\varepsilon).$$

We will see that $I_1(\varepsilon)$ and $I_2(\varepsilon)$ are small because the integrand is exponentially small in ε, while $I_3(\varepsilon)$ because the domain of integration is small. Indeed, observe that

$$\begin{aligned} I_1(\varepsilon) &\leq \int_0^t\int_0^t ds_1ds_3\int_{\mathbb{R}}\int_{\mathbb{R}} ds_2ds_4\int dp_1dp_2 e^{-\varepsilon^{\kappa-1}\mathfrak{g}(p_1)/2}e^{-[\mathfrak{g}(p_1)|s_1-s_3|+\mathfrak{g}(p_2)|s_2-s_4|]/(2\varepsilon)}\hat{R}(p_1)\hat{R}(p_2) \\ &= (2t\varepsilon)^2\int e^{-\varepsilon^{\kappa-1}\mathfrak{g}(p_1)/2}\frac{\hat{R}(p_1)dp_1}{\mathfrak{g}(p_1)}\int\frac{\hat{R}(p_2)dp_2}{\mathfrak{g}(p_2)} \end{aligned}$$

and it follows from the Lebesgue dominated convergence theorem that

$$\lim_{\varepsilon\downarrow 0} \sup_{t\in[0,T]} \varepsilon^{-2}I_1(\varepsilon) = 0. \tag{3.14}$$

Similarly one can prove that (3.14) holds for $I_2(\varepsilon)$. On the other hand, we note that if $0\leq s_1-s_3\leq\varepsilon^\kappa$ and $0\leq s_2-s_4\leq\varepsilon^\kappa$ (so that $(s_1,s_2,s_3,s_4)\in A_3$) then (since $0\leq s_3\leq s_2$), we have $0\leq s_1-s_4\leq 2\varepsilon^\kappa$ as well. Hence,

$$I_3(\varepsilon) \leq Ct\varepsilon^{3\kappa}$$

and (3.14) follows for $I_3(\varepsilon)$, provided that $\kappa > 2/3$. We have shown in this way that

$$\lim_{\varepsilon\downarrow 0} \sup_{t\in[0,T]} \varepsilon^{-2}\mathcal{I}(t;\mathcal{F}_1) = 0.$$

A similar argument also yields an analogous statement for $\mathcal{I}(t;\mathcal{F}_2)$.

Assume now for the sake of the induction argument that (3.13) holds for some $n \geq 2$ and for all non-ladder diagrams with $2k$ vertices with $k \leq n$. Let $\mathcal{F}$ be a non-ladder diagram consisting of $n+1$ edges. As before, we choose $\kappa \in (0,1)$ that shall be specified later. For a given edge $e = (k_0, l_0)$ of a diagram $\mathcal{F}$ set

$$A(e) = [|s_{k_0} - s_{l_0}| \geq \varepsilon^{\kappa}] \subseteq \Delta_{2n+2}(t),$$

and $A(\mathfrak{e}) = \bigcup_{e\in\mathcal{F}} A^c(e)$. Define also, again for $e \in \mathcal{F}$,

$$I_e(\varepsilon) = \int\limits_{\Delta_{2n}(t)\cap A(e)} d\mathbf{s}^{(2n+2)} \int d\mathbf{p}^{(2n+2)} \prod_{(k,l)\in\mathcal{F}} e^{-\mathfrak{g}(p_k)(s_k-s_l)/\varepsilon}\delta(p_k+p_l)\hat{R}(p_k),$$

as well as

$$I_{\mathfrak{e}}(\varepsilon) = \int\limits_{\Delta_{2n}(t)\cap A(\mathfrak{e})} d\mathbf{s}^{(2n+2)} \int d\mathbf{p}^{(2n+2)} \prod_{(k,l)\in\mathcal{F}} e^{-\mathfrak{g}(p_k)(s_k-s_l)/\varepsilon}\delta(p_k+p_l)\hat{R}(p_k).$$

Note that for the first term, as in the computation for the diagram $\mathcal{F}_1$ with $n=2$, we have

$$\begin{aligned}
I_e(\varepsilon) &\leq \int_0^t \cdots \int_0^t d\mathbf{s}^{(2n+2)} \int d\mathbf{p}^{(2n+2)} e^{-\mathfrak{g}(p_{k_0})\varepsilon^{\kappa-1}/2} \prod_{(k,l)\in\mathcal{F}} e^{-\mathfrak{g}(p_k)|s_k-s_l|/(2\varepsilon)}\delta(p_k+p_l)\hat{R}(p_k) \\
&\leq \int d\mathbf{p}^{(2n+2)} e^{-\mathfrak{g}(p_{k_0})\varepsilon^{\kappa-1}/2} \prod_{(k,l)\in\mathcal{F}} \delta(p_k+p_l)\hat{R}(p_k) \prod_{(k,l)\in\mathcal{F}} \int_0^t ds_k \int_{\mathbb{R}} ds_l e^{-\mathfrak{g}(p_k)|s_k-s_l|/(2\varepsilon)} \\
&= (2t\varepsilon)^{n+1} \left[\int \frac{\hat{R}(p)dp}{\mathfrak{g}(p)}\right]^n \int e^{-\mathfrak{g}(p)\varepsilon^{\kappa-1}/2} \frac{\hat{R}(p)dp}{\mathfrak{g}(p)},
\end{aligned}$$

thus the term in the exponent is very large and negative, whence

$$\lim_{\varepsilon\to 0} \varepsilon^{-(n+1)} I_e(\varepsilon) = 0.$$

On the other hand, for $I_{\mathfrak{e}}(\varepsilon)$ we have two possibilities: either it splits into a union of two sub-diagrams or not. More precisely, either (1) there exists m_0 such that $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$, where $\mathcal{F}_i$, $i = 1,2$ are Feynman diagrams formed over $\{1,\ldots,2m_0\}$ and $\{2m_0+1,\ldots,2n+2\}$ respectively, or (2) there exists a sequence of edges $e_i = (k_i, l_i)$, $i = 1,\ldots,m$ such that $k_1 = 1$, $k_{i+1} < l_i < l_{i+1}$, for $i = 1,\ldots,m-1$, and $l_m = 2n+2$. In the first case we have

$$\begin{aligned}
I_{\mathfrak{e}}(\varepsilon) &\leq \int_0^t ds_1 \ldots \int_0^{s_{2m_0-1}} ds_{2m_0} \int dp_1 \ldots dp_{2m_0} \prod_{(k,l)\in\mathcal{F}_1} e^{-\mathfrak{g}(p_k)(s_k-s_l)/\varepsilon}\delta(p_k+p_l)\hat{R}(p_k) \\
&\times \left\{ \int\limits_{\Delta_{2(n+1-m_0)}(s_{2m_0})} ds_{2m_0+1} \ldots ds_{2n+2} \int dp_{2m_0+1} \ldots dp_{2n+2} \prod_{(k,l)\in\mathcal{F}_2} e^{-\mathfrak{g}(p_k)(s_k-s_l)/\varepsilon}\delta(p_k+p_l)\hat{R}(p_k). \right\}.
\end{aligned}$$

Hence,

$$I_{\mathfrak{e}}(\varepsilon) \leq \mathcal{I}_\varepsilon(t;\mathcal{F}_1)\mathcal{I}_\varepsilon(t;\mathcal{F}_2),$$

and thus (3.13) holds in light of the induction hypothesis. In the second case, for $\mathbf{s}^{(2n+2)} \in A(\mathfrak{e})$ we have $0 \le s_1 - s_{2n+2} \le m\varepsilon^\kappa$ therefore

$$\mathcal{I}_\varepsilon(t;\mathcal{F}) \le Ct\varepsilon^{(2n+1)\kappa}$$

and (3.13) holds (with n replaced by $n+1$), provided that $\kappa > (n+1)/(2n+1)$. $\square$

The contribution of the ladder diagrams

The last step in the proof of Proposition 3.3 is to consider the contribution of the ladder diagrams. We have shown so far that

$$\lim_{\varepsilon\downarrow 0} \mathbb{E}\hat{\zeta}^\varepsilon_{2n}(t,\xi) = J_n(t,\xi), \tag{3.15}$$

where

$$J_n(t,\xi) = \hat{\phi}_0(\xi) \lim_{\varepsilon\downarrow 0} \frac{(-1)^n}{[\varepsilon(2\pi)^d]^n} \int\limits_{\Delta_{2n}(t)} d\mathbf{s}^{(2n)} \int d\mathbf{p}^{(2n)} \prod_{k=1}^n \hat{R}(p_{2k-1})\delta(p_{2k-1}+p_{2k})$$
$$\times e^{-\mathfrak{g}(p_{2k-1})(s_{2k-1}-s_{2k})/\varepsilon} \exp\left\{ iG_n(\mathbf{s}^{(2n)},\mathbf{p}^{(2n)})/\varepsilon \right\} \tag{3.16}$$

where $G_n(\mathbf{s}^{(2n)},\mathbf{p}^{(2n)})$ is given by (2.4). For the ladder diagram, taking into account the delta-functions, we have

$$G_n(\mathbf{s}^{(2n)},\mathbf{p}^{(2n)}) = \sum_{m=1}^n \left[\xi\cdot p_{2m-1} - \frac{1}{2}|p_{2m-1}|^2 \right] (s_{2m-1}-s_{2m}).$$

Hence, (3.16) can be written as

$$J_n(t,\xi) = \hat{\phi}_0(\xi) \lim_{\varepsilon\downarrow 0} \frac{(-1)^n}{[\varepsilon(2\pi)^d]^n} \int\limits_{\Delta_{2n}(t)} d\mathbf{s}^{(2n)} \int d\mathbf{p}^{(2n)} \prod_{k=1}^n \hat{R}(p_{2k-1})\delta(p_{2k-1}+p_{2k}) e^{-Q(p_{2k-1})(s_{2k-1}-s_{2k})/\varepsilon}, \tag{3.17}$$

with

$$Q(p) = \mathfrak{g}(p) - i\left(\xi\cdot p - \frac{1}{2}|p|^2\right).$$

Changing variables $s'_{2m} = (s_{2m-1}-s_{2m})/\varepsilon$ we obtain after dropping the primes:

$$J_n(t,\xi) = \hat{\phi}_0(\xi) \lim_{\varepsilon\downarrow 0} \frac{(-1)^n}{(2\pi)^{nd}} \int_0^t ds_1 \int_0^{s_1/\varepsilon} ds_2 \int_0^{s_1-\varepsilon s_2} ds_3 \ldots \int_0^{s_{2n-3}-\varepsilon s_{2n-2}} ds_{2n-1} \int_0^{s_{2n-1}/\varepsilon} ds_{2n}$$
$$\times \int \ldots \int \prod_{k=1}^n \hat{R}(p_{2k-1}) dp_{2k-1} \prod_{k=1}^n e^{-Q(p_{2k-1})s_{2k}}. \tag{3.18}$$

One can now compute the limit in (3.18):

$$J_n(t,\xi) = \hat{\phi}_0(\xi) \frac{(-1)^n}{(2\pi)^{nd}} \int_0^t ds_1 \int_0^{s_1} ds_3 \ldots \int_0^{s_{2n-3}} ds_{2n-1} \int \ldots \int \prod_{k=1}^n \frac{\hat{R}(p_{2k-1})}{Q(p_{2k-1})} dp_{2k-1}$$
$$= \hat{\phi}_0(\xi) \frac{(-1)^n t^n}{(2\pi)^{nd} n!} \left(\int \frac{\hat{R}(p)}{Q(p)} dp \right)^n = \hat{\phi}_0(\xi) \frac{(-tD_\xi)^n}{2^n n!}, \tag{3.19}$$

where

$$D_\xi = 2\int \frac{\hat R(p)}{\mathfrak{g}(p) - i(\xi\cdot p - |p|^2/2)}\frac{dp}{(2\pi)^d}. \tag{3.20}$$

Hence, we have

$$\lim_{\varepsilon\to 0+}\mathbb{E}\hat\zeta^\varepsilon_{2n}(t,\xi) = \hat\phi_0(\xi)\frac{(-tD_\xi)^n}{2^n n!}. \tag{3.21}$$

This completes the proof of Proposition 3.3. □

The limit of the second moment: the proof of Proposition 3.5

We now identify the limit of $\mathbb{E}\left[\hat\zeta_\varepsilon(t,\xi)\right]^2$. Consider the expansion

$$\left[\hat\zeta_\varepsilon(t,\xi)\right]^2 = \sum_{n_1,n_2=0}^\infty \hat\zeta^\varepsilon_{n_1}(t,\xi)\hat\zeta^\varepsilon_{n_2}(t,\xi), \tag{3.22}$$

where each term $\hat\zeta^\varepsilon_n(t,\xi)$ is given by (2.3). Evaluating the expectation in (3.22) and using an argument as in the proof of part (ii) of Proposition 2.1 gives

$$\mathbb{E}\left[\hat\zeta_\varepsilon(t,\xi)\right]^2 = \sum_{n_1,n_2=0}^\infty J^\varepsilon_{n_1,n_2}(t,\xi), \tag{3.23}$$

where

$$J^\varepsilon_{n_1,n_2}(t,\xi) = \mathbb{E}\left[\hat\zeta^\varepsilon_{n_1}(t,\xi)\hat\zeta^\varepsilon_{n_2}(t,\xi)\right], \tag{3.24}$$

or, equivalently,

$$J^\varepsilon_{n_1,n_2}(t,\xi) = (-1)^n\left[\frac{1}{\varepsilon^{1/2}(2\pi)^d}\right]^{2n}\int\int_{D^t_{n_1,n_2}}\prod_{j=1}^N d\mathbf{s}_1 d\mathbf{s}_2\int\prod_{j=1}^2\left[\hat\phi_0(\xi - p_{i1} - \ldots - p_{in_i})e^{iG_{n_j}(\mathbf{s}_j,\mathbf{p}_j)/\varepsilon}\right]$$
$$\times\mathbb{E}\left[\hat V(\frac{s_{11}}{\varepsilon},dp_{11})\ldots\hat V(\frac{s_{1n_1}}{\varepsilon},dp_{1n_1})\hat V(\frac{s_{21}}{\varepsilon},dp_{21})\ldots\hat V(\frac{s_{2n_2}}{\varepsilon},dp_{2n_2})\right],$$

where $\mathbf{s}_j = (s_{j1},\ldots,s_{jn_j})$, $\mathbf{p}_j = (p_{j1},\ldots,p_{jn_j})$ and $D^t_{n_1,n_2} := \Delta_{n_1}(t)\times\Delta_{n_2}(t)$. We evaluate the expectation using the Feynman diagrams, as in (3.6), and get

$$J^\varepsilon_{n_1,n_2}(t,\xi) = \sum_{\mathcal{F}} J^\varepsilon_{n_1,n_2}(t,\xi;\mathcal{F}). \tag{3.25}$$

Here the summation extends over all Feynman diagrams formed over pairs of integers (jk), with $j = 1,2$, and $k = 1,\ldots,n_j$. We introduce a lexicographical ordering between pairs, that is, we say that $(jk)\prec(j'k')$ if $j<j'$, or if $j=j'$ then $k\le k'$. If (e,f) is an edge of a Feynman diagram we say that e is a left vertex if $e\prec f$. Also, given a vertex $e=(jk)$ we will use the notation $s(e)=s_{jk}$, $p(e)=p_{jk}$. The following analog of Proposition 3.1 holds.

Proposition 3.8 *There exist constants $J_{n_1,n_2}(t,\xi)$ such that*

$$\sup_{t\in[0,t]}|J^\varepsilon_{n_1,n_2}(t,\xi)| \le J_{n_1,n_2}(T,\xi),\quad \forall\,\varepsilon\in(0,1] \tag{3.26}$$

and

$$\sum_{n_1,n_2=0}^{+\infty} J_{n_1,n_2}(T,\xi) < +\infty.$$

Proof. Estimates following (3.6) essentially hold without changes, that is, we start with

$$
\begin{aligned}
&|J^\varepsilon_{n_1,n_2}(t,\xi)| \le \frac{C^n\|\hat\phi_0\|^2_\infty}{\varepsilon^n} \iint_{D^t_{n_1,n_2}} d\mathbf{s}_1 d\mathbf{s}_2 \\
&\times \int \left|\mathbb{E}\left[\hat V(\frac{s_{11}}{\varepsilon}, dp_{11})\dots \hat V(\frac{s_{1n_1}}{\varepsilon}, dp_{1n_1})\hat V(\frac{s_{21}}{\varepsilon}, dp_{21})\dots \hat V(\frac{s_{2n_2}}{\varepsilon}, dp_{2n_2})\right]\right| \\
&\le \frac{C^n\|\hat\phi_0\|^2_\infty}{n_1!n_2!\varepsilon^n}\int_0^t\int_0^t d\mathbf{s}_1 d\mathbf{s}_2 \int \left|\mathbb{E}\left[\hat V(\frac{s_{11}}{\varepsilon}, dp_{11})\dots \hat V(\frac{s_{1n_1}}{\varepsilon}, dp_{1n_1})\hat V(\frac{s_{21}}{\varepsilon}, dp_{21})\dots \hat V(\frac{s_{2n_2}}{\varepsilon}, dp_{2n_2})\right]\right|,
\end{aligned}
$$

with $2n = n_1 + n_2$. This can be estimated, as in (3.9), and we obtain

$$
|J^\varepsilon_{n_1,n_2}(t,\xi)| \le \frac{C_T^n\|\hat\phi_0\|^2_\infty}{n_1!n_2!}\#(\mathcal{F}),
$$

where $\#(\mathcal{F})$ is the total number of the Feynman diagrams, and is equal to

$$
\#(\mathcal{F}) = (n_1+n_2-1)!! = (2n-1)!!.
$$

We conclude that

$$
|J^\varepsilon_{n_1,n_2}(t,\xi)| \le \frac{C_T^n(2n-1)!!}{n_1!n_2!}\|\hat\phi_0\|^2_\infty. \tag{3.27}
$$

On the other hand, we have

$$
\sum_{n=0}^\infty \sum_{n_1+n_2=2n} \frac{C_T^n(2n-1)!!}{n_1!n_2!} = \sum_{n=0}^\infty \frac{C_T^n 2^{2n}(2n-1)!!}{(2n)!} = \sum_{n=0}^\infty \frac{(2C_T)^n}{n!} < +\infty,
$$

and the conclusion of Proposition 3.8 follows. □

As a consequence of the above proposition, we may pass to the limit $\varepsilon \downarrow 0$ term-wise in the series (3.23):

$$
\lim_{\varepsilon\downarrow 0}\mathbb{E}\left[\hat\zeta_\varepsilon(t,\xi)\right]^2 = \sum_{n_1,n_2=0}^\infty \sum_{\mathcal{F}} \lim_{\varepsilon\downarrow 0} J^\varepsilon_{n_1,n_2}(t,\xi;\mathcal{F}), \tag{3.28}
$$

where $J^\varepsilon_{n_1,n_2}(t,\xi;\mathcal{F})$ is given by

$$
\begin{aligned}
J^\varepsilon_{n_1,n_2}(t,\xi;\mathcal{F}) &= \frac{(-1)^n}{[(2\pi)^d\varepsilon]^n}\iint_{D^t_{n_1,n_2}} d\mathbf{s}_1 d\mathbf{s}_2 \int d\mathbf{p}_1 d\mathbf{p}_2 \prod_{(jk,j'm)\in\mathcal{F}}\left[e^{-\mathfrak{g}(p_{jk})|s_{jk}-s_{j'm}|/\varepsilon}\hat R(p_{jk})\delta(p_{jk}+p_{j'm})\right] \\
&\times \prod_{j=1}^2\left[e^{iG_{n_j}(\mathbf{s}_j,\mathbf{p}_j)/\varepsilon}\hat\phi_0(\xi - p_{j1} - \ldots - p_{jn_j})\right],
\end{aligned} \tag{3.29}
$$

and we only need to study the limit of $J^\varepsilon_{n_1,n_2}(t,\xi;\mathcal{F})$ for a fixed diagram $\mathcal{F}$. Recall that in the case of the first moment calculations that we addressed previously this limit did not vanish only for the ladder diagrams. We claim that only those diagrams that are ladder ones when restricted to both the first n_1 vertices and separately to the final n_2 vertices (thus both n_1 and n_2 must be even) contribute to the limit.

Let Π be the set of all permutations of the vertices $\{(1;1),(1;2),\dots(2;n_2)\}$. We divide the domain of integration $D^t_{n_1,n_2} = \Delta_{n_1}(t)\times\Delta_{n_2}(t)$ into the sets $\Delta(\sigma)$, $\sigma\in\Pi$ as follows: a point

$(s_{11},\ldots,s_{1n_1},s_{21},\ldots,s_{2n_2})\in\Delta(\sigma)$, if $s_{\sigma(1;1)}\ge s_{\sigma(1;2)}\ldots\ge s_{\sigma(2;n_2)}$ and $\mathbf{s}\in D$. This gives rise to a decomposition

$$D^t_{n_1,n_2}=\bigcup_{\sigma\in\Pi}\Delta(\sigma).$$

Note that the set $\Delta(\sigma)$ may be empty for some permutations σ because $s_{jk}\le s_{jk'}$ for all $j=1,2$ and $k\le k'$ if $\mathbf{s}\in D^t_{n_1,n_2}$, hence, for instance, $s_{12}>s_{11}$ is impossible. We can write then

$$J^\varepsilon_{n_1,n_2}(t,\xi;\mathcal{F})=\sum_{\sigma\in\Pi}J^\varepsilon_{n_1,n_2}(t,\xi;\mathcal{F},\sigma),$$

where $J^\varepsilon_{n_1,n_2}(t,\xi;\mathcal{F},\sigma)$ corresponds to the integration over $\Delta(\sigma)$. By the same argument as in the proof of Lemma 3.7 we can prove that that

$$\lim_{\varepsilon\downarrow 0}J^\varepsilon_{n_1,n_2}(t,\xi;\mathcal{F},\sigma)=0,$$

unless $\mathcal{F}=\mathcal{F}_\sigma:=(\sigma(1;1),\sigma(1;2))(\sigma(1;3),\sigma(1;4))\ldots(\sigma(2;n_2-1),\sigma(2;n_2))$, that is, for each domain $\Delta(\sigma)$ there is only one diagram that potentially may contribute to the limit, and such diagrams are the analogs of the ladder diagrams introduced before. It follows that

$$\bar{J}_{n_1,n_2}(t,\xi;\sigma)=\sum_{\mathcal{F}}\lim_{\varepsilon\downarrow 0}J^\varepsilon_{n_1,n_2}(t,\xi;\mathcal{F},\sigma)=\lim_{\varepsilon\downarrow 0}J^\varepsilon_{n_1,n_2}(t,\xi;\mathcal{F}_\sigma,\sigma).$$

Let $(\bar{e}_{2k-1},\bar{e}_{2k})$, $k=1,\ldots,n$ be the edges of a ladder diagram $\mathcal{F}_\sigma$ as above, that is, $\bar{e}_1=\sigma(1;1)$, $\bar{e}_2=\sigma(1;2)$, and so on. We claim that in order for the diagram to contribute to the limit all its edges must be of the form $(\bar{e}_{2k-1},\bar{e}_{2k})=((j;2k-1),(j;2k))$ for some $j=1,2$ and $k=1,\ldots,[n_j/2]$, that is, no vertices corresponding to two different simplices should be paired. This, in fact, forces both n_j, $j=1,2$ to be even. To prove the claim it suffices only to show that all diagrams $\mathcal{F}_\sigma$ containing an edge of the form $(\bar{e}_{2k-1},\bar{e}_{2k})=((1;i_1),(2;i_2))$ satisfy

$$\lim_{\varepsilon\downarrow 0}J^\varepsilon_{n_1,n_2}(t,\xi;\mathcal{F}_\sigma,\sigma)=0. \tag{3.30}$$

Suppose that the edge corresponds to the smallest values of such "mixed" s, that is, all smaller times come from the same simplex: $s(\bar{e}_{2k-1})\ge s(\bar{e}_{2k})\ge s_{j,r}\ge\ldots\ge s_{j,n_j}$. To fix our attention we let $j=2$ and $s_{1n_1}\ge s_{2,r-1}$. The other cases can be argued in the same way. Note that then $n_2-r+1=2n-2k$ (recall that $n_1+n_2=2n$) has to be even, and we should also have $i_2=r-1$ and $i_1=n_1$. Let us denote $d\mathbf{s}'_{j,m}=ds_{j,1}\ldots ds_{j,m}$, $d\mathbf{s}''_{j,m}=ds_{j,m}\ldots ds_{j,n_j}$, with $j=1,2$, and

$$\Delta'_m(t;\sigma)=[t\ge s(\bar{e}_1)\ge\ldots s(\bar{e}_{2m})\ge 0],$$

$$\Delta''_m(t;\sigma)=[t\ge s(\bar{e}_{2m})\ge\ldots s(\bar{e}_{2n})\ge 0].$$

Denote also $G_m(\mathbf{s},\mathbf{p})$ the expression (2.4), where the range of summation has been restricted to $k=1,\dots,m$ and by $G'_{nm}(\mathbf{s},\mathbf{p}) := G_n(\mathbf{s},\mathbf{p}) - G_m(\mathbf{s},\mathbf{p})$. Using (3.29) we can write

$$
\begin{aligned}
J^{\varepsilon}_{n_1,n_2}(t,\xi;\mathcal{F}_\sigma) &= \frac{(-1)^n}{[(2\pi)^d\varepsilon]^n}\int\prod_{m=1}^{n}\left[\hat{R}(p_{\bar{e}_{2m-1}})\delta(p_{\bar{e}_{2m-1}}+p_{\bar{e}_{2m}})\right]d\mathbf{p}_1 d\mathbf{p}_2 \\
&\times\prod_{j=1}^{2}\hat{\phi}_0(\xi-p_{j1}-\ldots-p_{jn_j})\int\limits_{\Delta'_{k-1}(t;\sigma)} d\mathbf{s}'_{1,n_1-1}d\mathbf{s}'_{2,r-2}\prod_{m=1}^{k-1}\left[e^{-\mathfrak{g}(p_{\bar{e}_{2m-1}})|s_{\bar{e}_{2m-1}}-s_{\bar{e}_{2m}}|/\varepsilon}\right] \\
&\times e^{iG_{n_1-1}(\mathbf{s}_1,\mathbf{p}_1)/\varepsilon}e^{iG_{r-2}(\mathbf{s}_2,\mathbf{p}_2)/\varepsilon}\int_0^{s(\bar{e}_{2k-2})}ds_{1,n_1}\int_0^{s_{1,n_1}}ds_{2,r-1}e^{-\mathfrak{g}(p_{1,n_1})(s_{1,n_1}-s_{2,r-1})/\varepsilon} \\
&\times\exp\left\{i\left[\xi\cdot p_{1,n_1}+\frac{1}{2}|p_{1,n_1}|^2-p_{1,n_1}\cdot\left(\sum_{m=1}^{n_1}p_{1,m}\right)\right](s_{1,n_1}-s_{2,r-1})/\varepsilon\right\} \\
&\times\exp\left\{ip_{1,n_1}\cdot\left(-\sum_{m=1}^{n_1}p_{1,m}+\sum_{m=1}^{r-2}p_{2,m}\right)s_{2,r-1}/\varepsilon\right\}\mathcal{I}_\varepsilon(s_{2,r-1},\mathbf{p}_2),
\end{aligned}
\tag{3.31}
$$

where

$$
\mathcal{I}_\varepsilon(s_{2,r-1},\mathbf{p}_2) := \int\limits_{\Delta''_{k-1}(s_{2,r-1};\sigma)} e^{iG'_{n_2,r}(\mathbf{s}''_{2,r},\mathbf{p}_2)/\varepsilon}\prod_{m=0}^{(n_2-r-1)/2}e^{-\mathfrak{g}(p_{2,r+2m})(s_{2,r+2m}-s_{2,r+2m+1})/\varepsilon}d\mathbf{s}''_{2,r}.
$$

Observe that

$$
G'_{n_2,r}(\mathbf{s}''_{2,r},\mathbf{p}_2) = \sum_{m=0}^{(n_2-r-1)/2}C_{r,m}(\mathbf{p}_2)(s_{2,r+2m}-s_{2,r+2m+1}),
$$

where

$$
C_{r,m}(\mathbf{p}_2) := (\xi\cdot p_{2,r+2m}) - \left(\sum_{j=1}^{r-1}p_{2,j}\right)\cdot p_{2,r+2m}-\frac{1}{2}|p_{2,r+2m}|^2.
$$

Performing the change of variables $s'_{2,r+l}=s_{2,r+l}/\varepsilon$, and then subsequently $s''_{2,r+2m+1} := s'_{2,r+2m}-s'_{2,r+2m+1}$, $s''_{2,r+2m} := s'_{2,r+2m}$ for the variables "following" the edge $(\bar{e}_{2k-1},\bar{e}_{2k})$, i.e. for $l=r,\dots,n_2$, we conclude that

$$
\begin{aligned}
\mathcal{I}_\varepsilon(s_{2,r-1},\mathbf{p}_2) &= \varepsilon^{n_2-r+1}\int\limits_{\Delta_{n_2-r+1}(s_{2,r-1}/\varepsilon)}\prod_{m=0}^{(n_2-r-1)/2}e^{[-\mathfrak{g}(p_{2,r+2m})+iC_{r,m}(\mathbf{p}_2)]s_{2,r+2m}}d\mathbf{s}''_{2,r} \\
&= \varepsilon^{2n-2k}|\Delta_{(n_2-r+1)/2}(s_{2,r-1}/\varepsilon)|\left\{\prod_{m=0}^{(n_2-r-1)/2}[\mathfrak{g}(p_{2,r+2m})-iC_{r,m}(\mathbf{p}_2)]\right\}^{-1}+o(1),
\end{aligned}
$$

as $\varepsilon\ll 1$ and $n_2-r+1=2n-2k$. Hence,

$$
J^{\varepsilon}_{n_1,n_2}(t,\xi;\mathcal{F}_\sigma,\sigma) = \tilde{J}^{\varepsilon}_{n_1,n_2}(t,\xi;\sigma)+o(1),
$$

where

$$\tilde{J}^{\varepsilon}_{n_1,n_2}(t,\xi;\sigma)=\frac{(-1)^n}{(2\pi)^{nd}(n-k)!}\int d\mathbf{p}_1 d\mathbf{p}_2\prod_{m=1}^{n}\delta(p_{\bar{e}_{2m-1}}+p_{\bar{e}_{2m}})\hat{R}(p_{\bar{e}_{2m-1}}) \tag{3.32}$$
$$\times\left\{\prod_{m=0}^{(n_2-r-1)/2}[\mathfrak{g}(p_{2,r+2m})-iC_{r,m}(\mathbf{p}_2)]\right\}^{-1}\prod_{j=1}^{2}\hat{\phi}_0(\xi-p_{j1}-\ldots-p_{jn_j})K_\varepsilon(t,\mathbf{p}_1,\mathbf{p}_2)$$

and

$$K_\varepsilon(t,\mathbf{p}_1,\mathbf{p}_2)=\varepsilon^{-k}\int\int_{\Delta'_t(\sigma)} d\mathbf{s}_1 d\mathbf{s}'_2\prod_{j=1}^{k-1}e^{-\mathfrak{g}(p_{\bar{e}_{2j-1}})|s_{\bar{e}_{2j-1}}-s_{\bar{e}_{2j}}|/\varepsilon}e^{iG_{n_1}(\mathbf{s}_1,\mathbf{p}_1)/\varepsilon}$$
$$\times e^{iG_{r-2}(\mathbf{s}_2,\mathbf{p}_2)/\varepsilon}\int_0^{s(\bar{e}_{2k-2})}ds_{1,n_1}\int_0^{s_{1,n_1}}s_{2,r-1}^{n-k}ds_{2,r-1}e^{-\mathfrak{g}(p_{1,n_1})(s_{1,n_1}-s_{2,r-1})/\varepsilon}$$
$$\times\exp\left\{i\left[\xi\cdot p_{1,n_1}+1/2|p_{1,n_1}|^2-p_{1,n_1}\cdot\left(\sum_{m=1}^{n_1}p_{1,m}\right)\right](s_{1,n_1}-s_{2,r-1})/\varepsilon\right\}$$
$$\times\exp\left\{-ip_{1,n_1}\cdot\left(\sum_{m=1}^{n_1}p_{1,m}-\sum_{m=1}^{r-1}p_{2,m}\right)s_{2,r-1}/\varepsilon\right\}.$$

Note that the expression in parentheses appearing in the last exponent equals $2\sum' p_{1,m}$, where the sum extends over all indices that correspond to the vertices $(1;m)$ that appear in the edges of the form $((1;m),(2;l))$.

We conclude from the Lebesgue dominated convergence theorem that

$$\lim_{\varepsilon\to 0}\tilde{J}^{\varepsilon}_{n_1,n_2}(t,\xi;\sigma):=\frac{(-1)^n}{(2\pi)^{2nd}(n-k)!}\int d\mathbf{p}_1 d\mathbf{p}_2\prod_{m=1}^{n}\delta(p_{\bar{e}_{2m-1}}+p_{\bar{e}_{2m}})\hat{R}(p_{\bar{e}_{2m-1}}) \tag{3.33}$$
$$\times\left\{\prod_{m=0}^{(n_2-r-1)/2}[\mathfrak{g}(p_{2,r+2m})-iC_{r,m}(\mathbf{p}_1,\mathbf{p}_2)]\right\}^{-1}\prod_{j=1}^{2}\hat{\phi}_0(\xi-p_{j1}-\ldots-p_{jn_j})\lim_{\varepsilon\downarrow 0}K_\varepsilon(t,\mathbf{p}_1,\mathbf{p}_2).$$

To compute $\lim_{\varepsilon\to 0}K_\varepsilon(t,\mathbf{p}_1,\mathbf{p}_2)$ we change the s variables according to $s'_{jm}:=s_{jm}/\varepsilon$ and then let again $s'_{1,r-1}:=s_{1,n_1}$, $s'_{2,r-1}:=s_{1,n_1}-s_{2,r-1}$. We obtain that

$$K_\varepsilon(t,\mathbf{p}_1,\mathbf{p}_2)=\varepsilon^n\int_{\Delta'_{t/\varepsilon}(\sigma)}d\mathbf{s}_1 d\mathbf{s}'_2\prod_{j=1}^{k-1}e^{-\mathfrak{g}(p_{\bar{e}_{2j-1}})|s_{\bar{e}_{2j-1}}-s_{\bar{e}_{2j}}|}e^{iG_{n_1}(\mathbf{s}_1,\mathbf{p}_1)}e^{iG_r(\mathbf{s}_2,\mathbf{p}_2)}\int_0^{s(\bar{e}_{2k-2})}ds_{2,r-1}$$
$$\times e^{-\mathfrak{g}(p_{1,n_1})s_{2,n_1}}\exp\left\{i\left[\xi\cdot p_{1,n_1}+1/2|p_{1,n_1}|^2-p_{1,n_1}\cdot\left(\sum_{m=1}^{n_1}p_{1,m}\right)\right]s_{2,r-1}\right\}$$
$$\times\int_{s_{2,r-1}}^{s(\bar{e}_{2k-2})}(s_{1,n_1}-s_{2,r-1})^{n-k}\exp\left\{2ip_{1,n_1}\cdot\left(\sum{}'p_{1,m}\right)(s_{1,n_1}-s_{2,r-1})\right\}ds_{1,n_1}. \tag{3.34}$$

Since for any $a\neq 0$, $T>0$ and an integer $m\geq 0$ we have an estimate

$$\sup_{0<A<B<T/\varepsilon}\left|\int_A^B s^m e^{ias}ds\right|\leq C_{T,a}\varepsilon^{-m},$$

where $C_{T,a} < +\infty$, the last integral on the right hand side of (3.34) can be estimated by $C(\mathbf{p}_1, \mathbf{p}_2)\varepsilon^{k-n}$, with $C(\mathbf{p}_1, \mathbf{p}_2) < +\infty$ except possibly for a set of zero measure and as a result we obtain that

$$K_\varepsilon(t, \mathbf{p}_1, \mathbf{p}_2) \le \varepsilon^k \int\limits_{\Delta'_{k-1}(t/\varepsilon;\sigma)} d\mathbf{s}_1 d\mathbf{s}'_2 \prod_{m=1}^{k-1} e^{-\mathfrak{g}(p_{\bar{e}_{2m-1}})s_{\bar{e}_{2m-1}}} \int_0^{s(\bar{e}_{2k-2})} e^{-\mathfrak{g}(p_{\bar{e}_{2k-1}})s_{2,n_1}} ds_{2,r-1}.$$

Thus $K_\varepsilon(t, \mathbf{p}_1, \mathbf{p}_2) \le C'(\mathbf{p}_1, \mathbf{p}_2)\varepsilon$, where constant $C'(\mathbf{p}_1, \mathbf{p}_2) < +\infty$ except possibly for a set of zero measure and (3.30) follows.

We have shown therefore that

$$\lim_{\varepsilon\downarrow 0} J^\varepsilon_{n_1,n_2}(t, \xi; \mathcal{F}) = J_{n_1}(t, \xi)J_{n_2}(t, \xi), \tag{3.35}$$

where $\mathcal{F}$ is a Feynman diagram that is the union of two ladder diagrams formed over the sets $\{(1;1), \ldots, (1;n_1)\}$ and $\{(2;1), \ldots, (2;n_2)\}$. In all other cases

$$\lim_{\varepsilon\downarrow 0} J^\varepsilon_{n_1,n_2}(t, \xi; \mathcal{F}) = 0.$$

4 Proof of Theorem 1.2

The overall steps in the proof of Theorem 1.2 are similar to that of Theorem 1.1: we expand $\hat{\zeta}_\varepsilon(t, \xi)$ into the Duhamel expansion series (2.2) and then, first, use Proposition 2.1 to establish convergence of $\mathbb{E}(\hat{\zeta}_\varepsilon(t, \xi))$, and, second, address convergence of the higher moments of $\hat{\zeta}_\varepsilon(t, \xi)$. The main difference with the proof of Theorem 1.1 is that now not only the ladder diagrams contribute in the limit $\varepsilon \to 0$ but rather all Feynman diagrams have a non-trivial contribution. This leads to a non-Markovian limit. Moreover, the limit is no longer deterministic, hence one has to find the limit of $\mathbb{E}(\hat{\zeta}_\varepsilon(t, \xi)^N)$ for all $N \ge 1$.

4.1 Convergence of the expectation

We first establish the analog of Corollary 3.4.

Proposition 4.1 *We have*

$$\lim_{\varepsilon\to 0} \mathbb{E}\hat{\zeta}_\varepsilon(t, \xi) = \hat{\phi}_0(\xi)\mathbb{E}\left[e^{iB_\kappa(t;\xi)}\right], \tag{4.1}$$

for all $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^d \setminus \{0\}$. Here $B_\kappa(t)$ is the fractional Brownian motion with the exponent $\kappa = (\alpha + 2\beta - 1)/(2\beta)$ and the diffusion coefficient given by (1.15) for $\beta < 1/2$ and by (1.16) for $\beta = 1/2$.

Outline of the proof

The strategy of the proof of Proposition 4.1 is similar to what we have done previously to arrive at Corollary 3.4. First, we will establish the following uniform bound:

Proposition 4.2 *For all $T > 0$, $n \ge 0$ and all $\xi \in \mathbb{R}^d \setminus \{0\}$ there exists a constant $C(T; \xi)$ such that*

$$\sup_{t\in[0,T]} |\mathbb{E}\hat{\zeta}^\varepsilon_n(t, \xi)| \le \frac{C^n(T; \xi)}{n!} \tag{4.2}$$

for all $\varepsilon \in (0, 1]$.

As before, this allows us to interchange the limit $\varepsilon \to 0$ and the summation in n.

Corollary 4.3 *We have*

$$\lim_{\varepsilon\downarrow 0} \mathbb{E}\hat{\zeta}_\varepsilon(t,\xi) = \sum_{n=0}^{\infty} \lim_{\varepsilon\downarrow 0} \mathbb{E}\hat{\zeta}_n^\varepsilon(t,\xi), \tag{4.3}$$

for all $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^d \setminus \{0\}$.

This corollary is an immediate consequence of the estimate (4.2). The last step in the proof of Proposition 4.1 is to identify the limit of the individual terms in the right side of (4.3).

Proposition 4.4 *We have*

$$\lim_{\varepsilon\downarrow 0} \mathbb{E}\hat{\zeta}_n^\varepsilon(t,\xi) = \hat{\phi}_0(\xi)\mathbb{E}\left[\frac{(iB_\kappa(t;\xi))^n}{n!}\right], \tag{4.4}$$

for all $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^d \setminus \{0\}$.

This implies the conclusion of Proposition 4.1.

The proof of Proposition 4.2

We suppose that $\mathfrak{g}(p) = \mu|p|^{2\beta}$ and $\hat{R}(p) = a(p)/|p|^{2\alpha+d-2}$ for parameters μ, β, α and a function $a(\cdot)$ as in the statement of Theorem 1.2. As in (3.6), we have the estimate

$$\begin{aligned} |\mathbb{E}\hat{\zeta}_{2n}^\varepsilon(t,\xi)| &\le \frac{\|\hat{\phi}_0\|_\infty}{(2n)!}\left[\frac{\gamma}{\varepsilon(2\pi)^d}\right]^{2n}\int_0^t\cdots\int_0^t d\mathbf{s}^{(2n)}\int\left|\mathbb{E}\left[\hat{V}(\frac{s_1}{\varepsilon},dp_1)\ldots\hat{V}(\frac{s_{2n}}{\varepsilon},dp_{2n})\right]\right| \\ &\le \frac{C^n\|\hat{\phi}_0\|_\infty}{(2n)!}\frac{\gamma^{2n}}{\varepsilon^{2n}}\sum_{\mathcal{F}}\int_0^t\cdots\int_0^t d\mathbf{s}^{(2n)}\int d\mathbf{p}^{(2n)}\prod_{(k,l)\in\mathcal{F}} e^{-\mu|p_k|^{2\beta}|s_k-s_l|/\varepsilon}\delta(p_k+p_l)\frac{a(p_k)}{|p_k|^{2\alpha+d-2}}, \end{aligned} \tag{4.5}$$

where the summation extends over all Feynman diagrams formed over vertices $\{1,\ldots,2n\}$. Changing variables $p_k' := p_k/\varepsilon^{1/(2\beta)}$ and setting $\kappa = (\alpha+2\beta-1)/(2\beta)$ we rewrite (4.5) as

$$\begin{aligned} |\mathbb{E}\hat{\zeta}_{2n}^\varepsilon(t,\xi)| &\le \frac{C^n\|\hat{\phi}_0\|_\infty}{(2n)!}\left(\frac{\gamma}{\varepsilon^\kappa}\right)^{2n}\sum_{\mathcal{F}}\int_0^t\cdots\int_0^t d\mathbf{s}^{(2n)}\int d\mathbf{p}^{(2n)}\prod_{(k,l)\in\mathcal{F}} e^{-\mu|p_k|^{2\beta}|s_k-s_l|} \\ &\times\delta(p_k+p_l)\frac{a(\varepsilon^{1/(2\beta)}p_k)}{|p_k|^{2\alpha+d-2}} = \frac{C^n\|\hat{\phi}_0\|_\infty}{2^n n!}\left(\frac{\gamma}{\varepsilon^\kappa}\right)^{2n}\left[\int_0^t\int_0^t\int e^{-\mu|p|^{2\beta}|s_1-s_2|}\frac{a(\varepsilon^{1/(2\beta)}p)}{|p|^{2\alpha+d-2}}ds_1ds_2dp\right]^n. \end{aligned} \tag{4.6}$$

We used the fact that the total number of the Feynman diagrams is $(2n-1)!!$ in the last step above. As $\varepsilon = \gamma^{1/\kappa}$, we may recast (4.6) as

$$\begin{aligned} |\mathbb{E}\hat{\zeta}_{2n}^\varepsilon(t,\xi)| &\le \frac{C^n\|\hat{\phi}_0\|_\infty}{n!}\left[\int_0^t\int_0^t\int e^{-\mu|p|^{2\beta}|s_1-s_2|}\frac{a(\varepsilon^{1/(2\beta)}p)}{|p|^{2\alpha+d-2}}ds_1ds_2dp\right]^n \\ &\le \frac{C^n\|\hat{\phi}_0\|_\infty}{n!}\left[\int_0^t ds_1\int_0^{s_1}ds_2\int\frac{e^{-\mu|p|^{2\beta}s_2}}{|p|^{2\alpha+d-2}}dp\right]^n = \frac{C^n(M_*(t))^n\|\hat{\phi}_0\|_\infty}{n!}, \end{aligned} \tag{4.7}$$

where

$$M_*(t) = \int\frac{e^{-\mu|p|^{2\beta}t}-1+\mu|p|^{2\beta}t}{\mu^2|p|^{2\alpha+4\beta+d-2}}dp < +\infty$$

for $\alpha < 1$ and $\alpha+\beta > 1$. Estimate (4.2) now follows. □

The proof of Proposition 4.4

We proceed now with the limit identification. As we have mentioned, the fundamental difference with the rapidly de-correlating case considered in Section 3 lies in the fact that the terms corresponding to an arbitrary Feynman diagram may have a non-vanishing limit, as $\varepsilon \downarrow 0$ – recall that in the previous case only those corresponding to the ladder diagrams have non vanishing limits. As before, starting with (2.3) we have

$$\mathbb{E}\zeta^{\varepsilon}_{2n}(t,\xi) = \hat{\phi}_0(\xi) \sum_{\mathcal{F}} \mathcal{I}^{(\varepsilon)}_{2n}(t;\mathcal{F}),$$

where

$$\mathcal{I}^{(\varepsilon)}_{2n}(t;\mathcal{F}) = \left[\frac{\gamma}{i\varepsilon(2\pi)^{d/2}}\right]^{2n} \int\limits_{\Delta_{2n}(t)} d\mathbf{s}^{(2n)} \int d\mathbf{p}^{(2n)} \prod_{(k,l)\in\mathcal{F}} e^{-\mu|p_k|^{2\beta}(s_k-s_l)/\varepsilon} \frac{a(p_k)\delta(p_k+p_l)}{|p_k|^{2\alpha+d-2}} e^{iG_{2n}(\mathbf{s}^{(2n)},\mathbf{p}^{(2n)})/\varepsilon}. \tag{4.8}$$

Thanks to estimate (4.3) what remains yet to do is to identify the limits

$$\mathcal{I}_{2n}(t;\mathcal{F}) = \lim_{\varepsilon\downarrow 0} \mathcal{I}^{(\varepsilon)}_{2n}(t;\mathcal{F}).$$

An upper bound for the integrand

We now proceed to re-write $\mathcal{I}^{\varepsilon}_{2n}(t;\mathcal{F})$ in such a form that the Lebesgue dominated convergence theorem could be applied to the integrand in the limit $\varepsilon \downarrow 0$. To begin, we make a change of variables $s_i = \sum_{j=i}^{2n} \tau_j$. Consider the phase $G_{2n}(\mathbf{s}^{(2n)},\mathbf{p}^{(2n)})$ and the decomposition (2.4)-(2.5). Note that the terms corresponding to A_{2n} and B_{2n}, after the change of variables, equal, respectively,

$$\tilde{A}(\tau^{(n)},\mathbf{p}^{(n)}) = \sum_{m=1}^{n} \left(\xi \cdot \sum_{j=1}^{m} p_j\right) \tau_m, \tag{4.9}$$

with $\tau^{(2n)} = (\tau_1,\ldots,\tau_{2n}) \in \mathbb{R}^{2n}$ and

$$\tilde{B}(\tau^{(n)},\mathbf{p}^{(n)}) = \sum_{m=1}^{n} \tau_m Q_m(\mathbf{p}^{(n)}), \tag{4.10}$$

where

$$Q_m(\mathbf{p}^{(n)}) = \frac{1}{2}\left|\sum_{j=1}^{m} p_j\right|^2. \tag{4.11}$$

Using the new variables, and introducing an additional variable τ_0, we can rewrite (4.8) in the following way

$$\mathcal{I}^{(\varepsilon)}_{2n}(t;\mathcal{F}) = e^{t}\left[\frac{\gamma}{i\varepsilon(2\pi)^{d/2}}\right]^{2n} \int_0^{+\infty} \cdots \int_0^{+\infty} d\tau^{(2n+1)} \int d\mathbf{p}^{(2n)} \delta(t-\tau_0-\ldots-\tau_{2n}) \tag{4.12}$$

$$\times \prod_{(km)\in\mathcal{F}} \frac{a(p_k)}{|p_k|^{2\alpha+d-2}} \delta(p_k+p_m) \exp\left\{-\mu|p_k|^{2\beta}\sum_{j=k}^{m-1}\tau_j/\varepsilon\right\} \exp\left\{-\sum_{j=0}^{2n}\tau_j + i\tilde{G}_{2n}(\tau^{(2n)},\mathbf{p}^{(2n)})/\varepsilon\right\},$$

where $\tau^{(2n+1)} = (\tau_0,\ldots,\tau_{2n})$. Next, using the fact that

$$\delta(t) = \int e^{-itz}\frac{dz}{2\pi},$$

we obtain

$$\mathcal{I}_{2n}^{(\varepsilon)}(t;\mathcal{F}) = \frac{(-1)^n e^t}{2\pi}\left[\frac{\gamma}{(2\pi)^{d/2}\varepsilon}\right]^{2n}\int_0^{+\infty}\dots\int_0^{+\infty} d\tau^{(2n+1)}\int d\mathbf{p}^{(2n)}\int dz e^{-izt} \tag{4.13}$$

$$\times\prod_{(km)\in\mathcal{F}}\left(\frac{a(p_k)}{|p_k|^{2\alpha+d-2}}\delta(p_k+p_m)\exp\left[-\mu|p_k|^{2\beta}\sum_{j=k}^{m-1}\tau_j/\varepsilon-(1-iz)\sum_{j=0}^{2n}\tau_j+i\tilde{G}_{2n}(\tau^{(2n)},\mathbf{p}^{(2n)})/\varepsilon\right]\right)$$

$$= \frac{(-1)^n e^t}{2\pi}\left[\frac{\gamma}{(2\pi)^{d/2}\varepsilon}\right]^{2n}\int_0^{+\infty}\dots\int_0^{+\infty} d\tau^{(2n+1)}\int d\mathbf{p}^{(2n)}\int dz e^{-izt}\left(\prod_{(km)\in\mathcal{F}}\frac{a(p_k)\delta(p_k+p_m)}{|p_k|^{2\alpha+d-2}}\right)$$

$$\times\exp\left\{-(1-iz)\tau_0-\sum_{j=1}^{2n}\left[\sum_{(km)\in\mathcal{F}}1_{[k,m)}(j)\mu|p_k|^{2\beta}/\varepsilon+1-iz\right]\tau_j+i\tilde{G}_{2n}(\tau^{(2n)},\mathbf{p}^{(2n)})/\varepsilon\right\}.$$

Integrating out the τ-variables gives

$$\mathcal{I}_{2n}^{(\varepsilon)}(t;\mathcal{F}) = \frac{(-1)^n e^t}{2\pi}\left[\frac{\gamma}{(2\pi)^{d/2}\varepsilon}\right]^{2n}\int d\mathbf{p}^{(2n)}\int\frac{e^{-izt}dz}{1-iz}\left(\prod_{(km)\in\mathcal{F}}\frac{a(p_k)\delta(p_k+p_m)}{|p_k|^{2\alpha+d-2}}\right)$$

$$\times\left\{\prod_{j=1}^{2n}\left[\mu\sum_{(km)\in\mathcal{F}}1_{[k,m)}(j)|p_k|^{2\beta}/\varepsilon+1+i\left(\tilde{Q}_j(\mathbf{p}^{(2n)})/\varepsilon-\sum_{k=1}^{j}\xi\cdot p_k/\varepsilon-z\right)\right]\right\}^{-1}. \tag{4.14}$$

Substituting $p_k' := p_k/\varepsilon^{1/(2\beta)}$, as in the passage from (4.5) to (4.6), and using the relation $\gamma=\varepsilon^{\kappa}$ leads to

$$\mathcal{I}_{2n}^{(\varepsilon)}(t;\mathcal{F}) = \frac{(-1)^n e^t}{(2\pi)^{nd+1}}\int d\mathbf{p}^{(2n)}\int\frac{e^{-izt}dz}{1-iz}\prod_{(km)\in\mathcal{F}}\frac{a(\varepsilon^{1/(2\beta)}p_k)}{|p_k|^{2\alpha+d-2}}\delta(p_k+p_m) \tag{4.15}$$

$$\times\left\{\prod_{j=1}^{2n}\left[\mu\sum_{(km)\in\mathcal{F}}1_{[k,m)}(j)|p_k|^{2\beta}+1+i\left(\tilde{Q}_j(\mathbf{p}^{(2n)})\varepsilon^{1/\beta-1}-\sum_{k=1}^{j}\xi\cdot p_k\varepsilon^{1/(2\beta)-1}-z\right)\right]\right\}^{-1}.$$

Let us denote by $\mathcal{L}(\mathcal{F})$ the set of all left vertices of $\mathcal{F}$, and for an edge $e=(km)\in\mathcal{F}$ set $\ell(m)=k$. The expression under the multiple integral on the right side can be majorized by

$$\frac{\|a\|_\infty^n}{(1+|z|)|p_{\ell(2n)}|^{2\alpha+d-2}\left(\mu|p_{\ell(2n)}|^{2\beta}+1+|z|\right)}\prod_{\substack{j\in\mathcal{L}(\mathcal{F})\\ j\neq\ell(2n)}}\left\{\frac{1}{|p_j|^{2\alpha+d-2}}\left[\mu\sum_{(km)\in\mathcal{F}}1_{[k,m)}(j)|p_k|^{2\beta}+1\right]^{-1}\right\}$$

$$\le\frac{\|a\|_\infty^n}{(1+|z|)|p_{\ell(2n)}|^{2\alpha+d-2}\left(\mu|p_{\ell(2n)}|^{2\beta}+1+|z|\right)}\prod_{\substack{j\in\mathcal{L}(\mathcal{F})\\ j\neq\ell(2n)}}\left\{\frac{1}{|p_j|^{2\alpha+d-2}}\left[\mu|p_j|^{2\beta}+1\right]^{-1}\right\}. \tag{4.16}$$

We used the simple fact that for a vertex $j\in\mathcal{L}(\mathcal{F})$ we have $1_{[k,m)}(j)=1$ if we take the edge with $k=j$ in the summation over the edges of $\mathcal{F}$ above. Now, the expression in the right side of (4.16) is integrable with respect to the measure $d\mu = dz\prod_{j\in\mathcal{L}(\mathcal{F})}dp_j$, since $\alpha+\beta>1$ and $\alpha\in(1/2,1)$.

Computation of the limit of $\mathcal{I}^{(\varepsilon)}_{2n}(t;\mathcal{F})$

The integrability of expression (4.16) allows us to apply the dominated convergence theorem in the expression (4.15) for $\mathcal{I}^{(\varepsilon)}_{2n}(t;\mathcal{F})$ and pass to the limit under the integral sign, concluding that for $\beta < 1/2$ we have, as both $1/\beta > 1$ and $1/(2\beta) > 1$:

$$\mathcal{I}_{2n}(t;\mathcal{F}) = \lim_{\varepsilon\downarrow 0}\mathcal{I}^{(\varepsilon)}_{2n}(t;\mathcal{F}) = \frac{(-1)^n e^t}{2\pi}\left[\frac{a(0)}{(2\pi)^d}\right]^n \hat{\phi}_0(\xi)\int d\mathbf{p}^{(2n)}\int \frac{e^{-izt}dz}{1-iz}\prod_{(km)\in\mathcal{F}}\frac{\delta(p_k+p_m)}{|p_k|^{2\alpha+d-2}}$$
$$\times\left\{\prod_{j=1}^{2n}\left(\mu\sum_{(km)}1_{[k,m)\in\mathcal{F}}(j)|p_k|^{2\beta}+1-iz\right)\right\}^{-1}, \tag{4.17}$$

while for $\beta = 1/2$ we get

$$\mathcal{I}_{2n}(t;\mathcal{F}) = \lim_{\varepsilon\downarrow 0}\mathcal{I}^{(\varepsilon)}_{2n}(t;\mathcal{F}) = \frac{(-1)^n e^t}{2\pi}\left[\frac{a(0)}{(2\pi)^d}\right]^n \hat{\phi}_0(\xi)\int d\mathbf{p}^{(2n)}\int \frac{e^{-izt}dz}{1-iz}\prod_{(km)\in\mathcal{F}}\frac{\delta(p_k+p_m)}{|p_k|^{2\alpha+d-2}}$$
$$\times\left\{\prod_{j=1}^{2n}\left[\mu\sum_{(km)\in\mathcal{F}}1_{[k,m)}(j)|p_k|^{2\beta}+1-i\left(z+\sum_{k=1}^{j}\xi\cdot p_k\right)\right]\right\}^{-1}. \tag{4.18}$$

To unify the notation we introduce $\zeta(\beta) := 0$ for $\beta < 1/2$ and $\zeta(1/2) := 1$. Then, retracing our steps above, we may re-write both (4.17) and (4.18) as (compare to (4.8))

$$\mathcal{I}_{2n}(t;\mathcal{F}) = \left[\frac{-a(0)}{(2\pi)^d}\right]^n \int_{\Delta_{2n}(t)} d\mathbf{s}^{(2n)}\int d\mathbf{p}^{(2n)}\prod_{(k,l)\in\mathcal{F}} e^{-\mu|p_k|^{2\beta}(s_k-s_l)}\frac{\delta(p_k+p_l)}{|p_k|^{2\alpha+d-2}}e^{i\zeta(\beta)\sum_{j=1}^{2n}s_j\xi\cdot p_j}. \tag{4.19}$$

The case $\beta < 1/2$. Now, we relate expression (4.19) to the fractional Brownian motion. Consider first the case $\beta < 1/2$. Then, after integrating out the p-variables (4.19) becomes

$$\mathcal{I}_{2n}(t;\mathcal{F}) = \left[\frac{-a(0)K_1(\alpha,\beta,\mu)}{(2\pi)^d}\right]^n \int_{\Delta_{2n}(t)} d\mathbf{s}^{(2n)}\prod_{(k,l)\in\mathcal{F}}|s_k-s_l|^{(\alpha-1)/\beta}, \tag{4.20}$$

with

$$K_1(\alpha,\beta,\mu) = \Omega_d\int_0^{+\infty}e^{-\mu\rho^{2\beta}}\frac{d\rho}{\rho^{2\alpha-1}},$$

as in (1.12). Here Ω_d is the surface area of the unit sphere in $\mathbb{R}^d$. Let us recall the representation

$$\sum_{\mathcal{F}}\prod_{(pq)\in\mathcal{F}}|s_p-s_q|^{2\kappa-2} = c_\kappa^{2n}\mathbb{E}\left[\prod_{p=1}^{2n}\int_{-\infty}^{\infty}\frac{e^{ik_ps_p}}{|k_p|^{\kappa-1/2}}w(dk_p)\right], \tag{4.21}$$

where $w(dk)$ is a Gaussian white noise and $c_\kappa > 0$ is given by

$$c_\kappa = \left(\frac{\Gamma(2\kappa-1)\sin(\pi\kappa)}{\pi}\right)^{1/2}. \tag{4.22}$$

Then, (4.20) with

$$\kappa = (\alpha+2\beta-1)/(2\beta), \tag{4.23}$$

can be restated as

$$\sum_{\mathcal{F}} \mathcal{I}_{2n}(t;\mathcal{F}) = \left[\frac{-a(0)K_1(\alpha,\beta,\mu)c_\kappa^2}{(2\pi)^d}\right]^n \int\limits_{\Delta_{2n}(t)} d\mathbf{s}^{(2n)} \mathbb{E}\left[\prod_{p=1}^{2n}\int_{-\infty}^{\infty}\frac{e^{ik_p s_p}}{|k_p|^{\kappa-1/2}} w(dk_p)\right]. \quad (4.24)$$

Taking into account the symmetry in the s_j-variables of the expression in the right hand side of (4.24) we obtain that

$$\sum_{\mathcal{F}} \mathcal{I}_{2n}(t;\mathcal{F}) = \frac{1}{(2n)!}\left[\frac{-a(0)K_1(\alpha,\beta,\mu)c_\kappa^2}{(2\pi)^d}\right]^n \int\limits_0^t \dots \int\limits_0^t d\mathbf{s}^{(2n)} \mathbb{E}\left[\prod_{p=1}^{2n}\int_{-\infty}^{\infty}\frac{e^{ik_p s_p}}{|k_p|^{\kappa-1/2}} w(dk_p)\right]$$

$$= \frac{1}{(2n)!}\left[\frac{-a(0)K_1(\alpha,\beta,\mu)c_\kappa^2}{(2\pi)^d}\right]^n \mathbb{E}\left[\prod_{p=1}^{2n}\int_{-\infty}^{+\infty}\frac{e^{ik_p t}-1}{ik_p|k_p|^{\kappa-1/2}} w(dk_p)\right]. \quad (4.25)$$

Using the harmonizable representation of the standard fractional Brownian motion, see Proposition 7.2.8, p. 328 of [23], we deduce that (4.25) can be reformulated as

$$\sum_{\mathcal{F}} \mathcal{I}_{2n}(t;\mathcal{F}) = \frac{1}{(2n)!}\left[\frac{-a(0)K_1(\alpha,\beta,\mu)c_\kappa^2 d_\kappa^2}{(2\pi)^d}\right]^n \mathbb{E}B_\kappa^{2n}(t).$$

Here $B_\kappa(t)$ is the standard (that is, of zero mean and variance one) fractional Brownian motion with the Hurst exponent κ and

$$d_\kappa = \left(\frac{\pi}{\kappa\Gamma(2\kappa)\sin(\kappa\pi)}\right)^{1/2} = \left(\frac{\pi}{2\kappa^2\Gamma(2\kappa-1)\sin(\kappa\pi)}\right)^{1/2}.$$

Observe that, fortunately, $c_\kappa d_\kappa = 1/(\sqrt{2}\kappa)$. To summarize, we we have shown that for $\beta < 1/2$

$$\lim_{\varepsilon\downarrow 0}\mathbb{E}\hat{\zeta}_\varepsilon(t,\xi) = \hat{\phi}_0(\xi)\mathbb{E}e^{i\sqrt{D}B_\kappa(t)}, \quad (4.26)$$

where

$$D = \frac{a(0)K_1(\alpha,\beta,\mu)}{2\kappa(2\pi)^d}. \quad (4.27)$$

The case $\beta = 1/2$. For $\beta = 1/2$ the calculation is very similar. Then, the Hurst exponent κ given by (4.23) is equal to α, and the right side of (4.25) equals

$$\sum_{\mathcal{F}} \lim_{\varepsilon\downarrow 0} \mathcal{I}_{2n}^{(\varepsilon)}(t;\mathcal{F}) = \frac{1}{(2n)!}\mathbb{E}[i\sqrt{D(\xi)}B_\alpha(t)]^{2n}$$

and

$$D(\xi) = \frac{a(0)K_2(\xi;\alpha,\mu)}{2\alpha(2\pi)^d}, \quad (4.28)$$

with

$$K_2(\xi;\alpha,\mu) = \int_0^{+\infty} e^{-\mu\rho}\frac{d\rho}{\rho^{2\alpha-1}}\int_{\mathbb{S}^{d-1}} e^{i|\xi|\rho\omega\cdot e_1} S(d\omega),$$

as in (1.13). This finishes the proof of Proposition 4.4. □

The limit of the higher moments

The last step in the proof of Theorem 1.2 is to show that

$$\lim_{\varepsilon\downarrow 0}\mathbb{E}[\hat{\zeta}_\varepsilon(t,\xi)]^N=[\hat{\phi}_0(\xi)]^N\mathbb{E}e^{iN\sqrt{D(\xi)}B_\kappa(t)} \tag{4.29}$$

for all integers $N\geq 1$. Consider the expansion

$$\left[\hat{\zeta}_\varepsilon(t,\xi)\right]^N=\sum_{n_1,\ldots,n_N=0}^{\infty}\hat{\zeta}^\varepsilon_{n_1}(t,\xi)\ldots\hat{\zeta}^\varepsilon_{n_N}(t,\xi), \tag{4.30}$$

where each term $\hat{\zeta}^\varepsilon_n(t,\xi)$ is given by (2.3). Evaluating the expectation in (4.30) and using an argument as in the proof of part (ii) of Proposition 2.1 gives

$$\mathbb{E}\left[\hat{\zeta}_\varepsilon(t,\xi)\right]^N=\sum_{n_1,\ldots,n_N=0}^{\infty}J^\varepsilon_{n_1,\ldots,n_N}(t,\xi), \tag{4.31}$$

where

$$J^\varepsilon_{n_1,\ldots,n_N}(t,\xi)=\mathbb{E}\left[\hat{\zeta}^\varepsilon_{n_1}(t,\xi)\ldots\hat{\zeta}^\varepsilon_{n_N}(t,\xi)\right], \tag{4.32}$$

or, equivalently,

$$\begin{aligned}J^\varepsilon_{n_1,\ldots,n_N}(t,\xi)&=(-1)^n\left[\frac{\gamma}{\varepsilon(2\pi)^{d/2}}\right]^{2n}\int_{\Delta_{n_1}(t)}\cdots\int_{\Delta_{n_N}(t)}\prod_{j=1}^{N}d\mathbf{s}_1\ldots d\mathbf{s}_N\\&\times\int\mathbb{E}\left[\hat{V}(\frac{s_{11}}{\varepsilon},dp_{11})\ldots\hat{V}(\frac{s_{1n_1}}{\varepsilon},dp_{1n_1})\ldots\hat{V}(\frac{s_{N1}}{\varepsilon},dp_{N1})\ldots\hat{V}(\frac{s_{Nn_N}}{\varepsilon},dp_{Nn_N})\right]\\&\times\hat{\phi}_0(\xi-p_{11}-\ldots-p_{1n_1})\ldots\hat{\phi}_0(\xi-p_{N1}-\ldots-p_{Nn_N})\prod_{j=1}^{N}e^{iG_{n_j}(\mathbf{s}_j,\mathbf{p}_j)/\varepsilon},\end{aligned} \tag{4.33}$$

where $\mathbf{s}_j=(s_{j1},\ldots,s_{jn_j})$ and $\mathbf{p}_j=(p_{j1},\ldots,p_{jn_j})$. We evaluate the expectation using the Feynman diagrams and get

$$J^\varepsilon_{n_1,\ldots,n_N}(t,\xi)=\sum_{\mathcal{F}}J^\varepsilon_{n_1,\ldots,n_N}(t,\xi;\mathcal{F}). \tag{4.34}$$

Here the summation extends over all Feynman diagrams formed over pairs of integers (jk), with $j=1,\ldots,N$, and $k=1,\ldots,n_j$. We introduce a lexicographical ordering between pairs, that is, we say that $(jk)\prec(j'k')$ if $j<j'$, or if $j=j'$ then $k\leq k'$. If (e,f) is an edge of a Feynman diagram we say that e is a left vertex if $e\prec f$. Also, given a vertex $e=(jk)$ we will use the notation $s(e)=s_{jk}$, $p(e)=p_{jk}$. The following bound holds.

Proposition 4.5 *There exist constants $J_{n_1,\ldots,n_N}(t,\xi)$ such that*

$$\sup_{t\in[0,t]}|J^\varepsilon_{n_1,\ldots,n_N}(t,\xi)|\leq J_{n_1,\ldots,n_N}(T,\xi),\quad\forall\,\varepsilon\in(0,1] \tag{4.35}$$

and

$$\sum_{n_1,\ldots,n_N=0}^{+\infty}J_{n_1,\ldots,n_N}(T,\xi)<+\infty.$$

Proof. Using the relation $\varepsilon = \gamma^{1/\kappa}$ in (4.33), dropping the phases and symmetrizing gives

$$|J^{\varepsilon}_{n_1,\dots,n_N}(t,\xi)| \le \frac{C^n \|\hat{\phi}_0\|_{\infty}^N}{n_1!\dots n_N!} \frac{1}{\varepsilon^{2n(1-\kappa)}} \int_0^t \dots \int_0^t \prod_{j=1}^N d\mathbf{s}_1 \dots d\mathbf{s}_N$$

$$\times \left| \int \mathbb{E}\left[\hat{V}(\frac{s_{11}}{\varepsilon}, dp_{11}) \dots \hat{V}(\frac{s_{1n_1}}{\varepsilon}, dp_{1n_1}) \dots \hat{V}(\frac{s_{N1}}{\varepsilon}, dp_{N1}) \dots \hat{V}(\frac{s_{Nn_N}}{\varepsilon}, dp_{Nn_N}) \right] \right|.$$

The right hand side can be estimated essentially in the same way as in (4.6) and (4.7) and we obtain that

$$|J^{\varepsilon}_{n_1,\dots,n_N}(t,\xi)| \le \frac{\|\hat{\phi}_0\|_{\infty}^N}{n_1!\dots n_N!} C_*^n \|a\|_{\infty}^n$$

for any $n_1, \dots, n_N$ such that $n_1 + \dots + n_N = 2n$. The right hand side is summable over n_j-s. Thus, we conclude that the conclusion of Proposition 4.5 holds. $\square$

Proposition 4.5 leads to the following.

Corollary 4.6 *We have*

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}[\hat{\zeta}_\varepsilon(t,\xi)]^N = \sum_{n_1,\dots,n_N=0}^{\infty} \lim_{\varepsilon \downarrow 0} J^{\varepsilon}_{n_1,\dots,n_N}(t,\xi). \tag{4.36}$$

Hence, it remains only to evaluate the individual limits of $J^{\varepsilon}_{n_1,\dots,n_N}(t,\xi)$ as $\varepsilon \downarrow 0$.

Computation of $\lim_{\varepsilon\downarrow 0} J^{\varepsilon}_{n_1,\dots,n_N}(t,\xi)$

In order to re-write $\lim_{\varepsilon\downarrow 0} J^{\varepsilon}_{n_1,\dots,n_N}(t,\xi)$ in a form more convenient for the subsequent analysis we will once again use the variables τ_j, with $s_i = \sum_{j=i}^n \tau_j$ to express $\hat{\zeta}_\varepsilon(t,\xi)$. Then the phase function $\tilde{G} = \tilde{A} - \tilde{B}$ with $\tilde{A}$ and $\tilde{B}$ as in (4.9)-(4.11). Then the domain of integration in the τ-variables is the set

$$D_n(t) = \{(\tau_1, \dots, \tau_n) : \ \tau_j \ge 0 \text{ for all } 1 \le j \le n \text{ and } \tau_1 + \dots + \tau_n \le t\}.$$

We will also use the spectral representation of the stationary field $V(t,x)$:

$$V(t,x) = \int e^{i(\omega t + p\cdot x)} \frac{\hat{V}(d\omega, dp)}{(2\pi)^{d+1}},$$

where $\hat{V}(d\omega, dp)$ is a Gaussian spectral measure with the structure measure given by

$$\mathbb{E}[\hat{V}(d\omega,dp)\hat{V}^*(d\omega',dp')] = (2\pi)^{d+1}\delta(\omega-\omega')\delta(p-p')\frac{2\mu a(p)|p|^{2\beta}}{(\omega^2+\mu|p|^{4\beta})|p|^{2\alpha+d-2}}.$$

We can now transform expression (2.3) into

$$\hat{\zeta}_n^\varepsilon(t,\xi) = \left(\frac{\gamma}{i\varepsilon}\right)^n \int\limits_{D_n(t)} d\tau^{(n)} \int \frac{\hat{V}(d\omega_1,dp_1)\hat{V}(d\omega_2,dp_2)\dots\hat{V}(d\omega_n,dp_n)}{(2\pi)^{(n+1)d}} \hat{\phi}_0(\xi - p_1 - p_2 - \dots - p_n)$$

$$\times e^{i[\omega_1(\tau_1+\dots+\tau_n)+\omega_2(\tau_2+\dots+\tau_n)+\dots+\omega_n\tau_n]/\varepsilon} e^{i\tilde{G}_n(\tau^{(n)},\mathbf{p}^{(n)})/\varepsilon}, \tag{4.37}$$

and further rewrite (4.37) in the following way:

$$\hat{\zeta}_n^\varepsilon(t,\xi) = e^t \left(\frac{\gamma}{i\varepsilon}\right)^n \int_0^{+\infty} \ldots \int_0^{+\infty} d\tau_0 \ldots d\tau_n \int \delta(t-\tau_0-\ldots-\tau_n) e^{i\sum_{j=1}^n \tau_j (\sum_{k=1}^j \omega_k)/\varepsilon}$$
$$\times \frac{\hat{V}(d\omega_1, dp_1)\hat{V}(d\omega_2, dp_2)\ldots\hat{V}(d\omega_n, dp_n)}{(2\pi)^{n(d+1)}} \hat{\phi}_0(\xi - p_1 - \ldots - p_n) e^{-\sum_{j=0}^n \tau_j} e^{i\tilde{G}_n(\tau^{(n)}, \mathbf{p}^{(n)})/\varepsilon}.$$

Since

$$\delta(t-\tau_0-\ldots\tau_n) = \int_{\mathbb{R}} e^{-iz(t-\tau_0-\ldots\tau_n)} \frac{dz}{2\pi},$$

integrating out the τ variables we obtain:

$$\frac{1}{i^n}\int_0^\infty \ldots \int_0^\infty d\tau_0 \ldots d\tau_n e^{-(1-iz)(\tau_0+\cdots+\tau_n)} e^{i\sum_{m=1}^n \tau_m (\sum_{j=1}^m [(\xi\cdot p_j)+\omega_j] - Q_m(\mathbf{p}^{(n)})/2)/\varepsilon}$$
$$= \frac{1}{1-iz}\prod_{m=1}^n \left\{ i + z + \frac{1}{2\varepsilon}\left[2\sum_{j=1}^m [(\xi\cdot p_j)+\omega_j] - Q_m(\mathbf{p}^{(n)})\right]\right\}^{-1},$$

so that (4.37) becomes

$$\hat{\zeta}_n^\varepsilon(t,\xi) = e^t \left(\frac{\gamma}{\varepsilon}\right)^n \int dz e^{-izt} \int \frac{\hat{V}(dp_1, d\omega_1)\hat{V}(dp_2, d\omega_2)\ldots\hat{V}(dp_n, d\omega_n)}{(2\pi)^{n(d+1)+1}} \hat{\phi}_0(\xi - p_0 - \ldots - p_n)$$
$$\times \frac{1}{1-iz}\left\{\prod_{m=1}^n \left[z + \frac{1}{2\varepsilon}\left(2\sum_{j=1}^m [(\xi\cdot p_j) + \omega_j - Q_m(p)\right) + i\right]\right\}^{-1}. \tag{4.38}$$

This expression for $\hat{\zeta}_n^\varepsilon(t,\xi)$ will be the starting point for our analysis of $J^\varepsilon_{n_1,\ldots,n_N}(t,\xi)$, that is, $J^\varepsilon_{n_1,\ldots,n_N}(t,\xi)$ is given by

$$J^\varepsilon_{n_1,\ldots,n_N}(t,\xi) = \frac{(-1)^n e^{tN}}{(2\pi)^{2n(d+1)+1}}\left(\frac{\gamma}{\varepsilon}\right)^{2n} \int \ldots \int \prod_{j=1}^N \left\{\frac{\exp\{-itz_j\}}{1-iz_j}\right\} dz_1 \ldots dz_N$$
$$\times \int \mathbb{E}\left[\hat{V}(d\omega_{11}, dp_{11})\ldots\hat{V}(d\omega_{1n_1}, dp_{1n_1})\ldots\hat{V}(d\omega_{N1}, dp_{N1})\ldots\hat{V}(d\omega_{N,n_N}, dp_{Nn_N})\right]$$
$$\times \hat{\zeta}_0(\xi - p_{11} - \ldots - p_{1n_1})\ldots\hat{\zeta}_0(\xi - p_{N1} - \ldots - p_{Nn_N})$$
$$\times \prod_{j=1}^N \left\{\prod_{m=1}^{n_j}\left[z_j + \frac{1}{2\varepsilon}\left\{2\sum_{k=1}^m [(\xi\cdot p_{jk}) + \omega_{jk}] - Q_m(p_j)\right\} + i\right]\right\}^{-1},$$

with $2n = \sum_{j=1}^N n_j$. We evaluate the expectation above using Feynman diagrams and get

$$J^\varepsilon_{n_1,\ldots,n_N}(t,\xi) = \frac{(-1)^n e^{tN}}{(2\pi)^{n(d+1)+1}}\left(\frac{\gamma}{\varepsilon}\right)^{2n} \sum_{\mathcal{F}} \int d\mathbf{p}_1 \ldots d\mathbf{p}_N \int d\omega_1 \ldots d\omega_N \tag{4.39}$$
$$\times \prod_{(jk,j'm)\in\mathcal{F}} \frac{2\mu|p_{jk}|^{2\beta+2-2\alpha-d} a(p_{jk})}{\omega_{jk}^2 + \mu^2|p_{jk}|^{4\beta}} \delta(p_{jk} + p_{j'm})\delta(\omega_{jk} + \omega'_{j'm})$$
$$\times \hat{\zeta}_0(\xi - p_{11} - \ldots - p_{1n_1})\ldots\hat{\zeta}_0(\xi - p_{N1} - \ldots - p_{Nn_N})$$
$$\times \int d\mathbf{z} \prod_{j=1}^N \left\{\frac{\exp\{-itz_j\}}{1-iz_j} \prod_{m=1}^{n_j}\left[z_j + \frac{1}{2\varepsilon}\left[2\sum_{k=1}^m (\xi\cdot p_{jk} + \omega_{jk}) - Q_m(p_j)\right] + i\right]^{-1}\right\}.$$

Here $d\mathbf{p}_m := dp_{m1} \dots dp_{mn_m}$ and, once again, the summation extends over all Feynman diagrams formed over elements that are pairs of integers (jk), $j = 1, \dots, N$, $k = 1, \dots, n_j$.

We change variables setting $p' = p/\varepsilon^{1/(2\beta)}$, $\omega' = \omega/\varepsilon$ and using the relation $\gamma = \varepsilon^\kappa$ we get, after dropping the primes

$$J^\varepsilon_{n_1,\dots,n_N}(t,\xi) = \frac{(-1)^n e^{tN}}{(2\pi)^{n(d+1)+1}} \sum_{\mathcal{F}} \int d\mathbf{p}_1 \dots d\mathbf{p}_N \int d\omega_1 \dots d\omega_N \tag{4.40}$$
$$\times \prod_{(jk,j'm)\in\mathcal{F}} \frac{2\mu |p_{jk}|^{2\beta+2-2\alpha-d} a(\varepsilon^{1/(2\beta)} p_{jk})}{\omega_{jk}^2 + \mu^2 |p_{jk}|^{4\beta}} \delta(p_{jk} + p_{j'm}) \delta(\omega_{jk} + \omega'_{j'm})$$
$$\times \hat{\zeta}_0(\xi - \varepsilon^{1/(2\beta)}(p_{11} + \dots + p_{1n_1})) \dots \hat{\zeta}_0(\xi - \varepsilon^{1/(2\beta)}(p_{N1} + \dots + p_{Nn_N}))$$
$$\times \int d\mathbf{z} \prod_{j=1}^N \left\{ \frac{\exp\{-itz_j\}}{1 - iz_j} \prod_{m=1}^{n_j} \left[z_j + \frac{1}{2} \left[2 \sum_{k=1}^m (\xi \cdot p_{jk} \varepsilon^{1/(2\beta)-1} + \omega_{jk}) - Q_m(p_j) \varepsilon^{1/\beta - 1} \right] + i \right]^{-1} \right\}.$$

One can majorize the integrand above by an integrable function, when $\alpha + \beta > 1$, as we did in the proof of Proposition 4.4, and obtain that:

$$\lim_{\varepsilon \downarrow 0} J^\varepsilon_{n_1,\dots,n_N}(t,\xi) = \frac{(-1)^n e^{tN} (\hat{\zeta}_0(\xi))^N (2\mu a(0))^n}{(2\pi)^{n(d+1)+1}} \sum_{\mathcal{F}\in\mathfrak{F}(2n)} \int d\mathbf{p}_1 \dots d\mathbf{p}_N \int d\omega_1 \dots d\omega_N$$
$$\times \prod_{(jk,j'm)\in\mathcal{F}} \frac{|p_{jk}|^{2\beta+2-2\alpha-d}}{\omega_{jk}^2 + \mu^2 |p_{jk}|^{4\beta}} \delta(p_{jk} + p_{j'm}) \delta(\omega_{jk} + \omega'_{j'm}) \tag{4.41}$$
$$\times \int d\mathbf{z} \prod_{j=1}^N \left\{ \frac{\exp\{-itz_j\}}{1 - iz_j} \prod_{m=1}^{n_j} \left[z_j + \sum_{k=1}^m (\zeta(\beta)(\xi \cdot p_{jk}) + \omega_{jk}) + i \right] \right\}^{-1}.$$

Recall that $\zeta(\beta) = 1$ for $\beta = 1/2$ and $\zeta(\beta) = 0$ for $\beta < 1/2$. Now, we need to relate the right side of (4.41) to the fractional Brownian motion. We do it separately for $\beta < 1/2$ and $\beta = 1/2$.

The case $\beta < 1/2$**.** When $\beta < 1/2$ the limit in (4.41) equals

$$J_{n_1,\dots,n_N}(t,\xi) = \frac{(-1)^n e^{tN} (2\mu a(0))^n [\hat{\zeta}_0(\xi)]^N}{(2\pi)^{n(d+1)+1}} \sum_{\mathcal{F}\in\mathfrak{F}(2n)} \int d\mathbf{p}_1 \dots d\mathbf{p}_N \int d\omega_1 \dots d\omega_N \tag{4.42}$$
$$\times \prod_{(jk,j'm)\in\mathcal{F}} \frac{|p_{jk}|^{2\beta+2-2\alpha-d} \delta(p_{jk} + p_{j'm}) \delta(\omega_{jk} + \omega'_{j'm})}{\omega_{jk}^2 + \mu^2 |p_{jk}|^{4\beta}} \int d\mathbf{z} \prod_{j=1}^N \left\{ \frac{\exp\{-itz_j\}}{1 - iz_j} \prod_{m=1}^{n_j} \left[z_j + \sum_{k=1}^m \omega_{jk} + i \right]^{-1} \right\}.$$

Integrating out the z and ω variables and reverting back to the s-variables time we obtain that

$$J_{n_1,\dots,n_N}(t,\xi) = \frac{(-1)^n a^n(0) [\hat{\zeta}_0(\xi)]^N}{(2\pi)^{nd}} \sum_{\mathcal{F}} \int d\mathbf{p}_1 \dots d\mathbf{p}_N \int_{\Delta_{n_1}(t)} \dots \int_{\Delta_{n_N}(t)} d\mathbf{s}_1 \dots d\mathbf{s}_N$$
$$\times \prod_{(jk,j'm)\in\mathcal{F}} \frac{e^{-\mu |p_{jk}|^{2\beta} |s_{jk} - s_{j'm}|}}{|p_{jk}|^{2\alpha+d-2}} \delta(p_{jk} + p_{j'm}). \tag{4.43}$$

Integrating out also the p variables gives

$$J_{n_1,\dots,n_N}(t,\xi) = \frac{[-a(0) K_1(\alpha,\beta,\mu)]^n}{(2\pi)^{nd}} [\hat{\zeta}_0(\xi)]^N \sum_{\mathcal{F}} \int_{\Delta_{n_1}(t)} \dots \int_{\Delta_{n_N}(t)} d\mathbf{s}_1 \dots d\mathbf{s}_N |s_{jk} - s_{j'm}|^{(\alpha-1)/\beta} \tag{4.44}$$

It remains now to relate $\bar{J}_{n_1,\dots,n_N}(t,\xi)$ to the fractional Brownian motion and sum all these terms. Note that the function

$$f(s_1,\dots,s_{2n}) := \sum_{\mathcal{F}} \prod_{\widehat{km}\in\mathcal{F}} |s_k - s_m|^{2\alpha-2}$$

is symmetric in all of its arguments, that is, $f(s_1,\dots,s_{2n}) = f(s_{\pi(1)},\dots,s_{\pi(2n)})$, where π is an arbitrary permutation of $\{1,2.\dots,2n\}$. Using this fact we can rewrite (4.44) in the form

$$\bar{J}_{n_1,\dots,n_N}(t,\xi) = \frac{[-a(0)K_1(\alpha,\beta,\mu)]^n[\hat{\zeta}_0(\xi)]^N}{(2\pi)^{nd}n_1!\dots n_N!} \sum_{\mathcal{F}} \underbrace{\int_0^t \dots \int_0^t}_{n_1\text{-times}} d\mathbf{s}_1 \dots \underbrace{\int_0^t \dots \int_0^t}_{n_N\text{-times}} d\mathbf{s}_N \prod_{(jk,j'm)\in\mathcal{F}} |s_{jk}-s_{j'm}|^{2\alpha-2}. \tag{4.45}$$

Recall that, as (4.21),

$$\sum_{\mathcal{F}} \prod_{(jk,j'm)\in\mathcal{F}} |s_{jk}-s_{j'm}|^{2\kappa-2} = c_\kappa^{2n}\mathbb{E}\left[\prod_{j=1}^N\prod_{m=1}^{n_j}\int_{-\infty}^{\infty}\frac{e^{ik_{jm}s_{jm}}}{|k_{jm}|^{\alpha-1/2}}w(dk_{jm})\right],$$

where $w(dk)$ is a Gaussian white noise and $c_\kappa > 0$ is given by (4.22). Hence,

$$\bar{J}_{n_1,\dots,n_N}(t,\xi) = \frac{(-1)^n[\hat{\zeta}_0(\xi)]^N(a(0)K_1(\alpha,\beta,\mu)c_\kappa^2)^n}{(2\pi)^{nd}n_1!\dots n_N!} \tag{4.46}$$

$$\times \underbrace{\int_0^t \dots \int_0^t}_{n_1\text{-times}} d\mathbf{s}_1 \dots \underbrace{\int_0^t \dots \int_0^t}_{n_N\text{-times}} d\mathbf{s}_N \mathbb{E}\left[\prod_{j=1}^N\prod_{m=1}^{n_j}\int_{-\infty}^{\infty}\frac{e^{ik_{jm}s_{jm}}}{|k_{jm}|^{\alpha-1/2}}w(dk_{jm})\right].$$

Performing the integrations with respect to s_i and then subsequently the summation over $n_1,\dots,n_N$ we obtain

$$\sum_{n_1=0,\dots,n_N=0}^{+\infty} \bar{J}_{n_1,\dots,n_N}(t,\xi) = [\hat{\zeta}_0(\xi)]^N\mathbb{E}\left[\exp\left\{iND^{1/2}B_\kappa(t)\right\}\right], \tag{4.47}$$

where $B_\kappa(t)$ is the fractional Brownian motion with the Hurst exponent κ and variance 1.

The case $\beta = 1/2$. The computation for $\beta = 1/2$ is very similar to that for $\beta < 1/2$. Here the limit in (4.41) equals

$$J_{n_1,\dots,n_N}(t,\xi) = \frac{(-1)^n(2\mu)a^n(0)[\hat{\zeta}_0(\xi)]^Ne^{tN}}{(2\pi)^{n(d+1)+1}} \sum_{\mathcal{F}\in\mathfrak{F}(2n)} \int d\mathbf{p}_1\dots d\mathbf{p}_N \int d\omega_1\dots d\omega_N \tag{4.48}$$

$$\times \prod_{(jk,j'm)\in\mathcal{F}} \frac{|p_{jk}|^{3-2\alpha-d}\delta(p_{jk}+p_{j'm})\delta(\omega_{jk}+\omega'_{j'm})}{\omega_{jk}^2+\mu^2|p_{jk}|^2}$$

$$\times \int d\mathbf{z}\prod_{j=1}^N\left\{\frac{\exp\{-itz_j\}}{1-iz_j}\prod_{m=1}^{n_j}\left[z_j+\sum_{k=1}^m(\xi\cdot p_{jk}+\omega_{jk})+i\right]^{-1}\right\}.$$

Integrating out the z and ω variables and reverting to s coordinates for time we obtain that

$$J_{n_1,\dots,n_N}(t,\xi) = \frac{(-1)^na^n(0)[\hat{\zeta}_0(\xi)]^N}{(2\pi)^{nd}} \sum_{\mathcal{F}\in\mathfrak{F}(2n)} \int d\mathbf{p}_1\dots d\mathbf{p}_N \int_{\Delta_{n_1}(t)}\dots\int_{\Delta_{n_N}(t)} d\mathbf{s}_1\dots d\mathbf{s}_N$$

$$\times \prod_{(jk,j'm)\in\mathcal{F}} |p_{jk}|^{2-2\alpha-d}e^{-\mu|p_{jk}||s_{jk}-s_{j'm}|+i\xi\cdot p_{jk}(s_{jk}-s_{j'm})}\delta(p_{jk}+p_{j'm}). \tag{4.49}$$

Integrating out the p variables we obtain that

$$J_{n_1,\dots,n_N}(t,\xi) = \frac{[-a(0)K_2(\xi;\mu)]^n[\hat{\zeta}_0(\xi)]^N}{(2\pi)^{nd}} \sum_{\mathcal{F}} \int_{\Delta_{n_1}(t)} \dots \int_{\Delta_{n_N}(t)} d\mathbf{s}_1 \dots d\mathbf{s}_N |s_{jk} - s_{j'm}|^{2(\alpha-1)}, \quad (4.50)$$

with the constant K_2 as in (1.13). From here on we conduct the calculation as in the previous case.

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