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Kinetic models for wave propagation in random media

I. Derivation of Radiative Transfer Equations

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Outline for Lecture I.

1. Waves in heterogeneous media

2. High Frequency regime and Geometrical optics

3. Wigner transforms

4. Radiative Transfer model in the weak coupling regime

5. Random Liouville, paraxial and Itô-Schrödinger approximations

6. More general Radiative Transfer models

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1. Time Reversal in random media

2. Statistical stability

3. Validity of Radiative Transfer Models

4. Applications to Detection and Imaging

Outline

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Models for acoustic wave propagation

The linear system of acoustic wave equations for the pressure $p(t, \mathbf{x})$ and the velocity field $\mathbf{v}(t, \mathbf{x})$ takes the form of the following first-order hyperbolic system

$$\rho(\mathbf{x})\frac{\partial \mathbf{v}}{\partial t} + \nabla p = 0, \quad \kappa(\mathbf{x})\frac{\partial p}{\partial t} + \nabla \cdot \mathbf{v} = 0, \quad p(0, \mathbf{x}) = p_0(\mathbf{x}), \mathbf{v}(0, \mathbf{x}) = \mathbf{v}_0(\mathbf{x}),$$

where $\rho(\mathbf{x}) = \rho_0$ (to simplify notation) is density and $\kappa(\mathbf{x})$ compressibility. Energy conservation is characterized by

$$\mathcal{E}_B(t) = \frac{1}{2}\int_{\mathbb{R}^d} \Big(\rho(\mathbf{x})|\mathbf{v}|^2(t, \mathbf{x}) + \kappa(\mathbf{x})p^2(t, \mathbf{x})\Big) d\mathbf{x} = \mathcal{E}_B(0).$$

We know that total energy is conserved. The role of a kinetic model is to describe its spatial distribution (at least asymptotically).

Scalar model

The pressure $p(t, \mathbf{x})$ also solves following closed form scalar equation

$$\frac{\partial^2 p}{\partial t^2} = c^2(\mathbf{x})\Delta p, \qquad c^2(\mathbf{x}) = \frac{1}{\rho_0 \kappa(\mathbf{x})}.$$

Moreover

$$\mathcal{E}_H(t) = \frac{1}{2}\int_{\mathbb{R}^d} \Big(\kappa(\mathbf{x})\Big(\frac{\partial p}{\partial t}\Big)^2(t,\mathbf{x}) + \frac{|\nabla p|^2(t,\mathbf{x})}{\rho_0}\Big) d\mathbf{x} = \mathcal{E}_H(0).$$

The latter conservation law is equivalent to the previous one: let $\phi(t, \mathbf{x})$ be a solution of the above equation, then $(\mathbf{v}, p) = (-\rho^{-1}\nabla\phi, \partial_t \phi)$ solves the first-order hyperbolic system and $\mathcal{E}_H[\phi](t) = \mathcal{E}_B[\mathbf{v}, p](t)$. It is thus natural that the kinetic models for the energy distributions associated to both conservations agree.

Another system model

Finally let us define $q(t, \mathbf{x}) = c^{-2}(\mathbf{x})\frac{\partial p}{\partial t}(t, \mathbf{x})$. Then $\mathbf{u} = (p, q)$ solves the following 2×2 system

$$\frac{\partial \mathbf{u}}{\partial t} + A\mathbf{u} \;=\; 0, \qquad A = -\begin{pmatrix} 0 & c^2(\mathbf{x}) \\ \Delta & 0 \end{pmatrix},$$

with appropriate initial conditions. Note that

$$A = J\,\Lambda(\mathbf{x}), \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Lambda(\mathbf{x}) = \begin{pmatrix} -\Delta & 0 \\ 0 & c^2(\mathbf{x}) \end{pmatrix} \text{ symmetric },$$

and that energy conservation may be recast as

$$\mathcal{E}(t) = \frac{1}{2\rho_0}\int_{\mathbb{R}^d} \mathbf{u}\Lambda\mathbf{u}\,d\mathbf{x} = \mathcal{E}(0).$$

Kinetic models associated to each acoustic equation must therefore agree and provide the same spatial energy distribution.

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High Frequency scaling

Consider the framework where the typical distance of propagation L of the waves is much larger than the typical wavelength λ in the system. We introduce the small adimensionalized parameter $\varepsilon = \dfrac{\lambda}{L} \ll 1$. We thus rescale space $\mathbf{x} \to \varepsilon^{-1}\mathbf{x}$ and since $l = c \times t$ rescale time accordingly $t \to \varepsilon^{-1} t$ to obtain the two model equations

$$\varepsilon^2 \frac{\partial^2 p_\varepsilon}{\partial t^2} = c_\varepsilon^2(\mathbf{x}) \varepsilon^2 \Delta p_\varepsilon, \qquad p_\varepsilon(0,\mathbf{x}) = p_{0\varepsilon}(\varepsilon^{-1}\mathbf{x})$$

$$\varepsilon \frac{\partial \mathbf{u}_\varepsilon}{\partial t} + A_\varepsilon \mathbf{u}_\varepsilon = 0, \qquad A_\varepsilon = -\begin{pmatrix} 0 & c_\varepsilon^2(\mathbf{x}) \\ \varepsilon^2 \Delta & 0 \end{pmatrix}, \qquad \mathbf{u}_\varepsilon(0,\mathbf{x}) = \mathbf{u}_{0\varepsilon}(\varepsilon^{-1}\mathbf{x}).$$

Energy conservation implies

$$\mathcal{E}_H(t) = \frac{1}{2\rho_0} \int_{\mathbb{R}^d} \Big(c_\varepsilon^{-2}(\mathbf{x}) \Big(\varepsilon \frac{\partial p_\varepsilon}{\partial t} \Big)^2 (t,\mathbf{x}) + |\varepsilon \nabla p_\varepsilon|^2 (t,\mathbf{x}) \Big) d\mathbf{x} = \mathcal{E}_H(0),$$

$$\mathcal{E}(t) = \frac{1}{2\rho_0} \int_{\mathbb{R}^d} \Big(|\varepsilon \nabla p_\varepsilon|^2 (t,\mathbf{x}) + c_\varepsilon^2(\mathbf{x}) q_\varepsilon^2(t,\mathbf{x}) \Big) d\mathbf{x} = \mathcal{E}(0).$$

Geometrical optics

In the high frequency regime and for "low frequency" media, i.e. $c_\varepsilon(\mathbf{x}) = c(\mathbf{x})$ independent of ε, wave propagation can be approximated by looking at solutions of the form

$$p_\varepsilon(t,\mathbf{x}) = \Big(p(t,\mathbf{x}) + \varepsilon p_{1\varepsilon}(t,\mathbf{x})\Big) e^{i\dfrac{S(t,\mathbf{x})}{\varepsilon}}.$$

Then $S(t,\mathbf{x})$ solves the eikonal equation

$$\left(\frac{\partial S}{\partial t}\right)^2 = c^2(\mathbf{x})|\nabla_{\mathbf{x}} S|^2,$$

and $p(t,\mathbf{x})$ the transport equation

$$\frac{\partial S}{\partial t}\frac{\partial p}{\partial t} - c^2(\mathbf{x})\nabla_{\mathbf{x}} S \cdot \nabla_{\mathbf{x}} p + \left(\frac{\partial^2 S}{\partial t^2} - c^2(\mathbf{x})\Delta_{\mathbf{x}} S\right) p_0 = 0,$$

with appropriate initial conditions so that $p_\varepsilon(0,\mathbf{x}) = p(0,\mathbf{x}) e^{i\dfrac{S(0,\mathbf{x})}{\varepsilon}}$.

Limitations of Geometrical optics

The eikonal equation admits a unique (physical) solution only for sufficiently short times that are very small in highly heterogeneous media.

When such caustics occur, the geometrical optics decomposition need to be generalized as a superposition of propagating fronts:

$$p_\varepsilon(t,\mathbf{x}) = \sum_{n=1}^{N} \Big(p_0^n(t,\mathbf{x}) + \varepsilon p_{1\varepsilon}^n(t,\mathbf{x})\Big) e^{i\frac{S^n(t,\mathbf{x})}{\varepsilon}}.$$

It is unclear how such decompositions can be used to model wave propagation in very heterogeneous media.

It is more natural to replace the physical description of high frequency waves (S and p_0 depend on time and space only) by a phase space description, which also accounts for the direction in which waves propagate.

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Theory of Wigner transforms (I)

[**L.P.** RMI-1993; **G.M.M.P** CPAM-1997]. Define the Wigner transform

$$W_\varepsilon[\psi,\phi](\mathbf{x},\mathbf{k}) = \int_{\mathbb{R}^d} e^{i\mathbf{y}\cdot\mathbf{k}}\psi(\mathbf{x}-\varepsilon\frac{\mathbf{y}}{2})\phi^*(\mathbf{x}+\varepsilon\frac{\mathbf{y}}{2})\frac{d\mathbf{y}}{(2\pi)^d}.$$

For ϕ and ψ in $L^2(\mathbb{R}^d)$, W_ε is bounded in $\mathcal{A}'(\mathbb{R}^{2d})$ defined as the dual of functions $\eta(\mathbf{x},\mathbf{k})$ such that $\int_{\mathbb{R}^d}\sup_{\mathbf{x}}\|\hat{\eta}(\mathbf{x},\mathbf{y})\|d\mathbf{y}$ is bounded. This subset of $\mathcal{S}'(\mathbb{R}^{2d})$ includes bounded measures on $\mathbb{R}^{2d}$. The Wigner transform has "*bounded*" $L^2(\mathbb{R}^{2d})-$norm of order $\varepsilon^{-d/2}$.

For bounded sequences $\psi_\varepsilon,\phi_\varepsilon$ in $L^2(\mathbb{R}^d)$, we can extract convergent subsequences of $W_\varepsilon[\psi_\varepsilon,\phi_\varepsilon]$ in $\mathcal{A}'(\mathbb{R}^{2d})$. The limits of W^0 of $W_\varepsilon[\phi_\varepsilon,\phi_\varepsilon]$ are positive measures.

Theory of Wigner transforms (II)

Let ψ_ε be a (scalar) bounded family in $L^2(\mathbb{R}^d)$ which is ε-oscillatory and compact at infinity and such that the Wigner transform $W_\varepsilon[\psi_\varepsilon, \psi_\varepsilon]$ converges to the Wigner measure $W^0[\psi_\varepsilon]$. Then if $|\psi_\varepsilon|^2 \to \nu$ as measures on $\mathbb{R}^d_{\mathbf{x}}$, we have

$$\int_{\mathbb{R}^d} W^0[\psi_\varepsilon](\cdot, d\mathbf{k}) = \nu, \qquad \int_{\mathbb{R}^d} W^0[\psi_\varepsilon](d\mathbf{x}, d\mathbf{k}) = \overline{\lim_{\varepsilon \to 0}} \int_{\mathbb{R}^d} |\psi_\varepsilon|^2(\mathbf{x}) d\mathbf{x}.$$

The first equality shows that the Wigner measure may be interpreted as a probability density (energy density for classical waves) in the phase space. The second equality shows that provided that the field ψ_ε oscillates at the scale ε not too far from the origin, the limiting Wigner measure captures the whole probability density (energy density for classical waves).

Otherwise both equalities above are inequalities $\leq$.

Noteworthy properties

The Wigner transform of vector fields is defined by:

$$W_\varepsilon[\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k}) = \int_{\mathbb{R}^d} e^{i\mathbf{y}\cdot\mathbf{k}}\mathbf{u}(\mathbf{x}-\varepsilon\frac{\mathbf{y}}{2})\mathbf{v}^*(\mathbf{x}+\varepsilon\frac{\mathbf{y}}{2})\frac{d\mathbf{y}}{(2\pi)^d}.$$

It is the inverse Fourier transform of the product:

$$W_\varepsilon[\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k}) = \mathcal{F}^{-1}\Big(\mathbf{u}(\mathbf{x}+\varepsilon\frac{\mathbf{y}}{2})\mathbf{v}^*(\mathbf{x}-\varepsilon\frac{\mathbf{y}}{2})\Big).$$

We verify that

$$\begin{aligned}
\int_{\mathbb{R}^d} W[\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k})d\mathbf{k} &= (\mathbf{u}\mathbf{v}^*)(\mathbf{x}) \\
\int_{\mathbb{R}^d} \mathbf{k}W[\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k})d\mathbf{k} &= \frac{i\varepsilon}{2}(\mathbf{u}\nabla\mathbf{v}^* - \nabla\mathbf{u}\mathbf{v}^*)(\mathbf{x}) \\
\int_{\mathbb{R}^{2d}} |\mathbf{k}|^2 W[\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k})d\mathbf{k}d\mathbf{x} &= \varepsilon^2\int_{\mathbb{R}^d}\nabla\mathbf{u}\cdot\nabla\mathbf{v}^*d\mathbf{x}.
\end{aligned}$$

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Equations for the Wigner transform

Consider two field equations and the Wigner transform:

$$\varepsilon\frac{\partial \mathbf{u}_\varepsilon^\varphi}{\partial t}+A_\varepsilon^\varphi \mathbf{u}_\varepsilon^\varphi=0, \quad \varphi=1,2, \qquad W_\varepsilon(t,\mathbf{x},\mathbf{k})=W[\mathbf{u}_\varepsilon^1(t,\cdot),\mathbf{u}_\varepsilon^2(t,\cdot)](\mathbf{x},\mathbf{k}).$$

Here $\mathbf{u}_\varepsilon^\varphi=(p_\varepsilon^\varphi,(c_\varepsilon^\varphi)^{-2}(\mathbf{x})\partial_t p_\varepsilon^\varphi)$. Then we verify that

$$\varepsilon\frac{\partial W_\varepsilon}{\partial t}+W[A_\varepsilon^1\mathbf{u}_\varepsilon^1,\mathbf{u}_\varepsilon^2]+W[\mathbf{u}_\varepsilon^1,A_\varepsilon^2\mathbf{u}_\varepsilon^2]=0.$$

Calculations of the type

$$W[P(\mathbf{x},\varepsilon\mathbf{D})\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k})=\int_{\mathbb{R}^{2d}} e^{-i\mathbf{y}\cdot\boldsymbol{\xi}}P(\mathbf{y},i\mathbf{k}+\frac{\varepsilon\mathbf{D}_\mathbf{x}}{2})\Big[e^{i\boldsymbol{\xi}\cdot\mathbf{x}}\underline{W[\mathbf{u},\mathbf{v}]}(\mathbf{x},\mathbf{k}-\frac{\varepsilon\boldsymbol{\xi}}{2})\Big]\frac{d\boldsymbol{\xi}d\mathbf{x}}{(2\pi)^d}$$

$$W[V(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k})=\int_{\mathbb{R}^{2d}} e^{i\frac{\mathbf{x}\cdot\mathbf{p}}{\varepsilon}}e^{i\mathbf{x}\cdot\mathbf{q}}\hat{V}(\mathbf{q},\mathbf{p})\underline{W[\mathbf{u},\mathbf{v}]}(\mathbf{x},\mathbf{k}-\frac{\mathbf{p}}{2}-\frac{\varepsilon\mathbf{q}}{2})\frac{d\mathbf{p}d\mathbf{q}}{(2\pi)^{2d}},$$

allow us to obtain an explicit equation for W_ε. The above formulas are amenable to asymptotic expansions in ε.

A priori bounds

The Wigner transform $W_\varepsilon(t,\cdot,\cdot)$ is uniformly bounded in $\mathcal{A}'(\mathbb{R}^{2d})$ by construction. For the Schrödinger equation, we can show that the following quantities are conserved:

$$\int_{\mathbb{R}^{2d}} W_\varepsilon(t,\mathbf{x},\mathbf{k})d\mathbf{x}d\mathbf{k}, \quad \int_{\mathbb{R}^{2d}} W_\varepsilon^2(t,\mathbf{x},\mathbf{k})d\mathbf{x}d\mathbf{k}, \quad \frac{1}{2}\int_{\mathbb{R}^{2d}} W_\varepsilon\Big(|\mathbf{k}|^2 + V_\varepsilon(\mathbf{x})\Big)d\mathbf{k}d\mathbf{x}.$$

W_ε is not non negative in general, although its limit is. So the first and third conservations provide little a priori information.

The second $L^2(\mathbb{R}^{2d})$ a priori bound is much more useful, but only in the case of a mixture of states

$$W_\varepsilon(t,\mathbf{x},\mathbf{k}) = \int_S W[\psi_\varepsilon(t,\cdot;\omega)\phi_\varepsilon(t,\cdot;\omega)]d\mu(\omega),$$

where (S,μ) are such that $W_\varepsilon(0,\cdot,\cdot) \in L^2(\mathbb{R}^{2d})$ is bounded independent of ε. This holds in the Time Reversal framework. For pure states (i.e., when $d\mu(\omega) = \delta(\omega-\omega_0)$), the a priori L^2 bound is $O(\varepsilon^{-d})$.

Weak-Coupling Regime

In the weak coupling regime, the random fluctuations of the media are modeled by

$$(c_\varepsilon^\varphi)^2(\mathbf{x}) = c_0^2 - \sqrt{\varepsilon} V^\varphi(\frac{\mathbf{x}}{\varepsilon}), \quad \varphi = 1, 2.$$

where c_0 is the background speed assumed to be constant to simplify. We consider two random media $V^\varphi(\varepsilon^{-1}\mathbf{x})$, $\varphi = 1, 2$ and fields propagating in these media, i.e., solving

$$\varepsilon \frac{\partial \mathbf{u}_\varepsilon^\varphi}{\partial t} + A_\varepsilon^\varphi \mathbf{u}_\varepsilon^\varphi = 0, \quad A_\varepsilon^\varphi = -\begin{pmatrix} 0 & c_0^2 \\ p(\varepsilon \mathbf{D}) & 0 \end{pmatrix} + \sqrt{\varepsilon} V^\varphi(\frac{\mathbf{x}}{\varepsilon}) K, \; K = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

$p(\mathbf{D}) = -\Delta$ for the wave equation. $V^\varphi(\mathbf{x})$ for $\varphi = 1, 2$ is a statistically homogeneous mean-zero random field with correlation function and power spectra:

$$\begin{aligned} c_0^4 R^{\varphi\psi}(\mathbf{x}) &= \langle V^\varphi(\mathbf{y}) V^\psi(\mathbf{y} + \mathbf{x}) \rangle, \qquad 1 \le \varphi, \psi \le 2, \\ (2\pi)^d c_0^4 \hat{R}^{\varphi\psi}(\mathbf{p}) \delta(\mathbf{p} + \mathbf{q}) &= \langle \hat{V}^\varphi(\mathbf{p}) \hat{V}^\psi(\mathbf{q}) \rangle. \end{aligned}$$

Equation for the Wigner Transform

Recalling that

$$W_\varepsilon(t,\mathbf{x},\mathbf{k}) = W[\mathbf{u}_\varepsilon^1(t,\cdot),\mathbf{u}_\varepsilon^2(t,\cdot)](\mathbf{x},\mathbf{k})$$

and that

$$\varepsilon\frac{\partial W_\varepsilon}{\partial t} + W[A_\varepsilon^1\mathbf{u}_\varepsilon^1,\mathbf{u}_\varepsilon^2] + W[\mathbf{u}_\varepsilon^1, A_\varepsilon^2\mathbf{u}_\varepsilon^2] = 0,$$

we obtain after (simple) pseudo-differential calculus that W_ε solves the following equation:

$$\varepsilon\frac{\partial W_\varepsilon}{\partial t} + P(i\mathbf{k}+\frac{\varepsilon\mathbf{D}}{2})W_\varepsilon + W_\varepsilon P^*(i\mathbf{k}-\frac{\varepsilon\mathbf{D}}{2}) + \sqrt{\varepsilon}\Big(\mathcal{K}_\varepsilon^1 K W_\varepsilon + \mathcal{K}_\varepsilon^{2*} W_\varepsilon K^*\Big) = 0,$$

$$P(i\mathbf{k}+\frac{\varepsilon\mathbf{D}}{2}) = -\begin{pmatrix} 0 & c_0^2 \\ p(i\mathbf{k}+\frac{\varepsilon\mathbf{D}}{2}) & 0 \end{pmatrix}, \; \mathcal{K}_\varepsilon^\varphi W = \int_{\mathbb{R}^d} e^{i\frac{\mathbf{x}\cdot\mathbf{p}}{\varepsilon}}\hat{V}^\varphi(\mathbf{p})W(\mathbf{k}-\frac{\mathbf{p}}{2})\frac{d\mathbf{p}}{(2\pi)^d}.$$

Multiple scale expansion

Because of the presence of a highly-oscillatory phase $\exp(i(\mathbf{x}/\varepsilon)\cdot\mathbf{k})$ in the operator $\mathcal{K}^{\varphi}_{\varepsilon}$, direct asymptotic expansions on W_ε do not provide the correct limit. Instead we introduce the following two-scale version of W_ε:

$$W_\varepsilon(t,\mathbf{x},\mathbf{k}) = W_\varepsilon(t,\mathbf{x},\frac{\mathbf{x}}{\varepsilon},\mathbf{k}),$$

and using that $\mathbf{D} \to \mathbf{D}_{\mathbf{x}} + \varepsilon^{-1}\mathbf{D}_{\mathbf{y}}$, find the equation

$$\varepsilon\frac{\partial W_\varepsilon}{\partial t} + P(i\mathbf{k} + \frac{\mathbf{D}_{\mathbf{y}}}{2} + \frac{\varepsilon\mathbf{D}_{\mathbf{x}}}{2})W_\varepsilon + W_\varepsilon P^*(i\mathbf{k} - \frac{\mathbf{D}_{\mathbf{y}}}{2} - \frac{\varepsilon\mathbf{D}_{\mathbf{x}}}{2})$$
$$+\sqrt{\varepsilon}\Big(\mathcal{K}^1 K W_\varepsilon + \mathcal{K}^{2*} W_\varepsilon K^*\Big) = 0,$$
$$\mathcal{K}^{\varphi} W = \int_{\mathbb{R}^d} e^{i\mathbf{y}\cdot\mathbf{p}}\,\hat{V}^{\varphi}(\mathbf{p}) W(\mathbf{k} - \frac{\mathbf{p}}{2})\frac{d\mathbf{p}}{(2\pi)^d}.$$

Asymptotic expansions

We can now use standard asymptotic techniques:

$$\begin{aligned} P &= P_0 + \varepsilon P_1 + O(\varepsilon^2), \\ W_\varepsilon(t,\mathbf{x},\mathbf{y},\mathbf{k}) &= W_0(t,\mathbf{x},\mathbf{k}) + \sqrt{\varepsilon} W_1(t,\mathbf{x},\mathbf{y},\mathbf{k}) + \varepsilon W_2(t,\mathbf{x},\mathbf{y},\mathbf{k}) \end{aligned}$$

plug them into the equation for W_ε, equate like powers of ε, and obtain three successive equations. The leading equation is

$$P_0(i\mathbf{k})W_0 + W_0 P_0^*(i\mathbf{k}) = 0; \quad P_0 = -J\Lambda_0,\ \Lambda_0 = \begin{pmatrix} -p(i\mathbf{k}) & 0 \\ 0 & c_0^2 \end{pmatrix},\ J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Define $q_0(i\mathbf{k}) = \sqrt{-p(i\mathbf{k})}$, $\lambda_\pm(\mathbf{k}) = \pm i c_0 q_0(i\mathbf{k})$, and observe that the following spectral decomposition holds:

$$P_0 = \lambda_+ \mathbf{b}_+ \mathbf{c}_+^* + \lambda_- \mathbf{b}_- \mathbf{c}_-^*,$$

for some vectors $\mathbf{b}_\pm$ and $\mathbf{c}_\pm = \Lambda_0 \mathbf{b}_\pm$.

Leading order term

The leading order equation $P_0(i\mathbf{k})W_0 + W_0 P_0^*(i\mathbf{k}) = 0$ imposes that

$$\underline{W_0 = a_+\mathbf{b}_+\mathbf{b}_+^* + a_-\mathbf{b}_-\mathbf{b}_-^*;} \qquad a_\pm = \mathbf{c}_\pm^* W_0 \mathbf{c}_\pm.$$

Because all the components of $\mathbf{u}_\varepsilon^\varphi$ are real-valued we verify that

$$\bar{a}_\pm(-\mathbf{k}) = a_\mp(\mathbf{k}).$$

It is thus sufficient to find an equation for the mode $a_+(\mathbf{k})$.

When $\mathbf{u}_\varepsilon^1 = \mathbf{u}_\varepsilon^2$, we verify that:

$$\underline{\mathcal{E}(t) = \int_{\mathbb{R}^{2d}} a_+(t,\mathbf{x},\mathbf{k})d\mathbf{k}d\mathbf{x}.}$$

Thus a_+ can be given the interpretation of an energy density in the phase-space.

First-order corrector

To summarize lenghtly calculations, after solving the next-order equation, we find that $\widehat{W}_1(t,\mathbf{x},\mathbf{p},\mathbf{k})$ the Fourier transform $\mathbf{y}\to\mathbf{p}$ of the first-order corrector W_1 may be decomposed as

$$\widehat{W}_1(\mathbf{p},\mathbf{k}) = \sum_{i,j=\pm} \alpha_{ij}(\mathbf{p},\mathbf{k})\mathbf{b}_i(\mathbf{k}+\frac{\mathbf{p}}{2})\mathbf{b}_j^*(\mathbf{k}-\frac{\mathbf{p}}{2}),$$

where

$$\alpha_{mn}(\mathbf{p},\mathbf{k}) = \frac{1}{2c_0^2}\frac{\widehat{V}^1(\mathbf{p})\lambda_m(\mathbf{k}+\frac{\mathbf{p}}{2})a_n(\mathbf{k}-\frac{\mathbf{p}}{2}) - \widehat{V}^2(\mathbf{p})\lambda_n(\mathbf{k}-\frac{\mathbf{p}}{2})a_m(\mathbf{k}+\frac{\mathbf{p}}{2})}{\lambda_m(\mathbf{k}+\frac{\mathbf{p}}{2}) - \lambda_n(\mathbf{k}-\frac{\mathbf{p}}{2}) + \theta}.$$

$a_\pm$ are the coefficients of the leading order term W_0. So we find that W_1 is linear in the fields V^φ and the leading-order term W_0.

Getting close to Radiative transfer equations

The third equation in the expansion is

$$P_0(i\mathbf{k}+\frac{\mathbf{D_y}}{2})W_2 + W_2 P_0^*(i\mathbf{k}-\frac{\mathbf{D_y}}{2}) + \mathcal{K}_1 K W_1 + \mathcal{K}_2^* W_1 K^*$$
$$+\frac{\partial W_0}{\partial t} + P_1(i\mathbf{k})W_0 + W_0 P_1^*(i\mathbf{k}) = 0.$$

After multiplying the above equation by $\mathbf{c}_+^*$ on the left and $\mathbf{c}_+$ on the right (recall that $a_+ = \mathbf{c}_+^* W_0 \mathbf{c}_+$), taking ensemble averaging, and invoking a few (non-rigorous) arguments, one finds that

$$\frac{\partial a_+}{\partial t} - \nabla_{\mathbf{k}}\omega_+(\mathbf{k})\cdot\nabla_{\mathbf{x}} a_+(\mathbf{x},\mathbf{k}) + \langle \mathbf{c}_+^* \mathcal{L}_1 W_1 \mathbf{c}_+\rangle = 0,$$

where $\omega_+(\mathbf{k}) = i\lambda_+(i\mathbf{k}) = -c_0 q_0(i\mathbf{k})$. Note that $-\nabla_{\mathbf{k}}\omega_+(\mathbf{k}) = c_0\hat{\mathbf{k}}$ for the wave equation. We find that energy propagates along straight lines (since c_0 is constant) is when $W_1 \equiv 0$.

Getting closer to Radiative transfer equations

It turns out that the missing term is given by

$$\langle \mathbf{c}_+^*(\mathbf{k})\mathcal{L}_1 W_1(\mathbf{k})\mathbf{c}_+(\mathbf{k})\rangle = \frac{\lambda_+(\mathbf{k})}{4(2\pi)^d}\int_{\mathbb{R}^d}\Big(\frac{-\widehat{R}^{11}(\mathbf{k}-\mathbf{q})\lambda_i(\mathbf{q})a_+(\mathbf{k})}{\lambda_i(\mathbf{q})-\lambda_+(\mathbf{k})+\theta}$$
$$+\frac{\widehat{R}^{12}(\mathbf{k}-\mathbf{q})\lambda_+(\mathbf{k})a_i(\mathbf{q})}{\lambda_i(\mathbf{q})-\lambda_+(\mathbf{k})+\theta}+\frac{\widehat{R}^{12}(\mathbf{k}-\mathbf{q})\lambda_+(\mathbf{k})a_j(\mathbf{q})}{\lambda_+(\mathbf{k})-\lambda_j(\mathbf{q})+\theta}+\frac{-\widehat{R}^{22}(\mathbf{k}-\mathbf{q})\lambda_j(\mathbf{q})a_+(\mathbf{k})}{\lambda_+(\mathbf{k})-\lambda_j(\mathbf{q})+\theta}\Big)d\mathbf{q}.$$

Here we have used the definition of the power spectrum:

$$(2\pi)^d c_0^4 \widehat{R}^{\varphi\psi}(\mathbf{p})\delta(\mathbf{p}+\mathbf{q}) = \langle \widehat{V}^{\varphi}(\mathbf{p})\widehat{V}^{\psi}(\mathbf{q})\rangle.$$

We use the summation over repeated indices i,j and $\theta > 0$ is a regularization parameters ensuring causality. Since $\lambda_j(\mathbf{k})$ is purely imaginary, we deduce from the relation $\frac{1}{ix+\varepsilon} \to \frac{1}{ix} + \pi\text{sign}(\varepsilon)\delta(x)$, as $\varepsilon \to 0$, that

$$\lim_{0<\theta\to 0}\Big(\frac{1}{\lambda_j(\mathbf{q})-\lambda_+(\mathbf{k})+\theta}+\frac{1}{\lambda_+(\mathbf{q})-\lambda_j(\mathbf{k})+\theta}\Big) = 2\pi\delta(i\lambda_j(\mathbf{q})-i\lambda_+(\mathbf{k})).$$

Finally there

Summing up all the previous calculations, we find that a_+ satisfies the following radiative transfer equation:

$$\boxed{\begin{aligned} &\frac{\partial a_+}{\partial t} - \nabla_{\mathbf{k}}\omega_+ \cdot \nabla_{\mathbf{x}} a_+ + (\Sigma(\mathbf{k}) + i\Pi(\mathbf{k}))a_+ \\ = \ & \int_{\mathbb{R}^d} \sigma(\mathbf{k},\mathbf{q}) a_+(\mathbf{q}) \delta\Big(\omega_+(\mathbf{q}) - \omega_+(\mathbf{k})\Big) d\mathbf{q}. \end{aligned}}$$

with

$$\begin{aligned} \Sigma(\mathbf{k}) &= \frac{\pi\omega_+^2(\mathbf{k})}{2(2\pi)^d} \int_{\mathbb{R}^d} \frac{\widehat{R}^{11} + \widehat{R}^{22}}{2}(\mathbf{k}-\mathbf{q}) \delta\Big(\omega_+(\mathbf{q}) - \omega_+(\mathbf{k})\Big) d\mathbf{q}, \\ i\Pi(\mathbf{k}) &= \frac{1}{4(2\pi)^d} \text{ p.v.} \int_{\mathbb{R}^d} (\widehat{R}^{11} - \widehat{R}^{22})(\mathbf{k}-\mathbf{q}) \sum_{i=\pm} \frac{\lambda_+(\mathbf{k})\lambda_i(\mathbf{q})}{\lambda_+(\mathbf{k}) - \lambda_i(\mathbf{q})} d\mathbf{q}, \\ \sigma(\mathbf{k},\mathbf{q}) &= \frac{\pi\omega_+^2(\mathbf{k})}{2(2\pi)^d} \widehat{R}^{12}(\mathbf{k}-\mathbf{q}). \end{aligned}$$

Recall that $\omega_+(\mathbf{k}) = i\lambda_+(i\mathbf{k}) = -c_0 q_0(i\mathbf{k}) = -c_0|\mathbf{k}|$ for the wave equation.

Rigorous derivations of radiative transfer

Theorem[Erdös-Yau-2000]. Consider the Schrödinger equation in the weak-coupling regime: $i\varepsilon\frac{\partial\psi_\varepsilon}{\partial t}+\frac{\varepsilon^2}{2}\Delta\psi_\varepsilon-\sqrt{\varepsilon}V(\frac{\mathbf{x}}{\varepsilon})\psi_\varepsilon=0$, with smooth WKB-type initial conditions in dimension $d\geq 2$, and where $V(\mathbf{x})$ is a mean-zero real Gaussian field with smooth power spectrum $\hat{R}(\mathbf{p})$. Then $\mathbb{E}\{W_\varepsilon(t,\mathbf{x},\mathbf{k})\}$, the expectation of the Wigner transform of ψ_ε converges weakly in $\mathcal{S}'(\mathbb{R}^{2d})$ to the solution of the kinetic equation

$$\frac{\partial W}{\partial t}+\mathbf{k}\cdot\nabla_{\mathbf{x}}W=2\pi\int_{\mathbb{R}^d}\hat{R}(\mathbf{k}-\mathbf{q})\Big(W(\mathbf{q})-W(\mathbf{k})\Big)\delta\Big(\frac{|\mathbf{k}|^2}{2}-\frac{|\mathbf{q}|^2}{2}\Big)d\mathbf{q}.$$

The proof is based on diagrammatic expansions in the Duhamel formula $\psi_\varepsilon(t)=e^{-iH_\varepsilon t}\psi_\varepsilon(0)$. The law of the limiting measure is not characterized.

A similar result was recently obtained for (a discrete version of) the wave equation by Jani Lukkarinen and Herbert Spohn (Kinetic Limit for Wave Propagation in a Random Medium; math-ph/0505075).

Outline for Lecture I.

1. Waves in heterogeneous media

2. High Frequency regime and Geometrical optics

3. Wigner transforms

4. Radiative Transfer model in the weak coupling regime

5. **Paraxial, Itô-Schrödinger, and Random Liouville approximations**

6. More general Radiative Transfer models

Analysis for the Paraxial Equation

The pressure field $p(z, \mathbf{x}, t)$ satisfies the scalar wave equation

$$\frac{1}{c^2(z, \mathbf{x})} \frac{\partial^2 p}{\partial t^2} - \Delta p = 0. \tag{1}$$

The parabolic approximation consists of positing that

$$p(z, \mathbf{x}, t) \approx \int_{\mathbb{R}} e^{i(-c_0 \kappa t + \kappa z)} \psi(z, \mathbf{x}, \kappa) c_0 d\kappa,$$

where ψ satisfies the Schrödinger equation

$$2i\kappa \frac{\partial \psi}{\partial z}(z, \mathbf{x}, \kappa) + \Delta_{\mathbf{x}} \psi(z, \mathbf{x}, \kappa) + \kappa^2 (n^2(z, \mathbf{x}) - 1) \psi(z, \mathbf{x}, \kappa) = 0,$$
$$\psi(z = 0, \mathbf{x}, \kappa) = \psi_0(\mathbf{x}, \kappa)$$

with $\Delta_{\mathbf{x}}$ the transverse Laplacian in the variable $\mathbf{x}$. The refraction index $n(z, \mathbf{x}) = c_0 / c(z, \mathbf{x})$, and c_0 is a reference speed.

Scaling and random medium

The scaled Schrödinger equation in the *weak coupling regime* is

$$2i\kappa\varepsilon\frac{\partial\psi_\varepsilon}{\partial z} + \varepsilon^2\Delta_{\mathbf{x}}\psi_\varepsilon + \kappa^2\sqrt{\varepsilon}V(\frac{\mathbf{x}}{\varepsilon},\frac{z}{\varepsilon})\psi_\varepsilon = 0,$$
$$\psi_\varepsilon(z=0,\mathbf{x},\kappa) = \psi_0(\mathbf{x},\kappa).$$

The random field $V(z,\mathbf{x})$ is a Markov process in z with infinitesimal generator Q. It is stationary in z and $\mathbf{x}$ with correlation function $R(z,\mathbf{x})$

$$\mathbb{E}\{V(s,\mathbf{y})V(z+s,\mathbf{x}+\mathbf{y})\} = R(z,\mathbf{x}) \quad \text{for all } \mathbf{x},\mathbf{y}\in\mathbb{R}^d, \text{ and } z,s\in\mathbb{R}.$$

The generator Q is a chosen conveniently, e.g. as a bounded operator on $L^\infty(\mathcal{V})$ with a unique invariant measure $\pi(\hat{V})$.

Equation for the Wigner Transform

Let us define the Wigner transform as the following mixture of states

$$W_\varepsilon(t,\mathbf{x},\mathbf{k};\kappa) = \int_S \int_{\mathbb{R}^d} e^{i\mathbf{k}\cdot\mathbf{y}} \psi_\varepsilon(t,\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2};\kappa;\omega)\psi_\varepsilon(t,\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2};\kappa;\omega)\frac{d\mathbf{y}}{(2\pi)^d}d\mu(\omega),$$

where ψ_ε solves the paraxial equation and where (S,μ) is such that $W_\varepsilon(0,\mathbf{x},\mathbf{k};\kappa)$ is uniformly bounded in $L^2(\mathbb{R}^d\times\mathbb{R}^d)$ and converges as $\varepsilon\to 0$ to $W^0(\mathbf{x},\mathbf{k};\kappa)$.

Then the Wigner transform W_ε solves the following equation:

$$\frac{\partial W_\varepsilon}{\partial z} + \frac{1}{\kappa}\mathbf{k}\cdot\nabla_{\mathbf{x}} W_\varepsilon = \kappa \mathcal{L}_\varepsilon W_\varepsilon$$

$$W_\varepsilon(0,\mathbf{x},\mathbf{k};\kappa) = W_\varepsilon^0(\mathbf{x},\mathbf{k};\kappa),$$

$$\mathcal{L}_\varepsilon W_\varepsilon = \frac{1}{i\sqrt{\varepsilon}}\int_{\mathbb{R}^d}\frac{d\tilde{V}(\frac{z}{\varepsilon},\mathbf{p})}{(2\pi)^d}e^{i\mathbf{p}\cdot\mathbf{x}/\varepsilon}\Big[W_\varepsilon(\mathbf{x},\mathbf{k}-\frac{\mathbf{p}}{2}) - W_\varepsilon(\mathbf{x},\mathbf{k}+\frac{\mathbf{p}}{2})\Big].$$

Moreover, $W_\varepsilon(z;\kappa)$ is uniformly bounded in $L^2(\mathbb{R}^d\times\mathbb{R}^d)$ for $z>0$.

Main stability result

The Wigner distribution W_ε converges *in probability and weakly* in $L^2(\mathbb{R}^{2d})$ to the solution $\overline{W}$ of the transport equation

$$\frac{\partial \overline{W}}{\partial z} + \frac{1}{\kappa}\mathbf{k} \cdot \nabla_{\mathbf{x}} \overline{W} = \kappa \mathcal{L} \overline{W},$$

with initial data $W_0(\mathbf{x}, \mathbf{k}; \kappa)$ and operator $\mathcal{L}$ defined by

$$\mathcal{L}\lambda = \int_{\mathbb{R}^d} \frac{d\mathbf{p}}{(2\pi)^d} \hat{R}(\frac{|\mathbf{p}|^2 - |\mathbf{k}|^2}{2}, \mathbf{p} - \mathbf{k})(\lambda(\mathbf{p}) - \lambda(\mathbf{k})),$$

where $\hat{R}(\omega, \mathbf{p})$ is the Fourier transform of the correlation function of V.

More precisely, for any test function $\lambda \in L^2(\mathbb{R}^{2d})$ the process $\langle W_\varepsilon(z), \lambda \rangle$ converges to $\langle \overline{W}(z), \lambda \rangle$ in probability as $\varepsilon \to 0$, uniformly on finite intervals $0 \le z \le L$. Here, $\langle \cdot, \cdot \rangle$ is the usual scalar product in $L^2(\mathbb{R}^{2d})$.

Itô Schrödinger equations

Let us come back to the (rescaled) parabolic approximation

$$\frac{\partial \psi}{\partial z} + \frac{-iL_z}{2kL_x^2}\Delta_{\mathbf{x}}\psi = \frac{ikL_z\nu}{2}V(\frac{L_x\mathbf{x}}{l_x}, \frac{L_z z}{l_z})\psi.$$

We now assume that the random fluctuations are very fast in z: $l_z \ll \lambda$. Then we can formally replace

$$\frac{kL_z\nu}{2}V(\frac{L_x\mathbf{x}}{l_x}, \frac{L_z z}{l_z})dz \qquad \text{by} \qquad \kappa B(\frac{L_x\mathbf{x}}{l_x}, dz),$$

where $B(\mathbf{x}, dz)$ is the usual Wiener measure in z with statistics

$$\langle B(\mathbf{x}, z)B(\mathbf{y}, z')\rangle = Q(\mathbf{y}-\mathbf{x})z \wedge z'.$$

Itô Schrödinger equation

The parabolic equation in this regime becomes then

$$d\psi(\mathbf{x}, z) = \frac{iL_z}{2kL_x^2}\Delta_{\mathbf{x}}\psi(\mathbf{x}, z)dz + i\kappa\psi(\mathbf{x}, z) \circ B(\frac{L_x\mathbf{x}}{l_x}, dz).$$

Here $\circ$ means that the stochastic equation is understood in the Stratonovich sense. In the Itô sense it becomes the Itô-Schrödinger equation:

$$\boxed{d\psi(\mathbf{x}, z) = \frac{1}{2}\left(\frac{iL_z}{kL_x^2}\Delta_{\mathbf{x}} - \kappa^2 Q(\mathbf{0})\right)\psi(\mathbf{x}, z)dz + i\kappa\psi(\mathbf{x}, z)B(\frac{L_x\mathbf{x}}{l_x}, dz).}$$

Advantage: Closed equations for the statistical moments.

Second Moment

Introduce the Wigner transform

$$W(\mathbf{x},\mathbf{p},z) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\mathbf{p}\cdot\mathbf{y}}\psi(\mathbf{x}-\frac{\eta\mathbf{y}}{2},z)\psi^*(\mathbf{x}+\frac{\eta\mathbf{y}}{2},z)d\mathbf{y}.$$

By application of the Itô formula:

$$\begin{aligned} d(\psi(\mathbf{x}_1,z)\psi^*(\mathbf{x}_2,z)) &= \psi(\mathbf{x}_1,z)d\psi^*(\mathbf{x}_2,z) \\ &+ d\psi(\mathbf{x}_1,z)\psi^*(\mathbf{x}_2,z) + d\psi(\mathbf{x}_1,z)d\psi^*(\mathbf{x}_2,z), \end{aligned}$$

we find that

$$\boxed{\frac{\partial\langle W\rangle}{\partial z} + \frac{L_z}{kL_x^2\eta}\mathbf{p}\cdot\nabla_{\mathbf{x}}\langle W\rangle = \int_{\mathbb{R}^d}\left[\widehat{Q}(\mathbf{p}-\mathbf{p}') - Q(\mathbf{0})\delta(\mathbf{p}-\mathbf{p}')\right]\langle W\rangle(\mathbf{p}')d\mathbf{p}'.}$$

We thus get an equation for the average Wigner transform for free.

Scintillation = second moment for the WT

Define $\mathcal{W}(\mathbf{x},\mathbf{p},\boldsymbol{\xi},\mathbf{q},z) = W(\mathbf{x},\mathbf{p},z)W(\boldsymbol{\xi},\mathbf{q},z)$. Its statistical average can be related to the fourth statistical moment of ψ and we find that

$$\boxed{\frac{\partial\langle\mathcal{W}\rangle}{\partial z} + \frac{L_z}{kL_x^2\eta}(\mathbf{p}\cdot\nabla_{\mathbf{x}} + \mathbf{q}\cdot\nabla_{\boldsymbol{\xi}})\langle\mathcal{W}\rangle = \mathcal{R}_2\langle\mathcal{W}\rangle + K_{12}\langle\mathcal{W}\rangle}$$

$$K_{12}\mathcal{W} = \int_{\mathbb{R}^d}\hat{Q}(\mathbf{u})e^{i\frac{(\mathbf{x}-\boldsymbol{\xi})\cdot\mathbf{u}}{\eta}}\Big(\mathcal{W}(\mathbf{p}-\tfrac{\mathbf{u}}{2},\mathbf{q}-\tfrac{\mathbf{u}}{2}) + \mathcal{W}(\mathbf{p}+\tfrac{\mathbf{u}}{2},\mathbf{q}+\tfrac{\mathbf{u}}{2})$$
$$-\mathcal{W}(\mathbf{p}-\tfrac{\mathbf{u}}{2},\mathbf{q}+\tfrac{\mathbf{u}}{2}) - \mathcal{W}(\mathbf{p}+\tfrac{\mathbf{u}}{2},\mathbf{q}-\tfrac{\mathbf{u}}{2})\Big)d\mathbf{u}$$
$$K_2\mathcal{W} = \int_{\mathbb{R}^{2d}}\Big[\hat{Q}(\mathbf{p}-\mathbf{p}')\delta(\mathbf{q}-\mathbf{q}') + \hat{Q}(\mathbf{q}-\mathbf{q}')\delta(\mathbf{p}-\mathbf{p}')\Big]\mathcal{W}(\mathbf{p}',\mathbf{q}')d\mathbf{p}'d\mathbf{q}'$$
$$\mathcal{R}_2\mathcal{W} = K_2\mathcal{W} - 2Q(\mathbf{0})\mathcal{W}.$$

When the phase term cancels so that "$|K_{12}\mathcal{W}| \ll 1$", we obtain that

$$\boxed{J_\eta(\mathbf{x},\mathbf{p},\boldsymbol{\xi},\mathbf{q},z) = \langle\mathcal{W}(\mathbf{x},\mathbf{p},\boldsymbol{\xi},\mathbf{q},z)\rangle - \langle W(\mathbf{x},\mathbf{p},z)\rangle\,\langle W(\boldsymbol{\xi},\mathbf{q},z)\rangle},$$

the scintillation function, is small. The energy is then statistically stable.

Smallness of the scintillation function

Theorem. Let us assume that $W_\eta(\mathbf{x},\mathbf{p},0)$ is deterministic and such that

$$\int_{\mathbb{R}^{2d}} |W_\eta(\mathbf{x},\mathbf{p},0)|^2 d\mathbf{x}d\mathbf{p} + \int_{\mathbb{R}^d} \sup_{\mathbf{x}} |W_\eta(\mathbf{x},\mathbf{p},0)|^2 d\mathbf{p} \le C,$$

where C is a constant independent of η. Assume also that the correlation function $Q(\mathbf{x}) \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Then

$$\|J_\eta\|_2(z) \le C\eta^{d/2},$$

uniformly in z on compact intervals.

Weak statistical stability

Theorem. Under the assumptions of the previous theorem and $\lambda \in L^2(\mathbb{R}^{2d})$, we obtain that

$$\left\langle \left\{ \Big((W_\eta, \lambda) - (\langle W_\eta \rangle, \lambda) \Big)^2 \right\} \right\rangle \leq C \eta^{d/2} \|\lambda\|_2^2.$$

Also (W_η, λ) becomes deterministic in the limit of small values of η as

$$P\Big(\Big| (W_\eta, \lambda) - (\langle W_\eta \rangle, \lambda) \Big| \geq \alpha \Big) \leq \frac{C \eta^{d/2} \|\lambda\|_2^2}{\alpha^2} \to 0 \qquad \text{as } \eta \to 0.$$

The Wigner transform W_η of the stochastic field ψ_η converges weakly and in probability to the deterministic solution $\overline{W}(\mathbf{x}, \mathbf{p}, z)$ of a Radiative Transfer Equation.

Scintillation may appear and not disappear

Theorem. Assume that $\boxed{W_\eta(\mathbf{x},\mathbf{p},0)=\delta(\mathbf{x}-\mathbf{x}_0)\delta(\mathbf{p}-\mathbf{p}_0)}$. Then the scintillation function J_η is composed of a singular term of the form (with $Q=Q(\mathbf{0})$):

$$\delta(\mathbf{x}-\boldsymbol{\xi})\delta(\mathbf{p}-\mathbf{q})\Big(\alpha(\mathbf{x},\mathbf{p},z)-e^{-2Qz}\alpha(\mathbf{x}-z\mathbf{p},\mathbf{p},0)\Big)$$

plus other contributions that are mutually singular with respect to this term. Moreover the density $\alpha(\mathbf{x},\mathbf{p},z)$ solves the radiative transfer equation with initial condition $a_0(\mathbf{x},\mathbf{p})=\delta(\mathbf{x}-\mathbf{x}_0)\delta(\mathbf{p}-\mathbf{p}_0)$:

$$\frac{\partial\alpha}{\partial z}+\mathbf{p}\cdot\nabla_{\mathbf{x}}\alpha+2Q\alpha=\int_{\mathbb{R}^d}\hat{Q}(\mathbf{u})\Big(\alpha(\mathbf{x},\mathbf{p}+\frac{\mathbf{u}}{2},z)+\alpha(\mathbf{x},\mathbf{p}-\frac{\mathbf{u}}{2},z)\Big)d\mathbf{u}.$$

The total intensity of this scintillation is $(1-e^{-2Qz})$ (so it grows in z though it vanishes at $z=0$).

In this case *Energy* is NOT statistically stable.

Random Liouville model

We come back to the full wave equation and $\mathbf{w}_\varepsilon(t,\mathbf{x}) = A_\varepsilon^{1/2}(\mathbf{x})\mathbf{u}_\varepsilon(t,\mathbf{x})$ ($\mathbf{u}_\varepsilon = (\mathbf{v}_\varepsilon, p_\varepsilon)$) which solves the first-order symmetrized system:

$$\frac{\partial \mathbf{w}_\varepsilon}{\partial t} + A_\varepsilon^{-1/2}(\mathbf{x}) D^j \frac{\partial}{\partial x^j}\Big(A_\varepsilon^{-1/2}(\mathbf{x})\mathbf{w}_\varepsilon(\mathbf{x})\Big) = 0.$$

Define $P_\varepsilon(\mathbf{x},\mathbf{k}) = P_0(\mathbf{x},\mathbf{k}) + \varepsilon P_1(\mathbf{x})$, where

$$P_0(\mathbf{x},\mathbf{k}) = iA_\varepsilon^{-\frac{1}{2}}(\mathbf{x}) D^j A_\varepsilon^{-\frac{1}{2}}(\mathbf{x}) k_j = ic_\varepsilon(\mathbf{x}) k_j D^j$$

$$2P_1(\mathbf{x}) = A_\varepsilon^{-\frac{1}{2}}(\mathbf{x}) D^j \frac{\partial}{\partial x_j}\Big(A_\varepsilon^{-\frac{1}{2}}(\mathbf{x})\Big) - \frac{\partial}{\partial x_j}\Big(A_\varepsilon^{-\frac{1}{2}}(\mathbf{x})\Big) D^j A_\varepsilon^{-\frac{1}{2}}(\mathbf{x}).$$

The Wigner transform $W_\varepsilon(t,\mathbf{x},\mathbf{k})$ satisfies the evolution equation

$$\varepsilon\frac{\partial W_\varepsilon}{\partial t} + \mathcal{L}_\varepsilon W_\varepsilon = 0$$

$$\mathcal{L}_\varepsilon f(\mathbf{x},\mathbf{k}) = \int \Big(P_\varepsilon(\mathbf{y},\mathbf{q}) e^{i\phi} f(\mathbf{z},\mathbf{p}) - f(\mathbf{z},\mathbf{p}) e^{-i\phi} P_\varepsilon(\mathbf{y},\mathbf{q})\Big) \frac{d\mathbf{z}d\mathbf{p}d\mathbf{y}d\mathbf{q}}{(\pi\varepsilon)^{2d}},$$

$$\phi(\mathbf{x},\mathbf{z},\mathbf{k},\mathbf{p},\mathbf{y},\mathbf{q}) = \tfrac{2}{\varepsilon}((\mathbf{p}-\mathbf{k})\cdot\mathbf{y} + (\mathbf{q}-\mathbf{p})\cdot\mathbf{x} + (\mathbf{k}-\mathbf{q})\cdot\mathbf{z}).$$

The Liouville equations

Consider the leading-order term for the Wigner transform. The matrix $-iP_0$ has eigenvalues $\lambda_0 = 0$ of multiplicity $d-1$ and $\lambda^\varepsilon_{1,2}(\mathbf{x},\mathbf{k}) = \pm c_\varepsilon(\mathbf{x})|\mathbf{k}|$:

$$-iP_0(\mathbf{x},\mathbf{k}) = \sum_{q=0}^{2} \lambda^\varepsilon_q(\mathbf{x},\mathbf{k})\Pi_q(\mathbf{x},\mathbf{k}), \qquad \text{where} \qquad \sum_{q=0}^{2} \Pi_q(\mathbf{x},\mathbf{k}) = I.$$

The Liouville approximation to the Wigner transform is given by

$$U_\varepsilon(t,\mathbf{x},\mathbf{k}) = \sum_q u^\varepsilon_q(t,\mathbf{x},\mathbf{k})\Pi_q(\mathbf{k}),$$

where the coefficients u^ε_q solve the Liouville equation

$$\begin{aligned} &\frac{\partial u^\varepsilon_q}{\partial t} + \nabla_{\mathbf{k}}\lambda^\varepsilon_q \cdot \nabla_{\mathbf{x}} u^\varepsilon_q - \nabla_{\mathbf{x}}\lambda^\varepsilon_q \cdot \nabla_{\mathbf{k}} u^\varepsilon_q = 0 \\ &u^\varepsilon_q(0,\mathbf{x},\mathbf{k}) = \mathrm{Tr}\Pi_q W_0(\mathbf{x},\mathbf{k})\Pi_q \end{aligned}$$

Here, the coefficients λ^ε_q depend on $\delta(\varepsilon)$ and W_0 is chosen *independent* of ε.

Approximation of W_ε by Liouville equation

Theorem. Let $\rho_\varepsilon(\mathbf{x}) = \rho_0 + \sqrt{\delta}\rho_1(\frac{\mathbf{x}}{\delta})$ and $\kappa_\varepsilon(\mathbf{x}) = \kappa_0 + \sqrt{\delta}\kappa_1(\frac{\mathbf{x}}{\delta})$, with all terms sufficiently smooth. Then we have

$$\|W_\varepsilon(t,\mathbf{x},\mathbf{k}) - U_\varepsilon(t,\mathbf{x},\mathbf{k})\|_2 \leq C\frac{\varepsilon}{\delta^m}\exp(\frac{Ct}{\delta^{3/2}})\|W_0\|_{H^3} + \|W_\varepsilon^0 - W_0\|_{L^2},$$

for some m independent of ε.

In other words, assuming that W_ε^0 converges strongly to W_0 and that $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ with the constraint $\delta(\varepsilon) \gg |\ln\varepsilon|^{-2/3+\eta}$, then the difference $\|W_\varepsilon(t,\mathbf{x},\mathbf{k}) - U_\varepsilon(t,\mathbf{x},\mathbf{k})\|_{L^2} \to 0$ uniformly on finite intervals $t \in (0,T)$.

The convergence is uniform in the realization of the random medium (the statistics of ρ_1 and κ_1 have not been defined yet). So we *safely replace the analysis* of W_ε by that of U_ε, the solution of a Liouville equation with random coefficients.

Stability of the Wigner Transform

Theorem. Let u_ε be a propagating mode associated to U_ε. Then:

$$\mathbb{E}\{u_\varepsilon(t,\mathbf{x},\mathbf{k})\} \to F(t,\mathbf{x},\mathbf{k}) \quad \text{weakly as} \quad \delta(\varepsilon) \to 0,$$

where F satisfies the following Fokker-Planck equation

$$\boxed{\frac{\partial F}{\partial t} + c_0 \hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} F - \mathcal{L} F = 0},$$

$$\mathcal{L}F(\mathbf{k}) = \sum_{p,q=1}^{d} |\mathbf{k}|^2 D_{p,q}(\hat{\mathbf{k}}) \partial^2_{k_p,k_q} F(\mathbf{k}) + \sum_{p=1}^{d} |\mathbf{k}| E_p(\hat{\mathbf{k}}) \partial_{k_p} F(\mathbf{k}).$$

The coefficients $D_{p,q}$ and E_p are related to the power spectra of κ_1 and ρ_1. Moreover, we obtain the stability result

$$\mathbb{E}\left\{ \int \left| \left\langle u_\varepsilon(T,\mathbf{x}_0,\mathbf{k}) - F(T,\mathbf{x}_0,\mathbf{k}), \lambda(\mathbf{k}) \right\rangle \right|^2 d\mathbf{x}_0 \right\} \to 0 \quad \text{as} \quad \delta(\varepsilon) \to 0,$$

which implies that u_ε converges in probability to the deterministic solution F. This in turn implies the stability of the refocused signal $\mathbf{u}^B$.

Summary of radiative transfer models

We have obtained several transport models of the form

$$\frac{\partial a}{\partial t} + c_0 \hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} a + \mathcal{S} a = 0,$$

where the scattering operator $\mathcal{S}$ is given respectively by

$$\begin{aligned}
\text{Radiative Transfer:} \quad \mathcal{S}a &= \int_{\mathbb{R}^d} \hat{R}(\mathbf{p}-\mathbf{k})(a(\mathbf{k})-a(\mathbf{p}))\delta\Big(c_0|\mathbf{p}|-c_0|\mathbf{k}|\Big)d\mathbf{k} \\
\text{Paraxial:} \quad \mathcal{S}a &= \int_{\mathbb{R}^{d-1}} \hat{R}(\frac{|\mathbf{p}'|^2-|\mathbf{k}'|^2}{2}, \mathbf{p}'-\mathbf{k}')(a(\mathbf{k}')-a(\mathbf{p}'))d\mathbf{k}' \\
\text{Itô-Schrödinger:} \quad \mathcal{S}a &= \int_{\mathbb{R}^{d-1}} \hat{R}(0, \mathbf{p}'-\mathbf{k}')(a(\mathbf{k}')-a(\mathbf{p}'))d\mathbf{k}' \\
\text{Fokker-Planck:} \quad \mathcal{S}a &= -D(|\mathbf{k}|)\Delta_{\hat{\mathbf{k}}} a.
\end{aligned}$$

Note that Radiative Transfer and Fokker-Planck admit a diffusion limit for small mean free paths. This can be arranged for the paraxial approximation when $\hat{R}(t,\cdot) \approx \delta(t)\hat{R}'(\cdot)$, but *not* for Itô-Schrödinger.

Outline

1. Waves in heterogeneous media

2. High Frequency regime and Geometrical optics

3. Wigner transforms

4. Radiative Transfer model in the weak coupling regime

5. Random Liouville, paraxial and Itô-Schrödinger approximations

6. **More general Radiative Transfer models**

Equation for spatial Wigner transform

So far, all models start with an equation for the Wigner transform, which requires the field equation to be first-order in the time variable:

$$\varepsilon\frac{\partial \mathbf{u}_\varepsilon^\varphi}{\partial t} + A_\varepsilon^\varphi \mathbf{u}_\varepsilon^\varphi = 0, \qquad \varphi = 1, 2,$$

so that the Wigner transform of the two fields defined as

$$W_\varepsilon(t, \mathbf{x}, \mathbf{k}) = W[\mathbf{u}_\varepsilon^1(t, \cdot), \mathbf{u}_\varepsilon^2(t, \cdot)](\mathbf{x}, \mathbf{k}),$$

solves the equation

$$\varepsilon\frac{\partial W_\varepsilon}{\partial t} + W[A_\varepsilon^1 \mathbf{u}_\varepsilon^1, \mathbf{u}_\varepsilon^2] + W[\mathbf{u}_\varepsilon^1, A_\varepsilon^2 \mathbf{u}_\varepsilon^2] = 0.$$

Some pseudo-differential calculus allows us to write $W[A_\varepsilon^1 \mathbf{u}_\varepsilon^1, \mathbf{u}_\varepsilon^2]$ in terms of $W_\varepsilon(t, \mathbf{x}, \mathbf{k})$.

This method *does not* allow us to obtain kinetic models for e.g. second-order equations or time-discretizations of the wave equation.

Spatio-temporal Wigner transform

To handle more general differential or pseudo-differential opeators in the time variable, we introduce the *spatio-temporal* Wigner transform

$$W[\mathbf{u},\mathbf{v}](t,\omega,\mathbf{x},\mathbf{k}) = \int_{\mathbb{R}^{d+1}} e^{i\mathbf{k}\cdot\mathbf{y}+i\tau\omega}\mathbf{u}(t-\frac{\varepsilon\tau}{2},\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2})\mathbf{v}^*(t+\frac{\varepsilon\tau}{2},\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2})\frac{d\mathbf{y}d\tau}{(2\pi)^{d+1}}.$$

Let us illustrate the use of the spatio-temporal Wigner transform by considering the following constant coefficient equation

$$R(\varepsilon D_t)\mathbf{u}_\varepsilon(t,\mathbf{x}) + P(\varepsilon \mathbf{D_x})\mathbf{u}_\varepsilon(t,\mathbf{x}) = 0.$$

For $R(i\omega) = i\omega$, we are back to first-order equations in time. Then clearly,

$$W[R(\varepsilon D_t)\mathbf{u}_\varepsilon,\mathbf{u}_\varepsilon] + W[P(\varepsilon \mathbf{D_x})\mathbf{u}_\varepsilon,\mathbf{u}_\varepsilon] = 0.$$

The same calculus as earlier gives for $W_\varepsilon = W[\mathbf{u}_\varepsilon,\mathbf{u}_\varepsilon]$ the equations

$$\begin{aligned}\Big(R(i\omega+\frac{\varepsilon D_t}{2}) + P(i\mathbf{k}+\frac{\varepsilon\mathbf{D_x}}{2})\Big)W_\varepsilon(t,\omega,\mathbf{x},\mathbf{k}) &= 0,\\ W_\varepsilon(t,\omega,\mathbf{x},\mathbf{k})\Big(R^*(i\omega-\frac{\varepsilon D_t}{2}) + P^*(i\mathbf{k}-\frac{\varepsilon\mathbf{D_x}}{2})\Big) &= 0.\end{aligned}$$

Application to discrete wave equations

Consider the wave equation with dispersive effects:

$$R(\varepsilon D_t)\mathbf{u}_\varepsilon^\varphi + A_\varepsilon^\varphi \mathbf{u}_\varepsilon^\varphi = 0, \qquad \varphi = 1,2,$$

where $\bar{R}(i\omega) = -R(i\omega)$. For instance $i\Delta^{-1}\sin(\omega\Delta)$ corresponds to second-order time discretization. Then the energy density (or correlation function) associated to the above field equation is still modeled by a kinetic model. The radiative transfer equation for the propagating mode a_+ is

$$\frac{\partial a_+}{\partial t} - \nabla_{\mathbf{k}}\omega_+ \cdot \nabla_{\mathbf{x}} a_+ + (\tilde{\Sigma}(\mathbf{k}) + i\tilde{\Pi}(\mathbf{k}))a_+ = \int_{\mathbb{R}^d} \tilde{\sigma}(\mathbf{k},\mathbf{q}) a_+(\mathbf{q}) \delta\Big(\omega_+(\mathbf{q}) - \omega_+(\mathbf{k})\Big) d\mathbf{q},$$

where the above coefficients are related those with $R(i\omega) = i\omega$ by

$$\tilde{\Sigma}(\mathbf{k}) = \frac{\Sigma(\mathbf{k})}{|R'(i\omega_+(\mathbf{k}))|^2}, \quad \tilde{\sigma}(\mathbf{k},\mathbf{q}) = \frac{\sigma(\mathbf{k},\mathbf{q})}{|R'(i\omega_+(\mathbf{k}))|^2}, \quad \tilde{\Pi}(\mathbf{k}) = \frac{\Pi(\mathbf{k})}{R'(i\omega_+(\mathbf{k}))}.$$

This quantifies the effects of e.g. numerical discretizations on the kinetic parameters of a random media.

Application to scalar equations

We can apply the theory to general *scalar* equations of the form

$$R(\varepsilon D_t)p_\varepsilon^\varphi + \mathcal{H}_\varepsilon^\varphi p_\varepsilon^\varphi = 0, \quad 1 \le \varphi \le 2,$$
$$\mathcal{H}_\varepsilon^\varphi = b_\varepsilon^\varphi(\mathbf{x})\beta(\varepsilon\mathbf{D_x})d_\varepsilon^\varphi(\mathbf{x})\gamma(\varepsilon\mathbf{D_x}), \quad 1 \le \varphi \le 2.$$

For instance $R(i\omega) = -\omega^2$, $\beta(i\mathbf{k}) = -i\mathbf{k}\cdot$ and $\gamma(i\mathbf{k}) = i\mathbf{k}$, $b_0(\mathbf{x}) = \kappa^{-1}(\mathbf{x})$ and $d_0(\mathbf{x}) = \rho^{-1}(\mathbf{x})$ is the second-order scalar wave equation.

Kinetic models can be obtained this way for the following equations:

Schrödinger	$i\varepsilon\dfrac{\partial\psi_\varepsilon}{\partial t} + \dfrac{\varepsilon^2}{2}\Delta\psi_\varepsilon - V_\varepsilon(\mathbf{x})\psi_\varepsilon = 0$
Klein Gordon	$\varepsilon^2\dfrac{\partial^2\psi_\varepsilon}{\partial t^2} - \varepsilon^2\Delta\psi_\varepsilon + \alpha^2\psi_\varepsilon - \sqrt{\varepsilon}V_1(\frac{\mathbf{x}}{\varepsilon})\psi_\varepsilon = 0$
E&M	$\dfrac{\partial^2\mathbf{E}_\varepsilon}{\partial t^2} - \nabla\cdot c_\varepsilon^2(\mathbf{x})\nabla\mathbf{E}_\varepsilon = 0, \ \nabla\cdot\mathbf{E}_\varepsilon = 0.$

Conclusions

The Wigner transform is a very useful tool in the derivation of radiative transfer equations to model energy densities or correlation functions of waves in random media.

Kinetic equations model the correlation of two fields possibly propagating in two different (though hopefully correlated) media.

Though most derivations are formal in the weak coupling regime for wave equations, rigorous theories can be obtained for connected models of wave propation (e.g. paraxial approximation and random Liouville models).

The spatio-temporal Wigner transform is very useful to derive kinetic models from field equations that are more general than first-order differentiations in time.

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