Time reversal and waves in random media

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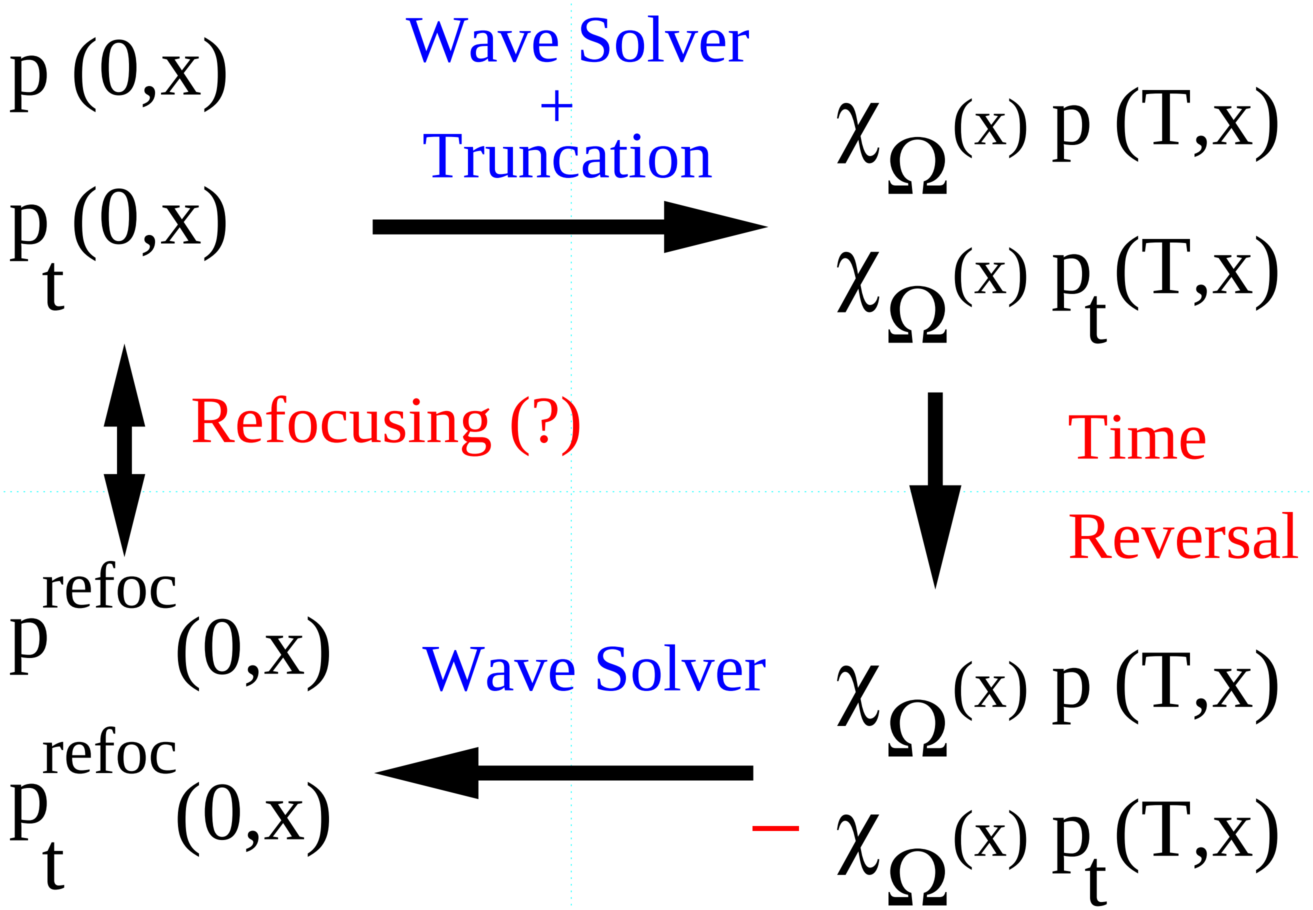
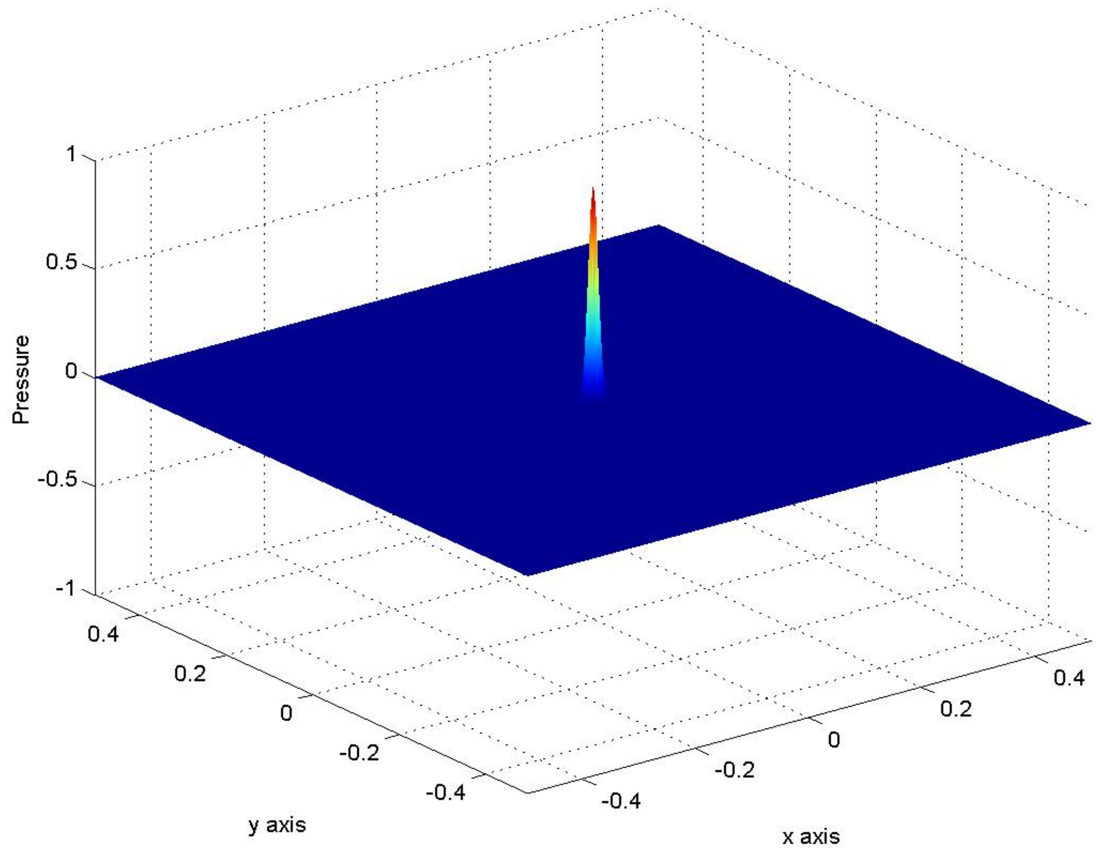
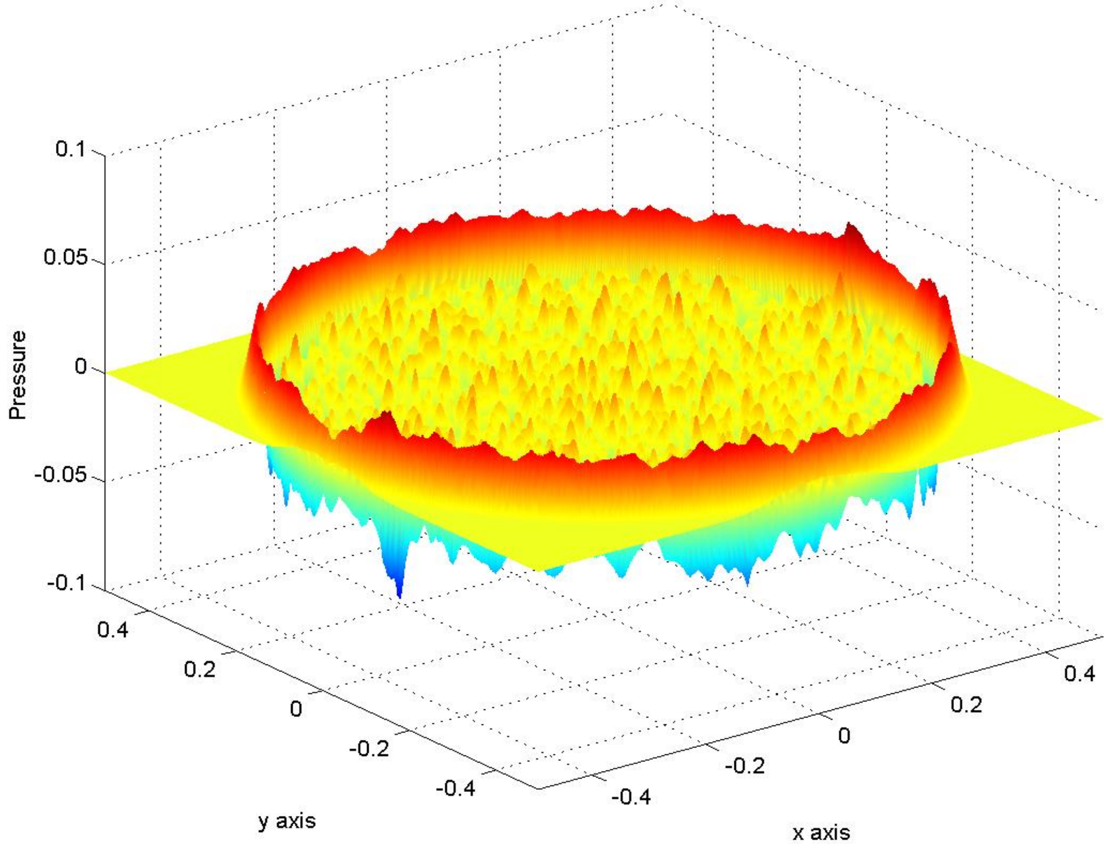
Single Time Time-Reversal



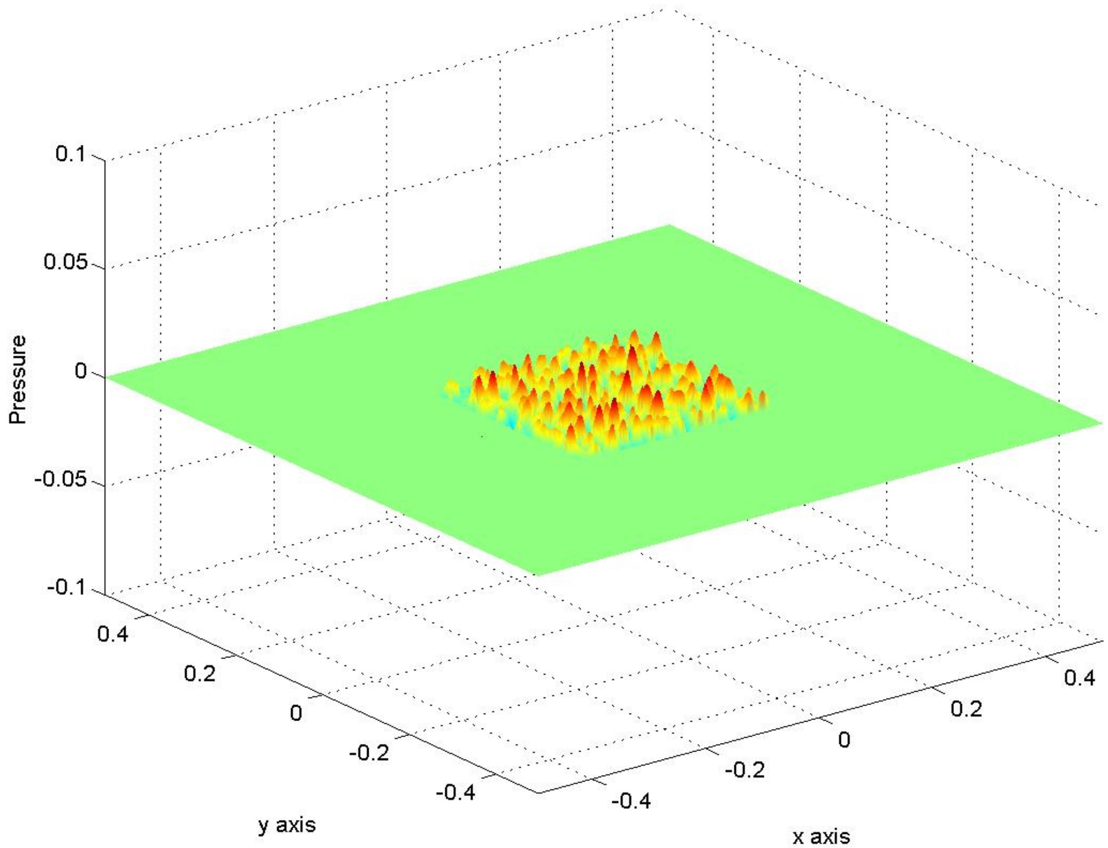
Numerical Experiment: Initial Data



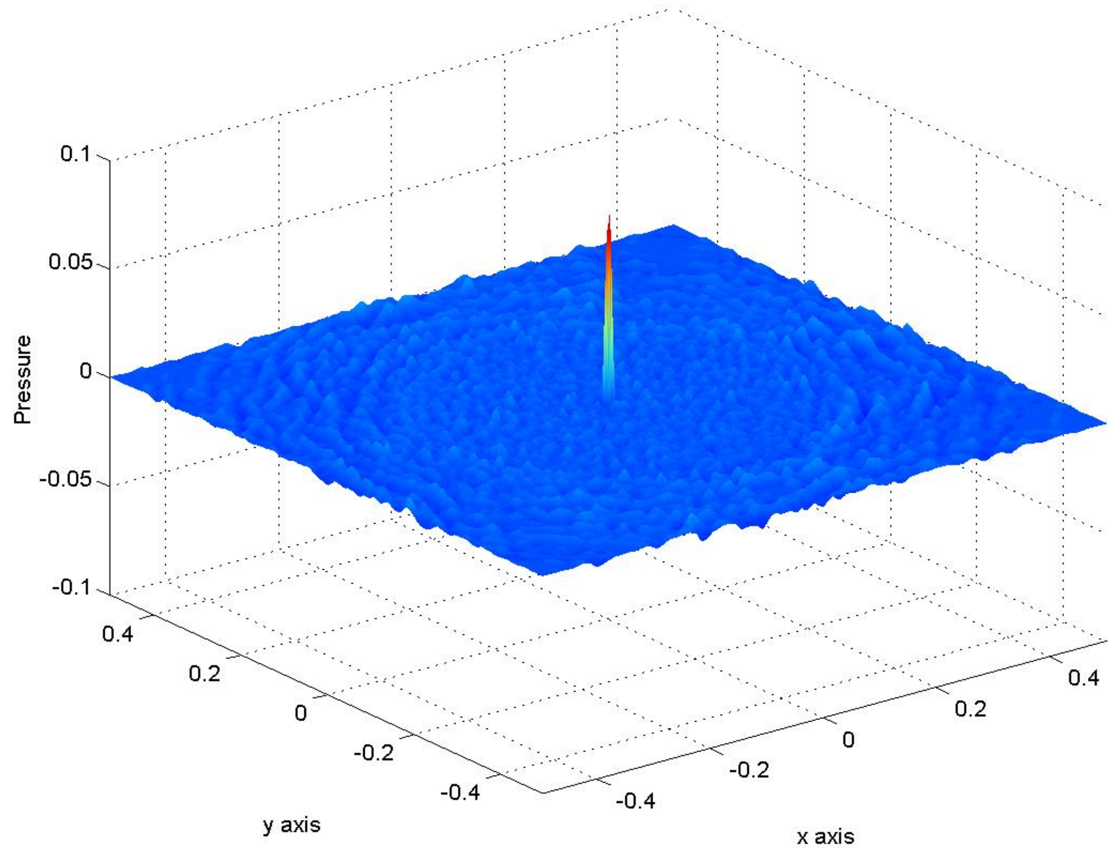
Numerical Experiment: Forward Solution



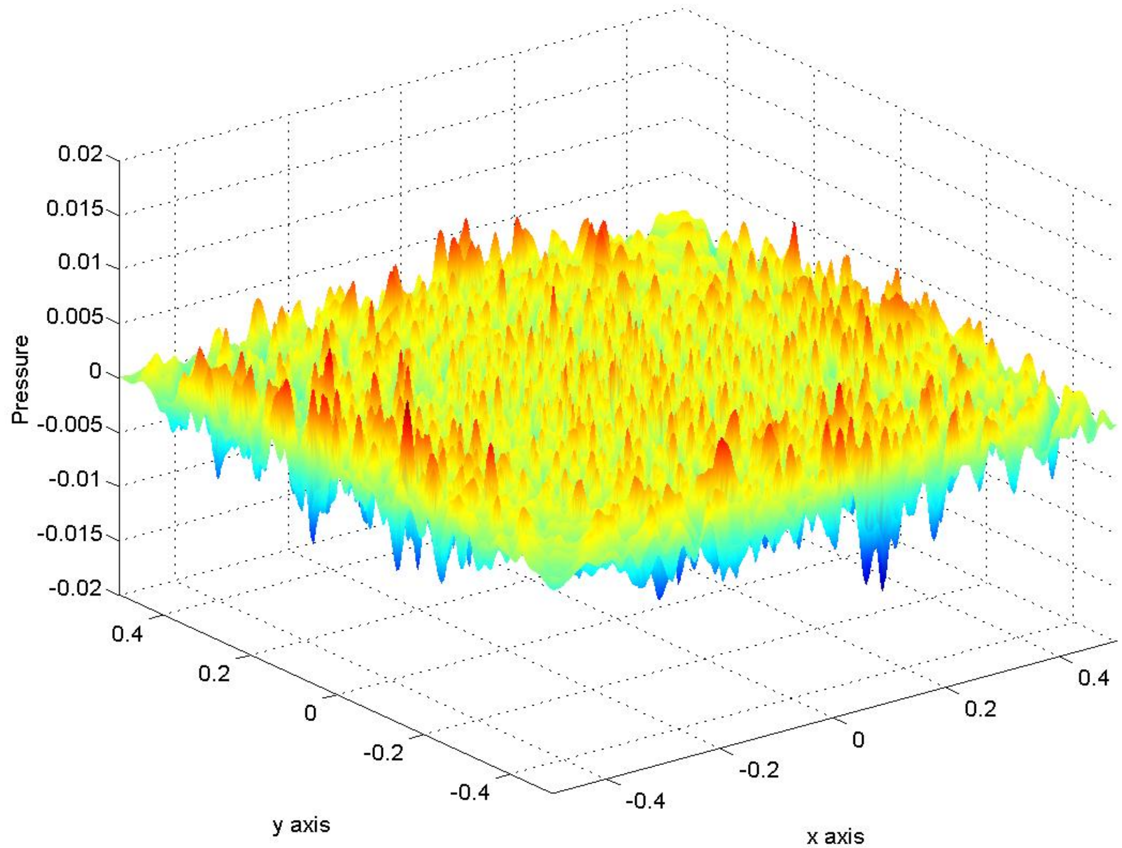
Numerical Experiment: Truncated Solution



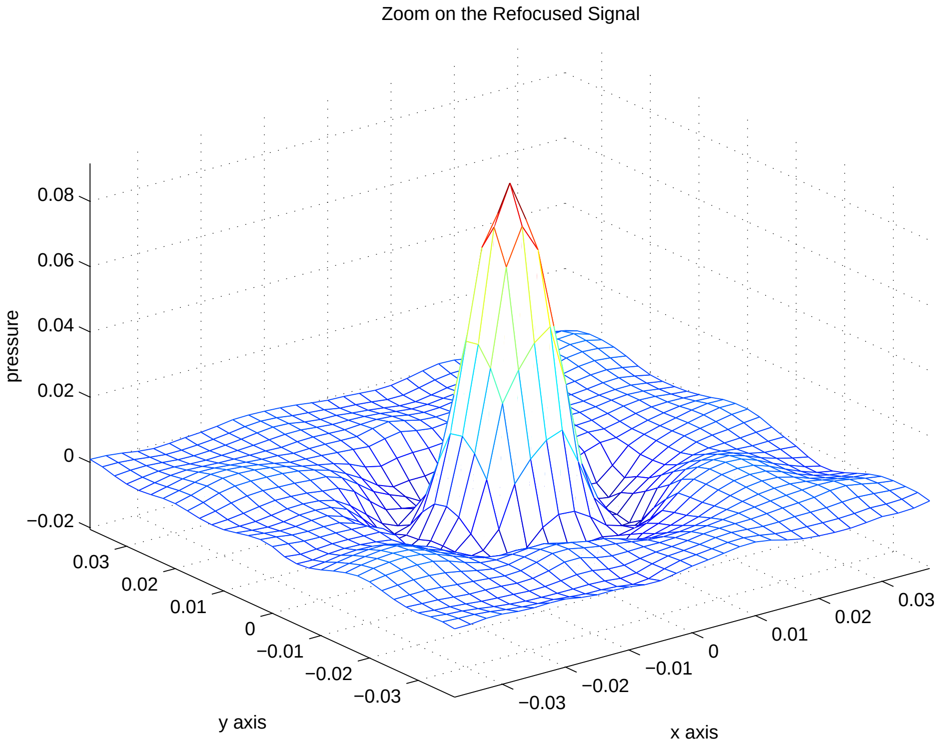
Numerics: Time-reversed Solution

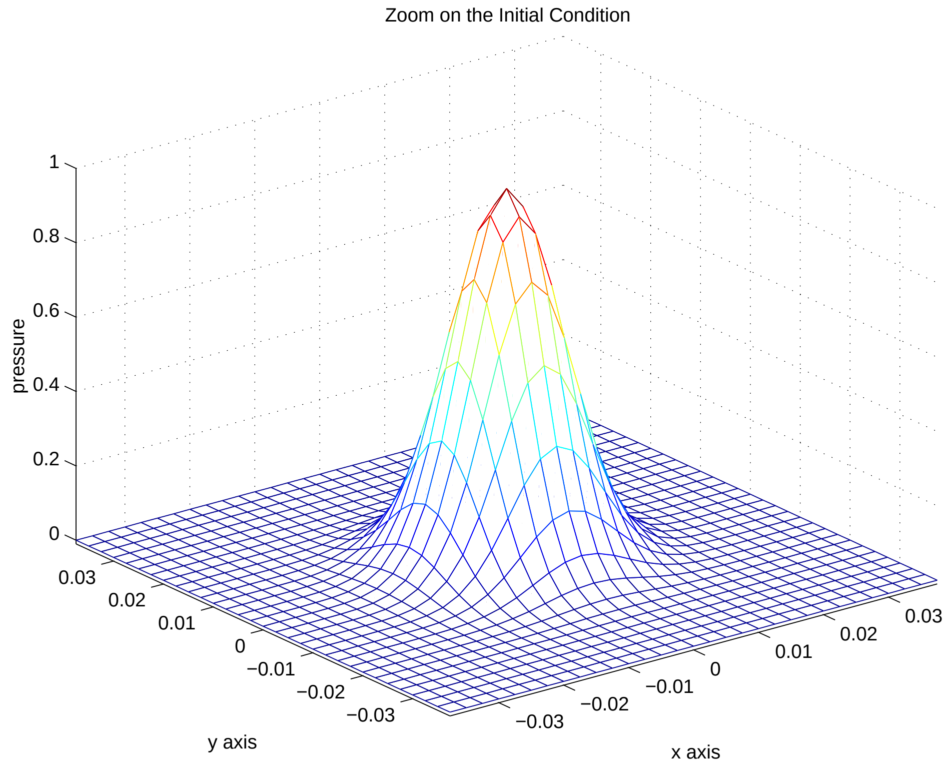


Numerics: Solution pushed forward



Zoom on Refocused and Original Signals





PART I: FORMAL THEORIES

Theory of time-reversal refocusing in 3D

The forward problem for $\mathbf{u} = (\mathbf{v}, p) = (v_1, v_2, v_3, p)$ is

$$A(\mathbf{x})\frac{\partial \mathbf{u}(t,\mathbf{x})}{\partial t} + D^j \frac{\partial \mathbf{u}(t,\mathbf{x})}{\partial x_j} = 0, \quad \mathbf{x} \in \mathbb{R}^3,$$

with initial conditions $\mathbf{u}(t=0) = \mathbf{u}_0$.

The back-propagated signal can be written using the Green's propagator $G(t, \mathbf{x}; \mathbf{y})$ as

$$\mathbf{u}^B(\mathbf{x}) = \int_{\mathbb{R}^9} \Gamma G(t, \mathbf{x}; \mathbf{y}) \Gamma G(t, \mathbf{y}'; \mathbf{z}) \chi_\Omega(\mathbf{y}) \chi_\Omega(\mathbf{y}') f(\mathbf{y} - \mathbf{y}') \mathbf{u}_0(\mathbf{z}) d\mathbf{y} d\mathbf{y}' d\mathbf{z}.$$

Here, Γ is a matrix that models the time reversal process. It is given by $\Gamma = \text{Diag}(-1, -1, -1, 1)$, so that the velocity field $\mathbf{v}$ is replaced by $-\mathbf{v}$ and the pressure field p remains unchanged; $\chi_\Omega(\mathbf{y})$ is the indicatrix function of Ω; $f(\mathbf{y})$ is a filtering function (possibly modeling some blurring).

Theory of time-reversal refocusing (II)

First step: Introduce the adjoint Green's matrix G_*, solution of

$$\frac{\partial G_*(t,\mathbf{x};\mathbf{y})}{\partial t} + \frac{\partial}{\partial x_j}(G_*(t,\mathbf{x};\mathbf{y}))D^j A^{-1}(\mathbf{x}) = 0,$$

with IC: $G_*(0,\mathbf{x};\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y})A^{-1}(\mathbf{x})$, so that $\Gamma G_*(t,\mathbf{x};\mathbf{y})A(\mathbf{x})\Gamma = G(t,\mathbf{y};\mathbf{x})$.

Second step: Rescale problem with $\mathbf{u}_0(\mathbf{x}) = \mathbf{S}\left(\frac{\mathbf{x}-\mathbf{x}_0}{\varepsilon}\right)$ and a filter $\frac{1}{\varepsilon^d}f(\frac{\mathbf{y}-\mathbf{y}'}{\varepsilon})$. An observation point $\mathbf{x}$ is close to $\mathbf{x}_0$ and we write it as $\mathbf{x} = \mathbf{x}_0 + \varepsilon\boldsymbol{\xi}$, so that

$$\mathbf{u}^B_\varepsilon(\boldsymbol{\xi};\mathbf{x}_0) = \int_{\mathbb{R}^9} \Gamma G(T,\mathbf{x}_0+\varepsilon\boldsymbol{\xi};\mathbf{y})G_*(T,\mathbf{x}_0+\varepsilon\mathbf{z};\mathbf{y}')A(\mathbf{x}_0+\varepsilon\mathbf{z})\Gamma \\ \mathbf{S}(\mathbf{z})\chi_\Omega(\mathbf{y})\chi_\Omega(\mathbf{y}')f(\frac{\mathbf{y}-\mathbf{y}'}{\varepsilon})d\mathbf{y}d\mathbf{y}'d\mathbf{z}.$$

Theory of time-reversal refocusing (III)

Third step: introduce the Wigner transform

$$W_\varepsilon(t,\mathbf{x},\mathbf{k}) = \int_{\mathbb{R}^6} \left[\int_{\mathbb{R}^3} e^{i\mathbf{k}\cdot\mathbf{z}} G(t,\mathbf{x}-\frac{\varepsilon\mathbf{z}}{2};\mathbf{y}) G_*(t,\mathbf{x}+\frac{\varepsilon\mathbf{z}}{2};\mathbf{y}')\frac{d\mathbf{z}}{(2\pi)^3}\right] \chi_\Omega(\mathbf{y})\chi_\Omega(\mathbf{y}') f(\frac{\mathbf{y}-\mathbf{y}'}{\varepsilon}) d\mathbf{y}d\mathbf{y}'.$$

Fourth step: write the refocused signal in terms of the WT

$$\mathbf{u}^B_\varepsilon(\boldsymbol{\xi};\mathbf{x}_0) = \int_{\mathbb{R}^6} \Gamma W_\varepsilon(t,\mathbf{x}_0+\varepsilon\frac{\boldsymbol{\xi}+\mathbf{z}}{2},\mathbf{k}) e^{-i\mathbf{k}\cdot(\mathbf{z}-\boldsymbol{\xi})} A(\mathbf{x}_0+\varepsilon\mathbf{z})\Gamma\mathbf{S}(\mathbf{z}) d\mathbf{z}d\mathbf{k}.$$

Refocusing is then obtained by analyzing the limit of W_ε as $\varepsilon \to 0$.

Wigner Transform Theory (I)

Let $\mathcal{A}(\mathbb{R}^6)$ be the subset of $\mathcal{S}'(\mathbb{R}^6)$ of matrix-valued distributions $\eta(\mathbf{x},\mathbf{k})$ such that $\int_{\mathbb{R}^3}\sup_{\mathbf{x}}\|\widehat{\eta}(\mathbf{x},\mathbf{y})\|d\mathbf{y}$ is bounded, and $\mathcal{A}'$ its dual space.

Lemma. *The Wigner transform* $W_\varepsilon(t,\mathbf{x},\mathbf{k})$ *is bounded in* $\mathcal{C}^0((0,T);\mathcal{A}'(\mathbb{R}^6))$ *independent of* ε *provided that* $\widehat{f}(\mathbf{k})\in L^1(\mathbb{R}^3)$. *As a consequence, it converges weakly along a subsequence* $\varepsilon_k\to 0$ *to a distribution* $W(t,\mathbf{x},\mathbf{k})\in\mathcal{C}^0(0,T;\mathcal{A}'(\mathbb{R}^6))$.

Wigner Transform Theory (II)

The Wigner distribution at time $t = 0$ is given by

$$\boxed{W(0, \mathbf{x}, \mathbf{k}) = |\chi_\Omega(\mathbf{x})|^2 \widehat{f}(\mathbf{k}) A^{-1}(\mathbf{x})}.$$

The dispersion matrix $L(\mathbf{x}, \mathbf{k}) = A^{-1}(\mathbf{x}) k_j D^j$ has a double eigenvalue $\omega_0 = 0$ and simple eigenvalues $\omega_{1,2} = \pm c(\mathbf{x})|\mathbf{k}|$, $c(\mathbf{x}) = 1/\sqrt{\rho(\mathbf{x})\kappa(\mathbf{x})}$, with are eigenvectors $\mathbf{b}$.

The limit Wigner distribution can be decomposed as

$$W(t, \mathbf{x}, \mathbf{k}) = \sum_{j=1}^{2} a_{ij}^0(t, \mathbf{x}, \mathbf{k}) \mathbf{b}_i^0 \mathbf{b}_j^{0*} + a_1(t, \mathbf{x}, \mathbf{k}) \mathbf{b}^1 \mathbf{b}^{1*} + a_2(t, \mathbf{x}, \mathbf{k}) \mathbf{b}^2 \mathbf{b}^{2*}.$$

High frequency limit of refocused signal

In the limit $\varepsilon \to 0$, the back-propagated signal is given by

$$\mathbf{u}^B(\boldsymbol{\xi};\mathbf{x}_0) = \int_{\mathbb{R}^3} F(t,\boldsymbol{\xi}-\mathbf{z};\mathbf{x}_0)\mathbf{S}(\mathbf{z})d\mathbf{z} = (F(t,\cdot;\mathbf{x}_0) * \mathbf{S})(\boldsymbol{\xi}).$$

The quality of the refocusing of the back-propagated signal is determined by the decay properties in $\boldsymbol{\xi}$ of the kernel

$$F(t,\boldsymbol{\xi};\mathbf{x}_0) = \sum_{m=1}^{2} \int_{\mathbb{R}^3} a_m(t,\mathbf{x}_0,\mathbf{k})\Gamma \mathbf{b}^m(\mathbf{x}_0,\mathbf{k})\mathbf{b}^{m*}(\mathbf{x}_0,\mathbf{k})e^{i\mathbf{k}\cdot\boldsymbol{\xi}}A(\mathbf{x}_0)\Gamma d\mathbf{k},$$

with $a_1(0,\mathbf{x},\mathbf{k}) = a_2(0,\mathbf{x},\mathbf{k}) = |\chi_\Omega(\mathbf{x})|^2\hat{f}(\mathbf{k})$.

• When $f=\delta$ and $\Omega=\mathbb{R}^3$, we have $\mathbf{u}^B(\boldsymbol{\xi};\mathbf{x}_0) = \mathbf{S}(\boldsymbol{\xi})$. All the information propagates back and the refocusing is perfect.

• In homogeneous medium with $c(\mathbf{x}) = c_0$, the amplitudes $a_{1,2}(t,\mathbf{x}_0,\mathbf{k}) = |\chi_\Omega(\mathbf{x}_0 \mp c_0\hat{\mathbf{k}}t)|^2\hat{f}(\mathbf{k})$ become increasingly singular in $\mathbf{k}$ as time grows.

Application to diffusive media (formal theory)

Assume random fluctuations of the density and compressibility of the form $\rho_\varepsilon(\mathbf{x}) = \rho_0 + \sqrt{\varepsilon}\rho_1(\frac{\mathbf{x}}{\varepsilon})$, $\kappa_\varepsilon(\mathbf{x}) = \kappa_0 + \sqrt{\varepsilon}\kappa_1(\frac{\mathbf{x}}{\varepsilon})$, with $\rho_1(\mathbf{x})$, $\kappa_1(\mathbf{x})$ mean-zero stationary random processes. In the limit of large distances of propagation we can show (formally) that

$$\boxed{\hat{\mathbf{u}}^B(\mathbf{k};\mathbf{x}_0) = \widehat{F}(t,\mathbf{k};\mathbf{x}_0)\hat{\mathbf{S}}(\mathbf{k})} \quad \Big(\text{ or } \ \mathbf{u}^B(\cdot;\mathbf{x}_0) = F(t,\cdot;\mathbf{x}_0) \star \mathbf{S}(\cdot)\Big),$$

where $\widehat{F}(t,\mathbf{k};\mathbf{x}_0) = \psi(t,\mathbf{x}_0,k)\widehat{f}(k)I_4$, with $k = |\mathbf{k}|$ and

$$\partial_t\psi(t,\mathbf{x},k) - D(k)\Delta_{\mathbf{x}}\psi(t,\mathbf{x},k) = 0, \qquad \psi(0,\mathbf{x},k) = |\chi_\Omega(\mathbf{x})|^2.$$

Here, $D(k)$ is a diffusion coefficient that depends on the power spectrum of the random fluctuations. Qualitatively, as t grows, ψ becomes un-localized, so that F is localized. Refocusing is greatly improved by the presence of the random medium.

Robustness of Time Reversal

The refocusing is extremely sensitive to modifications in the "random" medium. It is however very robust when other operations than time reversal are performed at the receivers.

Let us assume that the usual time reversal operation represented by $\Gamma_0 = \text{Diag}(-1,-1,-1,1)$ is replaced by multiplication by an (almost) arbitrary $\Gamma(\mathbf{x})$. The initial conditions for the Wigner transform are then

$$W(0,\mathbf{x},\mathbf{k}) = |\chi(\mathbf{x})|^2\Gamma(\mathbf{x})\Gamma_0 A^{-1}(\mathbf{x})\hat{f}(\mathbf{k}).$$

The rest of the theory stays unchanged.

Robustness of Time Reversal (II)

The initial conditions for the acoustic modes are then

$$a_\pm(0,\mathbf{x},\mathbf{k}) = |\chi(\mathbf{x})|^2 \widehat{f}(\mathbf{k})\Big(A(\mathbf{x})\Gamma(\mathbf{x})\mathbf{b}_\mp(\mathbf{x},\mathbf{k})\cdot\mathbf{b}_\pm(\mathbf{x},\mathbf{k})\Big).$$

When $\Gamma(\mathbf{x}) = \Gamma_0$ we get back full time reversal results. When $\Gamma = Id$, we obtain that $a_\pm(0,\mathbf{x},\mathbf{k}) = 0$ by orthogonality of the eigenvectors $\mathbf{b}_j$. When only pressure is measured, $\Gamma = \mathrm{Diag}(0,0,0,1)$, we obtain

$$a_\pm(0,\mathbf{x},\mathbf{k}) = \frac{1}{2}|\chi(\mathbf{x})|^2 \widehat{f}(\mathbf{k}).$$

When only the first component of the velocity field is measured with $\Gamma = \mathrm{Diag}(-1,0,0,0)$, the initial data is

$$a_\pm(0,\mathbf{x},\mathbf{k}) = |\chi(\mathbf{x})|^2 \widehat{f}(\mathbf{k})\frac{k_1^2}{2|\mathbf{k}|^2}.$$

Time Reversal in Changing Environment

Consider the simpler case of a Schrödinger equation in the weak coupling regime

$$\frac{\partial \psi_\varepsilon}{\partial t}(t,\mathbf{x}) = \frac{i\varepsilon}{2}\Delta\psi_\varepsilon(t,\mathbf{x}) - \frac{i}{\sqrt{\varepsilon}}V_j(\frac{\mathbf{x}}{\varepsilon})\psi_\varepsilon(t,\mathbf{x}).$$

$j = 1$ corresponds to the medium during the forward propagation and $j = 2$ to the medium during the backward propagation after time reversal. $V_{1,2}$ play the same role as $\kappa_{1,2}$. They are mean-zero stationary random processes such that

$$R_{mn}(\mathbf{x}) = \langle V_m(\mathbf{y})V_n(\mathbf{y}+\mathbf{x})\rangle, \qquad m,n = 1,2.$$

The Fourier Transforms $\hat{R}_{mn}(\mathbf{k})$ are the power spectra.

Wigner Transform

The Wigner transform is defined by

$$W_\varepsilon(t,\mathbf{x},\mathbf{k}) = \int_{\mathbb{R}^3} e^{i\mathbf{k}\cdot\mathbf{y}} \psi_{f\varepsilon}(t,\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2})\psi^*_{b\varepsilon}(t,\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2})\frac{d\mathbf{y}}{(2\pi)^3}.$$

$\psi_{f\varepsilon}$ and $\psi_{b\varepsilon}$ solve the Schrödinger with forward potential $V = V_1$ and backward potential $V = V_2$.

W_ε solves the following equation:

$$\frac{\partial W_\varepsilon}{\partial t} + \mathbf{k}\cdot\nabla W_\varepsilon = \int_{\mathbb{R}^3} K_\varepsilon(\mathbf{x},\mathbf{k}-\mathbf{p})W_\varepsilon(t,\mathbf{x},\mathbf{p})d\mathbf{p},$$

$$K_\varepsilon(\mathbf{x},\mathbf{p}) = \frac{1}{i\pi^3\sqrt{\varepsilon}}\Big(\hat{V}_1(2\mathbf{p})e^{i2\mathbf{p}\cdot\mathbf{x}/\varepsilon} - \hat{V}_2(-2\mathbf{p})e^{-i2\mathbf{p}\cdot\mathbf{x}/\varepsilon}\Big).$$

Asymptotic limit for W_ε (I)

Inverting the free transport operator $\partial_t + \mathbf{k}\cdot\nabla$ we obtain that

$$W_\varepsilon(t,\mathbf{x},\mathbf{k}) = W_\varepsilon(0,\mathbf{x}-t\mathbf{k},\mathbf{k}) + \int_0^t \int K_\varepsilon(\mathbf{x}-s\mathbf{k},\mathbf{k}-\mathbf{p}) W_\varepsilon(t-s,\mathbf{x}-s\mathbf{k},\mathbf{p}) d\mathbf{p} ds.$$

After one more iteration we have

$$\begin{aligned}
&W_\varepsilon(t,\mathbf{x},\mathbf{k}) = W_\varepsilon(0,\mathbf{x}-t\mathbf{k},\mathbf{k})\\
&+ \int_0^t \int K_\varepsilon(\mathbf{x}-s\mathbf{k},\mathbf{k}-\mathbf{p}) W_\varepsilon(0,\mathbf{x}-s\mathbf{k}-(t-s)\mathbf{p},\mathbf{p}) d\mathbf{p} ds\\
&+ \int_0^t \int K_\varepsilon(\mathbf{x}-s\mathbf{k},\mathbf{k}-\mathbf{p}) \int_0^{t-s} \int K_\varepsilon(\mathbf{x}-s\mathbf{k}-u\mathbf{p},\mathbf{p}-\mathbf{q})\\
&\qquad \times W_\varepsilon(t-s-u,\mathbf{x}-s\mathbf{k}-u\mathbf{p},\mathbf{q}) d\mathbf{q} du d\mathbf{p} ds.
\end{aligned}$$

Asymptotic limit for W_ε (II)

Assume that $\langle K_\varepsilon \otimes K_\varepsilon W_\varepsilon \rangle = \langle K_\varepsilon \otimes K_\varepsilon \rangle \langle W_\varepsilon \rangle$ and that W_ε is sufficiently smooth. Such assumptions cannot be justified at this level although they are known to provide the correct results!

Using that $\langle \hat{V}_m(\mathbf{p}) \hat{V}_n(\mathbf{q}) \rangle = (2\pi)^3 \hat{R}_{mn}(\mathbf{p}) \delta(\mathbf{p}+\mathbf{q})$, $m, n = 1, 2$, after Fourier transforms, we deduce that

$$\begin{aligned}\langle K_\varepsilon(\mathbf{y}, \mathbf{k}-\mathbf{p}) K_\varepsilon(\mathbf{y}-u\mathbf{p}, \mathbf{p}-\mathbf{q}) \rangle &= \frac{-1}{\pi^3 \varepsilon} \\ \times \Big(\quad & \hat{R}_{11}(2(\mathbf{k}-\mathbf{p})) e^{2i(\mathbf{k}-\mathbf{p})\cdot\mathbf{p}u/\varepsilon} \delta(\mathbf{k}-\mathbf{q}) \\ - \quad & \hat{R}_{12}(2(\mathbf{k}-\mathbf{p})) e^{2i(\mathbf{k}-\mathbf{p})\cdot\mathbf{p}u/\varepsilon} \delta(\mathbf{k}+\mathbf{q}-2\mathbf{p}) \\ - \quad & \hat{R}_{21}(-2(\mathbf{k}-\mathbf{p})) e^{-2i(\mathbf{k}-\mathbf{p})\cdot\mathbf{p}u/\varepsilon} \delta(\mathbf{k}+\mathbf{q}-2\mathbf{p}) \\ + \quad & \hat{R}_{22}(2(\mathbf{k}-\mathbf{p})) e^{-2i(\mathbf{k}-\mathbf{p})\cdot\mathbf{p}u/\varepsilon} \delta(\mathbf{k}-\mathbf{p}) \quad \Big).\end{aligned}$$

Asymptotic limit for W_ε (III)

The power spectrum $\hat{R}_{mn}$ is a 2×2 positive definite matrix such that $\hat{R}_{mn}(-\mathbf{p}) = \hat{R}_{nm}(\mathbf{p})$, $m, n = 1, 2$. After the changes of variables $2\mathbf{p}-\mathbf{k} \to \mathbf{p}$ and $u \to \varepsilon u$ and replacing $W_\varepsilon(t-s-\varepsilon u, \mathbf{x}-s\mathbf{k}-\varepsilon u\mathbf{p}, \mathbf{q})$ by $W_\varepsilon(t-s, \mathbf{x}-s\mathbf{k}, \mathbf{q})$ we deduce that the ensemble average of the scattering term is approximated by

$$\int_0^t \int \int_0^{(t-s)/\varepsilon}$$
$$\times\Big(- e^{iu\frac{|\mathbf{k}|^2-|\mathbf{p}|^2}{2}} R_{11}(\mathbf{p}-\mathbf{k}) - e^{-iu\frac{|\mathbf{k}|^2-|\mathbf{p}|^2}{2}} R_{22}(\mathbf{p}-\mathbf{k})\Big)\langle W_\varepsilon\rangle(t-s, \mathbf{x}-s\mathbf{k}, \mathbf{k})$$
$$+\Big(e^{iu\frac{|\mathbf{k}|^2-|\mathbf{p}|^2}{2}} + e^{-iu\frac{|\mathbf{k}|^2-|\mathbf{p}|^2}{2}}\Big) R_{21}(\mathbf{p}-\mathbf{k})\langle W_\varepsilon\rangle(t-s, \mathbf{x}-s\mathbf{k}, \mathbf{p})\frac{du\,d\mathbf{p}\,ds}{(2\pi)^3}.$$

Asymptotic limit for W_ε (IV)

Pass to the limit $\varepsilon \to 0$ and replace $\langle W_\varepsilon \rangle$ by its limit W using

$$\int_0^\infty e^{\pm iu\omega} du = \pi\delta(\omega) \pm \frac{i}{\omega}.$$

We find the transport equation in integral form

$$W(t,\mathbf{x},\mathbf{k}) = W(0,\mathbf{x}-t\mathbf{k},\mathbf{k}) + \int_0^t$$

$$\Big(\int \hat{R}_{21}(\mathbf{p}-\mathbf{k}) W(t-s,\mathbf{x}-s\mathbf{k},\mathbf{p}) \delta(\frac{|\mathbf{k}|^2 - |\mathbf{p}|^2}{2}) \frac{d\mathbf{p}}{(2\pi)^2}$$

$$-(\Sigma(\mathbf{k}) + i\Pi(\mathbf{k})) W(t-s,\mathbf{x}-s\mathbf{k},\mathbf{k}) \Big) ds,$$

$$\Sigma(\mathbf{k}) = \int \frac{\hat{R}_{11}(\mathbf{p}-\mathbf{k}) + \hat{R}_{22}(\mathbf{p}-\mathbf{k})}{2} \delta(\frac{|\mathbf{k}|^2 - |\mathbf{p}|^2}{2}) \frac{d\mathbf{p}}{(2\pi)^2}$$

$$\Pi(\mathbf{k}) = -\!\!\!\!\!\!\int \Big(\hat{R}_{11}(\mathbf{p}-\mathbf{k}) - \hat{R}_{22}(\mathbf{p}-\mathbf{k}) \Big) \frac{2}{|\mathbf{k}|^2 - |\mathbf{p}|^2} \frac{d\mathbf{p}}{(2\pi)^3}.$$

Asymptotic limit for W_ε (V)

Assume that $\hat{V}_1(\mathbf{k}) = \hat{V}(\mathbf{k})$ and $\hat{V}_2(\mathbf{k}) = \phi(\mathbf{k})\hat{V}_1(\mathbf{k})$, where $\phi(\mathbf{k})$ is deterministic. We have the Radiative Transfer Equation

$$\frac{\partial W}{\partial t} + \mathbf{k}\cdot\nabla W + (\sigma_a(\mathbf{k}) + i\Pi(\mathbf{k}))W = QW,$$

$$QW(t,\mathbf{x},\mathbf{k}) = \int \hat{R}(\mathbf{p}-\mathbf{k})\phi(\mathbf{p}-\mathbf{k})\Big[W(t,\mathbf{x},\mathbf{p}) - W(t,\mathbf{x},\mathbf{k})\Big]\delta(\frac{|\mathbf{k}|^2-|\mathbf{p}|^2}{2})\frac{d\mathbf{p}}{(2\pi)^2},$$

$$\sigma_a(\mathbf{k}) = \int \hat{R}(\mathbf{p}-\mathbf{k})\Big(\frac{1+|\phi(\mathbf{p}-\mathbf{k})|^2}{2} - \phi(\mathbf{p}-\mathbf{k})\Big)\delta(\frac{|\mathbf{k}|^2-|\mathbf{p}|^2}{2})\frac{d\mathbf{p}}{(2\pi)^2},$$

$$\Pi(\mathbf{k}) = ⨍ \hat{R}(\mathbf{p}-\mathbf{k})\Big[1-|\phi(\mathbf{p}-\mathbf{k})|^2\Big]\frac{2}{|\mathbf{k}|^2-|\mathbf{p}|^2}\frac{d\mathbf{p}}{(2\pi)^3}.$$

Back to acoustics

Replacing V_j by κ_j (with ρ constant) we get that the propagating modes satisfy the RTE (Derivation still formal and much more difficult)

$$\frac{\partial a_\pm}{\partial t} \pm c_0 \hat{\mathbf{k}} \cdot \nabla a_\pm + \Big(\sigma_a(\mathbf{k}) \pm i\Pi(\mathbf{k})\Big) a_\pm = Q a_\pm,$$
$$a_\pm(0,\mathbf{x},\mathbf{k}) = |\chi(\mathbf{x})|^2$$

$$Qa(\mathbf{k}) = \int_{\mathbb{R}^3} \sigma(\mathbf{k},\mathbf{p})\phi(\mathbf{x},\mathbf{p}-\mathbf{k})\Big(a(\mathbf{p}) - a(\mathbf{k})\Big)\delta(c_0(|\mathbf{k}|-|\mathbf{p}|))d\mathbf{p}$$
$$\Pi(\mathbf{k}) = \fint_{\mathbb{R}^3} (1-|\phi(\mathbf{x},\mathbf{p}-\mathbf{k})|^2)\frac{c_0}{2}\frac{|\mathbf{k}||\mathbf{p}|^2}{|\mathbf{k}|^2-|\mathbf{p}|^2}\frac{\hat{R}(\mathbf{k}-\mathbf{p})}{(2\pi)^3}d\mathbf{p}$$
$$\sigma_a(\mathbf{k}) = \int_{\mathbb{R}^3} \sigma(\mathbf{k},\mathbf{p})\Big(\frac{1+|\phi(\mathbf{x},\mathbf{p}-\mathbf{k})|^2}{2} - \phi(\mathbf{x},\mathbf{p}-\mathbf{k})\Big)\delta(c_0(|\mathbf{k}|-|\mathbf{p}|))d\mathbf{p}.$$

Diffusion Approximation

Classical. Assume $\Sigma = O(\eta^{-1})$, $\sigma_a = O(\eta)$, and $|\phi| = (1 + \eta\psi)$. Use $a = a_0 + \eta a_1 + \eta^2 a_2$, plug Ansatz into transport equation, equate like powers of η and deduce that a_0 solves the following diffusion equation:

$$\frac{\partial a_0}{\partial t} + \frac{\Sigma(|\mathbf{k}|)\psi^2}{2} a_0 - D(|\mathbf{k}|)\Delta a_0 = 0,$$

$$e^{-i\Pi(|\mathbf{k}|)t/\eta^2} a_0(0, \mathbf{x}) = |\chi(\mathbf{x})|^2 \frac{1}{4\pi} \int_{S^2} e^{i\boldsymbol{\tau}\cdot\mathbf{k}} d\hat{\mathbf{k}} = |\chi(\mathbf{x})|^2 \frac{\sin|\boldsymbol{\tau}||\mathbf{k}|}{|\boldsymbol{\tau}||\mathbf{k}|}$$

$$D(|\mathbf{k}|) = \frac{c_0^2}{3[\Sigma(|\mathbf{k}|) - \lambda(|\mathbf{k}|)]},$$

$$\lambda(|\mathbf{k}|)\hat{\mathbf{k}} = \frac{c_0^2|\mathbf{k}|^2}{(4\pi)^2} \int_{\mathbb{R}^3} \hat{R}(\mathbf{p}-\mathbf{k})\hat{\mathbf{p}}\delta(c_0(|\mathbf{k}| - |\mathbf{p}|))d\mathbf{p}.$$

Application to Filters in Time Reversal

The back-propagated signal in the diffusive regime takes the form

$$\hat{\mathbf{u}}^B(\mathbf{k};\mathbf{x}_0) = \left[\begin{pmatrix} \sin(\Pi_s T)\sqrt{\frac{\kappa_0}{\rho}} i\hat{\mathbf{k}} \\ \cos(\Pi_s T) \end{pmatrix} \hat{p}_0(\mathbf{k}) + \begin{pmatrix} \cos(\Pi_s T) i\hat{\mathbf{k}} \\ -\sin(\Pi_s T)\sqrt{\frac{\rho}{\kappa_0}} \end{pmatrix} |\mathbf{k}|\hat{\varphi}(\mathbf{k}) \right]$$
$$\times\ e^{-i\boldsymbol{\tau}\cdot\mathbf{k}}\ \frac{\sin|\boldsymbol{\tau}||\mathbf{k}|}{|\boldsymbol{\tau}||\mathbf{k}|}\ e^{-\Sigma\psi^2 T/2}\ a(T,\mathbf{x}_0,|\mathbf{k}|).$$

This is to be compared to the case where $\Pi_s = \psi = |\boldsymbol{\tau}| = 0$ when the medium remains the same during the forward and backward propagations. The above formula emphasizes two principal distinct effects that appear when the back-propagation occurs in a different underlying medium.

2D Numerical simulations

In two space dimensions the filter is given by

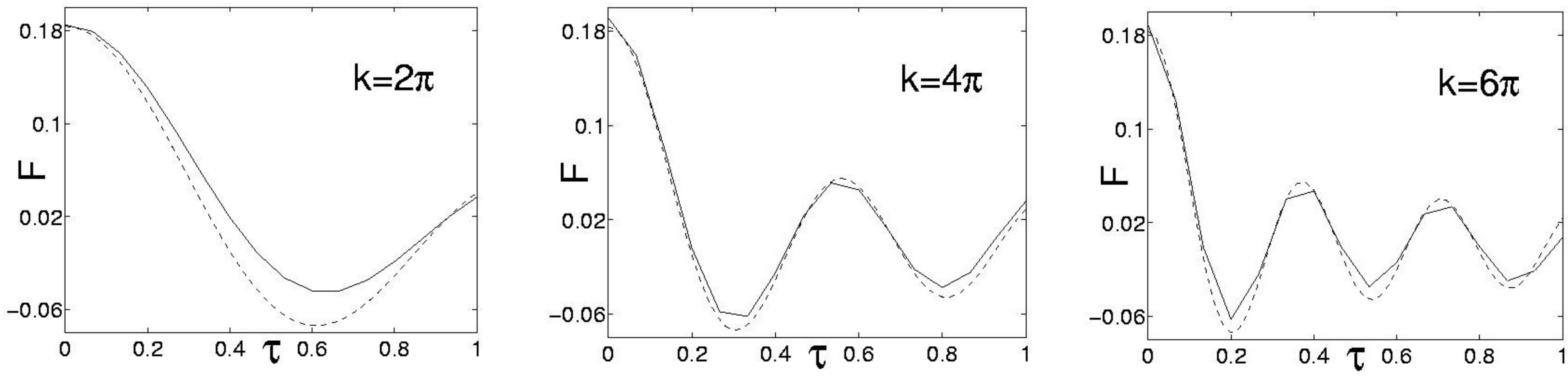
$$F(\psi, |\boldsymbol{\tau}|, |\mathbf{k}|, T, L, k_{\mathsf{max}}, \kappa) = \bar{a}\, J_0(|\boldsymbol{\tau}||\mathbf{k}|)\ \cos(2\psi\Pi_0 T)\, e^{-\frac{\Sigma}{2}\psi^2 T}.$$

It should be compared to the numerical simulation

$$F_{\mathsf{data}} = \frac{(p^B(\mathbf{x}+\boldsymbol{\tau}), p_0(\mathbf{x}))}{\|p_0(\mathbf{x})\|^2}.$$

We consider two simulations with varying $|\boldsymbol{\tau}|$ (shifting medium) and varying ψ (change in fluctuations intensity).

2D Numerical simulations (II)



Comparison of F_{data} (solid lines) and the theoretical prediction F (dashed lines) as a function of τ with $\psi = 0$. Periodic box of size $L = 20$, propagation time $T = 200$, number of modes in power spectrum: 50.

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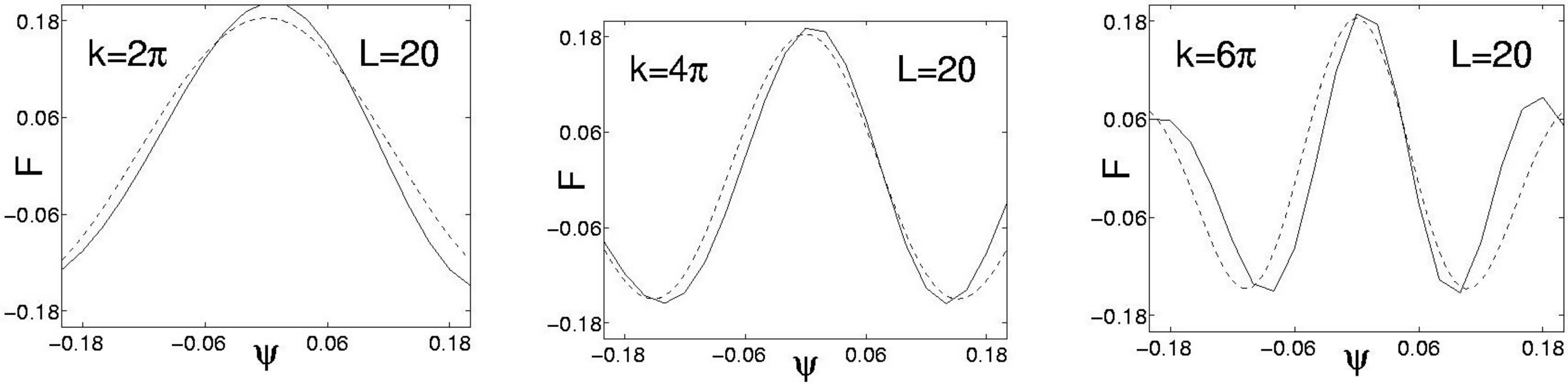


Comparison of F_{data} (solid lines) and the theoretical prediction F (dashed lines) as a function of ψ with $\tau = 0$. Periodic box of size $L = 20$, propagation time $T = 200$, number of modes in power spectrum: 50.

PART II: RIGOROUS THEORIES

Two models where stability can be proved

• Paraxial (a.k.a. Parabolic) Approximation. Here, we obtain a (quantum) wave equation with *mixing* time dependent coefficients. For a typical wavelength (width of initial pulse) of order $\varepsilon \ll 1$, the fluctuations are of the form

$$\sqrt{\varepsilon} V(\frac{\mathbf{x}}{\varepsilon}, \frac{z}{\varepsilon}).$$

• Random Liouville Equations. Here the high frequency limit of the wave equation (Liouville equation) with *random Hamiltonian* is used to show that the Wigner transform solves in the limit $\varepsilon \to 0$ a Fokker-Planck equation. For a typical wavelength of order $\varepsilon \ll 1$, the fluctuations are of the form

$$\sqrt{\delta(\varepsilon)} V(\frac{\mathbf{x}}{\delta(\varepsilon)}), \quad C|\ln \varepsilon|^{-2/3+\eta} \ll \delta(\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0.$$

PART II/1: PARAXIAL APPROXIMATION

Analysis for the Paraxial Equation

The pressure field $p(z, \mathbf{x}, t)$ satisfies the scalar wave equation

$$\frac{1}{c^2(z, \mathbf{x})} \frac{\partial^2 p}{\partial t^2} - \Delta p = 0. \qquad (1)$$

The parabolic approximation consists of

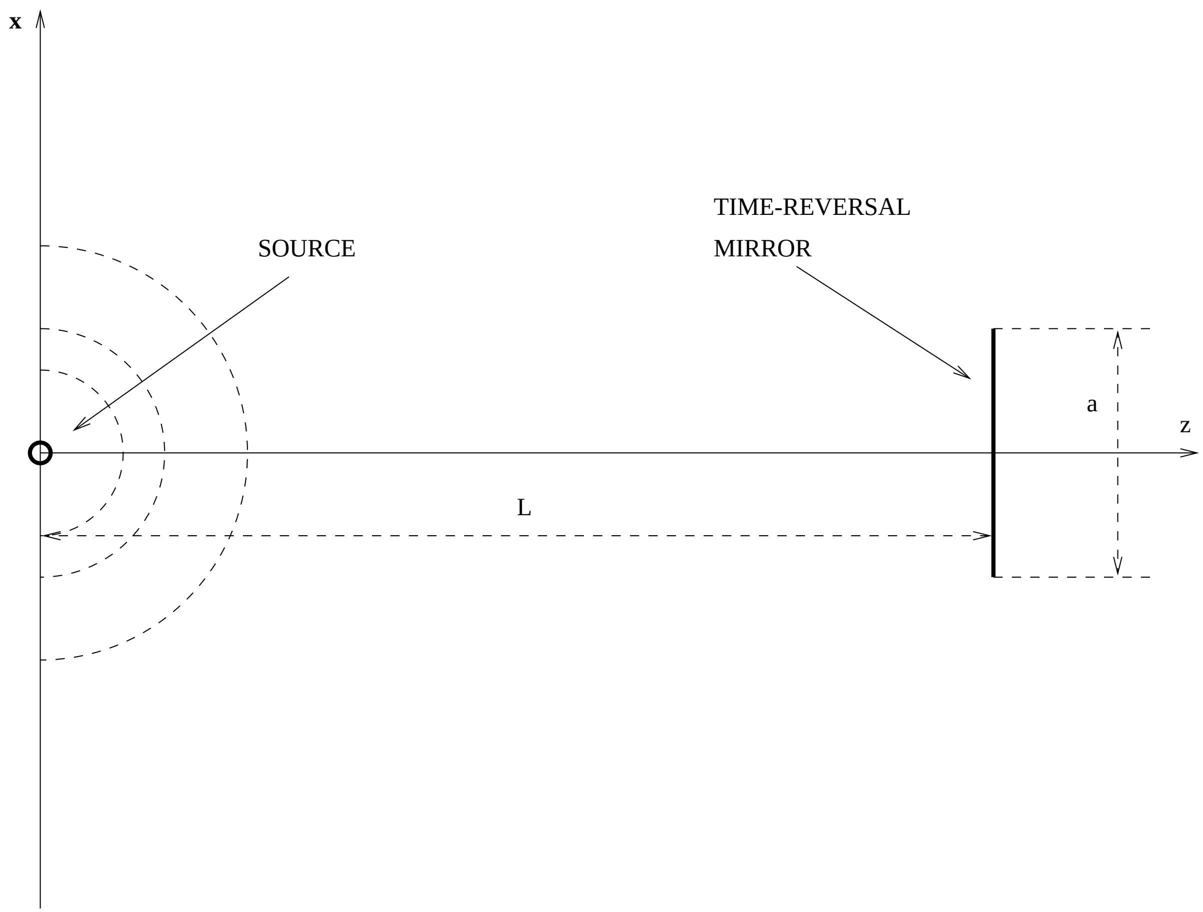
$$p(z, \mathbf{x}, t) \approx \int_{\mathbb{R}} e^{i(-c_0 \kappa t + \kappa z)} \psi(z, \mathbf{x}, \kappa) c_0 d\kappa,$$

where ψ satisfies the Schrödinger equation

$$2i\kappa \frac{\partial \psi}{\partial z}(z, \mathbf{x}, \kappa) + \Delta_{\mathbf{x}} \psi(z, \mathbf{x}, \kappa) + \kappa^2 (n^2(z, \mathbf{x}) - 1) \psi(z, \mathbf{x}, \kappa) = 0,$$
$$\psi(z = 0, \mathbf{x}, \kappa) = \psi_0(\mathbf{x}, \kappa)$$

with $\Delta_{\mathbf{x}}$ the transverse Laplacian in the variable $\mathbf{x}$. The refraction index $n(z, \mathbf{x}) = c_0 / c(z, \mathbf{x})$, and c_0 is a reference speed.

Cartoon of Paraxial Approximation



Time Reversal within Paraxial Approximation

The back-propagated signal can be written as

$$\psi^B(\mathbf{x},\kappa) = \int_{\mathbb{R}^{3d}} G^*(L,\mathbf{x},\kappa;\boldsymbol{\eta})G(L,\mathbf{y},\kappa;\mathbf{y}')\chi(\boldsymbol{\eta})\chi(\mathbf{y})f(\boldsymbol{\eta}-\mathbf{y})\psi_0(\mathbf{y}',\kappa)d\mathbf{y}d\mathbf{y}'d\boldsymbol{\eta}.$$

After introduction of the Wigner Transform and scaling, we get

$$\psi^B_\varepsilon(\boldsymbol{\xi},\kappa;\mathbf{x}_0) = \int_{\mathbb{R}^{2d}} e^{i\mathbf{k}\cdot(\boldsymbol{\xi}-\mathbf{y})} W_\varepsilon(L,\mathbf{x}_0+\varepsilon\frac{\mathbf{y}+\boldsymbol{\xi}}{2},\mathbf{k},\kappa)\psi_0(\mathbf{y},\kappa)\frac{d\mathbf{y}d\mathbf{k}}{(2\pi)^d}.$$

The above formula shows that the asymptotic behavior of $\psi^B_\varepsilon(\boldsymbol{\xi},\kappa;\mathbf{x}_0)$ as $\varepsilon\to 0$ is characterized by that of the Wigner transform $W_\varepsilon(L,\mathbf{x},\mathbf{k},\kappa)$.

Scaling and random medium

The scaled Schrödinger equation is

$$2i\kappa\varepsilon\frac{\partial\psi_\varepsilon}{\partial z}+\varepsilon^2\Delta_{\mathbf{x}}\psi_\varepsilon+\kappa^2\sqrt{\varepsilon}V(\frac{\mathbf{x}}{\varepsilon},\frac{z}{\varepsilon})\psi_\varepsilon=0,$$
$$\psi_\varepsilon(z=0,\mathbf{x},\kappa)=\psi_0(\mathbf{x},\kappa).$$

The random field $V(z,\mathbf{x})$ is a Markov process in z with infinitesimal generator Q. It is stationary in z and $\mathbf{x}$ with correlation function $R(z,\mathbf{x})$

$$\mathbb{E}\left\{V(s,\mathbf{y})V(z+s,\mathbf{x}+\mathbf{y})\right\}=R(z,\mathbf{x})\quad\text{for all } \mathbf{x},\mathbf{y}\in\mathbb{R}^d,\text{ and } z,s\in\mathbb{R}.$$

The generator Q is a bounded operator on $L^\infty(\mathcal{V})$ with a unique invariant measure $\pi(\hat{V})$, i.e. $Q^*\pi=0$, and there exists $\alpha>0$ such that if $\langle g,\pi\rangle=0$ then

$$\|e^{rQ}g\|_{L^\infty_{\mathcal{V}}}\leq C\|g\|_{L^\infty_{\mathcal{V}}}e^{-\alpha r}.$$

Equation for the Wigner Transform

$$\frac{\partial W_\varepsilon}{\partial z} + \frac{1}{\kappa}\mathbf{k}\cdot\nabla_{\mathbf{x}} W_\varepsilon = \kappa \mathcal{L}_\varepsilon W_\varepsilon$$
$$W_\varepsilon(0,\mathbf{x},\mathbf{k};\kappa) = W_\varepsilon^0(\mathbf{x},\mathbf{k};\kappa),$$

$$\mathcal{L}_\varepsilon W_\varepsilon = \frac{1}{i\sqrt{\varepsilon}}\int_{\mathbb{R}^d} \frac{d\tilde{V}(\frac{z}{\varepsilon},\mathbf{p})}{(2\pi)^d} e^{i\mathbf{p}\cdot\mathbf{x}/\varepsilon}\left[W_\varepsilon(\mathbf{x},\mathbf{k}-\frac{\mathbf{p}}{2}) - W_\varepsilon(\mathbf{x},\mathbf{k}+\frac{\mathbf{p}}{2})\right].$$

The initial condition is given by

$$W_\varepsilon^0(\mathbf{x},\mathbf{k};\kappa) = \int_{\mathbb{R}^d} \frac{e^{i(\mathbf{k}+\mathbf{q})\cdot\mathbf{y}}}{(2\pi)^d}\chi(\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2})\chi(\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2})\widehat{f}(\mathbf{q})d\mathbf{y}d\mathbf{q}.$$

It is uniformly bounded in $L^2(\mathbb{R}^d\times\mathbb{R}^d)$ (hence so is $W_\varepsilon(z;\kappa)$) and converges as $\varepsilon\to 0$ to $W^0(\mathbf{x},\mathbf{k};\kappa) = |\chi(\mathbf{x})|^2\widehat{f}(\mathbf{k})$.

Main stability result

Let the array $\chi(\mathbf{y})$ and the *filter* $f(\mathbf{y})$ *be in* $L^1 \cap L^\infty(\mathbb{R}^d)$, while $\psi_0 \in L^2(\mathbb{R}^d)$ for a given $\kappa \in \mathbb{R}$. The refraction index $n(z, \mathbf{x})$ satisfies assumptions given above. Then for each $\boldsymbol{\xi} \in \mathbb{R}^d$ the back-propagated signal $\psi_\varepsilon^B(\boldsymbol{\xi}, \mathbf{x}_0, \kappa)$ converges in probability and weakly in $L^2_{\mathbf{x}_0}(\mathbb{R}^d)$ as $\varepsilon \to 0$ to the deterministic

$$\psi^B(\boldsymbol{\xi}, \kappa; \mathbf{x}_0) = \int_{\mathbb{R}^{2d}} e^{i\mathbf{k}\cdot(\boldsymbol{\xi}-\mathbf{y})} \overline{W}(L, \mathbf{x}_0, \mathbf{k}, \kappa) \psi_0(\mathbf{y}, \kappa) \frac{d\mathbf{y}\, d\mathbf{k}}{(2\pi)^d}.$$

The function $\overline{W}$ satisfies the transport equation

$$\frac{\partial \overline{W}}{\partial z} + \frac{1}{\kappa} \mathbf{k} \cdot \nabla_{\mathbf{x}} \overline{W} = \kappa \mathcal{L} \overline{W},$$

with initial data $\overline{W}_0(\mathbf{x}, \mathbf{k}) = \hat{f}(\mathbf{k}) |\chi(\mathbf{x})|^2$ and operator $\mathcal{L}$ defined by

$$\mathcal{L}\lambda = \int_{\mathbb{R}^d} \frac{d\mathbf{p}}{(2\pi)^d} \hat{R}\left(\frac{|\mathbf{p}|^2 - |\mathbf{k}|^2}{2}, \mathbf{p} - \mathbf{k}\right)(\lambda(\mathbf{p}) - \lambda(\mathbf{k})),$$

where $\hat{R}(\omega, \mathbf{p})$ is the Fourier transform of the correlation function of V.

Result on the Wigner transform

Under the same assumptions, the Wigner distribution W_ε converges *in probability and weakly* in $L^2(\mathbb{R}^{2d})$ to the solution $\overline{W}$ of the above transport equation. More precisely, for any test function $\lambda \in L^2(\mathbb{R}^{2d})$ the process $\langle W_\varepsilon(z), \lambda\rangle$ converges to $\langle \overline{W}(z), \lambda\rangle$ in probability as $\varepsilon \to 0$, uniformly on finite intervals $0 \leq z \leq L$.

Here, $\langle \cdot, \cdot \rangle$ is the usual scalar product in $L^2(\mathbb{R}^{2d})$.

Details of the proofs

The scaling of the random fluctuations is supposed to be $\sqrt{\varepsilon}V(\frac{\mathbf{x}}{\varepsilon},\frac{z}{\varepsilon})$.

We then have the following equation for the scaled W_ε:

$$\frac{\partial W_\varepsilon}{\partial z}+\mathbf{k}\cdot\nabla_{\mathbf{x}}W_\varepsilon=\mathcal{L}_\varepsilon W_\varepsilon$$
$$W_\varepsilon(0,\mathbf{x},\mathbf{k})=W_\varepsilon^0(\mathbf{x},\mathbf{k}),$$

with

$$\mathcal{L}_\varepsilon W_\varepsilon=\frac{1}{i\sqrt{\varepsilon}}\int_{\mathbb{R}^d}\frac{d\tilde{V}(\frac{z}{\varepsilon},\mathbf{p})}{(2\pi)^d}e^{i\mathbf{p}\cdot\mathbf{x}/\varepsilon}\Big[W_\varepsilon(\mathbf{x},\mathbf{k}-\frac{\mathbf{p}}{2})-W_\varepsilon(\mathbf{x},\mathbf{k}+\frac{\mathbf{p}}{2})\Big].$$

Thanks to the blurring at the detectors, we obtain uniform bounds in L^2 for the Wigner transform W_ε independently of the realization of the random medium.

Construction of approximate martingales

Let us define P_ε as the probability measure on the space of paths $C([0,L];X)$ generated by V_ε and W_ε. Let $\lambda(z,\mathbf{x},\mathbf{k})$ be a deterministic test function. We use the Markovian property of the random field $V(z,\mathbf{x})$ in z to construct a first functional $G_\lambda : C([0,L];X) \to C[0,L]$ by

$$G_\lambda[W](z) = \langle W, \lambda\rangle(z) - \int_0^z \langle W, \frac{\partial \lambda}{\partial z} + \mathbf{k}\cdot\nabla_{\mathbf{x}}\lambda + \mathcal{L}\lambda\rangle(\zeta)d\zeta$$

and show that it is an approximate P_ε-martingale, more precisely

$$\left|\mathbb{E}^{P_\varepsilon}\{G_\lambda[W](z)|\mathcal{F}_s\} - G_\lambda[W](s)\right| \leq C_{\lambda,L}\sqrt{\varepsilon}$$

uniformly for all $W \in C([0,L];X)$ and $0 \leq s < z \leq L$. Then there exists a subsequence $\varepsilon_j \to 0$ so that P_{ε_j} converges weakly to a measure P supported on $C([0,L];X)$. Weak convergence of P_ε and the above error estimate together imply that $G_\lambda[W](z)$ is a P-martingale so that

$$\mathbb{E}^P\{G_\lambda[W](z)|\mathcal{F}_s\} - G_\lambda[W](s) = 0.$$

Taking $s = 0$ above we obtain the transport equation for $\overline{W} = \mathbb{E}^{P}\{W(z)\}$ in its weak formulation.

The second step is to show that for every test function $\lambda(z, \mathbf{x}, \mathbf{k})$ the new functional

$$G_{2,\lambda}[W](z) = \langle W, \lambda\rangle^2(z) - 2\int_0^z \langle W, \lambda\rangle(\zeta)\langle W, \frac{\partial \lambda}{\partial z} + \mathbf{k}\cdot\nabla_{\mathbf{x}}\lambda + \mathcal{L}\lambda\rangle(\zeta)d\zeta$$

is also an approximate P_ε-martingale. We then obtain that $\mathbb{E}^{P_\varepsilon}\left\{\langle W, \lambda\rangle^2\right\} \to \langle\overline{W}, \lambda\rangle^2$, which implies convergence in probability. It follows that the limit measure P is unique and deterministic, and that the whole sequence P_ε converges.

That $G_{2,\lambda}[W](z)$ is an approximate P_ε-martingale uses very explicitly the uniform a priori L^2 bound on the Wigner distribution W_ε.

PART II/2: ITO SCHRÖDINGER APPROXIMATION

Itô Schrödinger equations

Let us come back to the parabolic approximation

$$\frac{\partial\psi}{\partial z}+\frac{-iL_z}{2kL_x^2}\Delta_{\mathbf{x}}\psi=\frac{ikL_z\nu}{2}\mu(\frac{L_x\mathbf{x}}{l_x},\frac{L_z z}{l_z})\psi.$$

We now assume that the variations in z are very fast: $l_z \ll \lambda$. Then we can formally replace

$$\frac{kL_z\nu}{2}\mu(\frac{L_x\mathbf{x}}{l_x},\frac{L_z z}{l_z})dz \qquad \text{by} \qquad \kappa B(\frac{L_x\mathbf{x}}{l_x},dz),$$

where $B(\mathbf{x},dz)$ is the usual Wiener measure in z with statistics

$$\langle B(\mathbf{x},z)B(\mathbf{y},z')\rangle = Q(\mathbf{y}-\mathbf{x})z\wedge z'.$$

Itô Schrödinger equation

The parabolic equation in this regime becomes then

$$d\psi(\mathbf{x},z) = \frac{iL_z}{2kL_x^2}\Delta_{\mathbf{x}}\psi(\mathbf{x},z)dz + i\kappa\psi(\mathbf{x},z)\circ B(\frac{L_x\mathbf{x}}{l_x},dz).$$

Here $\circ$ means that the stochastic equation is understood in the Stratonovich sense. In the Itô sense it becomes the Itô-Schrödinger equation:

$$d\psi(\mathbf{x},z) = \frac{1}{2}\Big(\frac{iL_z}{kL_x^2}\Delta_{\mathbf{x}} - \kappa^2 Q(\mathbf{0})\Big)\psi(\mathbf{x},z)dz + i\kappa\psi(\mathbf{x},z)B(\frac{L_x\mathbf{x}}{l_x},dz).$$

Advantage: Closed equations for the statistical moments.

First moment

The first moment defined by $m_1(\mathbf{x}, z) = \langle \psi(\mathbf{x}, z) \rangle$ satisfies

$$\frac{\partial m_1}{\partial z}(\mathbf{x}, z) = \frac{1}{2}\Big(\frac{iL_z}{kL_x^2}\Delta_{\mathbf{x}} - Q(\mathbf{0})\Big)m_1(\mathbf{x}, z).$$

The L^2 norm of the first moment

$$M_2(z) = \Big(\int_{\mathbb{R}^d} |m_1(\mathbf{x}, z)|^2 d\mathbf{x}\Big)^{1/2}.$$

is given by

$$M_2(z) = e^{-\frac{Q(\mathbf{0})}{2}z} M_2(0).$$

This shows that the coherent field m_1 decays exponentially in z. This exponential decay is *not* related to intrinsic absorption. Instead it describes the loss of coherence caused by multiple scattering.

Second Moment (I)

Energy propagation is better understood by looking at the second moment

$$\tilde{m}_2(\mathbf{x}_1, \mathbf{x}_2, z) = \langle \psi(\mathbf{x}_1, z)\psi^*(\mathbf{x}_2, z)\rangle.$$

By application of the Itô formula we have

$$\begin{aligned} d(\psi(\mathbf{x}_1, z)\psi^*(\mathbf{x}_2, z)) &= \psi(\mathbf{x}_1, z)d\psi^*(\mathbf{x}_2, z) \\ &\quad + d\psi(\mathbf{x}_1, z)\psi^*(\mathbf{x}_2, z) + d\psi(\mathbf{x}_1, z)d\psi^*(\mathbf{x}_2, z). \end{aligned}$$

This implies that

$$\frac{\partial \tilde{m}_2}{\partial z} = \frac{iL_z}{2kL_x^2}(\Delta_{\mathbf{x}_1} - \Delta_{\mathbf{x}_2})\tilde{m}_2 + \left(Q\left(\frac{L_x(\mathbf{x}_1 - \mathbf{x}_2)}{l_x}\right) - Q(\mathbf{0})\right)\tilde{m}_2.$$

Second Moment (II)

Introduce the rescaled variables: $\mathbf{x} = \dfrac{\mathbf{x}_1 + \mathbf{x}_2}{2}$, $\quad \mathbf{y} = \dfrac{\mathbf{x}_1 - \mathbf{x}_2}{\eta}$. Here the adimensionalized wavelength $\varepsilon \ll \eta \ll 1$. Defining $m_2(\mathbf{x}, \mathbf{y}) = \tilde{m}_2(\mathbf{x}_1, \mathbf{x}_2)$ we have

$$\frac{\partial m_2}{\partial z} = \frac{iL_z}{kL_x^2 \eta} \nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{y}} m_2(z) - \Big(Q(\mathbf{0}) - Q(\mathbf{y}) \Big) m_2(z).$$

Introduce the Wigner transform

$$W(\mathbf{x}, \mathbf{p}, z) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\mathbf{p}\cdot\mathbf{y}} \psi(\mathbf{x} - \frac{\eta \mathbf{y}}{2}, z) \psi^*(\mathbf{x} + \frac{\eta \mathbf{y}}{2}, z) d\mathbf{y}.$$

Then $m_2(\mathbf{x}, \mathbf{y}, z) = \int_{\mathbb{R}^d} e^{i\mathbf{p}\cdot\mathbf{y}} \langle W \rangle (\mathbf{x}, \mathbf{p}, z) d\mathbf{p}$ and

$$\frac{\partial \langle W \rangle}{\partial z} + \frac{L_z}{kL_x^2 \eta} \mathbf{p} \cdot \nabla_{\mathbf{x}} \langle W \rangle = \int_{\mathbb{R}^d} \Big[\hat{Q}(\mathbf{p} - \mathbf{p}') - Q(\mathbf{0}) \delta(\mathbf{p} - \mathbf{p}') \Big] \langle W \rangle (\mathbf{p}') d\mathbf{p}'.$$

We thus get an equation for the limiting Wigner transform for free.

Scintillation (moment of order 4)

We can similarly obtain an equation for the fourth moment:

$$\tilde{m}_4(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, z) = \langle \psi(\mathbf{x}_1, z)\psi^*(\mathbf{x}_2, z)\psi(\mathbf{x}_3, z)\psi^*(\mathbf{x}_4, z)\rangle.$$

We introduce the change of variables $m_4(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t}, z) = \tilde{m}_4(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, z)$, where $\mathbf{x} = \frac{\mathbf{x}_1+\mathbf{x}_2}{2}$, $\mathbf{y} = \frac{\mathbf{x}_1-\mathbf{x}_2}{\eta}$, $\boldsymbol{\xi} = \frac{\mathbf{x}_3+\mathbf{x}_4}{2}$, $\mathbf{t} = \frac{\mathbf{x}_3-\mathbf{x}_4}{\eta}$, $\eta = \frac{l_x}{L_x}$. We obtain

$$\frac{\partial m_4}{\partial z} = \frac{iL_z}{kL_x^2\eta}(\nabla_\mathbf{x}\cdot\nabla_\mathbf{y} + \nabla_{\boldsymbol{\xi}}\cdot\nabla_\mathbf{t})m_4(z) - \mathcal{Q}m_4(z),$$

$$\mathcal{Q}(\mathbf{x}, \mathbf{y}, \boldsymbol{\xi}, \mathbf{t}) = \Big(2Q(\mathbf{0}) - Q(\mathbf{y}) - Q(\mathbf{t}) + \sum_{\epsilon_i,\epsilon_j=\pm}\epsilon_i\epsilon_j Q(\frac{\mathbf{x}-\boldsymbol{\xi}}{\eta} + \epsilon_i\mathbf{y} - \epsilon_j\mathbf{t})\Big).$$

Scintillation = second moment for the WT

Define $\mathcal{W}(\mathbf{x},\mathbf{p},\boldsymbol{\xi},\mathbf{q},z) = W(\mathbf{x},\mathbf{p},z)W(\boldsymbol{\xi},\mathbf{q},z)$.
Its statistical average can be related to m_4 and we find that

$$\boxed{\frac{\partial\langle\mathcal{W}\rangle}{\partial z} + \frac{L_z}{kL_x^2\eta}(\mathbf{p}\cdot\nabla_{\mathbf{x}} + \mathbf{q}\cdot\nabla_{\boldsymbol{\xi}})\langle\mathcal{W}\rangle = \mathcal{R}_2\langle\mathcal{W}\rangle + K_{12}\langle\mathcal{W}\rangle}$$

$$K_{12}\mathcal{W} = \int_{\mathbb{R}^d}\hat{Q}(\mathbf{u})e^{i\frac{(\mathbf{x}-\boldsymbol{\xi})\cdot\mathbf{u}}{\eta}}\Big(\mathcal{W}(\mathbf{p}-\tfrac{\mathbf{u}}{2},\mathbf{q}-\tfrac{\mathbf{u}}{2}) + \mathcal{W}(\mathbf{p}+\tfrac{\mathbf{u}}{2},\mathbf{q}+\tfrac{\mathbf{u}}{2})$$
$$-\mathcal{W}(\mathbf{p}-\tfrac{\mathbf{u}}{2},\mathbf{q}+\tfrac{\mathbf{u}}{2}) - \mathcal{W}(\mathbf{p}+\tfrac{\mathbf{u}}{2},\mathbf{q}-\tfrac{\mathbf{u}}{2})\Big)d\mathbf{u}$$

$$K_2\mathcal{W} = \int_{\mathbb{R}^{2d}}\Big[\hat{Q}(\mathbf{p}-\mathbf{p}')\delta(\mathbf{q}-\mathbf{q}') + \hat{Q}(\mathbf{q}-\mathbf{q}')\delta(\mathbf{p}-\mathbf{p}')\Big]\mathcal{W}(\mathbf{p}',\mathbf{q}')d\mathbf{p}'d\mathbf{q}'$$
$$\mathcal{R}_2\mathcal{W} = K_2\mathcal{W} - 2Q(\mathbf{0})\mathcal{W}.$$

When the phase term cancels so that "$|K_{12}\mathcal{W}| \ll 1$", we obtain that

$$\boxed{J_\eta(\mathbf{x},\mathbf{p},\boldsymbol{\xi},\mathbf{q},z) = \langle\mathcal{W}(\mathbf{x},\mathbf{p},\boldsymbol{\xi},\mathbf{q},z)\rangle - \langle W(\mathbf{x},\mathbf{p},z)\rangle\,\langle W(\boldsymbol{\xi},\mathbf{q},z)\rangle},$$

the scintillation function, is small. The energy is then statistically stable.

Smallness of the scintillation function

Theorem. Let us assume that $W_\eta(\mathbf{x},\mathbf{p},0)$ is deterministic and such that

$$\int_{\mathbb{R}^{2d}} |W_\eta(\mathbf{x},\mathbf{p},0)|^2 d\mathbf{x} d\mathbf{p} + \int_{\mathbb{R}^d} \sup_{\mathbf{x}} |W_\eta(\mathbf{x},\mathbf{p},0)|^2 d\mathbf{p} \le C,$$

where C is a constant independent of η. Assume also that the correlation function $Q(\mathbf{x}) \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Then

$$\|J_\eta\|_2(z) \le C\eta^{d/2},$$

uniformly in z on compact intervals.

Weak statistical stability

Theorem. Under the assumptions of the previous theorem and $\lambda \in L^2(\mathbb{R}^{2d})$, we obtain that

$$\Big\langle \Big\{ \Big((W_\eta, \lambda) - (\langle W_\eta \rangle, \lambda) \Big)^2 \Big\} \Big\rangle \leq C\eta^{d/2} \|\lambda\|_2^2.$$

Also (W_η, λ) becomes deterministic in the limit of small values of η as

$$P\Big(\Big| (W_\eta, \lambda) - (\langle W_\eta \rangle, \lambda) \Big| \geq \alpha \Big) \leq \frac{C\eta^{d/2}\|\lambda\|_2^2}{\alpha^2} \to 0 \qquad \text{as } \eta \to 0.$$

The Wigner transform W_η of the stochastic field ψ_η converges weakly and in probability to the deterministic solution $\overline{W}(\mathbf{x}, \mathbf{p}, z)$ of a Radiative Transfer Equation.

Application to Time Reversal

Theorem. Assume that the initial condition $\psi_0(\mathbf{y}) \in L^2(\mathbb{R}^d)$, the filter $f(\mathbf{y}) \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, and the detector amplification $\chi(\mathbf{x})$ is sufficiently smooth. Then $\psi_\eta^B(\boldsymbol{\xi}; \mathbf{x}_0)$ converges weakly and in probability to the deterministic *back-propagated signal*

$$\psi^B(\boldsymbol{\xi}; \mathbf{x}_0) = \int_{\mathbb{R}^d} e^{i\mathbf{k}\cdot\boldsymbol{\xi}} \overline{W}(\mathbf{x}_0, \mathbf{k}, L)\hat{\psi}_0(\mathbf{k})d\mathbf{k},$$

where $\overline{W}(\mathbf{x}_0, \mathbf{k}, L)$ is the solution of a RTE with initial conditions $\overline{W}(\mathbf{x}, \mathbf{k}, 0) = \hat{f}(\mathbf{k})|\chi(\mathbf{x})|^2$. Moreover introducing $\lambda(\boldsymbol{\xi}, \mathbf{x}_0) = \tilde{\lambda}(\mathbf{x}_0)\mu(\boldsymbol{\xi})$ we have the following estimate

$$\left\langle (\psi_\eta^B - \langle \psi_\eta^B \rangle, \lambda)^2 \right\rangle \leq C\eta^d \|\psi_0\|_2^2 \|\lambda\|_2^2 = C\eta^d \|\psi_0\|_2^2 \|\mu\|_2^2 \|\tilde{\lambda}\|_2^2,$$

uniformly in L on compact intervals.

We *do not* have such an estimate for the *parabolic* approximation.

Scintillation may appear and not disappear

Theorem. Assume that $W_\eta(\mathbf{x},\mathbf{p},0) = \delta(\mathbf{x}-\mathbf{x}_0)\delta(\mathbf{p}-\mathbf{p}_0)$ [not physical in Time Reversal]. Then the scintillation function J_η is composed of a singular term of the form (with $Q = Q(\mathbf{0})$):

$$\delta(\mathbf{x}-\boldsymbol{\xi})\delta(\mathbf{p}-\mathbf{q})\Big(\alpha(\mathbf{x},\mathbf{p},z) - e^{-2Qz}\alpha(\mathbf{x}-z\mathbf{p},\mathbf{p},0)\Big)$$

plus other contributions that are mutually singular with respect to this term. Moreover the density $\alpha(\mathbf{x},\mathbf{p},z)$ solves the radiative transfer equation with initial condition $a_0(\mathbf{x},\mathbf{p}) = \delta(\mathbf{x}-\mathbf{x}_0)\delta(\mathbf{p}-\mathbf{p}_0)$:

$$\frac{\partial \alpha}{\partial z} + \mathbf{p}\cdot\nabla_{\mathbf{x}}\alpha + 2Q\alpha = \int_{\mathbb{R}^d} \hat{Q}(\mathbf{u})\Big(\alpha(\mathbf{x},\mathbf{p}+\frac{\mathbf{u}}{2},z) + \alpha(\mathbf{x},\mathbf{p}-\frac{\mathbf{u}}{2},z)\Big)d\mathbf{u}.$$

The total intensity of this scintillation is $(1-e^{-2Qz})$ (so it grows in z though it vanishes at $z=0$).

In this case *Energy* is NOT statistically stable.

PART II/3: RANDOM LIOUVILLE REGIME

Stability by Random Liouville

Let us come back to the full wave equation and introduce $\mathbf{v}_\varepsilon(t,\mathbf{x}) = A_\varepsilon^{1/2}(\mathbf{x})\mathbf{u}_\varepsilon(t,\mathbf{x})$ that satisfies the symmetrized system

$$\frac{\partial \mathbf{v}_\varepsilon}{\partial t} + A_\varepsilon^{-1/2}(\mathbf{x}) D^j \frac{\partial}{\partial x^j}\Big(A_\varepsilon^{-1/2}(\mathbf{x})\mathbf{v}_\varepsilon(\mathbf{x})\Big) = 0.$$

Define $P_\varepsilon(\mathbf{x},\mathbf{k}) = P_0(\mathbf{x},\mathbf{k}) + \varepsilon P_1(\mathbf{x})$, where

$$P_0(\mathbf{x},\mathbf{k}) = iA_\varepsilon^{-\frac{1}{2}}(\mathbf{x}) D^j A_\varepsilon^{-\frac{1}{2}}(\mathbf{x}) k_j = ic_\varepsilon(\mathbf{x}) k_j D^j$$
$$2P_1(\mathbf{x}) = A_\varepsilon^{-\frac{1}{2}}(\mathbf{x}) D^j \frac{\partial}{\partial x_j}\Big(A_\varepsilon^{-\frac{1}{2}}(\mathbf{x})\Big) - \frac{\partial}{\partial x_j}\Big(A_\varepsilon^{-\frac{1}{2}}(\mathbf{x})\Big) D^j A_\varepsilon^{-\frac{1}{2}}(\mathbf{x}).$$

The Wigner transform $W_\varepsilon(t,\mathbf{x},\mathbf{k})$ satisfies the evolution equation

$$\varepsilon\frac{\partial W_\varepsilon}{\partial t} + \mathcal{L}_\varepsilon W_\varepsilon = 0$$
$$\mathcal{L}_\varepsilon f(\mathbf{x},\mathbf{k}) = \int \Big(P_\varepsilon(\mathbf{y},\mathbf{q}) e^{i\phi} f(\mathbf{z},\mathbf{p}) - f(\mathbf{z},\mathbf{p}) e^{-i\phi} P_\varepsilon(\mathbf{y},\mathbf{q})\Big) \frac{d\mathbf{z}d\mathbf{p}d\mathbf{y}d\mathbf{q}}{(\pi\varepsilon)^{2d}},$$
$$\phi(\mathbf{x},\mathbf{z},\mathbf{k},\mathbf{p},\mathbf{y},\mathbf{q}) = \frac{2}{\varepsilon}((\mathbf{p}-\mathbf{k})\cdot\mathbf{y} + (\mathbf{q}-\mathbf{p})\cdot\mathbf{x} + (\mathbf{k}-\mathbf{q})\cdot\mathbf{z}).$$

The Liouville equations

The self-adjoint matrix $-iP_0$ has eigenvalues $\lambda_0 = 0$ of multiplicity $d-1$ and $\lambda^\varepsilon_{1,2}(\mathbf{x},\mathbf{k}) = \pm c_\varepsilon(\mathbf{x})|\mathbf{k}|$ and can be diagonalized as

$$-iP_0(\mathbf{x},\mathbf{k}) = \sum_{q=0}^{2} \lambda^\varepsilon_q(\mathbf{x},\mathbf{k})\Pi_q(\mathbf{x},\mathbf{k}), \qquad \text{where} \qquad \sum_{q=0}^{2} \Pi_q(\mathbf{x},\mathbf{k}) = I.$$

The Liouville approximation to the Wigner transform is given by

$$U_\varepsilon(t,\mathbf{x},\mathbf{k}) = \sum_q u^\varepsilon_q(t,\mathbf{x},\mathbf{k})\Pi_q(\mathbf{k}),$$

where the coefficients u^ε_q solve the Liouville equation

$$\begin{aligned} &\frac{\partial u^\varepsilon_q}{\partial t} + \nabla_{\mathbf{k}}\lambda^\varepsilon_q \cdot \nabla_{\mathbf{x}} u^\varepsilon_q - \nabla_{\mathbf{x}}\lambda^\varepsilon_q \cdot \nabla_{\mathbf{k}} u^\varepsilon_q = 0 \\ &u^\varepsilon_q(0,\mathbf{x},\mathbf{k}) = \mathrm{Tr}\Pi_q W_0(\mathbf{x},\mathbf{k})\Pi_q \end{aligned}$$

Here, the coefficients λ^ε_q depend on $\delta(\varepsilon)$ and W_0 is chosen *independent* of ε.

Approximation of W_ε by Liouville equation

Theorem. Let $\rho_\varepsilon(\mathbf{x}) = \rho_0 + \sqrt{\delta}\rho_1(\frac{\mathbf{x}}{\delta})$ and $\kappa_\varepsilon(\mathbf{x}) = \kappa_0 + \sqrt{\delta}\kappa_1(\frac{\mathbf{x}}{\delta})$, with all terms sufficiently smooth. Then we have

$$\|W_\varepsilon(t,\mathbf{x},\mathbf{k}) - U_\varepsilon(t,\mathbf{x},\mathbf{k})\|_2 \leq C\frac{\varepsilon}{\delta^m}\exp(\frac{Ct}{\delta^{3/2}})\|W_0\|_{H^3} + \|W_\varepsilon^0 - W_0\|_{L^2},$$

for some m independent of ε.

In other words, assuming that W_ε^0 converges strongly to W_0 and that $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ with the constraint $\delta(\varepsilon) \gg |\ln\varepsilon|^{-2/3+\eta}$, then the difference $\|W_\varepsilon(t,\mathbf{x},\mathbf{k}) - U_\varepsilon(t,\mathbf{x},\mathbf{k})\|_{L^2} \to 0$ uniformly on final intervals $t \in (0,T)$.

The convergence is uniform in the realization of the random medium (the statistics of ρ_1 and κ_1 have not been defined yet). So we *safely replace the analysis* of W_ε by that of U_ε, the solution of a Liouville equation with random coefficients.

Analysis of the random Liouville equation

The Liouville equation is of the form

$$\frac{\partial u_\varepsilon}{\partial t} + \Big(c_0 + \sqrt{\delta} c_1(\frac{\mathbf{x}}{\delta})\Big)\hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} u_\varepsilon - \frac{|\mathbf{k}|}{\sqrt{\delta}} \nabla_{\mathbf{x}} c_1(\frac{\mathbf{x}}{\delta}) \cdot \nabla_{\mathbf{k}} u_\varepsilon = 0,$$
$$u_\varepsilon(0, \mathbf{x}, \mathbf{k}) = u_0(\mathbf{x}, \mathbf{k}).$$

Its solution is given by $u_\varepsilon(t, \mathbf{x}, \mathbf{k}) = u_0(\mathbf{X}(t), \mathbf{K}(t))$, where

$$-\frac{d\mathbf{X}}{dt} = \Big(c_0 + \sqrt{\delta} c_1(\frac{\mathbf{X}(t)}{\delta})\Big)\hat{\mathbf{K}}, \qquad \mathbf{X}(0) = \mathbf{x},$$
$$-\frac{d\mathbf{K}}{dt} = -\frac{|\mathbf{K}(t)|}{\sqrt{\delta}} \nabla_{\mathbf{x}} c_1(\frac{\mathbf{X}(t)}{\delta}), \qquad \mathbf{K}(0) = \mathbf{k}.$$

Decorrelation of nearby particles

Let us assume that two particles satisfy the system for $j = 1, 2$,

$$\frac{d\mathbf{X}_j^{(\delta)}(t)}{dt} = \Big(c_0 + \sqrt{\delta}c_1(\frac{\mathbf{X}_j^{(\delta)}(t)}{\delta})\Big)\hat{\mathbf{K}}_j^{(\delta)}(t), \quad \mathbf{X}_j^{(\delta)}(0) = \mathbf{x}_j$$
$$\frac{d\mathbf{K}_j^{(\delta)}(t)}{dt} = \frac{1}{\sqrt{\delta}}\nabla_{\mathbf{x}}c_1(\frac{\mathbf{X}_j^{(\delta)}(t)}{\delta})|\mathbf{K}_j^{(\delta)}(t)|, \quad \mathbf{K}_j^{(\delta)}(0) = \mathbf{k}_j.$$

Under suitable mixing conditions for c_1 and for $\mathbf{k}_1 \neq \mathbf{k}_2$, the laws of the processes $(\mathbf{K}_1^{(\delta)}, \mathbf{X}_1^{(\delta)}, \mathbf{K}_2^{(\delta)}, \mathbf{X}_2^{(\delta)})$ converge weakly as $\delta \to 0$ to the law of $(\mathbf{K}_1, \mathbf{X}_1, \mathbf{K}_2, \mathbf{X}_2)$, where $\mathbf{X}_j(t) = \mathbf{x}_j + c_0 \int_0^t \hat{\mathbf{K}}_j(s)ds, \quad j = 1, 2$, and where $\mathbf{k}_j(\cdot)$, $j = 1, 2$ are independent symmetric diffusions in $\mathbb{R}^d \setminus \{\mathbf{0}\}$ starting at $\mathbf{k}_j$, $j = 1, 2$ correspondingly with common generator

$$\mathcal{L}F(\mathbf{k}) = \sum_{p,q=1}^{d} |\mathbf{k}|^2 D_{p,q}(\hat{\mathbf{k}})\partial^2_{k_p,k_q}F(\mathbf{k}) + \sum_{p=1}^{d} |\mathbf{k}|E_p(\hat{\mathbf{k}})\partial_{k_p}F(\mathbf{k}).$$

Stability of the Wigner Transform

We deduce from the previous result that

$$\mathbb{E}\{u_\varepsilon(t,\mathbf{x},\mathbf{k})\} \to F(t,\mathbf{x},\mathbf{k}) \quad \text{weakly as} \quad \delta(\varepsilon) \to 0,$$

where F satisfies the following Fokker-Planck equation

$$\frac{\partial F}{\partial t} + c_0 \hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} F - \mathcal{L}F = 0.$$

Moreover, we obtain the stability result

$$\mathbb{E}\left\{\int \left|\left\langle u_\varepsilon(T,\mathbf{x}_0,\mathbf{k}) - F(T,\mathbf{x}_0,\mathbf{k}), \lambda(\mathbf{k})\right\rangle\right|^2 d\mathbf{x}_0\right\} \to 0 \quad \text{as} \quad \delta(\varepsilon) \to 0,$$

which implies that u_ε converges in probability to the deterministic solution F. This in turn implies the stability of the refocused signal $\mathbf{u}^B$.

Conclusions

• We have a theory to express the high frequency limit of the refocused signal in Time Reversal experiments using a Wigner transform. In the scalar case, this expression is

$$\hat{u}^B(\mathbf{p};\mathbf{x}_0) = W(T,\mathbf{x}_0,\mathbf{p})\hat{S}(\mathbf{p};\mathbf{x}_0).$$

The filter can also be generalized to changing environments.

• In certain cases, we can rigorously characterize the high frequency limit of the Wigner transform and if possible (and true) obtain its stability. This has been done for the parabolic approximation and the Itô Schrödinger approximation, and in the random Liouville regime, where high frequency waves are approximated by particles propagating in random media.

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