Inverse Transport Problems and Applications

I. Problems in Integral Geometry

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Outline for the three lectures

I. Inverse problems in integral geometry

Radon transform and attenuated Radon transform

Ray transforms in hyperbolic geometry

II. Forward and Inverse problems in highly scattering media

Photon scattering in tissues within diffusion approximation

Inverse problems in Optical tomography

III. Inverse transport problems

Singular expansion of albedo operator

Perturbations about "scattering-free" problems

Unsolved practical inverse problems.

Outline for Lecture I

1. X-ray tomography and Radon transform

Radon transform as a transport source problem

2. SPECT and Attenuated Radon transform

Complexification of the transport equation

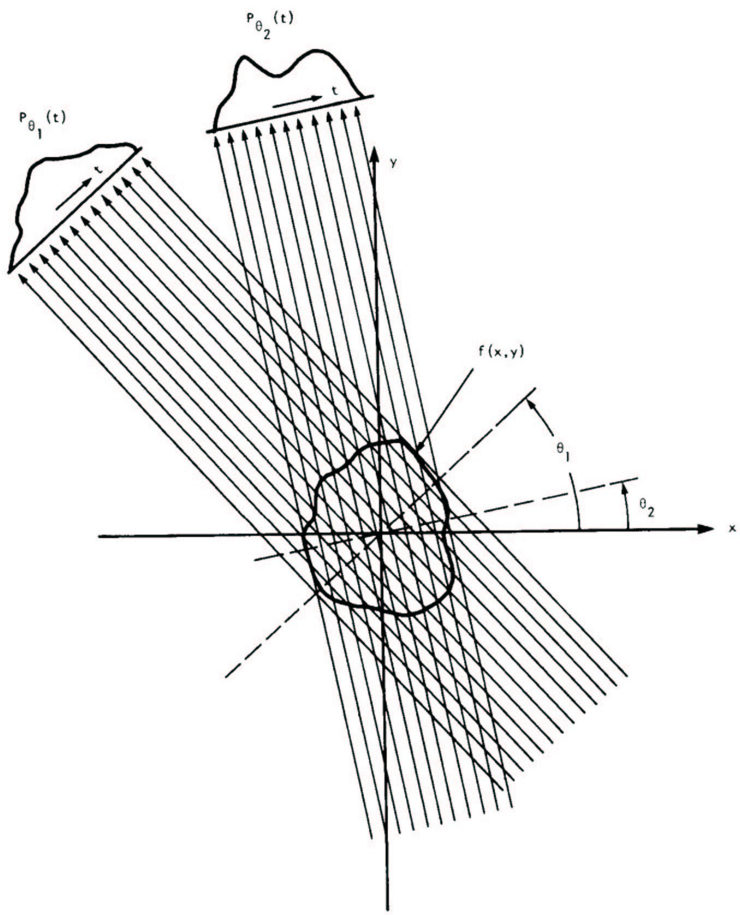
Explicit inversion formula (à la Novikov)

3. Source problem in geophysical imaging and hyperbolic geometry

Application in geophysical imaging

Complexification of geodesic vector field in hyperbolic geometry

Data Acquisition in CT-scan



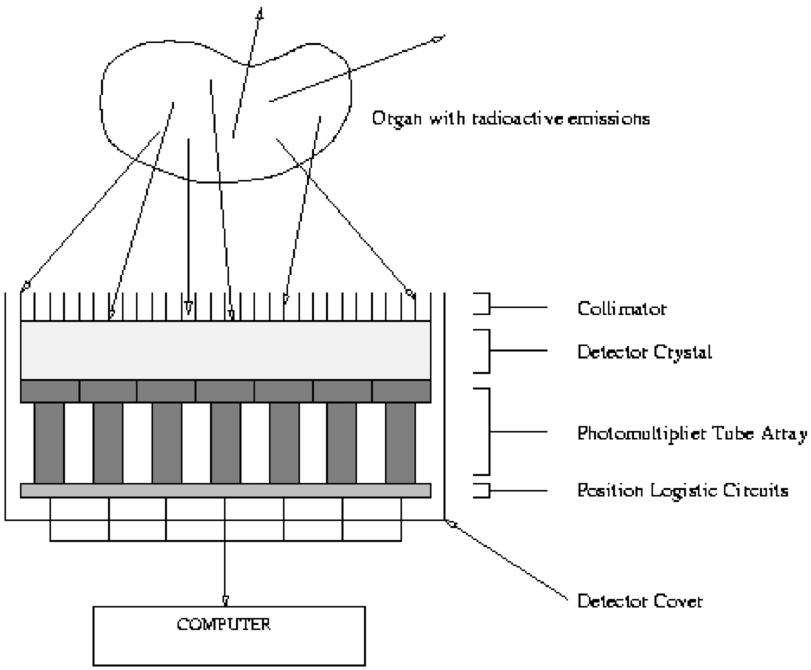
$a(\mathbf{x})$ is the unknown absorption coefficient. For each line in the plane (full measurements), the measured ratio $u_{\text{out}}(s,\theta)/u_{\text{in}}(s,\theta)$ is equal to $\exp\left(-\int_{\text{line(s},\theta)} a(\mathbf{x})dl\right)$. s =line-offset.

The X-ray density $u(\mathbf{x},\theta)$ solves the transport equation

$$\boldsymbol{\theta}\cdot\nabla u(\mathbf{x},\theta)+a(\mathbf{x})u(\mathbf{x},\theta)=0.$$

Here, $\mathbf{x}$ is position and $\boldsymbol{\theta} = (\cos\theta,\sin\theta)$ direction.

Radon transform and transport source problem



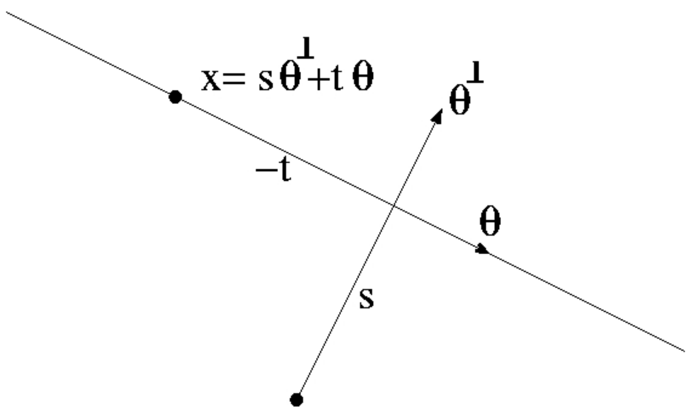
We recast the inverse problem for $a(\mathbf{x})$ as an inverse transport source problem

$$\boldsymbol{\theta} \cdot \nabla u(\mathbf{x}, \theta) = a(\mathbf{x}).$$

This time, the "measurement" $u_{\text{out}}(\mathbf{x}, \theta)$ provides $\int_{\text{line}} a(\mathbf{x}) dl$.

The inverse source problem consists of reconstructing $a(\mathbf{x})$ from all its line integrals, i.e., from the (unattenuated) Radon transform.

Radon transform. Notation and Inversion



We define the Radon transform

$$Ra(s,\theta) = R_\theta a(s) = \int_{\mathbb{R}} a(s\boldsymbol{\theta}^\perp + t\boldsymbol{\theta})dt$$

for $s \in \mathbb{R}$ and $\boldsymbol{\theta} \in S^1$.

Note the redundancy:

$Ra(-s,\theta+\pi) = Ra(s,\theta)$.

Introduce the adjoint operator and the Hilbert transform

$$R^* g(\mathbf{x}) = \int_0^{2\pi} g(\mathbf{x}\cdot\boldsymbol{\theta}^\perp, \theta)d\theta, \qquad Hf(t) = \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{f(s)}{t-s} ds.$$

Then we have (e.g. in the L^2-sense) the reconstruction formula

$$Id = \frac{1}{4\pi} R^* \frac{\partial}{\partial s} HR = \frac{1}{4\pi} R^* H \frac{\partial}{\partial s} R.$$

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2. SPECT and Attenuated Radon transform

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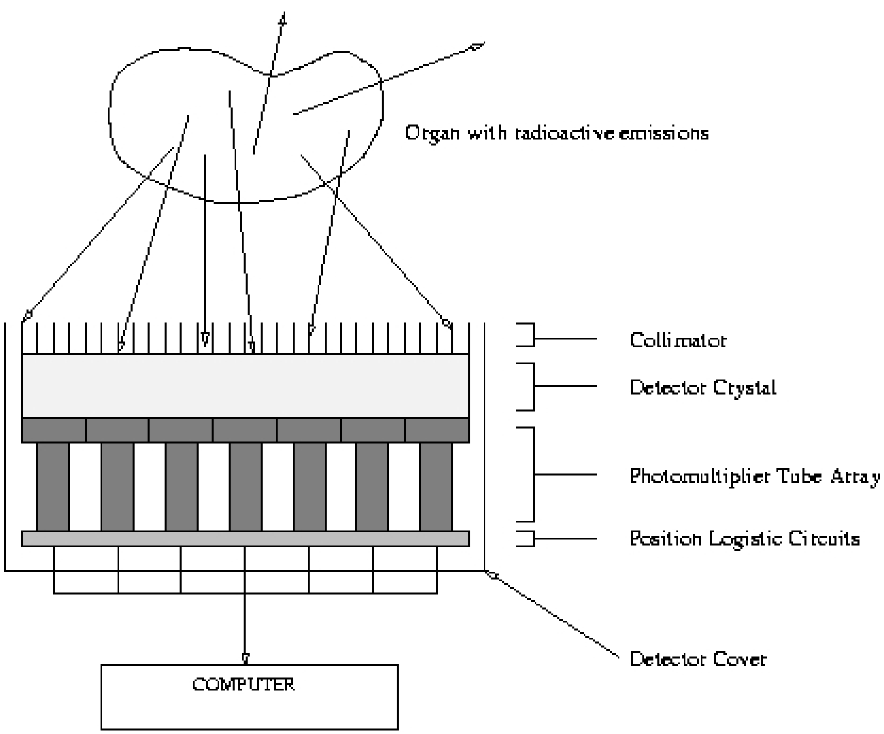
3. Source problem in geophysical imaging and hyperbolic geometry

    Application in geophysical imaging

    Complexification of geodesic vector field in hyperbolic geometry

Data Acquisition in SPECT (Single Photon Emission Computed Tomography)



Here $a(\mathbf{x})$ is a known absorption coefficient and $f(\mathbf{x})$ an unknown source term (radioactive particles are injected into the body and attach differently to different tissues). The $\gamma-$ ray density solves

$$\boldsymbol{\theta} \cdot \nabla u(\mathbf{x}, \theta) + a(\mathbf{x}) u(\mathbf{x}, \theta) = f(\mathbf{x}).$$

The inverse source problem consists of reconstructing $f(\mathbf{x})$ from $u_{\text{out}}(\mathbf{x}, \theta)$.

Doppler tomography

So far, the unknown quantities $a(\mathbf{x})$ or $f(\mathbf{x})$ are scalar quantities. There are applications where vector-valued functions are to be imaged.

For instance in Doppler tomography. Consider a fluid with sound speed $c(\mathbf{x})$ and velocity $\mathbf{v}(\mathbf{x})$. Ultrasounds propagating along the line $l(s,\theta)$ have effective speed $c(\mathbf{x}) + \boldsymbol{\theta}\cdot\mathbf{v}(\mathbf{x})$. Interchanging sources and receivers, we collect the two travel times

$$T_1 = \int_{l(s,\theta)} \frac{ds}{c(\mathbf{x}) + \boldsymbol{\theta}\cdot\mathbf{v}(\mathbf{x})}, \qquad T_2 = \int_{l(s,\theta)} \frac{ds}{c(\mathbf{x}) - \boldsymbol{\theta}\cdot\mathbf{v}(\mathbf{x})}.$$

Assuming $|\mathbf{v}| \ll c$, we thus measure

$$\int_{l(s,\theta)} \frac{ds}{c(\mathbf{x})}, \qquad \int_{l(s,\theta)} \frac{\boldsymbol{\theta}\cdot\mathbf{v}(\mathbf{x})}{c^2(\mathbf{x})}ds = \boldsymbol{\theta}\cdot\int_{l(s,\theta)} \frac{\mathbf{v}(\mathbf{x})}{c^2(\mathbf{x})}ds.$$

We thus reconstruct $c^{-1}(\mathbf{x})$ from its Radon transform and then want to reconstruct a field $\mathbf{F}(\mathbf{x})$ from its vectorial Radon transform $\boldsymbol{\theta}\cdot R\mathbf{F}(s,\theta)$.

Mathematical modeling

The transport equation with anisotropic source term is given by

$$\boldsymbol{\theta} \cdot \nabla \psi(\mathbf{x}, \theta) + a(\mathbf{x})\psi(\mathbf{x}, \theta) = f(\mathbf{x}, \theta) = \sum_{k=-N}^{N} f_k(\mathbf{x}) e^{ik\theta}, \quad \mathbf{x} \in \mathbb{R}^2, \ \boldsymbol{\theta} \in S^1.$$

We identify $\boldsymbol{\theta} = (\cos\theta, \sin\theta) \in S^1$ and $\theta \in (0, 2\pi)$. We assume that $f_{-k} = \overline{f}_k$ and $f_k(\mathbf{x})$ is compactly supported. The boundary conditions are such that for all $\mathbf{x} \in \mathbb{R}^2$,

$$\lim_{s \to +\infty} \psi(\mathbf{x} - s\boldsymbol{\theta}, \theta) = 0.$$

The absorption coefficient $a(\mathbf{x})$ is smooth and decays sufficiently fast at infinity. The above transport solution admits a unique solution and we can define the *symmetrized* beam transform as

$$D_\theta a(\mathbf{x}) = \frac{1}{2} \int_0^\infty [a(\mathbf{x} - t\boldsymbol{\theta}) - a(\mathbf{x} + t\boldsymbol{\theta})] dt.$$

Mathematical modeling (II)

The symmetrized beam transform satisfies $\boldsymbol{\theta} \cdot \nabla D_\theta a(\mathbf{x}) = a(\mathbf{x})$ so that the transport solution is given by

$$e^{D_\theta a(\mathbf{x})}\psi(\mathbf{x},\theta) = \int_0^\infty (e^{D_\theta a} f)(\mathbf{x} - t\boldsymbol{\theta},\theta)dt.$$

Upon defining $\boldsymbol{\theta}^\perp = (-\sin\theta, \cos\theta)$ and $\mathbf{x} = s\boldsymbol{\theta}^\perp + t\boldsymbol{\theta}$, we find that

$$\begin{aligned} \lim_{t\to+\infty} e^{D_\theta a(s\boldsymbol{\theta}^\perp + t\boldsymbol{\theta})}\psi(s\boldsymbol{\theta}^\perp + t\boldsymbol{\theta},\theta) &= \int_{\mathbb{R}} (e^{D_\theta a} f)(s\boldsymbol{\theta}^\perp + t\boldsymbol{\theta},\theta)dt \\ \text{Measurements} = \lim_{t\to+\infty} \psi(s\boldsymbol{\theta}^\perp + t\boldsymbol{\theta},\theta) &= e^{-(R_\theta a)(s)/2}(R_{a,\theta} f)(s), \end{aligned}$$

where R_θ is the Radon transform and $R_{a,\theta}$ the Attenuated Radon Transform (AtRT) defined by:

$$\begin{aligned} R_\theta f(s) &= \int_{\mathbb{R}} f(s\boldsymbol{\theta}^\perp + t\boldsymbol{\theta},\theta)dt = \int_{\mathbb{R}^2} f(\mathbf{x},\theta)\delta(\mathbf{x}\cdot\boldsymbol{\theta}^\perp - s)d\mathbf{x} \\ (R_{a,\theta} f)(s) &= (R_\theta(e^{D_\theta a} f))(s) = \text{Data} \end{aligned}$$

Inverse Problems

The inverse problem consists then in answering the following questions:

1. Knowing the AtRT $R_{a,\theta}f(s)$ for $\boldsymbol{\theta} \in S^1$ and $s \in \mathbb{R}$, and the absorption $a(\mathbf{x})$, what can we reconstruct in $f(\mathbf{x}, \theta)$?

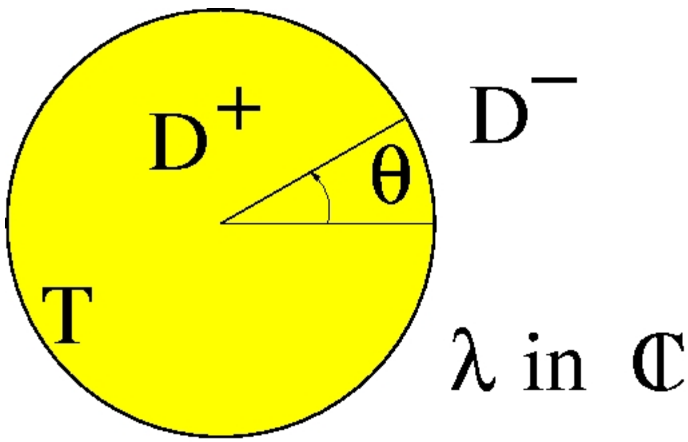
2. Assuming $f(\mathbf{x}, \theta) = f_0(\mathbf{x})$ or $f(\mathbf{x}, \theta) = \mathbf{F}(\mathbf{x}) \cdot \boldsymbol{\theta}$, can we obtain explicit formulas for the source term?

3. Can we reconstruct $f(\mathbf{x}, \theta) = f_0(\mathbf{x})$ from half of the measurements or do we at least have uniqueness of the reconstruction?

4. Do we have a reliable numerical technique to obtain fast reconstructions?

Reconstruction as a Riemann Hilbert problem

We recast the inversion as a Riemann Hilbert (RH) problem. Let us define $T = \{\lambda \in \mathbb{C}, |\lambda| = 1\}$, $D^+ = \{\lambda \in \mathbb{C}, |\lambda| < 1\}$, and $D^- = \{\lambda \in \mathbb{C}, |\lambda| > 1\}$. Let $\varphi(t)$ be a smooth function defined on T. Then there is a unique function $\phi(\lambda)$ such that:



(i) $\phi(\lambda)$ is analytic on D^+ and D^-

(ii) $\lambda\phi(\lambda)$ is bounded at infinity

(iii) $\varphi(t) = \phi^+(t) - \phi^-(t) \equiv [\phi](t)$
$$= \lim_{0<\varepsilon\to 0} (\phi((1-\varepsilon)t) - \phi((1+\varepsilon)t))$$

Moreover $\phi(\lambda)$ is given by the Cauchy formula

$$\phi(\lambda) = \frac{1}{2\pi i}\int_T \frac{\varphi(t)}{t-\lambda}dt, \qquad \lambda \in \mathbb{C}\backslash T.$$

Riemann-Hilbert problem for AtRT, a road map

1. Extend the transport equation to the complex plane (complex-valued directions of propagation $\boldsymbol{\theta} \to e^{i\theta} = \lambda \in \mathbb{C}$). Replace the transport solution $\psi(\mathbf{x}, \lambda)$ by $\phi(\mathbf{x}, \lambda)$, which is analytic on D^+ and D^- and $O(\lambda^{-1})$ at infinity by subtracting a finite number of analytic terms on $\mathbb{C}\backslash\{0\}$.

2. Verify that the jump of $\phi(\mathbf{x}, \lambda)$ at $\lambda \in T$ is a function of the measured data $R_{a,\theta}f(s)$.

3. Read off the constraints imposed on the source terms $f_k(\mathbf{x})$ obtained by Taylor expansion of $\phi(\mathbf{x}, \lambda)$ at $\lambda = 0$.

4. In simplified settings, reconstruct the $f_k(\mathbf{x})$ from the constraints.

Step 1: Complexification of transport equation

Define

$$\lambda = e^{i\theta}, \qquad z = x + iy \text{ with } \mathbf{x} = (x, y), \quad \frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big).$$

The <u>complexified</u> transport equation is then recast as

$$\Big(\lambda\frac{\partial}{\partial z} + \lambda^{-1}\frac{\partial}{\partial \bar{z}} + a(z)\Big)\psi(z,\lambda) = f(z,\lambda).$$

We consider the above equation for arbitrary complex values of λ. $\psi(z,\lambda)$ is analytic on $\lambda \in \mathbb{C}\backslash(T \cup \{0\})$ and is given by

$$\psi(z,\lambda) = e^{-h(z,\lambda)} \int_{\mathbb{C}} G(z-\zeta,\lambda) e^{h(\zeta,\lambda)} f(\zeta,\lambda) dm(\zeta),$$

where $h(z,\lambda) = \int_{\mathbb{C}} G(z-\zeta,\lambda) a(\zeta) dm(\zeta)$ and

$$\Big(\lambda\frac{\partial}{\partial z} + \lambda^{-1}\frac{\partial}{\partial \bar{z}}\Big) G(z,\lambda) = \delta(z), \qquad \text{so that} \qquad G(z,\lambda) = \frac{\text{sign}(|\lambda|-1)}{\pi(\lambda\bar{z} - \lambda^{-1}z)}.$$

The source term is given by $f(z,\lambda)=\sum_{k=-N}^{N} f_k(z)\lambda^k$. On D^+ we have

$$G(z,\lambda)=\frac{1}{\pi z}\sum_{m=0}^{\infty}\left(\frac{\overline{z}}{z}\right)^m \lambda^{2m+1}, \qquad \text{and} \qquad \psi(\cdot,\lambda)=\sum_{m=1}^{\infty}(\mathcal{H}_m f(\cdot,\lambda))\lambda^m,$$

where the operators $\mathcal{H}_m$ are explicitly computable with

$$\mathcal{H}_1=\left(\frac{\partial}{\partial \overline{z}}\right)^{-1}, \quad \mathcal{H}_2=-\mathcal{H}_1 a\mathcal{H}_1, \quad \frac{\partial}{\partial \overline{z}}\mathcal{H}_{k+2}+a\mathcal{H}_{k+1}+\frac{\partial}{\partial z}\mathcal{H}_k=0.$$

Using a similar expression on D^-, we find that

$$\phi(z,\lambda)=\psi(z,\lambda)-\sum_{n=-\infty}^{-1}\lambda^n\sum_{m=1}^{\infty}(\mathcal{H}_m f_{n-m})(z)-\sum_{n=-\infty}^{0}\lambda^{-n}\sum_{m=1}^{\infty}(\overline{\mathcal{H}_m} f_{m-n})(z),$$

satisfies the hypotheses of the RH problem: it is analytic on $D^+\cup D^-$ and of order $O(\lambda^{-1})$ at infinity. Its jump across T is the same as that of ψ since the difference $\psi-\phi$ is analytic in $\mathbb{C}\backslash\{0\}$. On D^+ it is given by

$$\phi(z,\lambda)=\sum_{n=0}^{\infty}\lambda^n\sum_{m=1}^{\infty}(\mathcal{H}_m f_{n-m}-\overline{\mathcal{H}_m} f_{n+m})(z).$$

Simplifications when $f(z,\lambda) = f_0(z)$

When $f(z,\lambda) = f_0(z)$, the solution

$$\psi(z,\lambda) = e^{-h(z,\lambda)} \int_{\mathbb{C}} G(z-\zeta,\lambda) e^{h(\zeta,\lambda)} f_0(\zeta) dm(\zeta),$$

of the complexified transport equation

$$\Big(\lambda \frac{\partial}{\partial z} + \lambda^{-1} \frac{\partial}{\partial \bar{z}} + a(z)\Big)\psi(z,\lambda) = f_0(z),$$

is analytic on $\lambda \in \mathbb{C}\backslash T$ and $\lambda\psi(z,\lambda)$ is bounded at infinity since $G(z,\lambda)$ satisfies these properties. So $\phi(z,\lambda) = \psi(z,\lambda)$ meets the conditions of the Riemann-Hilbert problem (sectionally analytic and right behavior at infinity).

Moreover we find using the transport equation and the Cauchy formula:

$$f_0(z) = \lim_{\lambda\to 0} \frac{1}{\lambda}\frac{\partial}{\partial \bar{z}}\psi(z,\lambda) = \lim_{\lambda\to 0} \frac{1}{\lambda}\frac{\partial}{\partial \bar{z}}\frac{1}{2\pi}\int_T \frac{[\psi](t)}{t-\lambda} dt.$$

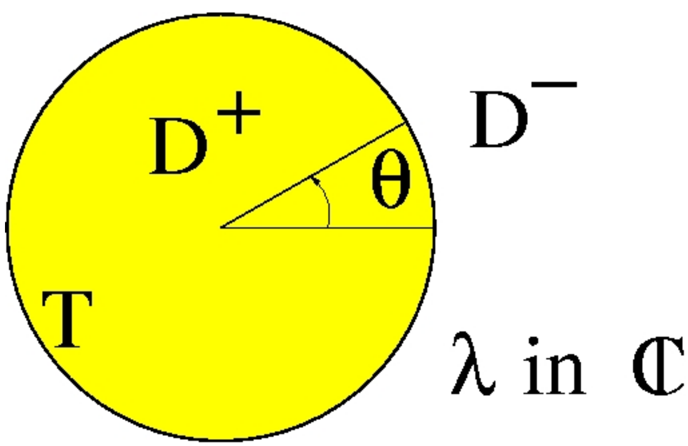
Step 2: jump conditions

Recall that

$$G(z,\lambda) = \frac{\mathrm{sign}(|\lambda|-1)}{\pi(\lambda\bar{z} - \lambda^{-1}z)}, \qquad h(z,\lambda) = \int_{\mathbb{C}} G(z-\zeta,\lambda)a(\zeta)dm(\zeta),$$

and that the transport solution is given by

$$\psi(z,\lambda) = e^{-h(z,\lambda)} \int_{\mathbb{C}} G(z-\zeta,\lambda)e^{h(\zeta,\lambda)} f(\zeta,\lambda)dm(\zeta).$$



Writing $\lambda = re^{i\theta}$

and sending $r-1$ to ± 0,

we calculate $G_{\pm}(z,\theta)$ and $\psi_{\pm}(z,\theta)$.

Step 2: jump conditions (ii)

Writing $\lambda = re^{i\theta}$ and sending $r-1$ to ± 0, we obtain

$$\begin{aligned} G_\pm(\mathbf{x},\theta) &= \frac{\pm 1}{2\pi i(\boldsymbol{\theta}^\perp\cdot\mathbf{x} \mp i0\text{sign}(\boldsymbol{\theta}\cdot\mathbf{x}))}, \\ h_\pm(\mathbf{x},\theta) &= \pm\frac{1}{2i}(HR_\theta a)(\mathbf{x}\cdot\boldsymbol{\theta}^\perp) + (D_\theta a)(\mathbf{x}), \qquad Hu(t) = \frac{1}{\pi}\int_{\mathbb{R}}\frac{u(s)}{t-s}ds. \end{aligned}$$

Here H is the Hilbert transform. We thus obtain that ψ converges on both sides of T parameterized by $\theta \in (0, 2\pi)$ to

$$\begin{aligned} \psi_\pm(\mathbf{x},\theta) = e^{-D_\theta a} e^{\frac{\mp 1}{2i}(HR_\theta a)(\mathbf{x}\cdot\boldsymbol{\theta}^\perp)} \frac{\mp 1}{2i} H\Big(e^{\frac{\pm 1}{2i}(HR_\theta a)(s)}\underline{R_\theta(e^{D_\theta a}f)}\Big)(\mathbf{x}\cdot\boldsymbol{\theta}^\perp) \\ + e^{-D_\theta a} D_\theta(e^{D_\theta a}f)(\mathbf{x}). \end{aligned}$$

Notice that $(\psi_+ - \psi_-)$ is a function of the measurements $R_{a,\theta}f(s) = R_\theta(e^{D_\theta a}f)(s)$ whereas $\psi_\pm$ individually are not.

Jump conditions (ii)

Let us define

$$\varphi(\mathbf{x},\theta) = (\psi^{+} - \psi^{-})(\mathbf{x},\theta).$$

It depends on the measured data and is given by

$$i\varphi(\mathbf{x},\theta) = [R^{*}_{-a,\theta} H_a R_{a,\theta} f](\mathbf{x}) = [R^{*}_{-a,\theta} H_a g(s,\theta)](\mathbf{x}),$$

where

$$R^{*}_{a,\theta} g(\mathbf{x}) = e^{D_\theta a(\mathbf{x})} g(\mathbf{x}\cdot\boldsymbol{\theta}^{\perp}), \qquad H_a = (C_c H C_c + C_s H C_s)$$
$$C_c g(s,\theta) = g(s,\theta)\cos\Big(\frac{HRa(s,\theta)}{2}\Big), \quad C_s g(s,\theta) = g(s,\theta)\sin\Big(\frac{HRa(s,\theta)}{2}\Big).$$

Here $R^{*}_{a,\theta}$ is the adjoint operator to $R_{a,\theta}$. We note that $i\varphi(\mathbf{x},\theta)$ is *real-valued* and that $\boldsymbol{\theta}\cdot\nabla\varphi + a\varphi = 0$.

Step 3: constraints on source terms

The function ϕ is sectionally analytic, of order $O(\lambda^{-1})$ at infinity and such that

$$\varphi(z,\theta) = \phi^+(z,\theta) - \phi^-(z,\theta) \quad \text{on } T.$$

So ϕ is the unique solution to the RH problem given by

$$\phi(z,\lambda) = \frac{1}{2\pi i}\int_T \frac{\varphi(z,t)}{t-\lambda}dt = \sum_{n=0}^{\infty} \lambda^n \frac{1}{2\pi i}\int_T \frac{\varphi(z,t)dt}{t^{n+1}}$$

on D^+ so that

$$\boxed{\sum_{m=1}^{\infty} (\mathcal{H}_m f_{n-m} - \overline{\mathcal{H}_m} f_{n+m})(z) = \frac{1}{2\pi i}\int_T \frac{\varphi(z,t)dt}{t^{n+1}} \equiv \varphi_n(z), \quad n \geq 0.}$$

Because $\frac{\partial}{\partial \bar{z}}\varphi_n + a\varphi_{n+1} + \frac{\partial}{\partial z}\varphi_{n+2} = 0$, there are actually only two independent constraints for $n = 0$ and $n = 1$. This *characterizes the redundancy* of order 2 of the AtRT.

Step 4: reconstruction in simplified setting.

Assume that $N=1$ so that $f(\mathbf{x},\lambda)=f_0(\mathbf{x})+\lambda f_1(\mathbf{x})+\lambda^{-1}f_{-1}(\mathbf{x})$. Then

$$\begin{aligned}\mathcal{H}_1 f_{-1}(z)-\overline{\mathcal{H}_1} f_1(z) &= \frac{1}{2\pi i}\int_T \frac{\varphi(z,t)dt}{t}=\varphi_0(z)\\ \mathcal{H}_2 f_{-1}(z)+\mathcal{H}_1 f_0(z) &= \frac{1}{2\pi i}\int_T \frac{\varphi(z,t)dt}{t^2}=\varphi_1(z).\end{aligned}$$

Define $\boldsymbol{\omega}=(\cos\omega,\sin\omega)\in S^1$ and impose for $\rho_1(z)$ real-valued:

$$f_1(z)=e^{i\omega}\rho_1(z),\qquad f_{-1}(z)=e^{-i\omega}\rho_1(z),$$
$$\text{so that } f_1(z)e^{i\theta}+f_{-1}(z)e^{-i\theta}=2\cos(\theta+\omega)\rho_1(z).$$

Since $\mathcal{H}_1$ is multiplication by $2/(i\xi_z)$ in the Fourier domain, we obtain

$$\boxed{\begin{aligned}f_1(\mathbf{x}) &= \frac{1}{4}D_{\boldsymbol{\omega}_s}\Delta(i\varphi_0)(\mathbf{x}),\qquad \boldsymbol{\omega}_s=(\sin\omega,\cos\omega),\\ f_0(\mathbf{x}) &= \frac{1}{4\pi}\int_0^{2\pi}\boldsymbol{\theta}^{\perp}\cdot\nabla(i\varphi)(\mathbf{x},\theta)d\theta+\frac{1}{2}D_{\boldsymbol{\omega}_s}\boldsymbol{\omega}_s^{\perp}\cdot\nabla(i\varphi_0)(\mathbf{x}).\end{aligned}}$$

When $\varphi_0\equiv 0$ this is the classical Novikov formula.

Step 4: reconstruction in simplest setting.

Assume that $N = 0$ so that $f(\mathbf{x}, \lambda) = f_0(\mathbf{x})$. Then

$$\begin{aligned} 0 &= \frac{1}{2\pi i} \int_T \frac{\varphi(z,t)dt}{t} = \varphi_0(z) \\ \mathcal{H}_1 f_0(z) &= \frac{1}{2\pi i} \int_T \frac{\varphi(z,t)dt}{t^2} = \varphi_1(z). \end{aligned}$$

Recall that $\mathcal{H}_1 = (\frac{\partial}{\partial \bar{z}})^{-1}$. We thus obtain

$$\boxed{\begin{aligned} 0 &= i\varphi_0(\mathbf{x}), \\ f_0(\mathbf{x}) &= \frac{1}{4\pi} \int_0^{2\pi} \boldsymbol{\theta}^{\perp} \cdot \nabla (i\varphi)(\mathbf{x}, \theta) d\theta. \end{aligned}}$$

This is the Novikov formula. The first equality is a compatibility conditions.

Step 4 bis: Application to Doppler tomography.

In Doppler tomography, the source term of interest is of the form

$$f(\mathbf{x},\theta) = \mathbf{F}(\mathbf{x})\cdot\boldsymbol{\theta} \qquad \mathbf{F}(\mathbf{x}) = (F_1(\mathbf{x}), F_2(\mathbf{x})).$$

So we define the source term $f_1(\mathbf{x}) = \frac{1}{2}(F_1(\mathbf{x}) - iF_2(\mathbf{x}))$ and $f_k(\mathbf{x}) \equiv 0$ for $|k| \neq 1$. The constraint $n = 0$ gives

$$\nabla \times \mathbf{F}(\mathbf{x}) = \frac{\partial F_2(\mathbf{x})}{\partial x} - \frac{\partial F_1(\mathbf{x})}{\partial y} = \frac{1}{2}\Delta(i\varphi_0)(\mathbf{x}).$$

The constraint $n = 1$ gives $\mathcal{H}_2 f_{-1}(z) = \varphi_1(z)$ so that

$$\frac{1}{2}\Big(F_1(z) + iF_2(z)\Big) = -\frac{\partial}{\partial \bar{z}}\frac{1}{a(z)}\frac{\partial \varphi_1(z)}{\partial \bar{z}}.$$

This *explicit* reconstruction formula is valid on the support of $a(\mathbf{x})$ and has *no equivalent* when $a \equiv 0$.

Reconstruction from partial measurements

Since we can reconstruct two functions from the AtRT, can we reconstruct one from half of the measurements? The answer is yes and we have an explicit reconstruction scheme under a smallness constraint on the variations of the absorption parameter.

The setting is as follows. We assume that $g(s,\theta)$ is available for all values of $s \in \mathbb{R}$ and for $\theta \in M \subset [0, 2\pi)$. The assumption on M is that $M^c = [0, 2\pi)\backslash M \subset \overline{M + \pi}$; for instance $M = [0, \pi)$ and $M^c = [\pi, 2\pi)$.

We also assume that the source term $f(\mathbf{x})$ is compactly supported in the unit ball B.

The derivation is based on decomposing the explicit reconstruction formula into skew-symmetric and symmetric components in $\mathcal{L}(L^2(B))$.

Reconstruction from partial measurements

Using the full-measurement inversion formula, we can recast the reconstruction problem as

$$f(\mathbf{x}) = d(\mathbf{x}) + F^a f(\mathbf{x}) + F^s f(\mathbf{x}), \qquad d(\mathbf{x}) = F^d f(\mathbf{x}),$$

where F^a is formally skew-symmetric and F^s is formally symmetric.

Theorem 1. The operators F^a and F^s are bounded in $\mathcal{L}(L^2(B))$ and F^s is compact in the same sense with range in $H^{1/2}(B)$.

Theorem 2. Provided that $\rho(F^s) < 1$, we can reconstruct $f(\mathbf{x})$ uniquely and explicitly from $g(s,\theta)$ for $\theta \in M$. Since F^s is compact we can always reconstruct the singular part of $f(\mathbf{x})$ that is not in the Range of F^s.

Theorem 3. [R. Novikov; H. Rullgård] The AtRT $g(s,\theta)$ on $\mathbb{R}\times\mathbf{\Theta}$, where $\mathbf{\Theta} \subset S^1$ has positive measure, uniquely determines $f(\mathbf{x})$. Moreover,

$$\|f\|_{L^2(\mathbb{R}^2)} \leq C\|g(s,\theta)\|_{H^{1/2}(\mathbb{R}\times M)}, \qquad \text{for some } C > 0.$$

Advantage of explicit reconstruction formulas: Fast numerical algorithms such as the slant stack algorithm

Joint work with Philippe Moireau, Ecole Polytechnique.

Let us represent $f(\mathbf{x})$ by an image with $n \times n$ pixels. The objectives are:

- to compute an accurate approximation of $g(s, \theta) = R_{a,\theta} f(s)$
- to compute it fast (with a cost of $O(n^2 \log n)$)
- to invert the AtRT accurately and fast from full or partial measurements.

Implementation of slant stack algorithm (RT)

1. We zero-pad the $n \times n$ image F to obtain the $n \times 2n$ image F^1,

2. We compute a Discrete Fourier Transform (DFT) on the columns,

3. We compute a fractional DFT on the rows,

4. We compute an inverse DFT (IDFT) on the columns.

Each of these operations can be performed in $O(n^2 \log n)$ operations. The discrete transform converges to the exact transform with spectral accuracy. The algorithm is based on a discretization of the Fourier slice theorem $\widehat{Rf}(\sigma, \theta) = \hat{f}(\sigma \boldsymbol{\theta}^\perp)$.

Classical phantom reconstruction: RT data

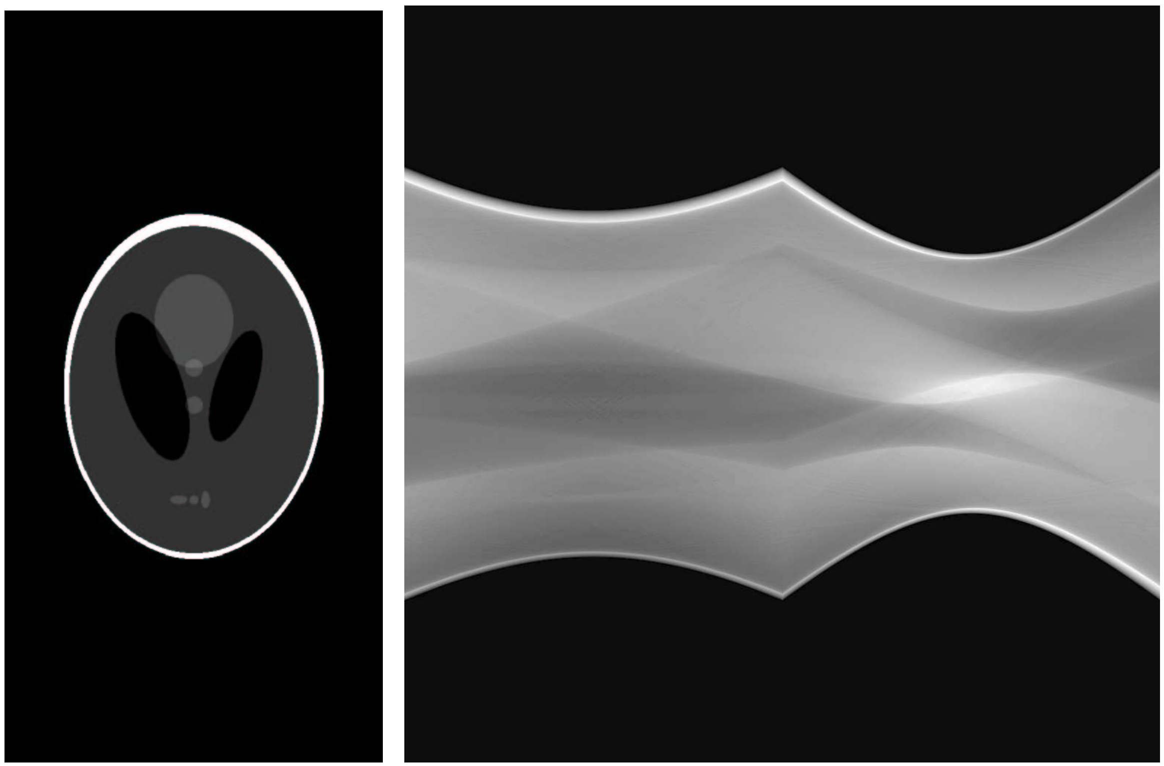
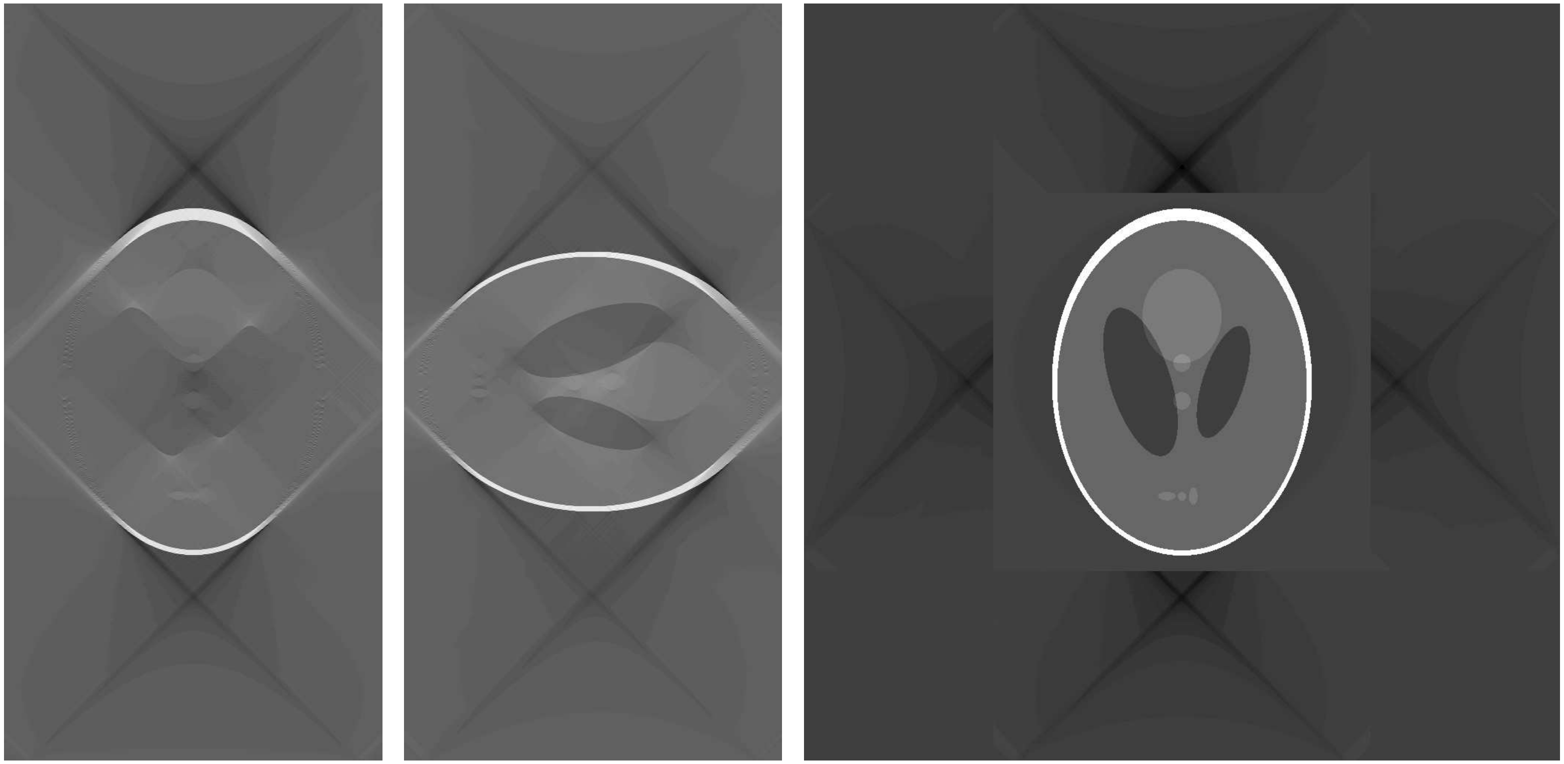


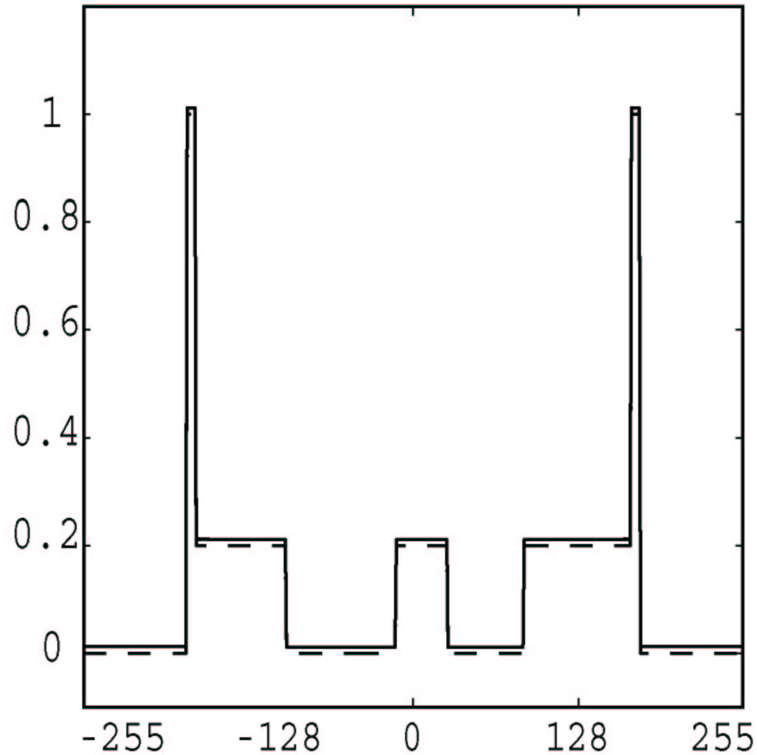
Image on left; slant-stack (lineogram) data on right.

Classical phantom reconstruction



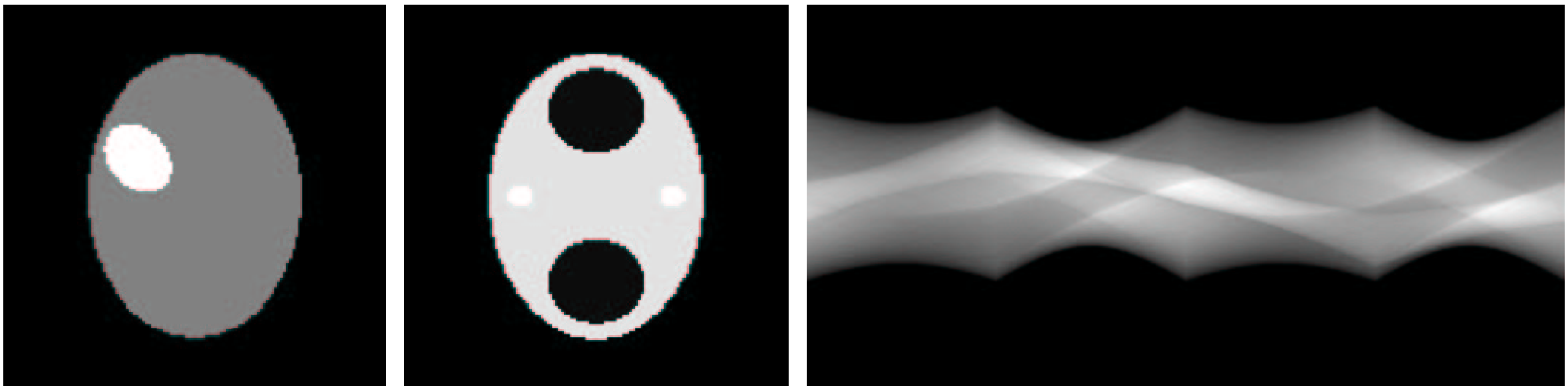
Left 2 pictures: reconstructions from partial data. Right: full reconstruction.

Classical phantom reconstruction (ii)



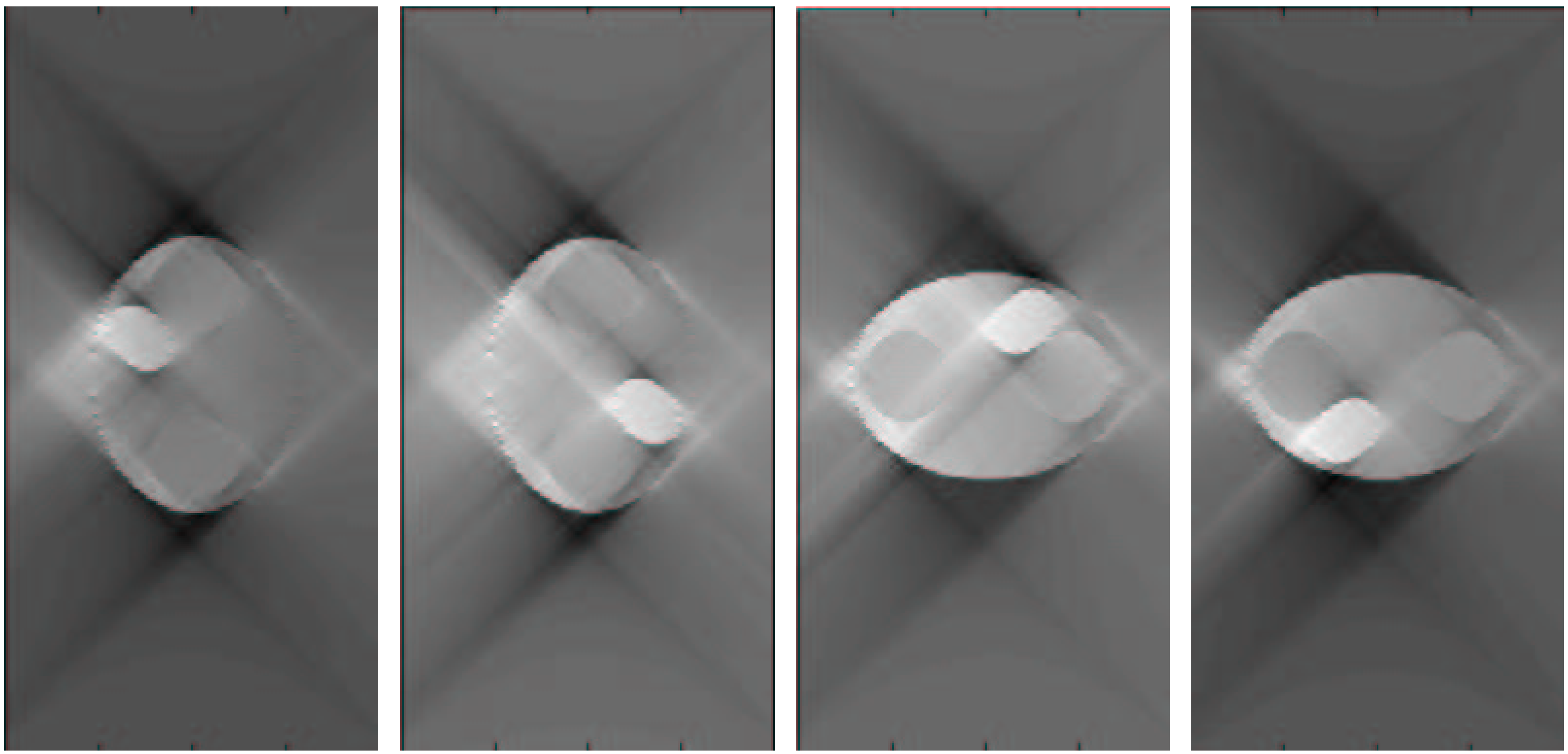
2D reconstruction and 1D cross-section.

Generalization to AtRT data



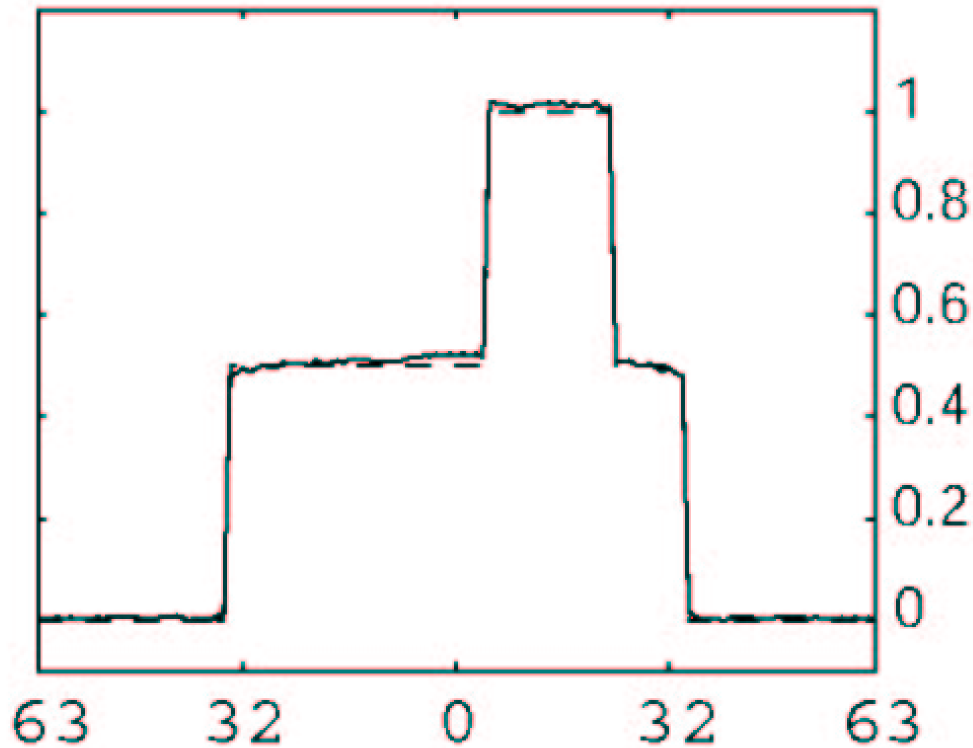
Left: source. Middle: absorption map. Right: AtRT data.

Example of AtRT reconstruction (i)



Reconstruction from $\pi/2$ data.

Example of reconstruction (ii)



Reconstruction from full data and 1D cross section.

Outline for Lecture I

1. X-ray tomography and Radon transform

   Radon transform as a transport source problem

2. SPECT and Attenuated Radon transform

   Complexification of the transport equation

   Explicit inversion formula (à la Novikov)

3. **Source problem in geophysical imaging and hyperbolic geometry**

   Application in geophysical imaging

   Complexification of geodesic vector field in hyperbolic geometry

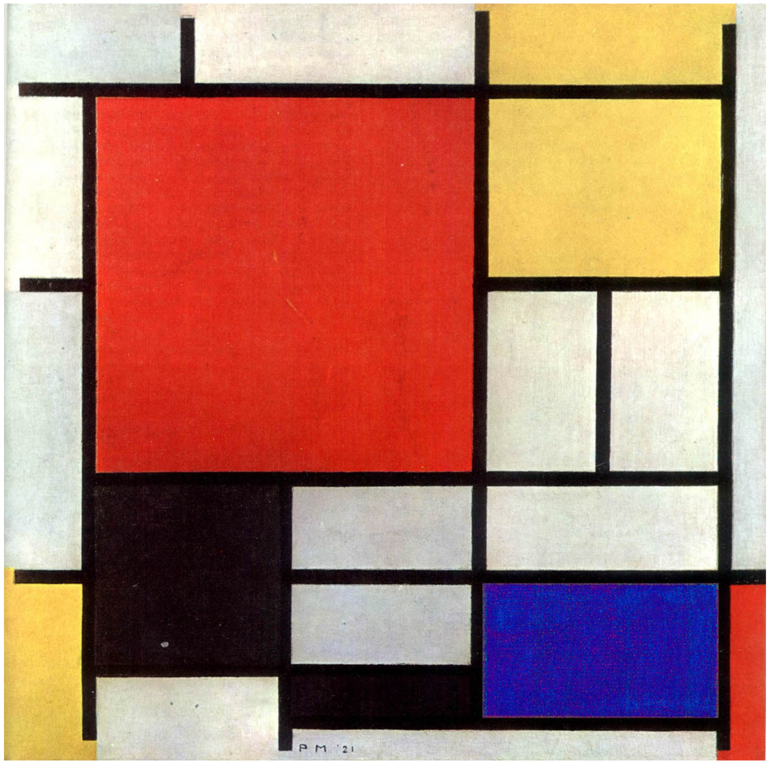
Ray transforms and inverse problems

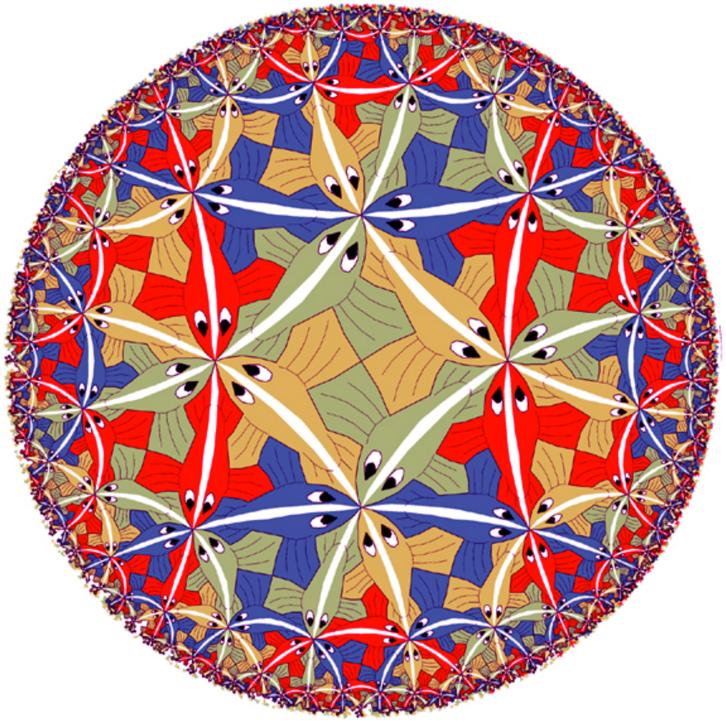
Many inverse problems involve integrations along geodesics.

In medical imaging, the geodesics are often lines: CT-scan (X-ray $\lambda = 0.1nm$), SPECT (gamma ray $159KeV$), PET (2 gamma rays $511KeV$): **Euclidean geometry** is fine.

Earth imaging is mostly based on reconstruction of quantities involving integration along geodesics. However the geodesics are almost always curved: **non-Euclidean geometry**.

Crash course in Dutch geometry





Geophysical imaging in hyperbolic geometry

Assuming that speed increases linearly in the Earth $C(z) \approx z \sim y$, energy propagates along the geodesics of the following Riemannian metric

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2).$$

(This is not a ridiculous assumption.)
Let X be the geodesic vector field and a a *known* absorption parameter. The forward problem is:

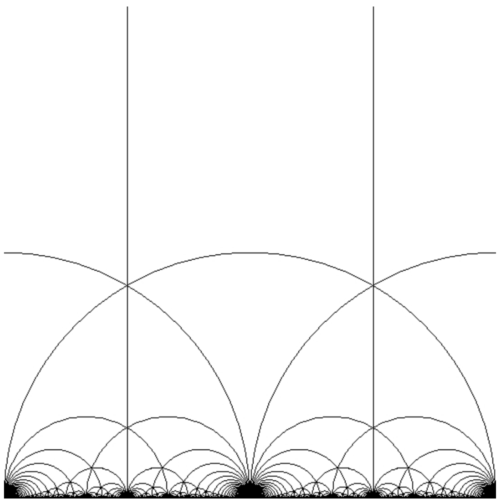
$$Xu + au = f, \quad f \text{ is the source term.}$$

The Inverse Problem is: reconstruct the source term f from boundary measurements of u (emission problem).

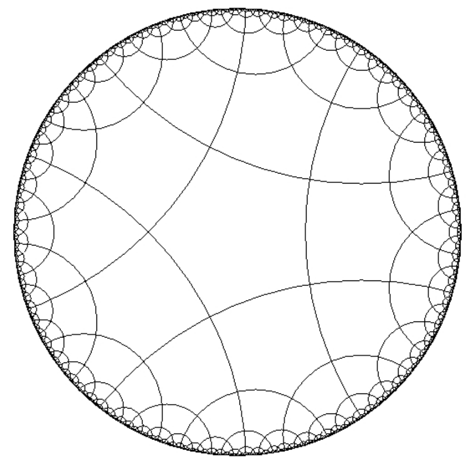
Below is a recently obtained *explicit* inversion formula for this problem.

What's the relationship with Escher?

There are various equivalent ways to look at hyperbolic geometry.



$$\Longleftrightarrow \quad z(w) = \frac{w - i}{1 - iw}$$



Poincaré $\frac{\text{plane}}{2}$; $ds^2 = \frac{dx^2 + dy^2}{y^2} \Longleftrightarrow$ Poincaré disc; $ds^2 = \frac{4dzd\bar{z}}{(1 - |z|^2)^2}$.

Euclidean geometry, summary

The inversion formula obtained earlier was based on the following *complexification*. The unit circle is parameterized as

$$\lambda = e^{i\theta}, \qquad \theta \in (0, 2\pi).$$

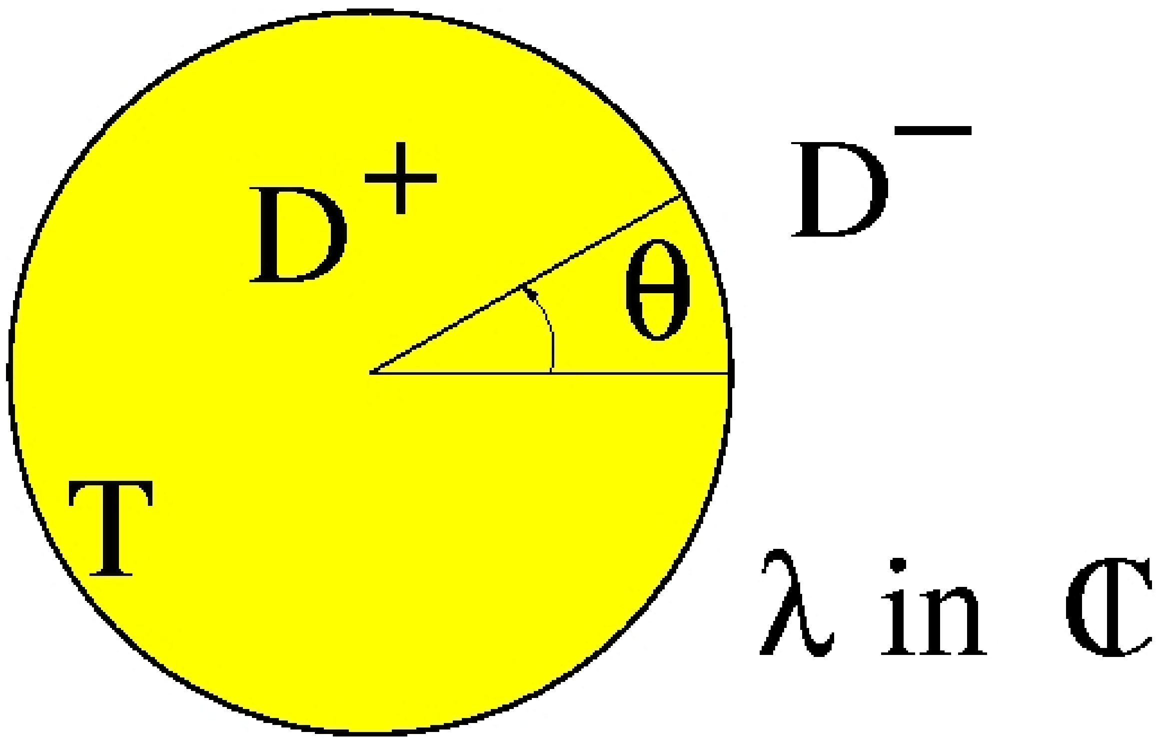
The parameter λ defined on the unit circle T is extended to the whole complex plane $\mathbb{C}$. The transport equation becomes

$$\Big(\lambda\frac{\partial}{\partial z} + \lambda^{-1}\frac{\partial}{\partial \overline{z}} + a(z)\Big)u(z,\lambda) = f(z), \qquad z \in \mathbb{C}, \quad \lambda \in \mathbb{C}.$$

We have used the classical parameterization

$$\frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big), \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big).$$

Geometry of the extension



$\lambda = e^{i\theta}$ is extended to the complex plane $\mathbb{C}$. Position $z \approx \mathbf{x}$ is a *fixed* parameter.

Novikov formula in Euclidean geometry (II)

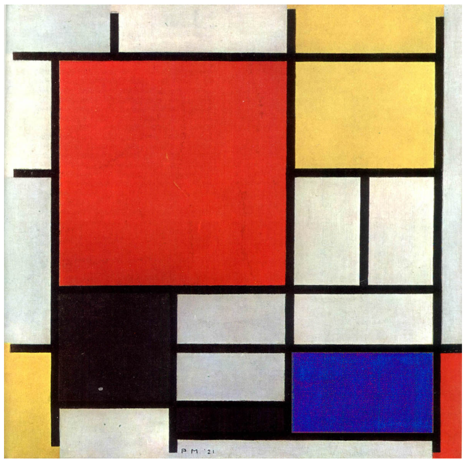
The reconstruction formula hinges on three ingredients:

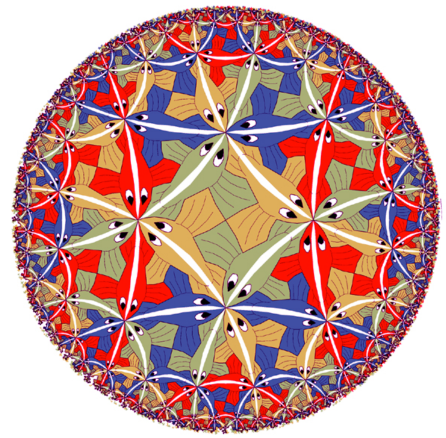
(i) We show that $u(z,\lambda)$ is *analytic* in $D^+ \cup D^- = \mathbb{C} \backslash T$ and that $\lambda u(z,\lambda)$ is bounded as $\lambda \to \infty$. This comes from the analysis of the fundamental solution of the $\bar{\partial}$ problem.

(ii) We verify that $\varphi(\mathbf{x},\theta) = u^+(\mathbf{x},\theta) - u^-(\mathbf{x},\theta)$, the jump of u at $\lambda = e^{i\theta}$ can be written as a function of the measured data $R_a f(s,\theta)$.

(iii) We solve the Riemann Hilbert problem using the Cauchy formula and evaluate the complexified transport equation $(Xu + au)(\lambda) = f$ at $\lambda = 0$ to obtain a *reconstruction formula* for $f(z) = f(\mathbf{x})$.

How about Hyperbolic Geometry?

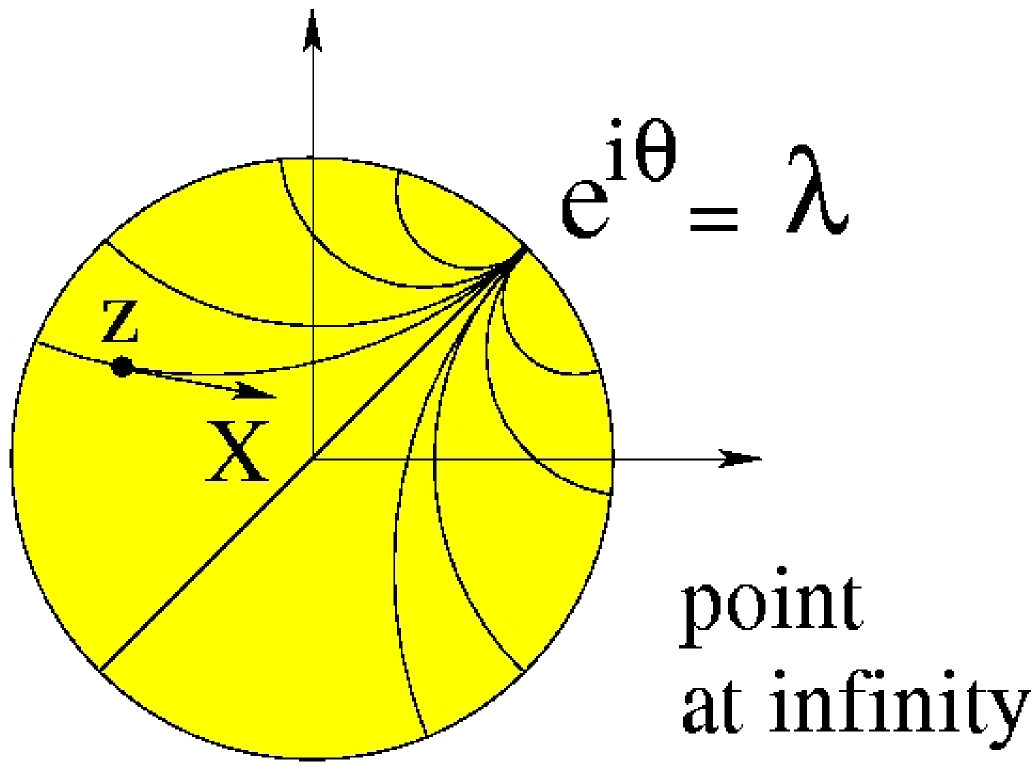


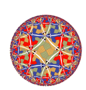
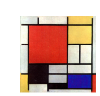


$\boldsymbol{\theta} \cdot \nabla$	Vector field	$\theta^i \frac{\partial}{\partial x^i} + \Gamma^k_{ij}\theta^i\theta^j \frac{\partial}{\partial \theta^k}$
$\lambda \frac{\partial}{\partial z} + \lambda^{-1} \frac{\partial}{\partial \bar{z}}$	Complexification	???

Suitable parameterization of geodesics

The vector field $X(e^{i\theta})$ at $z \in D$ is parameterized as:



The *point at infinity* in  corresponds to the *direction at infinity* in .

For the fun of it

On the hyperbolic disc, the geodesic vector field converging to $e^{i\theta}$ at infinity is

$$X(e^{i\theta}) = (1-|z|^2)\Big(\frac{1-e^{-i\theta}z}{1-e^{i\theta}\bar{z}}e^{i\theta}\partial + \frac{1-e^{i\theta}\bar{z}}{1-e^{-i\theta}z}e^{-i\theta}\bar{\partial}\Big).$$

It can be **complexified** for $\lambda \in \mathbb{C}$ as

$$X(\lambda) = (1-|z|^2)\Big(\frac{\lambda - z}{1-\lambda\bar{z}}\,\partial + \frac{1-\lambda\bar{z}}{\lambda - z}\,\bar{\partial}\Big).$$

It generates an elliptic operator for $\lambda \in \mathbb{C}\backslash T$ and thus admits a fundamental solution ($X(\lambda)G(z;\lambda,z_0) = \delta_g(z-z_0)$) of the form

$$G(z;\lambda,z_0) = \frac{-P(z_0,\lambda)}{2i\pi}\frac{1}{s(z,\lambda)-s(z_0,\lambda)}.$$

We deduce that the solution of the *complexified* transport equation

$$X(\lambda)u(z,\lambda) + a(z)u(z,\lambda) = f(z)$$

is given by

$$u(z,\lambda) = \int_D G(z;\lambda,\zeta) e^{h(\zeta,\lambda)-h(z,\lambda)} f(\zeta) dm_g(\zeta).$$

We have that G, hence u, is sectionally analytic. After an additional conformal mapping, it is given by the solution of a Riemann Hilbert problem via the following Cauchy formula

$$u(z,\lambda) = \tilde{u}(z,\mu) = \frac{1}{2\pi i} \int_T \frac{\tilde{\varphi}(z,\nu)}{\nu-\mu} d\nu,$$

where $\tilde{\varphi}$ depends explicitly only on the measured data.

Once $u(z,\lambda)$ is reconstructed, we apply the transport operator to it to obtain the source term $f(z)$.

Reconstruction formula

Let $R_a f(s,\theta)$ be the attenuated hyperbolic ray transform. Define

$$R_a f(s,\theta) = \int_{\xi(s,\theta)} e^{D_\theta a}(z,e^{i\theta}) f(z)\, dm_g(z)$$

$$\breve{X}^\perp(e^{i\theta}) = i(1-|z|^2)\Big(-\frac{1-e^{-i\theta}z}{1-e^{i\theta}\bar{z}} e^{i\theta}\partial + \frac{1-e^{i\theta}\bar{z}}{1-e^{-i\theta}z} e^{-i\theta}\bar{\partial}\Big)$$

$$Hf(t) = \frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}} \frac{f(s)}{t-s}\, ds$$

$$(R^*_{a,\theta} g)(z) = P(e^{-i\theta}z) e^{D_\theta a(z)} g(s(e^{-i\theta}z)), \qquad H_a = C_c H C_c + C_s H C_s$$

$$C_c g(s,\theta) = g(s,\theta)\cos\Big(\frac{H\hat{a}(s)}{2}\Big), \quad C_s g(s,\theta) = g(s,\theta)\sin\Big(\frac{H\hat{a}(s)}{2}\Big).$$

Then the source term is given by

$$f(z) = \frac{1}{4\pi}\int_0^{2\pi} \breve{X}^\perp(e^{i\theta})\Big(R^*_{-a,\theta} H_a [R_{a,\theta} f]\Big)(z,e^{i\theta})\, d\theta.$$

Vectorial ray transform

For a vector field $F(z)$, we can consider the vectorial ray transform

$$R_a F(s,\theta) \equiv R_{a,\theta}F(s) = \int_{\xi(s,\theta)} e^{D_\theta a}(z, e^{i\theta}) \langle X(e^{i\theta}), F\rangle dm_g(z).$$

Define $F^\flat = F_1 dx + F_2 dy$ such that $F^\flat X = \langle X(e^{i\theta}), F\rangle$. When $a = 0$,

$$\text{curl} F \equiv *dF^\flat = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{1}{2i}\Delta\varphi_0,$$

where φ_0 depends explicitly on $R_a F$. When $a > 0$,

$$F_1(z) + iF_2(z) = -2\bar\partial\Big(\frac{1-|z|^2}{a(z)}\bar\partial\varphi_1(z)\Big),$$

where φ_1 also depends explicitly on $R_a F$. As in Euclidean geometry, we can reconstruct the full vector field when $a > 0$ on its support.

Conclusions

Explicit inversion formulas in Euclidean and Hyperbolic geometry allow for efficient numerical inversions (whether in the form of a filtered-backprojection or in the form of a "faster" algorithm based on the fast Fourier transform).

The method of complexification of the geodesic vector field is somewhat *rigid*, which makes its extension to other problems difficult.

Yet several of the steps presented in the lecture extend to more general (Riemannian) geometries. This area of research may eventually provide algorithms to invert Radon transforms for arbitrary metrics (including the ones of interest in geophysical imaging).

The theory developed for scalar and vectorial source terms adapts to higher-order tensors with applications in anisotropic media.

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[5] Rullgård, H., Stability of the inverse problem for the attenuated Radon transform with 180^0 data, *Inverse Problems*, 2004

[6] Boman, J. and Strömberg, J. O., Novikov's inversion formula for the attenuated Radon transform–a new approach, *J. Geom. Anal.*, 2004

[7] Bal, G., Ray transforms in hyperbolic geometry, *J. Math. Pures Appl.*, 2005