Inverse Transport Problems and Applications

III. Full transport equation

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Outline for the three lectures

I. Inverse problems in integral geometry

Radon transform and attenuated Radon transform

Ray transforms in hyperbolic geometry

II. Forward and Inverse problems in highly scattering media

Photon scattering in tissues within diffusion approximation

Inverse problems in Optical tomography

III. Inverse transport problems

Singular expansion of albedo operator

Perturbations about “scattering-free” problems

Unsolved practical inverse problems.

Outline for Lecture III

1. Applications in imaging

Optical tomography and molecular imaging

Waves in random media

2. Inverse problems based on phase-space measurements

Singular decomposition of albedo (response) operator

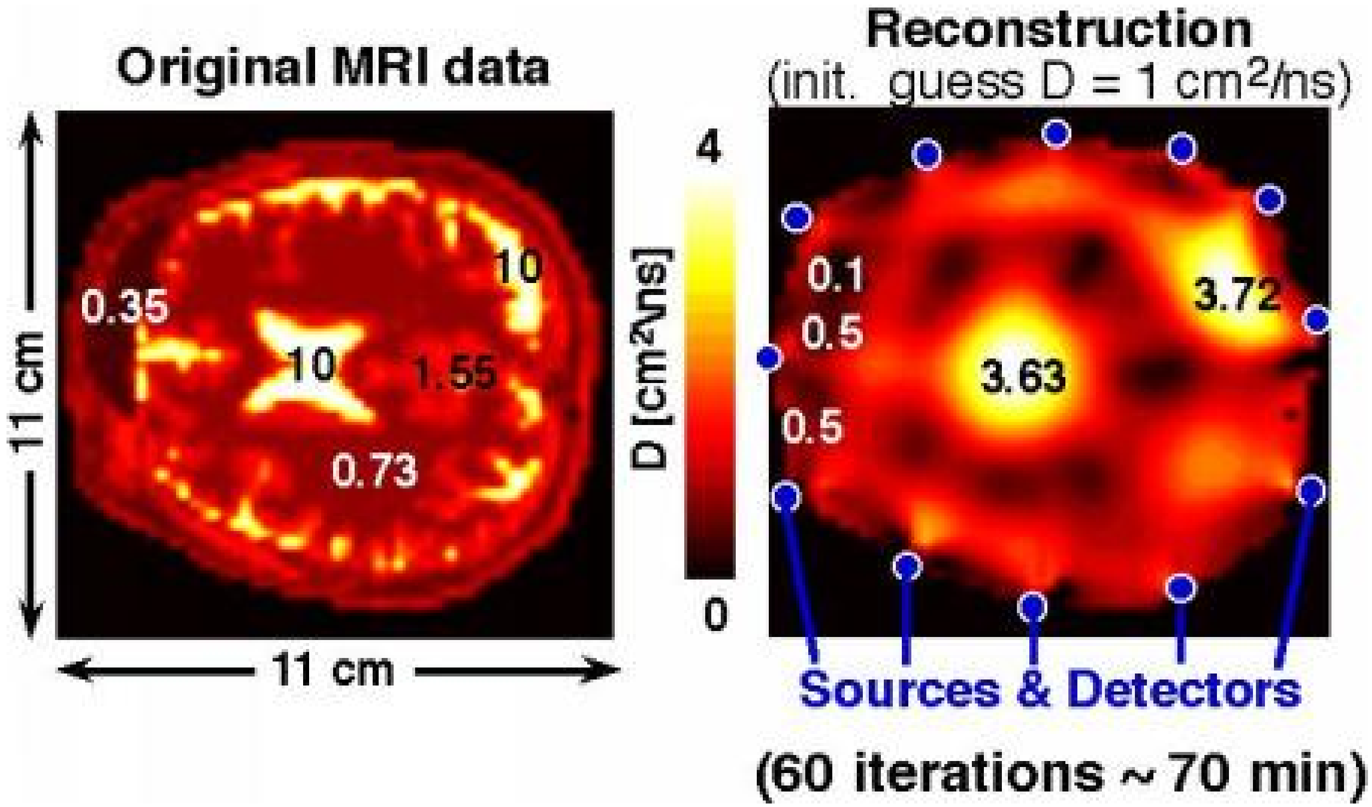
Perturbations of scattering-free problems

3. Inverse transport problem with "diffusion measurements"

What's wrong with full transport measurements?

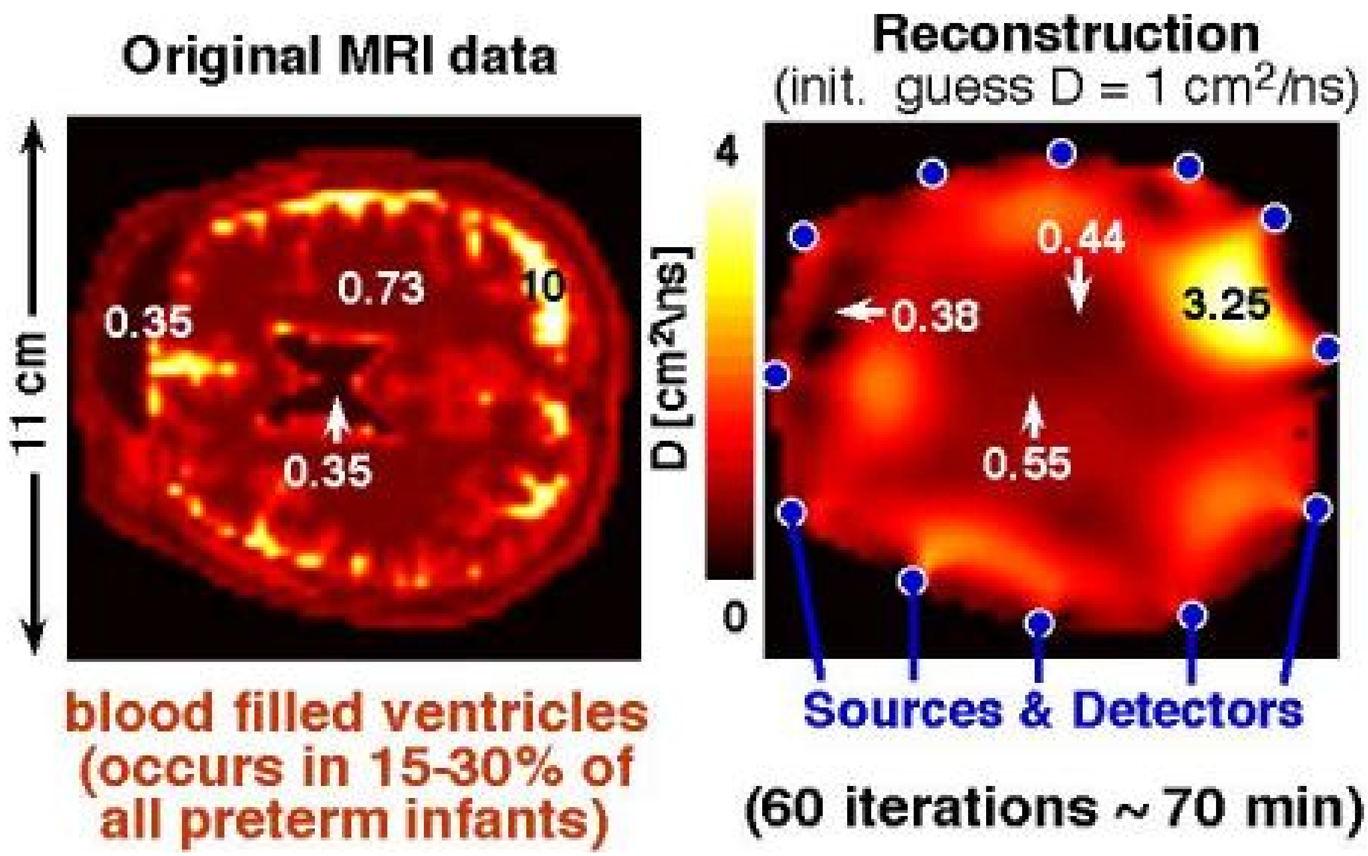
Ideas in diffusion theory that may work in transport.

Applications in Optical Tomography



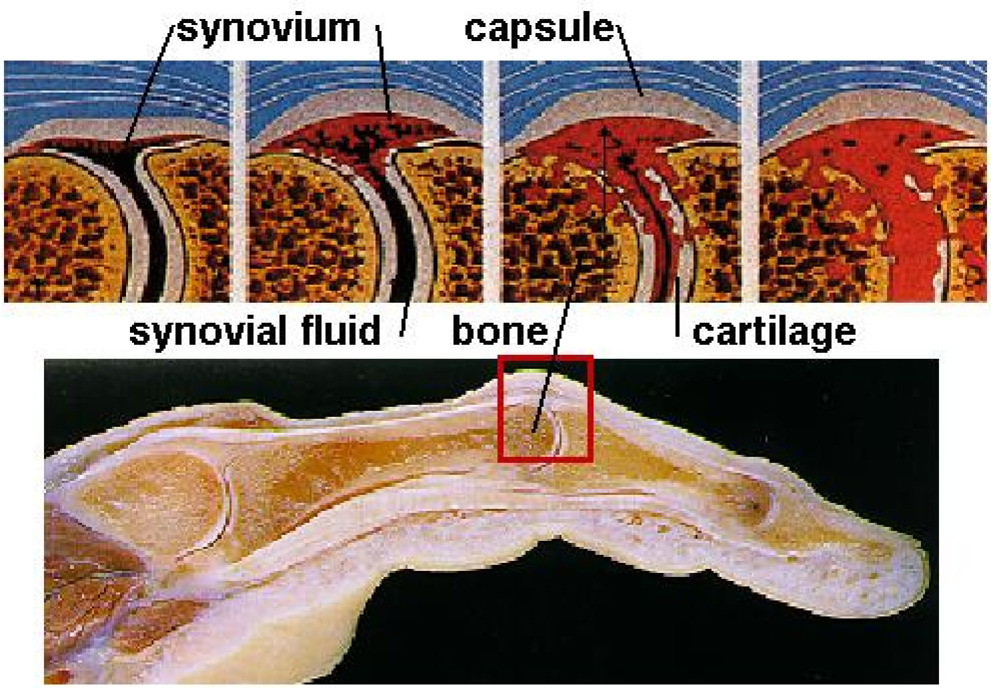
Brain with clear ventricle in neonate. (A.H.Hielscher, Columbia biomed.)

Applications in Optical Tomography



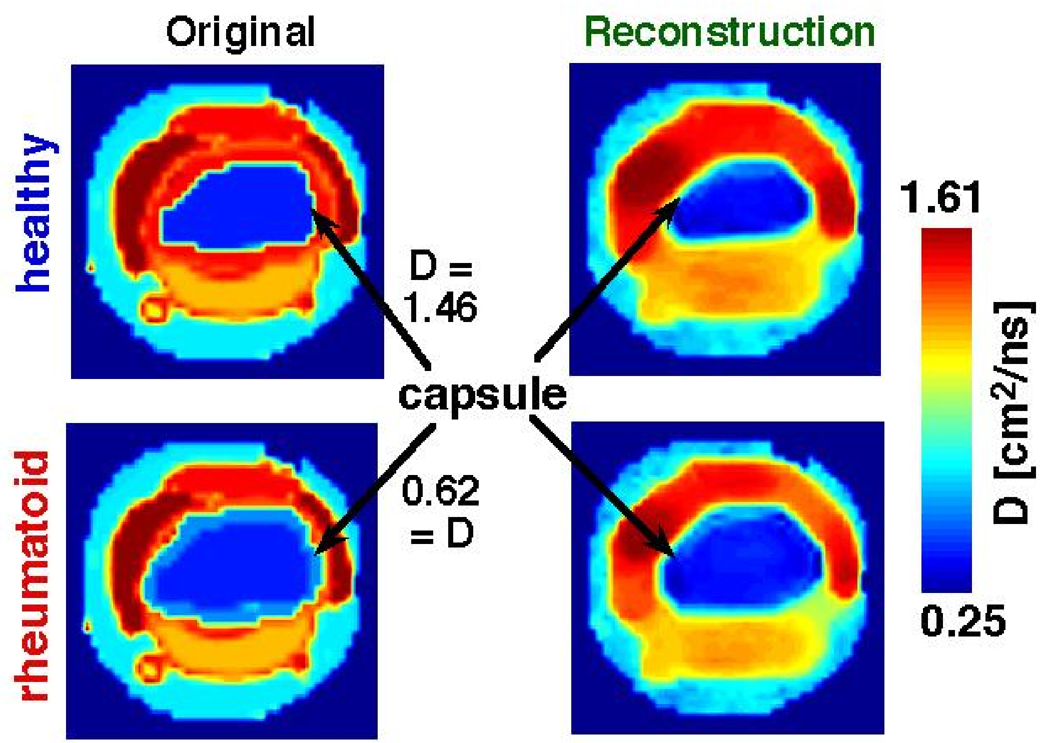
Brain with blood-filled ventricle in neonate. (A.H. Hielscher)

Applications in Optical Tomography



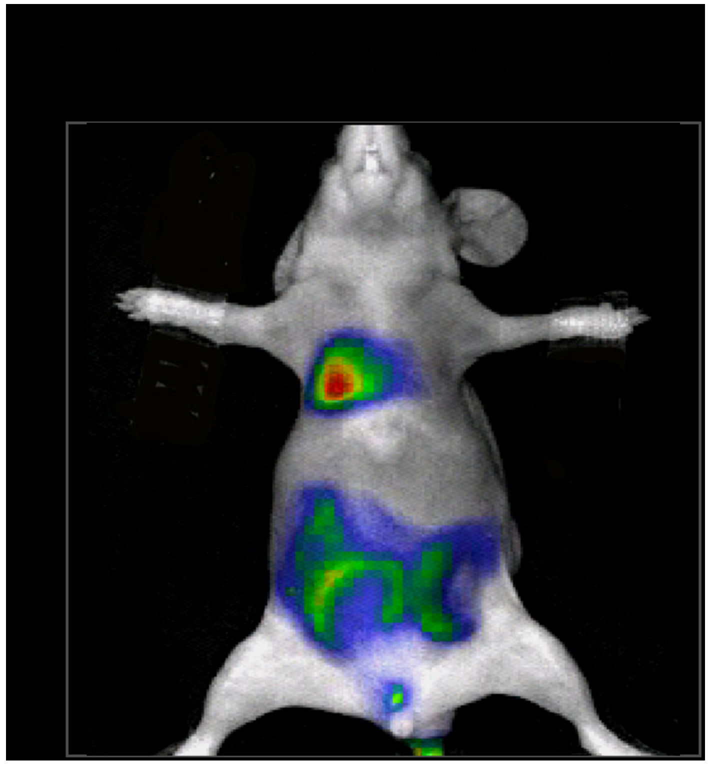
Detection of Rheumatoid arthritis. (A.H. Hielscher)

Applications in Optical Tomography



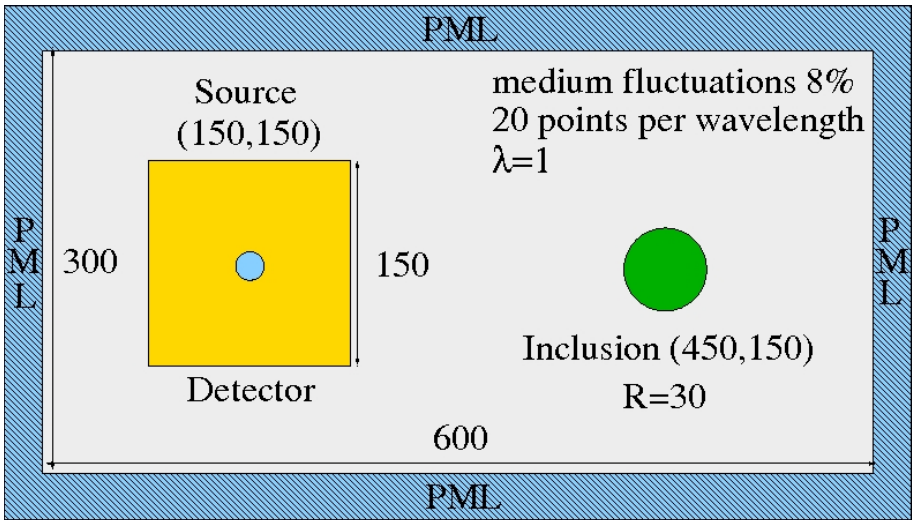
Optical imaging of Rheumatoid arthritis. (A.H. Hielscher)

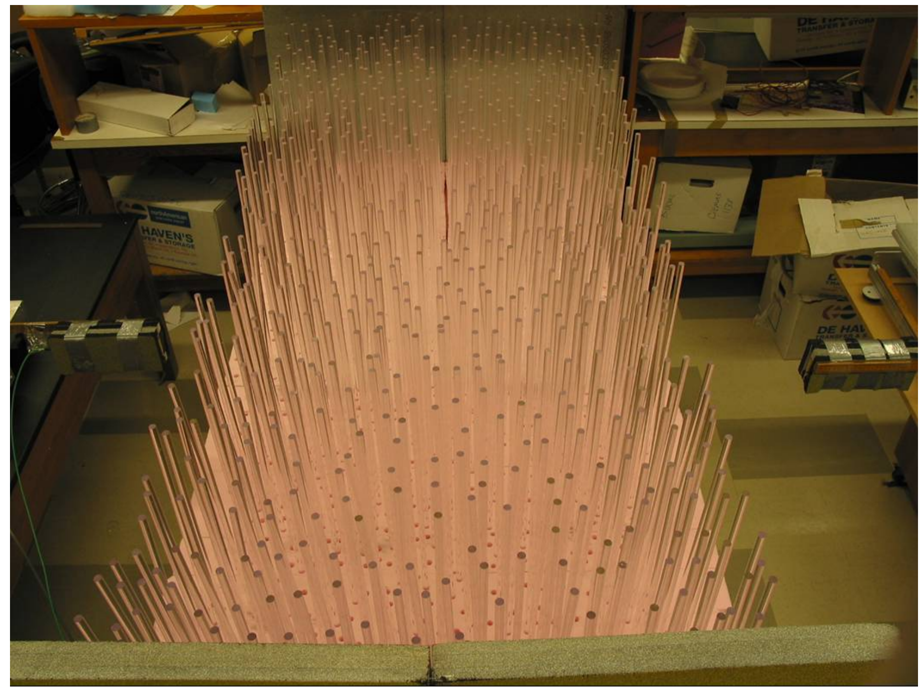
Applications in Optical Molecular Imaging



Optical fluorescence imaging in small animals.

Applications in Waves in random media





Numerical and experimental validations of radiative transfer

Outline for Lecture III

1. Applications in imaging

    Optical tomography and molecular imaging

    Waves in random media

2. **<u>Inverse problems based on phase-space measurements</u>**

    Singular decomposition of albedo (response) operator

    Perturbations of scattering-free problems

3. Inverse transport problem with “diffusion measurements”

    What’s wrong with full transport measurements?

    Ideas in diffusion theory that may work in transport.

Transport equations in optical tomography

In optical tomography the forward problem is

$$\begin{aligned} &\boldsymbol{\theta}\cdot\nabla u(\mathbf{x},\boldsymbol{\theta}) + a(\mathbf{x})u(\mathbf{x},\boldsymbol{\theta}) = \int_{S^2} k(\mathbf{x},\boldsymbol{\theta}',\boldsymbol{\theta})u(\mathbf{x},\boldsymbol{\theta}')d\boldsymbol{\theta}', \qquad (\mathbf{x},\boldsymbol{\theta})\in\Omega\times S^2\\ &u(\mathbf{x},\boldsymbol{\theta}) = g(\mathbf{x},\boldsymbol{\theta}), \qquad (\mathbf{x},\boldsymbol{\theta})\in\Gamma_-(\Omega),\end{aligned}$$

where the domains $\Gamma_\pm(\Omega)$ are defined by

$$\Gamma_\pm(\Omega) = \{(\mathbf{x},\boldsymbol{\theta})\in\partial\Omega\times S^2, \text{ such that } \pm\boldsymbol{\theta}\cdot\mathbf{n}(\mathbf{x})>0\},$$

where $\mathbf{n}(\mathbf{x})$ is the outward normal to Ω at $\mathbf{x}\in\partial\Omega$. The *Albedo* operator maps the incoming conditions to the outgoing radiation:

$$\mathcal{A}: g\mapsto u|_{\Gamma_+(\Omega)}.$$

The inverse problem consists of reconstructing $a(\mathbf{x})$ and $k(\mathbf{x},\boldsymbol{\theta}',\boldsymbol{\theta})$ from knowledge of $\mathcal{A}$.

Transport equations in optical molecular imaging

In optical molecular imaging the forward problem is

$$\boldsymbol{\theta} \cdot \nabla u(\mathbf{x}, \boldsymbol{\theta}) + a(\mathbf{x}) u(\mathbf{x}, \boldsymbol{\theta}) = \int_{S^{d-1}} k(\mathbf{x}, \boldsymbol{\theta}', \boldsymbol{\theta}) u(\mathbf{x}, \boldsymbol{\theta}') d\boldsymbol{\theta}' + f(\mathbf{x}), \ (\mathbf{x}, \boldsymbol{\theta}) \in \Omega \times S^{d-1}$$
$$u(\mathbf{x}, \boldsymbol{\theta}) = 0, \qquad (\mathbf{x}, \boldsymbol{\theta}) \in \Gamma_-(\Omega),$$

where $a(\mathbf{x})$ and $k(\mathbf{x}, \boldsymbol{\theta}', \boldsymbol{\theta})$ are assumed to be known.

The inverse problem consists of reconstructing $f(\mathbf{x})$ from knowledge of $u|_{\Gamma_+(\Omega)}$.

In both the optical tomography and the optical molecular imaging problems, $\int_{S^{d-1}} k(\mathbf{x}, \boldsymbol{\theta}', \boldsymbol{\theta}) d\boldsymbol{\theta}' \leq a(\mathbf{x})$ for the forward problem to be well-posed.

Singular decomposition of the Albedo operator

In the optical tomography framework, define $u_1(\mathbf{x},\boldsymbol{\theta};\mathbf{x}_0,\boldsymbol{\theta}_0)$ as

$$\begin{aligned}&\boldsymbol{\theta}\cdot\nabla u_1 + a(\mathbf{x})u_1 = 0 \qquad (\mathbf{x},\boldsymbol{\theta})\in\Omega\times S^{d-1}\\ &u_1(\mathbf{x},\boldsymbol{\theta};\mathbf{x}_0,\boldsymbol{\theta}_0) = \delta(\mathbf{x}-\mathbf{x}_0)\delta(\boldsymbol{\theta}-\boldsymbol{\theta}_0), \qquad (\mathbf{x},\boldsymbol{\theta})\in\Gamma_-(\Omega),\end{aligned}$$

and $u_2(\mathbf{x},\boldsymbol{\theta};\mathbf{x}_0,\boldsymbol{\theta}_0)$ as

$$\begin{aligned}&\boldsymbol{\theta}\cdot\nabla u_2 + a(\mathbf{x})u_2 = \int_{S^{d-1}} k(\mathbf{x},\boldsymbol{\theta}',\boldsymbol{\theta})u_1(\mathbf{x},\boldsymbol{\theta}')d\boldsymbol{\theta}', \qquad (\mathbf{x},\boldsymbol{\theta})\in\Omega\times S^{d-1}\\ &u_2(\mathbf{x},\boldsymbol{\theta};\mathbf{x}_0,\boldsymbol{\theta}_0) = 0, \qquad (\mathbf{x},\boldsymbol{\theta})\in\Gamma_-(\Omega),\end{aligned}$$

Let $u(\mathbf{x},\boldsymbol{\theta};\mathbf{x}_0,\boldsymbol{\theta}_0) = u_1 + u_2 + v$ be the solution of the full transport equation (replace u_2 and u_1 in the above equation by u and the boundary conditions by $\delta(\mathbf{x}-\mathbf{x}_0)\delta(\boldsymbol{\theta}-\boldsymbol{\theta}_0)$).

Knowledge of $\mathcal{A}$ is equivalent to that of $u(\mathbf{x},\boldsymbol{\theta};\mathbf{x}_0,\boldsymbol{\theta}_0)$. In any dimension, u_1 is more singular than u_2+v. In dimension $d\geq 3$, u_2 is more singular that v.

Inverse Transport Problem

Theorem[Choulli-Stefanov].
In any space dimension, knowledge of $\mathcal{A}$ implies that of $u_1(\mathbf{x},\boldsymbol{\theta};\mathbf{x}_0,\boldsymbol{\theta}_0)$ on $\Gamma_+ \times \Gamma_-$, which uniquely determines $a(\mathbf{x})$ by inverse Radon transform.

In dimension $d \geq 3$, knowledge of $\mathcal{A}$ implies that of $u_2(\mathbf{x},\boldsymbol{\theta};\mathbf{x}_0,\boldsymbol{\theta}_0)$ on $\Gamma_+ \times \Gamma_-$, which uniquely determines $k(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\theta}')$. More precisely, we have the formula

$$u_2(\mathbf{y}+s\boldsymbol{\theta},\boldsymbol{\theta};\mathbf{y}-t\boldsymbol{\theta}_0,\boldsymbol{\theta}_0) = \frac{\exp\Big(-\int_0^s a(\mathbf{y}+p\boldsymbol{\theta})dp - \int_0^t a(\mathbf{y}-p\boldsymbol{\theta}_0)dp\Big)}{\sqrt{1-(\boldsymbol{\theta}\cdot\boldsymbol{\theta}_0)^2}} k(\mathbf{y},\boldsymbol{\theta}_0,\boldsymbol{\theta}).$$

Here s and t are chosen so that u_2 is evaluated at the domain boundary.

The inverse transport problem for $d \geq 3$ is thus solved.

Two-dimensional inverse transport problem

In dimension $d = 2$, we can reconstruct $a(\mathbf{x})$ from the singularities of $\mathcal{A}$ but not $k(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\theta}')$. We may however uniquely reconstruct k provided that $k(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\theta}')$ is sufficiently small [Stefanov-Uhlmann]. The method is perturbative. We now propose an iterative method based on the same perturbative ideas.

Introduce some notation:

$$Tu = \boldsymbol{\theta} \cdot \nabla u + a(\mathbf{x})u, \qquad Ku = \int_{S^{d-1}} k(\mathbf{x}, \boldsymbol{\theta}', \boldsymbol{\theta}) u(\mathbf{x}, \boldsymbol{\theta}') d\boldsymbol{\theta}',$$

$$L : g_{|\Gamma_-} \mapsto u_{|\Omega \times V} \quad \text{solution of} \quad Tu = 0, \quad u = g \text{ on } \Gamma_-.$$

Then the transport solution of $\underline{Tu = Ku,\ u_{|\Gamma_-} = g}$ is given by

$$u = T^{-1}Ku + Lg = (I - T^{-1}K)^{-1}Lg = Lg + T^{-1}KLg + (T^{-1}K)^2 u.$$

The second term is linear in k while the last term is quadratic in k.

Two-dimensional inverse transport problem (ii)

The transport solution of $Tu = Ku$, $u_{|\Gamma_-} = g$ is given by

$$u = T^{-1}Ku + Lg = (I - T^{-1}K)^{-1}Lg = Lg + T^{-1}KLg + (T^{-1}K)^2 u.$$

Let $g_0 = \delta(\mathbf{x}-\mathbf{x}_0)\delta(\boldsymbol{\theta}-\boldsymbol{\theta}_0)$ on Γ_-. Then knowledge of $\mathcal{A}$ is equivalent to that of its kernel $d(\mathbf{x},\boldsymbol{\theta};\mathbf{x}_0,\boldsymbol{\theta}_0)$ and we have

$$\begin{aligned} &d(\mathbf{x},\boldsymbol{\theta};\mathbf{x}_0,\boldsymbol{\theta}_0) = \mathcal{F}T^{-1}KLg_0(\mathbf{x},\boldsymbol{\theta};\mathbf{x}_0,\boldsymbol{\theta}_0) \\ &+\mathcal{F}\Big(Lg_0 + (T^{-1}K)^2(I - T^{-1}K)^{-1}Lg_0\Big)(\mathbf{x},\boldsymbol{\theta};\mathbf{x}_0,\boldsymbol{\theta}_0), \end{aligned}$$

where $\mathcal{F}$ is restriction on Γ_+.

In $d \geq 3$, the first term (u_2) on the r.h.s. is more singular than the second term (v) of the r.h.s. for $\boldsymbol{\theta} \neq \boldsymbol{\theta}_0$. In $d = 2$, the latter is smaller than the former when k is sufficiently small.

Two-dimensional inverse transport problem (iii)

Recall that for $\boldsymbol{\theta} \neq \boldsymbol{\theta}_0$,

$$d(\mathbf{x}, \boldsymbol{\theta}; \mathbf{x}_0, \boldsymbol{\theta}_0) = \mathcal{F}T^{-1}KLg_0(\mathbf{x}, \boldsymbol{\theta}; \mathbf{x}_0, \boldsymbol{\theta}_0)$$
$$+\mathcal{F}(T^{-1}K)^2(I - T^{-1}K)^{-1}Lg_0(\mathbf{x}, \boldsymbol{\theta}; \mathbf{x}_0, \boldsymbol{\theta}_0).$$

Define $k = \mathcal{B}d$ the solution of $d(\mathbf{x}, \boldsymbol{\theta}; \mathbf{x}_0, \boldsymbol{\theta}_0) = \mathcal{F}T^{-1}KLg_0(\mathbf{x}, \boldsymbol{\theta}; \mathbf{x}_0, \boldsymbol{\theta}_0)$. (This is how k is reconstructed from u_2 in dimension $d \geq 3$.) Then we can recast the above expression as:

$$k = \mathcal{B}d - \mathcal{G}(k), \quad \mathcal{G} = \mathcal{B}\mathcal{F}\Big(\sum_{n=2}^{\infty}(T^{-1}K)^n\Big)Lg_0.$$

The leading term in $\mathcal{G}$ is quadratic in K so that *morally* $|\mathcal{G}(k_1) - \mathcal{G}(k_2)| \lesssim \rho\|k_1 - k_2\|$, where ρ is an a priori bound on k. This shows uniqueness of the reconstruction when ρ is small; see [Stefanov-Uhlmann].

An iterative reconstruction algorithm

Recall that

$$\underline{k = \mathcal{B}d - \mathcal{G}(k)}, \quad \mathcal{G} = \mathcal{B}\mathcal{F}\Big(\sum_{n=2}^{\infty}(T^{-1}K)^n\Big)Lg_0.$$

Let us assume that $0 \leq k(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\theta}') \leq 1$ (to simplify) and define

$$\mathcal{P}k = (0 \vee k) \wedge 1, \qquad \text{so that} \qquad k = \mathcal{P}(\mathcal{B}d - \mathcal{G}(k)) \equiv \mathcal{H}(k).$$

Define now the iterative algorithm

$$k^n = \mathcal{H}(k^{n-1}), \qquad k^0 = 0.$$

We can show that $\mathcal{H}$ is *continuous* on $L^\infty(\Omega \times S^1 \times S^1)$. Since k^n is bounded in that space and thus converges weakly (*) to k^∞, we find that $\mathcal{H}(k^{n-1})$ converges (weakly *) to $\mathcal{H}(k^\infty)$ so that

$$k^\infty = \mathcal{H}(k^\infty) \qquad \text{is a solution}.$$

More general Riemannian geometries

Both results (singular expansion in dimension $d \geq 3$ and perturbative argument in dimension $d = 2$) have been extended to the case of free transport along the geodesics of a Riemannian manifold [S. McDowall]. Let (M, g) be a Riemannian manifold with boundary ∂M and let u be the solution of the transport equation

$$Xu + a(\mathbf{x})u = \int_{\Omega_{\mathbf{x}} M} k(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\theta}')u(\mathbf{x}, \boldsymbol{\theta}')d\boldsymbol{\theta}'.$$

Then knowledge of the corresponding albedo operator uniquely determines $a(\mathbf{x})$ and $k(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\theta}')$ when the metric is known. In two space dimensions, simple metrics are also uniquely determined by the albedo operator (which determines the scattering relation).

This has interesting applications in geophysical imaging, as well in optical tomography when variations in the index of refraction are not neglected.

Inverse source problem in OMI (i)

Recall that the inverse problem in optical molecular imaging consists of reconstructing the source term $f(\mathbf{x})$ from u_{Γ_+}, where

$$\boldsymbol{\theta} \cdot \nabla u(\mathbf{x}, \boldsymbol{\theta}) + a(\mathbf{x}) u(\mathbf{x}, \boldsymbol{\theta}) = \int_{S^{d-1}} k(\mathbf{x}, \boldsymbol{\theta}', \boldsymbol{\theta}) u(\mathbf{x}, \boldsymbol{\theta}') d\boldsymbol{\theta}' + f(\mathbf{x}), \ (\mathbf{x}, \boldsymbol{\theta}) \in \Omega \times S^{d-1}$$

$$u(\mathbf{x}, \boldsymbol{\theta}) = 0, \qquad (\mathbf{x}, \boldsymbol{\theta}) \in \Gamma_-(\Omega).$$

No singularity analysis can be used for general $f(\mathbf{x})$. In the absence of scattering, the problem becomes

$$\boldsymbol{\theta} \cdot \nabla u(\mathbf{x}, \boldsymbol{\theta}) + a(\mathbf{x}) u(\mathbf{x}, \boldsymbol{\theta}) = f(\mathbf{x}),$$

which considered two-dimensional slice by two-dimensional slice, is the attenuated Radon transform, for which we have an inversion formula.

Inverse source problem in OMI (ii)

Since we can invert the source problem in the absence of scattering, we also should be able to do it in the presence of little scattering.

Recall the notation of the attenuated Radon transform. We define the *symmetrized* beam transform:

$$D_\theta a(\mathbf{x}) = \frac{1}{2}\int_0^\infty [a(\mathbf{x} - t\boldsymbol{\theta}) - a(\mathbf{x} + t\boldsymbol{\theta})]dt,$$

such that $\boldsymbol{\theta} \cdot \nabla D_\theta a = a$ and the attenuated Radon transform as

$$R_a f(s,\theta) = \int_{\mathbb{R}} (e^{D_\theta a} f)(s\boldsymbol{\theta}^\perp + t\boldsymbol{\theta}, \theta)dt$$

We recall the existence of an operator $\mathcal{N}$ (Novikov formula) such that

$$f(\mathbf{x}) = \mathcal{N}[R_a f(s,\theta)](\mathbf{x}).$$

Inverse source problem in OMI (iii)

In the presence of scattering, the attenuated Radon transform is replaced by the measurements

$$g(s,\theta) = R_a f(s,\theta) + Re^{Da} K e^{-Da} T f(s,\theta).$$

Above, R is the usual Radon transform. Applying the Novikov inversion operator, we thus obtain

$$\mathcal{N} g(\mathbf{x}) = f(\mathbf{x}) + \mathcal{N} R e^{Da} K e^{-Da} T f(\mathbf{x}) = (I - \mathcal{N}_K) f(\mathbf{x}).$$

Provided $\mathcal{N}_K$ is sufficiently small, the following algorithm converges

$$f^{(n)} = \mathcal{N} g(\mathbf{x}) + \mathcal{N}_K f^{(n-1)}, \quad f^0 = 0.$$

Remark on the Inverse source problem in OMI

The above algorithm can be improved upon by remarking that the Novikov formula can be used in the case of isotropic scattering. Recall the transport equation in that case

$$\boldsymbol{\theta} \cdot \nabla u + a(\mathbf{x})u = \sigma(\mathbf{x}) \int_{S^{n-1}} u(\mathbf{x}, \boldsymbol{\theta}')d\boldsymbol{\theta}' + f(\mathbf{x}) \equiv F(\mathbf{x}).$$

The Novikov formula allows to construct

$$F(\mathbf{x}) = \mathcal{N} g(\mathbf{x}).$$

We then solve the transport equation for $u(\mathbf{x}, \boldsymbol{\theta})$ and obtain

$$f(\mathbf{x}) = F(\mathbf{x}) - \sigma(\mathbf{x}) \int_{S^{n-1}} u(\mathbf{x}, \boldsymbol{\theta}')d\boldsymbol{\theta}'.$$

The perturbative algorithm shown earlier can be adapted [Bal-Tamasan] to solve the inverse transport source problem when *anisotropic* scattering is sufficiently small.

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   What's wrong with full transport measurements?

   Ideas in diffusion theory that may work in transport.

Diffusion-type measurements

In practice, angularly dependent measurements at the domain boundary may not be available. In OMI, angular measurements are necessary as the dimension of the source term is d whereas that of the measurements is $(d-1)\times(d-1)$.

However in optical tomography, the dimension of measurements is $(d-1)^4$, whereas $a(\mathbf{x})$ is typically d-dimensional and $k(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\theta}')$ typically $d\times(d-1)^2$-dimensional. Thus there is room for less accurate measurements to still uniquely define the coefficients a and k.

Diffusion-type measurements

In practice, the outgoing distribution $u_{|\Gamma_+}(\mathbf{x}, \boldsymbol{\theta})$ cannot be measured. Only the current

$$\int_{S^{d-1}} \boldsymbol{\theta} \cdot \mathbf{n}(\mathbf{x}) u(\mathbf{x}, \boldsymbol{\theta}) d\boldsymbol{\theta} \approx D(\mathbf{x}) \frac{\partial U}{\partial \mathbf{n}}(\mathbf{x})$$

is measured.

In the diffusive regime, both expressions above agree (for $\mathbf{x}$ away from the boundary) up to a error term $O(\varepsilon)$, i.e., proportional to the mean free path. Recall that for isotropic scattering ($k = k(\mathbf{x})$) we have

$$u_\varepsilon(\mathbf{x}, \boldsymbol{\theta}) = U(\mathbf{x}) - \varepsilon d D(\mathbf{x}) \boldsymbol{\theta} \cdot \nabla U(\mathbf{x}) + O(\varepsilon^2).$$

Diffusion-type measurements

For incoming boundary conditions of the form $g_0(\mathbf{x}, \boldsymbol{\theta}) = \delta(\mathbf{x}-\mathbf{x}_0)\delta(\boldsymbol{\theta}-\boldsymbol{\theta}_0)$, we have access to measurements of the form $J(\mathbf{x}; \mathbf{x}_0, \boldsymbol{\theta}_0)$. The latter kernel is still singular in the $\mathbf{x}$ variable. This singularity is sufficient to reconstruct $a(\mathbf{x})$ by inverse Radon transform. However the scattering kernel $k(\mathbf{x})$ (even assumed independent of angular variables) can no longer be obtained by the analysis of straightforward singularities of the kernel $J(\mathbf{x}; \mathbf{x}_0, \boldsymbol{\theta}_0)$.

We may even further restrict measurements by assuming that the illumination is either isotropic $g(\mathbf{x}, \boldsymbol{\theta}) = g(\mathbf{x})$ or that it is unidirectional $g(\mathbf{x}, \boldsymbol{\theta}) = g(\mathbf{x})\delta(\boldsymbol{\theta} - \boldsymbol{\theta}_0(\mathbf{x}))$, where $\boldsymbol{\theta}_0$ is normal ($\boldsymbol{\theta}_0 = -\mathbf{n}(\mathbf{x})$). Both situations are realistic (the latter more so than the former). There, even $a(\mathbf{x})$ may no longer be reconstructed from singularities of the measurement kernel $J(\mathbf{x}; \mathbf{x}_0)$. The latter measurements have the *same dimension* as in *diffusion theory*.

Inverse Transport versus inverse Diffusion

Consider the diffusion equation

$$i\omega U - \nabla \cdot D(\mathbf{x})\nabla U + \sigma_a(\mathbf{x})U = 0, \qquad \Omega$$

with all possible Cauchy data on $\partial\Omega$ known. When $\omega = 0$, theory says that *either* $D(\mathbf{x})$ or $\sigma_a(\mathbf{x})$ can be reconstructed, and that when $\omega \neq 0$, both can be reconstructed [e.g. Sylvester-Uhlmann].

In the diffusion approximation, the transport and diffusion coefficients are related by (in the isotropic scattering case)

$$D(\mathbf{x}) = \frac{1}{dk(\mathbf{x})}, \qquad \sigma_a(\mathbf{x}) = a(\mathbf{x}) - k(\mathbf{x}).$$

So we expect that diffusion-type measurements allow us to reconstruct both $a(\mathbf{x})$ and $k(\mathbf{x})$ when $\omega \neq 0$ and one of them when $\omega = 0$. *No such theory exists.*

Inverse transport, a summary of results

Knowledge of the full albedo operator $\mathcal{A}$ allows us to uniquely reconstruct $a(\mathbf{x})$ and $k(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\theta}')$ in dimension $d \geq 3$. In dimension $d = 2$, k is uniquely defined provided that it is sufficiently small. In OMI, the source term is also uniquely determined by boundary measurements provided that the the anisotropic part of the scattering coefficient $k(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\theta}')$ is sufficiently small.

All the results are based on the singular decomposition of the albedo operator (OT), the Novikov formula (OMI), and perturbative arguments.

There is no theory of uniqueness of reconstruction in the case of diffusion-type measurements (i.e., the ones available in practice).

Can the theories developed for the diffusion equation be extended to the transport equations?

Complex exponentials

Consider the solutions to two Schrödinger equations in $\mathbb{R}^d$, $d \geq 3$,

$$(\Delta + q_i)u_i = 0, \qquad \text{on } \Omega, \quad i = 1, 2$$

with identical Cauchy data so that

$$\int_\Omega (q_1 - q_2) u_1 u_2 d\mathbf{x} = \int_{\partial\Omega} \Big(\frac{\partial u_1}{\partial \boldsymbol{\nu}} u_2 - \frac{\partial u_2}{\partial \boldsymbol{\nu}} u_1 \Big) d\sigma(\mathbf{x}) = 0.$$

Then for all $\mathbf{k} \in \mathbb{R}^d$, we can find $u_1(\mathbf{k};\mathbf{l})$ and $u_2(\mathbf{k},\mathbf{l})$ such that

$$0 = \int_\Omega (q_1 - q_2) u_1 u_2 = \int_\Omega (q_1 - q_2) e^{i\mathbf{k}\cdot\mathbf{x}} + o(1) \qquad \text{as } |\mathbf{l}| \to \infty.$$

The reason is the existence of a sufficiently rich family of complex exponentials

$u(\mathbf{x};\mathbf{m}) = e^{i\mathbf{m}\cdot\mathbf{x}}$, $\mathbf{m} \in \mathbb{C}^d$, $\mathbf{m}\cdot\mathbf{m} = 0$, which are harmonic: $\Delta u(\mathbf{x};\mathbf{m}) = 0$.

Is there an *equivalent notion* in transport theory and which type of perturbations does it allow us to reconstruct?

Identification of boundary values; diffusion

Consider the problem $\nabla\gamma(\mathbf{x})\nabla u = 0$, $u_{|\partial\Omega} = \phi$ and the quadratic form

$$Q_\gamma(\phi) \equiv \int_\Omega \gamma|\nabla u|^2 d\mathbf{x} = \int_{\partial\Omega} u\gamma\frac{\partial u}{\partial \mathbf{n}} d\sigma(\mathbf{x}).$$

Knowledge of the Dirichlet-to-Neumann map is equivalent to that of $\phi \mapsto Q_\gamma(\phi)$. Moreover the solution u minimizes $\int_\Omega \gamma|\nabla u|^2 d\mathbf{x}$ for a given set of boundary conditions. This allows us to remark that for two conductivities with the same boundary measurements,

$$0 = Q_{\gamma_1}(\phi) - Q_{\gamma_2}(\phi) \geq \int_\Omega (\gamma_1 - \gamma_2)|\nabla u_1|^2 d\mathbf{x},$$

where $\nabla\gamma_k(\mathbf{x})\nabla u_k = 0$, $u_{k|\partial\Omega} = \phi$ $k = 1, 2$.

Appropriate choices of ϕ are used to show [Kohn-Vogelius] that

$$(\boldsymbol{\nu}\cdot\nabla)^l\gamma_1 = (\boldsymbol{\nu}\cdot\nabla)^l\gamma_2,$$

for all $l \in \mathbb{N}$ for which the above quantities make sense.

Identification of boundary values; transport (i)

A similar theory should work in the framework of transport equations. Consider isotropic transport in the even parity formulation. Let

$$\psi(\mathbf{x},\boldsymbol{\theta}) = \frac{1}{2}\Big(u(\mathbf{x},\boldsymbol{\theta}) + u(\mathbf{x},-\boldsymbol{\theta})\Big), \qquad \overline{\psi} = \int_V \psi(\boldsymbol{\theta})d\boldsymbol{\theta}.$$

We can show that for g such that $g(\mathbf{x},\boldsymbol{\theta}) = g(\mathbf{x},-\boldsymbol{\theta})$,

$$\begin{aligned} -\boldsymbol{\theta}\cdot\nabla\frac{1}{\Sigma(\mathbf{x})}\boldsymbol{\theta}\cdot\nabla\psi + \sigma_a(\mathbf{x})\psi + \sigma_s(\mathbf{x})(\psi-\overline{\psi}) &= 0, \quad \Omega\times V \\ \psi - \frac{1}{\Sigma}\boldsymbol{\theta}\cdot\nabla\psi &= g, \quad \Gamma_-. \end{aligned}$$

Here $\Sigma(\mathbf{x}) = \sigma_a(\mathbf{x}) + \sigma_s(\mathbf{x})$. For $\sigma = (\sigma_a, \sigma_s)$, define the quadratic form

$$\mathcal{Q}_\sigma(\psi) = \int_{\Omega\times V}\frac{1}{\Sigma}(\boldsymbol{\theta}\cdot\nabla\psi)^2 + \sigma_a\psi^2 + \sigma_s(\psi-\overline{\psi})^2 d\mathbf{x}d\boldsymbol{\theta} + \int_{\partial\Omega\times V}\psi^2|\boldsymbol{\theta}\cdot\boldsymbol{\nu}|d\sigma d\boldsymbol{\theta}.$$

We verify that for ψ solution of the above equation,

$$\mathcal{Q}_\sigma(\psi) = \int_{\partial\Omega\times V} g\psi|\boldsymbol{\theta}\cdot\mathbf{n}|d\sigma d\boldsymbol{\theta}.$$

Identification of boundary values; transport (ii)

When $g(\mathbf{x})$ is independent of $\boldsymbol{\theta}$, we observe that

$$Q_\sigma(g) \equiv \mathcal{Q}_\sigma(\psi) = \int_{\partial\Omega} g(\mathbf{x})\Big(\int_V \psi|\boldsymbol{\theta}\cdot\mathbf{n}|d\boldsymbol{\theta}\Big)d\sigma(\mathbf{x}).$$

So knowledge of $Q_\sigma(g)$ involves diffusion-type measurements only. We also verify that ψ minimizes

$$\frac{1}{2}\mathcal{Q}_\sigma(\psi) - \int_{\partial\Omega\times V} g\psi|\boldsymbol{\theta}\cdot\mathbf{n}|d\sigma d\boldsymbol{\theta},$$

so that $\mathcal{Q}_{\sigma^2}(\psi_2) \le \mathcal{Q}_{\sigma^1}(\psi_1)$. As a consequence, if diffusion-type boundary measurements of two configurations agree, we have

$$\begin{aligned} 0 &= Q_{\sigma^1}(g) - Q_{\sigma^2}(g) \\ &\ge \int_{\Omega\times V}\Big([\frac{1}{\Sigma^1}-\frac{1}{\Sigma^2}](\boldsymbol{\theta}\cdot\nabla\psi_1)^2 + [\sigma_a^1-\sigma_a^2]\psi_1^2 + (\sigma_s^1-\sigma_s^2)(Q\psi_1)^2\Big)d\mathbf{x}d\boldsymbol{\theta}.\end{aligned}$$

Identification of boundary values; transport (iii)

In the simplified setting where $\Sigma^1 = \Sigma^2$ and σ_s is unknown, we have

$$0 = Q_{\sigma^1}(g) - Q_{\sigma^2}(g) \geq \int_{\Omega\times V} [\sigma_s^2 - \sigma_s^1](\overline{\psi}_1)^2 d\mathbf{x} d\boldsymbol{\theta}.$$

When $\sigma_a \equiv 0$, we find that

$$0 = Q_{\sigma^1}(g) - Q_{\sigma^2}(g) \geq \int_{\Omega\times V} \Big([\frac{1}{\Sigma^1} - \frac{1}{\Sigma^2}](\boldsymbol{\theta}\cdot\nabla\psi_1)^2 + (\Sigma^1 - \Sigma^2)(Q\psi_1)^2 \Big) d\mathbf{x} d\boldsymbol{\theta}.$$

It remains to find a sequence of boundary conditions $g(\mathbf{x})$ such that $\overline{\psi}_1$ or $\boldsymbol{\theta}\cdot\nabla\psi_1$ localizes sufficiently well in the vicinity of $\mathbf{x}_0 \in \partial\Omega$ so that $0 \geq \sigma_s^2(\mathbf{x}_0) - \sigma_s^1(\mathbf{x}_0)$, whence $\underline{\sigma_s^2(\mathbf{x}_0) - \sigma_s^1(\mathbf{x}_0) = 0}$ since the reverse inequality holds; or $0 \geq \Sigma_1^{-1}(\mathbf{x}_0) - \Sigma_2^{-2}(\mathbf{x}_0)$, whence $\underline{\Sigma_1(\mathbf{x}_0) = \Sigma_2(\mathbf{x}_0)}$. Similar methods show the same results for partial derivatives of σ_s or Σ.

Note that the procedure solves the identification of real-analytic coefficients σ_s or Σ (as in [Kohn-Vogelius] for diffusion equations).

Ultimate inverse transport theory

In certain regimes of approximation of transport (such as highly peaked-forward scattering) or as a model for the energy density of waves propagating in random media, the radiative transfer equation takes the form of the following Fokker-Planck equation

$$\begin{aligned} &\boldsymbol{\theta} \cdot \nabla u + \sigma_a(\mathbf{x}) u - D(\mathbf{x}) \Delta_{\boldsymbol{\theta}} u = 0 \qquad \text{in } \Omega \times V, \\ &u = g \qquad \text{on } \Gamma_-. \end{aligned}$$

Here Δ_θ is the Laplace-Beltrami operator on the unit sphere S^{d-1}.
The inverse problem is to reconstruct $\sigma_a(\mathbf{x})$ and $D(\mathbf{x})$ from the albedo operator (boundary measurements).
One of the main difficulties is that $D(\mathbf{x})\Delta_{\boldsymbol{\theta}}$ smoothes out any singularity at the domain boundary so that the method of decomposition of the albedo operator into terms of decreasing singularity does not apply.
Can complex exponentials of some sort be useful there? Is there a variational formulation that one can use to evaluate coefficients on $\partial\Omega$?

Conclusions

There are many applications in inverse transport theory.

Many of these applications involve angularly averaged measurements that are not accounted for by existing transport theories.

Inverse transport with "diffusion-type" measurements precisely looks like diffusion (and *is* diffusion in the limit of small mean free paths). Can well-established techniques in diffusion theory be extended to transport and Fokker-Planck equations?

Expansions in singular terms are still very useful: when they apply, they show that the inverse problem is mildly ill-posed (i.e., noise is differentiated a finite number of times in the reconstruction).

When such expansions do not hold, chances are that the inverse problem is severely ill-posed (noise in differentiated an infinite number of times during reconstruction), which equally severely limits its practical usefulness.

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