The Annals of Applied Probability
2017, Vol. 27, No. 4, 2061–2097
DOI: 10.1214/16-AAP1249

CONTACT PROCESSES ON RANDOM REGULAR GRAPHS

BY STEVEN LALLEY AND WEI SU

University of Chicago

We show that the contact process on a random d-regular graph initiated by a single infected vertex obeys the "cutoff phenomenon" in its supercritical phase. In particular, we prove that, when the infection rate is larger than the lower critical value of the contact process on the infinite d-regular tree, there are positive constants C, p depending on the infection rate such that for any $\varepsilon > 0$, when the number n of vertices is large then (a) at times $t < (C - \varepsilon)\log n$ the fraction of infected vertices is vanishingly small, but (b) at time $(C + \varepsilon)\log n$ the fraction of infected vertices is within ε of p, with probability p.

1. Introduction. The contact process with infection rate $\lambda > 0$ on a connected, locally finite graph $G = (\mathcal{V}_G, \mathcal{E}_G)$ is a continuous-time Markov chain $(\xi_t)_{t\geq 0}$ with state space {subsets of $\mathcal{V}_G$} that evolves as follows:

(a) infected sites (i.e., vertices in ξ_t) recover at rate 1, and upon recovery are removed from ξ_t;

(b) healthy sites (i.e., vertices not in ξ_t) become infected at rate λ times the number of currently infected neighbors, and upon infection are added to ξ_t;

(c) the infection and recovery processes are independent, and independent from vertex to vertex.

See [13] for a formal construction, or alternatively [9] for the standard *graphical representation* using independent Poisson processes.

The behavior of the contact process on the infinite d-regular tree $G = \mathbb{T}_d$ is reasonably well understood. When $d = 2$ (where $\mathbb{T}_2 = \mathbb{Z}$), there is a single survival phase [13]. When $d \geq 3$, there are two survival phases: in particular, there are critical values $0 < \lambda_1(\mathbb{T}_d) < \lambda_2(\mathbb{T}_d) < \infty$ such that if $\lambda \leq \lambda_1$, then the contact process dies out almost surely; if $\lambda_1 < \lambda \leq \lambda_2$, then the contact process survives globally with positive probability but dies out locally almost surely; and if $\lambda > \lambda_2$ then the contact process survives locally with positive probability. (See [19] for the case $d \geq 4$ and [12, 20] for $d = 3$.) The parameter range $\lambda \in (\lambda_1, \lambda_2)$ is called the *weak survival phase*, and $\lambda > \lambda_2$ is the *strong survival phase*.

When G is finite there is no survival phase, since the absorbing state $\varnothing$ is accessible from every state ξ. Nevertheless, when the graph is large it will contain

Received April 2015; revised June 2016.
MSC2010 subject classifications. Primary 60K35; secondary 60K37, 05C80.
Key words and phrases. Contact process, random regular graph, cutoff phenomenon.

long linear chains, and so if the infection rate is above the critical value $\lambda_1(\mathbb{Z})$ the contact process will, with nonnegligible probability, survive for a long time in a quasi-stationary state before ultimately dying out. This suggests several problems of natural interest:

1. How does the survival time scale with the size of the graph?
2. What is the nature of the quasi-stationary state?
3. How does the process behave in its initial "explosive" stage?

These questions have been studied for several important families of graphs, notably the finite tori [17], finite regular trees of large depth [21] and versions of the "small worlds" networks of Watts and Strogatz [8]. Stacey has shown that when G_L is a finite d-homogeneous rooted tree of depth L, the extinction time of a contact process started from full occupancy in G_L grows linearly in L when $\lambda < \lambda_2(\mathbb{T}_d)$; but when $\lambda > \lambda_2(\mathbb{T}_d)$ it grows doubly exponentially in L, and almost exponentially in the number of vertices. In a more recent paper [6], it has been proved that the extinction time grows exponentially in the number of vertices.

In this paper, we consider a different class of graphs, the *random d-regular graphs*. These are of interest for a number of reasons: they are *expanders*, they are locally tree-like, and they are (unlike the finite trees) *statistically homogeneous* in a certain sense. See [22] for a survey. The behavior of several common stochastic models on random d-regular graphs has been studied. Lubetsky and Sly [14] have shown that the simple random walk on a large random d-regular graph undergoes *cutoff*, that is, the transition to stationarity occurs in a narrow time window. Bhamidi, Hofstad and Hooghiemstra [1] have shown that distance between two randomly chosen vertices in first passage percolation on a random d-regular graph is concentrated around $C \log n$. Chatterjee and Durrett [5] have shown that the *threshold* contact process on a random d-regular graph exhibits a phase transition in the infectivity parameter. More recently, Ding, Sly and Sun [7] have shown that the independence number of a random d-regular graph is sharply concentrated about its median.

Our main result is that the contact process on a random regular graph exhibits a cutoff phenomenon analogous to that for the simple random walk. We shall assume throughout that nd is even and $d \geq 3$. Let $G \sim \mathcal{G}(n, d)$ be a random graph uniformly distributed over the set of all d-regular graphs on the vertex set $\mathcal{V}_G = [n]$. Given G, for any subset $A \subset [n]$, let ξ_t^A be a contact process run on G with initial state $\xi_0^A = A$. (When $A = \{u\}$ is a singleton, we will write ξ_t^u instead of $\xi_t^{\{u\}}$, and when the initial state is the entire vertex set $\mathcal{V}_G$ we will omit the superscript and write $\bar{\xi}_t$ rather than $\xi_t^{\mathcal{V}_G}$.) Our primary interest is the behavior of the contact process in the "meta-stable" phase, where the infection rate λ exceeds $\lambda_1(\mathbb{T}_d)$, and our main focus will be the following question: for a typical pair of vertices, what is the time needed for a contact process started from one vertex to infect the other? Since the diameter of a typical random regular graph is on the order of $\log n$, we expect the infection time to be on the same order. The main result of this paper, Theorem 1.1, implies that this conjecture is true.

We say that a property holds *asymptotically almost surely* if the set of graphs in $\mathcal{G}(n,d)$ satisfying this property has probability approaching 1 as n goes to infinity. The word "typical" will mean "asymptotically almost every as $n \to \infty$". To denote conditional probabilities and expectations given the graph G, we will use a subscript: $\mathbb{P}_G$ and $\mathbb{E}_G$. The vertex degree $d \geq 3$ will be fixed throughout the discussion, and so we will use λ_1 and λ_2 as shorthand for $\lambda_1(\mathbb{T}_d)$ and $\lambda_2(\mathbb{T}_d)$. The infection rate $\lambda > \lambda_1$ will also be fixed throughout, and we will denote by p_λ and c_λ the survival probability and exponential growth rate (see Section 2), respectively, of the contact process on $\mathbb{T}$. Also, we shall adopt the convention that all $o(1)$ terms tend to 0 uniformly in n (independent of G). Our first main result is the following "cutoff" theorem.

THEOREM 1.1. *Fix vertices $u \neq v \in [n]$, and let $G \sim \mathcal{G}(n,d)$ be a random d-regular graph on the vertex set $\mathcal{V}_G = [n]$. For any $0 < \varepsilon < 1/32$, there exist constants $g_n(\varepsilon) \to 0$ as $n \to \infty$ such that for asymptotically almost every G,*

$$\mathbb{P}_G\{v \in \xi_s^u \text{ for some } s \leq (1-\varepsilon)c_\lambda^{-1}\log n\} \leq g_n(\varepsilon) \tag{1.1}$$

and

$$\mathbb{P}_G\{v \in \xi^u_{(1+\varepsilon)c_\lambda^{-1}\log n}\} \geq \bigl(1 - g_n(\varepsilon)\bigr)p_\lambda^2. \tag{1.2}$$

This result implies that, on the event where ξ_t^u does not die out quickly, it will not infect v much before time $c_\lambda^{-1}\log n$, but for any time $(1+\varepsilon)c_\lambda^{-1}\log n$ a bit larger than $c_\lambda^{-1}\log n$, vertex v will be infected with conditional probability $\approx p_\lambda$. Since this is true for each vertex v, this suggests that on the event where ξ_t^u does not die out quickly, it will enter a quasi-stationary state at time $\approx c_\lambda^{-1}\log n$ in which the fraction of infected vertices is approximately p_λ. The next theorem states that this is indeed the case.

THEOREM 1.2. *If $0 < \varepsilon < 1/32$, then for any $\delta > 0$ there exist constants $f_n(\delta) \to 0$ as $n \to \infty$ such that for asymptotically almost every G,*

$$\mathbb{P}_G\{(1-\delta)np_\lambda \leq |\bar{\xi}_{(1+\varepsilon)c_\lambda^{-1}\log n}| \leq (1+\delta)np_\lambda\} \geq 1 - f_n(\delta). \tag{1.3}$$

Moreover, for any sequence t_n of times such that $t_n > (1+\varepsilon)c_\lambda^{-1}\log n$,

$$\mathbb{P}_G\{(1-\delta)np_\lambda \leq |\bar{\xi}_{t_n}| \leq (1+\delta)np_\lambda \mid \bar{\xi}_{t_n} \neq \varnothing\} \geq 1 - f_n(\delta). \tag{1.4}$$

How long does the contact process remain in the quasi-stationary state? The next result asserts that it does so for not longer than $e^{\beta n}$, for some constant $\beta > 0$.

THEOREM 1.3. *There exists $\beta > 0$ such that for asymptotically almost every $G \sim \mathcal{G}(n, d)$, as $n \to \infty$,*

$$\mathbb{P}_G\{\bar{\xi}_{\exp(\beta n)} \neq \varnothing\} = 1 - o(1). \tag{1.5}$$

Assertion (1.1) will be proved in Section 3, and assertion (1.2) in Section 4. Theorem 1.3 will be proved in Section 5, and Theorem 1.2 in Section 6.

While preparing this paper, we learned that J.-C. Mourrat and D. Valesin [18] have independently established Theorem 1.3. Because our proof is somewhat different from theirs, and because Theorem 1.3 is a key complement to Theorem 1.2, we include it in this paper.

2. Preliminaries: Contact process on the infinite regular tree.

For the remainder of the paper, the degree $d \geq 3$ will be fixed, so henceforth we shall denote the infinite d-regular tree by $\mathbb{T}$ (or, when several different copies of $\mathbb{T}$ are involved in a single construction, by $\mathbb{T}^1, \mathbb{T}^2, \dots$). Where convenient, we shall view $\mathbb{T}$ as a rooted plane tree, so that every vertex other than the root has 1 adjacent "parent" edge and $d-1$ "offspring" edges, which are ordered left to right.

In this section, $\xi_t = \xi_t^O$ will denote a contact process started from a single vertex O (the *root*) on the infinite d-regular tree $\mathbb{T}$. We shall assume throughout that all contact processes on $\mathbb{T}$ are built using the graphical model of [9] (see also [11], Section 2.2); we will refer to the system of Poisson processes that determine the times of infection attempts and recoveries as the underlying *percolation structure*.

The d-regular tree is a non-amenable graph, in the sense that its Cheeger constant is positive. This can be quantified as follows. For a finite subset S of vertices of $\mathbb{T}$, call $v \in S$ a *border point* if among the d connected components obtained by removing v from $\mathbb{T}$, at least one of them contains no other vertices in S. Let $B(S)$ be the set of border points in S; then

$$|B(S)| \geq \left(1 - \frac{1}{d-1}\right)|S|.$$

See, for instance, Lemma 6.2 of [19] for a proof. We will denote by $h(\mathbb{T})$ the constant $1 - 1/(d-1)$.

2.1. *Contact process on $\mathbb{T}$: Growth estimates.*

The nonamenability of $\mathbb{T}$ implies that the supercritical contact process on $\mathbb{T}$ grows exponentially. Here is a precise formulation, proved in [15] and [16].

PROPOSITION 2.1. *There exist constants $c_\lambda > 0$ and $C_d > 0$ such that*

$$\exp(c_\lambda t) \leq \mathbb{E}|\xi_t| \leq C_d \exp(c_\lambda t). \tag{2.1}$$

COROLLARY 2.2. *There exists $B_1 = B_1(\lambda, d) < \infty$ such that*

$$\exp(c_\lambda t) \le \mathbb{E}\left|\bigcup_{s\le t}\xi_s\right| \le B_1 \exp(c_\lambda t) \qquad \forall t \ge 0. \tag{2.2}$$

Consequently, for any $\varepsilon > 0$,

$$\mathbb{P}\left\{\left|\bigcup_{s\le t}\xi_s\right| \ge B_1 \exp\{(c_\lambda + \varepsilon)t\}\right\} \le \exp\{-\varepsilon t\}. \tag{2.3}$$

PROOF. The lower bound $\exp(c_\lambda t) \le \mathbb{E}|\bigcup_{s\le t}\xi_s|$ follows directly from Proposition 2.1. It is also immediate from Proposition 2.1 that for any $T > 0$ large enough there exists $C_T < \infty$ such that

$$\mathbb{E}\left|\bigcup_{k=1}^{n}\xi_{kT}\right| \le C_d \exp(c_\lambda nT)/\bigl(1 - \exp(-c_\lambda T)\bigr) = C'_T \exp(c_\lambda nT).$$

The difference between the sets $\bigcup_{k=1}^{n}\xi_{kT}$ and $\bigcup_{t\le nT}\xi_t$ is the set of vertices that are infected by vertices in ξ_{kT} between times kT and $KT + T$; these can all be accounted for by running independent contact processes for time 1 starting from vertices in ξ_{kT} for some $k = 1, 2, \dots, N$. Thus,

$$\mathbb{E}\left|\bigcup_{t\le nT}\xi_t\right| \le \mathbb{E}\left|\bigcup_{k=1}^{n}\xi_{kT}\right|\mathbb{E}\left|\bigcup_{t\le T}\xi_t\right|.$$

That the expectation $\mathbb{E}|\bigcup_{t\le T}\xi_t|$ is finite follows by routine arguments, using the graphical construction of the contact process. Finally, (2.3) follows directly from (2.2), by the Markov inequality. □

2.2. *The severed contact process.* We will make frequent use of an auxiliary process, the *severed* contact process. We follow the terminology and notation of [16] and [19]. Define a branch $\mathcal{B}$ to be the connected component of the root in the subgraph obtained by removing a distinguished subset of $d - 1$ edges, each having an endpoint at the root O. The severed contact process is the contact process restricted to $\mathcal{B}$, that is, infection is not allowed to travel across any of the $d - 1$ removed edges. We will use the letter η to denote the severed contact process; in particular, η_t^S is the severed contact process with initial configuration $S \subset \mathcal{B}$, and $\eta_t = \eta_t^O$ the severed contact process started with O infected at time 0. Clearly, the severed contact process η_t is stochastically dominated by the contact process ξ_t. In the standard graphical representation [9] contact process ξ_t and the severed contact process are naturally coupled in such a way that $\eta_t^S \subset \xi_t^S$ for any initial configuration S and all $t \ge 0$. Hence, the expected cardinality of infected sites in the severed contact process is no larger than that of the original one. However, the severed process has comparable cardinality in expectation.

PROPOSITION 2.3. *There exists a constant* $A_1 = A_1(\lambda, d) > 0$ *such that*

$$\mathbb{E}|\eta_t| \geq A_1 \exp(c_\lambda t) \qquad \forall t \geq 0. \tag{2.4}$$

PROOF. This follows by an extension of the arguments of Morrow, Schinazi and Zhang [16], who proved the weaker bound $\mathbb{E}|\eta_t| \geq A_1 \exp(c_\lambda t)/t$. Clearly, the severed contact process η_t is dominated by the nonsevered contact process ξ_t, so by Proposition 2.1,

$$\mathbb{E}|\eta_s| \leq \mathbb{E}|\xi_s| \leq C_d \exp(c_\lambda s).$$

On the other hand, by inequality (5) of [16] and the lower bound in Proposition 2.1,

$$\int_0^t \mathbb{E}|\eta_s|\, ds \geq \frac{1}{\lambda}\left(\frac{1}{d}\mathbb{E}|\xi_t| - 1\right) \geq a_1 \exp(c_\lambda t) \qquad \forall t \geq 1$$

for a suitable constant $a_1 > 0$. This implies that for any $T > 0$,

$$\begin{aligned}
a_1 \exp(c_\lambda nT) &\leq \int_0^{nT} \mathbb{E}|\eta_s|\, ds \\
&\leq \int_0^{(n-1)T} C_d \exp(c_\lambda s)\, ds + \int_{(n-1)T}^{nT} \mathbb{E}|\eta_s|\, ds \\
&= C_d'(\exp(c_\lambda(n-1)T) + \int_{(n-1)T}^{nT} \mathbb{E}|\eta_s|\, ds.
\end{aligned}$$

Hence, for $T > 0$ sufficiently large we have

$$\int_{(n-1)T}^{nT} \mathbb{E}|\eta_s|\, ds \geq \frac{1}{2} a_1 \exp\{c_\lambda nT\}. \tag{2.5}$$

To complete the proof, we use the fact that for any $t, s > 0$,

$$\mathbb{E}|\eta_{t+s}| \geq \big(h(\mathbb{T})\mathbb{E}|\eta_s| - 1\big)\mathbb{E}|\eta_t|. \tag{2.6}$$

This holds because if we run the severed contact process up to time s, then discard those infected vertices that are not border points of η_s, and then run severed contact processes from each of the remaining (border) infected points for time t inside the unoccupied branch at time s, the resulting infection set is dominated by the original severed contact process at time $t + s$.[1] The inequality (2.4) now follows routinely

[1]The -1 on the right side of inequality (2.6) is to account for the possibility that for one of the border points of η_s, the adjacent empty branch might contain the root O. There cannot be more than one such border point, because by definition the adjacent empty branch cannot contain any other points of η_s.

from (2.5) and (2.6), because (2.6) implies that

$$\mathbb{E}|\eta_{nT}| \geq C \int_{nT-T}^{nT} \mathbb{E}|\eta_s|\, ds \quad \text{and}$$
$$\mathbb{E}|\eta_t| \geq C' \mathbb{E}|\eta_{nT-T}| \qquad \forall t \in [nT - T, nT]. \qquad \square$$

By following only those infection trails in the underlying percolation structure that remain in a given branch of $\mathbb{T}$, we obtain a severed contact process: thus, given a contact process ξ_t^A such that $x \in \xi_0$, there is a *severed* contact process η_t^x with initial configuration $\eta_0^x = \{x\}$ that is dominated by ξ_t^A. Clearly, if x, x' are distinct vertices and $B_x, B_{x'}$ are nonoverlapping branches of $\mathbb{T}$ rooted at x, x' then the severed contact processes $\eta_t^x, \eta_t^{x'}$ induced by the percolation structure are *independent*. The independence of severed contact processes in nonoverlapping branches of the tree allows the construction of *Galton–Watson* processes embedded in a contact process on $\mathbb{T}$. We next use such embedded Galton–Watson processes to prove the following.

PROPOSITION 2.4. *For any* $\delta > 0$,

$$\mathbb{P}\{|\eta_t| \geq \exp\{(c_\lambda - \delta)t\} \mid \eta_t \neq \varnothing\} \to 1 \qquad \text{as } t \to \infty. \tag{2.7}$$

PROOF. It suffices to prove this for times $t = nT$ that are integer multiples of a fixed $T > 0$, because for any fixed $T > 0$ the probability that an infected vertex remains infected for at least T time units is e^{-T}, so by the weak law of large numbers, if there are exponentially many vertices infected at time nT then with conditional probability $\to 1$, at least a fraction e^{-2T} will remain alive until time $(N+1)T$.

We will proceed by a comparison argument: in particular, we will show that the contact process $\{\xi_{nT}\}_{n\geq 0}$ at integer multiples of T dominates a Galton–Watson process $\{Z_n\}_{n\geq 0}$ with mean offspring number greater than $\exp(c_\lambda - \delta/2)T$ and finite variance. This will imply the desired result, because (i) on the event that the Galton–Watson process survives,

$$\liminf_{n\to\infty} Z_n / \exp\{n(c_\lambda - \delta/2)T\} > 0,$$

by the Kesten–Stigum theorem for finite variance Galton–Watson processes, and (ii) on the event that the severed contact process survives, at least one of its particles will engender a copy of the Galton–Watson process that survives.

Fix $T > 0$ so large that $h(\mathbb{T})E|\eta_T| \geq e^{c_\lambda T - \delta T/2}$; Proposition 2.3 ensures that this inequality holds for all large T. Let $\{\eta_t\}_{t\leq T}$ be the severed contact process, run for time T. At the terminal time T, discard all vertices of η_T that are not border points, and then run new severed contact processes for time T from each of the remaining infected vertices in the adjacent empty branches of the tree. Because these empty branches do not overlap, the resulting severed contact processes are independent, and up to automorphisms of $\mathbb{T}$ are copies of the original severed contact process $\{\eta_t\}_{t\le T}$. Let Z_2 be the cardinality of the union of these severed contact processes at time $2T$. Now proceed inductively, at each time nT discarding all vertices except border points and then running severed contact processes in the empty branches. If all of these severed contact processes are constructed on the same graphical structure as the original contact process ξ_t, then at any time nT the contact process ξ_{nT} will dominate the union of the severed contact processes, and so

$$|\xi_{nT}| \ge Z_n.$$

That $\{Z_n\}_{n\ge 0}$ is a Galton–Watson process follows by construction, since all of the severed contact process segments are independent copies of $\{\eta_t\}_{t\le T}$. To show that $\mathbb{E}Z_1^2 < \infty$ it suffices to show that $E|\xi_T|^2 < \infty$; this follows because the process $|\xi_t|$ is dominated by a Yule (binary fission) process of rate $d\lambda$. $\square$

REMARK 2.5. It follows that for any $\delta > 0$,

$$\mathbb{P}\bigl\{|\xi_t| \ge \exp\bigl\{(c_\lambda - \delta)t\bigr\} \mid \xi_t \ne \varnothing\bigr\} \to 1 \qquad \text{as } t \to \infty, \tag{2.8}$$

because on the event that the contact process ξ_t survives, at least one of its particles will engender a copy of the *severed* contact process that survives.

2.3. *Pioneer points*. Next, we introduce a concept that will figure prominently in the arguments of Sections 4 and 5. For any vertex $x \in \xi_t$, say that x is a *pioneer point* if $x \in B(\bigcup_{s\le t} \xi_s)$, in other words, there exists a branch of the tree connected to x which has been completely uninfected up to time t. We call such a branch a *free branch*. Clearly, a pioneer point is a border point of ξ_t. We will use ζ_t to denote the collection of pioneer points at time t.

PROPOSITION 2.6. *For any* $\delta > 0$,

$$\mathbb{P}\bigl\{|\zeta_t| \ge \exp\bigl(c_\lambda(1-\delta)t\bigr) \mid |\xi_t| > 0\bigr\} \to 1 \qquad \textit{as } t \to \infty.$$

PROOF. In view of Proposition 2.4 and Corollary 2.2, we may assume that on the event $\{|\xi_t| > 0\}$, both of the following events occur:

$$|\xi_t| \ge \exp\bigl(c_\lambda(1-\delta)t\bigr) \quad \text{and}$$
$$\Bigl|\bigcup_{s\le t} \xi_s\Bigr| \le \exp\bigl(c_\lambda(1+\delta)t\bigr).$$

Given these, it is easy to deduce that there is a subset $G_t \subset \xi_t$ of cardinality larger than

$$\frac{1}{2}\exp\bigl(c_\lambda(1-\delta)t\bigr)$$

such that adjacent to each vertex $y \in G_t$ is a branch $\mathcal{B}_y$ not containing the root which contains no more than $\exp(3c_\lambda \delta t)$ vertices in $\bigcup_{s \le t} \xi_s$; moreover, for distinct vertices $y, y' \in G_t$ the branches $\mathcal{B}_y, \mathcal{B}_{y'}$ do not overlap. Hence, for each vertex $y_0 \in G_t$ there exists a "downward" path $y_0 y_1 \cdots y_L$ in $\mathcal{B}_{y_0}$ (i.e., a path such that each vertex y_i is a "child" of y_{i-1}) of length $L \le \log_{d-1}(\exp(3c_\lambda \delta t)) \le 6\delta c_\lambda t$ such that y_L is adjacent to a branch with no vertices in $\bigcup_{s \le t} \xi_s$.

Fix such a path $y_0 y_1 \cdots y_L$ in $\mathcal{B}_{y_0}$, where y_0 is a vertex in G_t. Consider the event A_{y_0} that in the time interval $[t, t + 6\delta c_\lambda t]$ infection spreads along the path $y_0 y_1 \cdots y_L$ in such a way that at time $t + 6\delta c_\lambda t$ the vertex y_L is a pioneer point. This event has (conditional) probability at least $q^{6\delta c_\lambda t}$, where $q > 0$ is the probability that in a contact process initiated by a single infected vertex v infection spreads from v to a fixed neighbor vertex w *and to no other vertex* in the time interval $[0, 1]$. For distinct vertices $y, y' \in G_t$, the events A_y and $A_{y'}$ are independent, since the branches $\mathcal{B}_y, \mathcal{B}_{y'}$ do not overlap, so the number of vertices $y \in G_t$ for which A_y occurs stochastically dominates the Binomial distribution

$$W \sim \text{Binomial}\biggl(\frac{1}{2}\exp\bigl(c_\lambda(1-\delta)t\bigr), q^{6\delta c_\lambda t+1}\biggr),$$

and if δ is sufficiently small then with probability approaching 1 as $t \to \infty$,

$$W > \frac{1}{2}\exp\bigl(c_\lambda(1-\delta)t\bigr)q^{6\delta c_\lambda t+1} > \exp\bigl(c_\lambda(1-D\delta)t\bigr),$$

for some $D > 0$. Therefore, conditional on the event $|\xi_t| > 0$, with probability approaching 1,

$$|\zeta_{(1+6\delta c_\lambda)t}| > \exp\bigl(c_\lambda(1-D\delta)t\bigr).$$

Since $\delta > 0$ can be made arbitrarily small, the desired conclusion follows easily. $\square$

Similarly, for the *severed* contact process η_t, we define *pioneer points* to be those vertices in η_t that are also border points of $\bigcup_{s \le t} \eta_s$. Denote the set of such pioneer points by ψ_t. By the same argument as in the proof of Proposition 2.6, we obtain the following estimate.

PROPOSITION 2.7. *For any* $\delta > 0$,

$$\mathbb{P}\bigl\{|\psi_t| \ge \exp\bigl(c_\lambda(1-\delta)t\bigr) \mid |\eta_t| > 0\bigr\} \to 1 \qquad \textit{as } t \to \infty.$$

3. Contact process on a random regular graph: Constructions. There are two layers of randomness in our model: first, the randomness implicit in choosing a random d-regular graph G, and second, that involved in the evolution of the contact process on the chosen graph G. Although it is most natural to think of these two layers of randomness sequentially—first, choose the graph, then run the contact process—other constructions, in which G is built in stages as the contact process runs, will be needed for our arguments. In this section, we shall describe several constructions, which will couple the contact process (or several contact processes) on G with a "covering" contact process on an infinite d-regular tree $\mathbb{T}$. To force these constructions to produce random regular graphs, we will show that the induced constructions of a random graph G on the vertex set $[n]$ follow the so-called *configuration model*.

3.1. *The configuration model for random regular graphs*. *Assume henceforth that dn is even.* The *configuration model*, first introduced by Bollobás [2] (also see [3] and [22]), works as follows. To each of the n vertices u, associate d distinct half-edges (u, i), and perform a uniform perfect matching on these dn half-edges. Using this matching, construct a (multi-)graph by placing an edge between vertices u and v for every pair of half-edges (u, i) and (v, j) that are matched. The resulting graph need not be connected, and it might have multiple edges and self-loops; however, the probability that the configuration model produces a simple, connected graph is bounded away from 0 as $n \to \infty$ (cf. [22]). Moreover, given that the resulting graph is simple (that is, has no self-loops or multiple edges), it is uniformly distributed over $\mathcal{G}(n, d)$. Thus, whenever an event holds w.h.p. for the (multi-)graph obtained from the configuration model, it also holds w.h.p. under the uniform distribution on $\mathcal{G}(n, d)$.

An important feature of the configuration model is that the random matching of half-edges can be done in stages, one edge at a time, using any rule for choosing the first half-edge, as long as the second half-edge is chosen uniformly at random from the remaining half-edges (see [22]). In the constructions to follow, we will use the evolution of the contact process (or processes) to determine the schedule of pairings for half-edges.

Because the configuration model might produce a graph that is not simple or connected, we must make clear how a contact process on a d-regular multi-graph evolves. Our convention will be that infection attempts from an infected vertex v will always occur at rate λ; when such an attempt is made, one of the d edges incident to v is chosen at random, and the vertex at the other end of this edge is made the target of the infected attempt. Thus, if some vertex u is connected to v by $k \geq 2$ edges, the overall rate at which infection travels from v to u is $k\lambda$. If v has $k \geq 1$ self-loops, then the overall rate of infection out of v is $(d-k)\lambda$.

3.2. *Grow and explore*: *The base construction*. We begin by showing how to build the contact process and the random graph G in tandem in such a way that the contact process ξ_t^u has initial configuration $\xi_0^u = \{u\}$. We will later refer to this as the *base construction*. In this construction, edges will be added only at those times t when the contact process attempts a new infection from a vertex whose neighborhood structure has not yet been completely determined. Thus, at any time $t \geq 0$ there will be a (random) set U_t of unpaired half-edges; the set U_t

will decrease with t. At time 0, only vertex u is infected, and no edges are yet determined, so U_0 is full, that is, it contains all nd half-edges.

The recovery and infection times of the contact process are determined by a system of independent Poisson processes attached to the n vertices of the graph, two to each vertex (one for recoveries, the other for outgoing infections). At any time t when an infected vertex v attempts an infection, one of the d half-edges incident to v is selected uniformly at random. If this half-edge is already matched to a half-edge (w, j), then vertex w is infected, if it is currently healthy, or left infected if already infected, and the set U_t remains unchanged. (If $v = w$, then v remains infected.) If, on the other hand, one of the unmatched half-edges (v, i) incident to v is selected then one of the other remaining unmatched half-edges (w, j) is chosen at random from $U_t \setminus \{(v, i)\}$ and matched with (v, i), and vertex w is infected. After this, the half-edges (v, i) and (w, j) are removed from U_t. As $t \to T \leq \infty$, the set U_t will decrease to a limiting set U_T. $U_T = \varnothing$ then all half-edges will have been paired, and the resulting graph G will be connected. It is possible, though, that $U_T \neq \varnothing$. In this case, U_T will contain an even number of half-edges; these can then be randomly paired to complete the random graph G. The resulting graph G need not be connected.

PROPOSITION 3.1. *Conditional on the event that the resulting graph G is simple, the pair $(G, (\xi_t)_{0\leq t\leq T})$ will have the same joint distribution as for the contact process on a random regular graph.*

PROOF. First of all, G is uniform over $\mathcal{G}(n, d)$, because whenever we pair two unmatched half-edges, the second half-edge is always chosen uniformly at random from the unmatched pool.

Secondly, the interoccurrence times between recoveries are i.i.d. exponentials, as are the times between infection attempts. Moreover, in each infection attempt the active vertex chooses one of its incident half-edges, uniformly at random, as the direction of propagation, and so conditional on the final configuration of G the target vertex will be chosen uniformly among the d neighbors of the active vertex. Therefore, $(\xi_t)_{0\leq t\leq T}$ is a version of the contact process on G. $\square$

The base construction assumes a singleton initial configuration and that the graph is initially completely unexplored. It is clear that the construction can be easily modified so as to work with an arbitrary initial state $\xi_0 \subset [n]$ and with part of the graph G initially explored.

3.3. *Covering contact processes.* Next, we will describe several extensions of the base construction that will produce, along with the random graph G and the contact process ξ_t, a contact process $\tilde{\xi}_t$ (or several independent contact processes) on the infinite tree $\mathbb{T}$ and a covering map from $\mathbb{T}$ to G that (partially) projects $\tilde{\xi}_t$ to ξ_t.

3.3.1. *Cover tree version with singleton initial configuration.* As in the base construction, we shall only consider the initial condition, in which the contact process is initiated by a single infected vertex $u \in [n]$. We will use a contact process $\tilde{\xi}_t$ on the infinite cover tree $\mathbb{T}$, initiated at the root, and auxiliary randomization to assign labels $v = \phi(\tilde{v}) \in [n]$ to the vertices $\tilde{v}$ of $\mathbb{T}$. This assignment of labels to vertices of $\mathbb{T}$ will result in a (random) *labeling function*

$$\phi : \mathbb{T} \longrightarrow [n]$$

(here we abuse notation and use $\mathbb{T}$ to denote the vertex set of the tree $\mathbb{T}$) that will determine the covering map from $\mathbb{T}$ to G *and* the edge structure of G. Auxiliary randomization will also be used to partition the vertices in $\tilde{\xi}_t$ into two classes, so that at any time t,

$$\tilde{\xi}_t = \tilde{\xi}_{t,\mathrm{BLUE}} \cup \tilde{\xi}_{t,\mathrm{RED}}.$$

Only the BLUE vertices will figure into the evolution of ξ_t, which will be defined by the rule

$$\xi_t = \phi(\tilde{\xi}_{t,\mathrm{BLUE}}).$$

Where appropriate, we will denote vertices of $\mathbb{T}$ via a tilde, for example, $\tilde{x}$, and use x to denote the corresponding vertex $x \in [n]$, so that $\phi(\tilde{x}) = x$.

Fix a vertex $u \in [n]$; the singleton $\{u\}$ will be the initial configuration of the (projected) contact process on G. Denote the root vertex of the infinite tree $\mathbb{T}$ by $\tilde{u}$, and declare $\phi(\tilde{u}) = u$. Let $\tilde{\xi}_t$ be a contact process on $\mathbb{T}$ with initial configuration $\tilde{\xi}_0 = \tilde{u}$. Assume that this is constructed in the usual way, using a system of independent Poisson processes attached to the vertices $\tilde{v}$ of $\mathbb{T}$ to determine the times at which recoveries and attempted infections occur, and independent Uniform-[0, 1] random variables for auxiliary randomization that are used to determine the edges along which attempted infections are directed. Call this system of Poisson processes and auxiliary uniform random variables the *percolation structure* underlying the construction. Assume further that there are countably many Uniform-[0, 1] random variables, independent of the percolation structure, that can be used for the random choices to be made in the construction of the labeling function ϕ and the partitioning of the infected vertices $\tilde{\xi}_t$ into red and blue. At time $t = 0$, only the label $\phi(\tilde{u}) = u$ is determined. The function ϕ will be augmented only at those times when a *blue* vertex of $\tilde{\xi}_t$ attempts an infection, in such a way that at any such attempt the vertex $\tilde{x} \in \tilde{\xi}_{t,\mathrm{BLUE}}$ attempting the infection is labeled.

The rules for the modification of the function ϕ and assignment of colors (blue or red) to infected vertices are as follows. Suppose that at time t an infected vertex $\tilde{x} \in \tilde{\xi}_{t,\mathrm{BLUE}}$ attempts an infection. At this time t, some of the neighbors of $\tilde{x}$ might have been labeled, and others might not; because the infection at $\tilde{x}$ has resulted from a chain of infections by *blue* vertices, the parent of $\tilde{x}$ must also be labeled. Denote by $\tilde{y}_1, \dots, \tilde{y}_\ell$ the neighbors that have already been labeled, with $\phi(\tilde{y}_i) = y_i$, and by $\tilde{z}_1, \tilde{z}_2, \dots, \tilde{z}_{d-\ell}$ the neighbors that have not yet been labeled. At time t, some of the neighbors of x in G will have been determined, *including* $y_1, y_2, \dots, y_\ell$, but possibly also some others, which we denote by $y_{\ell+1}, y_{\ell+2}, \dots, y_{\ell+k}$, where $\ell + k \le d$. Because the infection attempt entails choosing one of the d neighbors of $\tilde{x}$ at random to serve as the target of the attempt, there are 3 possibilities:

(U1) With probability l/d, one of the vertices $\tilde{y}_1, \dots, \tilde{y}_\ell$ is chosen. In this case, ϕ is not augmented.

(U2) With probability k/d, one of the vertices $\tilde{z}_1, \dots, \tilde{z}_{d-\ell}$, say $\tilde{w}$, is chosen randomly and one of the labels $y_{\ell+1}, \dots, y_{\ell+k}$ is chosen uniformly at random to serve as the label $\phi(\tilde{w})$ for the vertex $\tilde{w}$.

(U3) With probability $1 - l/d - k/d$, an unused half-edge (x, i) incident to x is chosen randomly, and another unused half-edge (w, j) is then chosen randomly from among all remaining unused half-edges and matched with (x, i). Then one of the vertices $\tilde{z}_1, \dots, \tilde{z}_{d-\ell}$, say $\tilde{w}$, is randomly selected and labeled w [i.e., ϕ is augmented so that $\phi(\tilde{w}) = w$]. When this happens, we add an edge connecting x and w to G and remove the two half-edges (x, i) and (w, j) from the set U_t of unused half-edges.

To complete the construction, we must specify how the vertices of $\tilde{\xi}_t$ are to be colored (red or blue). This is done as follows. First, whenever a vertex $\tilde{v} \in \tilde{\xi}_t$ *recovers* (i.e., $\tilde{v}$ is removed from $\tilde{\xi}_t$) it loses its color. Second, when a *red* vertex attempts to infect a vertex $\tilde{v}$, the target vertex $\tilde{v}$ is assigned the color red unless it was already blue, in which case it remains blue. Third, when a *blue* vertex attempts to infect a vertex $\tilde{v}$, the target vertex $\tilde{v}$ is assigned the color blue unless $\tilde{v}$ is also assigned a label $v = \phi(\tilde{v})$ that is already assigned to another blue vertex; in this case $\tilde{v}$ is colored red. Thus, at any time t the mapping ϕ is one-to-one on $\tilde{\xi}_{t,\mathrm{BLUE}}$. This ensures that in the projected contact process $\xi = \phi(\tilde{\xi}_{\mathrm{BLUE}})$ the recovery and infection rates are 1 and λd, respectively, at any infected vertex.

As in the base construction, the pool U_t of unused half-edges decreases with t. If at time $T \le \infty$ the set U_T is empty, then the graph G will be completely determined; if on the other hand $U_T \ne \varnothing$ then the remaining half-edges in U_T can be randomly paired to complete the specification of G. In either case, the labeling function $\phi : \mathbb{T} \to G$ might not be completely determined by time T, but if so then the edge structure of G (which is completed at time T) will uniquely determine an extension of ϕ to the entire tree $\mathbb{T}$ that makes $\phi : \mathbb{T} \to G$ a covering transformation. By construction, when $\phi(\tilde{\xi}_{t,\mathrm{BLUE}})$ attempts an infection, any of its d neighbors is equally likely to be the target of the infection attempt, and when a new edge is added to G it follows the configuration model. Therefore, the projection obeys the same rules as in the base construction of grow and explore described above. This proves the following.

PROPOSITION 3.2. *The pair* $(G, (\phi(\tilde{\xi}_{t,\mathrm{BLUE}}))_{0 \le t \le T})$ *obtained by running the cover tree version of grow and explore has the same law as in the base construction of the grow and explore process.*

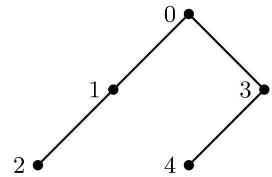

FIG. 1. *This graph records the order that vertices on a* 3*-regular cover tree are first infected.* 0 *indicates the vertex is in the initial configuration.*

The cover tree version of grow and explore that we have just described has two constituent processes, the contact process $\tilde{\xi}_t$ on the cover tree and the labeling process. We will refer to these as the *growth* process and the *exploration* process, respectively.

To emphasize the initial configuration $\{u\}$, its corresponding contact process on the cover tree is denoted as $\tilde{\xi}_t^u$. Later we will run several contact processes on multiple vertices, and adding the superscript will help us distinguish them.

The following figures illustrate concrete examples of how the construction works.

3.3.2. *Variation*: *Sector-priority labeling and coloring*. In the second-moment argument to be carried out in Section 4 below, a more complicated labeling and coloring scheme will be used, in which priority will be given to vertices x in one or more of the *severed contact processes* contained in the contact process $\tilde{\xi}_t$. Fix a time $T < \infty$, and let $\tilde{\xi}_t = \tilde{\xi}_t^u$ be a contact process on $\mathbb{T}$ initiated by a single infected vertex at the root $\tilde{u} \in \mathbb{T}$. Recall that $\tilde{\zeta}_T$ is the set of *pioneer points* of $\tilde{\xi}$ at time T. For each vertex $i \in \tilde{\zeta}_T$, there is a *free branch* of the tree adjacent to i, that is, a branch that has not been invaded by the contact process up to time T. Fix such a free branch $\mathcal{B}_i$ (there might be several) and let $(\tilde{\eta}_t^i)_{t \geq T}$ be the post-T *severed* contact process in this branch (thus, $\tilde{\eta}_t^i$ consists of all vertices $x \in \tilde{\xi}_t$ that are reached by infection trails in the percolation structure which lie entirely in the

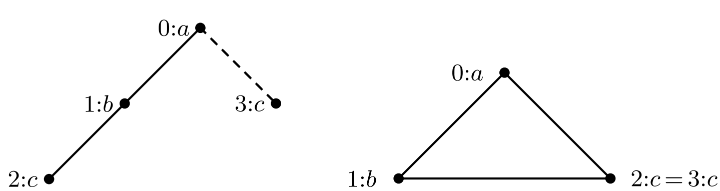

FIG. 2. *Given Figure* 1, *we try to project it onto the finite* 3*-regular graph. If the top vertex has already been labelled* a, *we sequentially label vertices* 1, 2, 3, 4 *according to the law described earlier. When we label vertex* 3 *we accidentally use label* c *again, so the finite graph immediately becomes the one on the right. Recall that we only allow one infection per vertex, so whenever we observe multiple infections at the same time on a vertex we will remove the whole infection trail except the chronologically first one.*

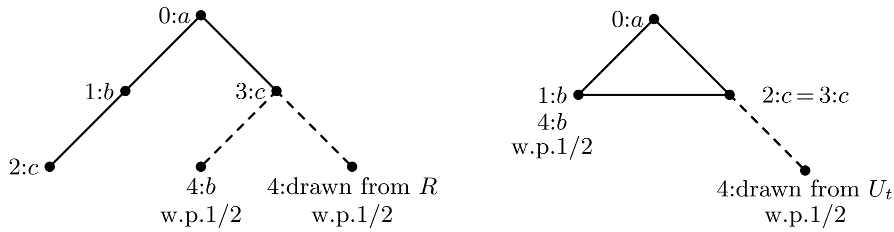

FIG. 3. *Given Figures 1 and 2, suppose the infection trail coming from vertex* 3 *to vertex* 4 *is not removed* (*which is possible if when* 3 *is infected, the infection at* 2 *already dies*), *then when we label vertex* 4, *according to our law it has chance* $1/2$ *to be b and chance* $1/2$ *to be drawn from* U_t, *the unused pool. The left graph is on the cover tree and the right one is on the finite graph.*

free branch $\mathcal{B}_i$). Keep in mind that the free branch might contain vertices of $\tilde{\xi}_t$ that are not in $\tilde{\eta}^i_t$; however, since the free branches attached to different pioneer points i, i' are disjoint, the severed contact processes $\tilde{\eta}^i$ and $\tilde{\eta}^{i'}$ will never collide.

The labeling function $\varphi : \mathbb{T} \to [n]$ is constructed in two stages, together with a partition of $\tilde{\xi}_t$ into *blue* and *red* vertices. The labeling rules are the same as in Section 3.3.1, and up to time T, the rules for coloring are also the same. Beginning at time T, however, the rules for assigning colors are modified as follows. At time T, each blue vertex in the set of pioneer points $\tilde{\zeta}_T$ is assigned *priority* 1; all other blue vertices in $\tilde{\xi}_T$ are assigned *priority* 2, and all red vertices are assigned *priority* 3. Uncolored vertices—that is, vertices not in $\tilde{\xi}_T$—-are assigned *priority* 4. At any time $t \geq T$ when a vertex $x \in \tilde{\xi}_t$ recovers, it loses its color and reverts to the lowest priority 4. When a vertex $x \in \tilde{\xi}_t$ attempts an infection of a vertex $\tilde{v} \in \tilde{\xi}_t$ with *the same or higher* priority, the status of $\tilde{v}$ is left unchanged. When a vertex $x \in \tilde{\xi}_t$ attempts an infection of a vertex $\tilde{v} \in \tilde{\xi}_t$ with *lower* priority, $\tilde{v}$ is assigned the color and priority of x *unless* x is blue and $\tilde{v}$ has been assigned a label $v = \phi(\tilde{v})$ that is also assigned to another blue vertex z. There are then two possibilities: (i) if z has priority 1 then $\tilde{v}$ is assigned the color *red* and priority 3; or (ii) if z has priority 2 (and, therefore, color blue) and the infecting vertex x has priority 1 then z becomes red and is given priority 3, while $\tilde{v}$ becomes blue and is given priority 1.

These rules ensure that when blue particles in any of the severed contact processes $\tilde{\eta}^i$ attempt infections, the target vertices will always become blue unless their labels already belong to other blue vertices in one of the severed contact processes $\tilde{\eta}^{i'}$. They also guarantee that at any time t the labeling function ϕ will be one-to-one on $\tilde{\xi}_t$. Finally, the labeling rules guarantee that the edge structure of the graph G will follow the configuration model, and so the pair $(G, (\phi(\tilde{\xi}_{t,\mathrm{BLUE}}))$ has the same law as in the base construction of the grow and explore process.

3.3.3. *Several independent contact processes.* In Section 4, we will find it necessary to run two independent contact processes simultaneously on the same random graph G. Furthermore, we will want the labeling and coloring done in two stages, the first in which labels and colors are assigned *synchronously*, but in the second *sequentially*, with *sector priority* rules as described in Section 3.3.2. In both stages, the construction of the graph G will follow the configuration model, and the contact processes on G will be (partial) projections of contact processes on covering trees.

We will construct several contact processes $\xi_t^1, \xi_t^2, \dots, \xi_t^k$ on G with singleton initial configurations $\xi_t^i = u_i$, where the vertices $u_i \in [n]$ are distinct, in such a way that conditional on the realization of the random graph G the processes ξ_t^i are independent. Let $\tilde{\xi}_t^i$ be independent contact processes on copies $\mathbb{T}^i$ of $\mathbb{T}$, each initiated at the root $\tilde{u}_i$ of $\mathbb{T}^i$. We will construct labeling functions $\phi_i : \mathbb{T}^i \to [n]$ for each cover tree and partitions of the infected sets $\tilde{\xi}_t^i$ into *red* and *blue* vertices, in such a way that the projections

$$\xi_t^i = \phi_i(\tilde{\xi}_{t,\mathrm{BLUE}}^i)$$

are (conditionally on G) independent contact processes. Begin by setting $\phi_i(\tilde{u}_i) = u_i$.

Stage 1: *Synchronous updating*. In the first stage of the construction, in which all contact processes will run from time $t = 0$ until time $t = T_1$, there is a single pool of unused edges U_t which changes only at times when one of the $\tilde{\xi}_t^i$ attempts an infection. The initial pool U_0 consists of all nd half-edges. The labeling functions ϕ_i are initially set so that only the root vertex of $\mathbb{T}^i$ is labeled (with the vertex label u_i), and ϕ_i is updated only when the process $\tilde{\xi}_{t,\mathrm{BLUE}}^i$ attempts an infection. At any such time, ϕ_i is updated by the same rule as in Section 3.3.1. [Note that if the infection attempt originates at $\tilde{x} \in \tilde{\xi}_t^i$, the set $\{y_r\}_{1 \le r \le \ell + k}$ of neighbors of $x = \phi_i(\tilde{x})$ that are already labeled might include vertices whose labels were assigned by infection attempts in some of the other covering contact processes $\tilde{\xi}^j$.] Thus, whenever the projected process ξ_t^i attempts an infection, any of its d neighbors is equally likely to be the target. If such an attempt calls for the use of new half-edges, these are deleted from the pool U_t, as in Section 3.3.1. The rule for assigning colors to vertices of $\tilde{\xi}_t^i$ is the same as in Section 3.3.1:

(i) vertices lose their colors when they recover (i.e., leave $\tilde{\xi}_t^i$);

(ii) an infection attempt by a red vertex changes the color of the target vertex to red *unless* the target vertex is blue, in which case it remains blue; and

(iii) an infection attempt by a blue vertex changes the color of the target vertex to blue unless the target vertex is assigned a label already assigned to another blue vertex, in which case the target becomes red.

Thus, each ϕ_i is bijective on $\tilde{\xi}_{t,\mathrm{BLUE}}^i$, but blue vertices in different contact processes $\tilde{\xi}^i$ and $\tilde{\xi}^j$ might project to the same vertex in G.

Stage 2: *Sequential updating with sector priority.* In stage 2, labeling and coloring will be done *sequentially*, first for $\tilde{\xi}^1$, then for $\tilde{\xi}^2$, and so on. As in Section 3.3.2, the labeling and coloring will be done using sector priority rules for each of the processes $\tilde{\xi}^i$.

At the end of stage 1, the pool of unused half-edges U_{T_1} will consist only of those half-edges not drawn during stage 1, and the labeling functions ϕ_i will have been at least partially specified. Beginning with $\tilde{\xi}^1$, the infected vertices at time T_1 are assigned *priorities*: blue vertices that are also pioneer points get priority 1, blue vertices that are not pioneer points priority 2, red vertices priority 3 and uncolored (i.e., uninfected) vertices priority 4. Then, during time interval $(T_1, T_{2,1}]$, coloring and labeling for vertices in $\tilde{\xi}^1$ are carried out according to the rules set out in Section 3.3.2. At the end of this period, the pool of unused half-edges will have been reduced to a subset $U_{T_1}^2$ of U_{T_1}, and thus more edges of the graph G will be determined. Beginning with this set of constraints, coloring and labeling for vertices in $\tilde{\xi}^2$ in the time interval $(T_1, T_{2,2})$ is then carried out, once again using sector-priority rules. At the end of this period the pool of unused half-edges will have been reduced even further, to a subset $U_{T_1}^3$ of $U_{T_1}^2$, and still more edges of the graph G will be determined. This process continues, sequentially, for $\tilde{\xi}^3$, then $\tilde{\xi}^4$, etc., until all of the covering processes have been colored and labeled (in time intervals $(T_1, T_{2,j}]$ that may differ for different j).

In this scheme, the contact processes $\tilde{\xi}_t^i$ are followed only for finite times $t \leq T_{2,i} < \infty$. Once the labeling and coloring in stages 1 and 2 are complete, the labeling functions φ_i and the edge structure of the graph G will be at least partially determined. To finish the construction of G, we then randomly pair the half-edges remaining in the set $U_{T_1}^{k+1}$ of half-edges not drawn during stages 1 and 2; this completes the specification of the graph G. Once G is determined, the labeling functions ϕ_i are extended to the full trees $\mathbb{T}_i$ in a manner consistent with the edge structure of G.

As in the earlier constructions, in both the synchronous and the sequential schemes the rules of labeling and assignment guarantee the following.

PROPOSITION 3.3. *The random graph G so constructed follows the same law as in the configuration model, and the processes ξ_t^i evolve as* (*conditionally*) *independent contact processes on G.*

It should be clear that more complicated constructions can also be used, in which synchronous and sequential updating are used alternately, or non-singleton initial configurations for the contact processes ξ^i are used.

3.4. *Proof of assertion* (1.1) *of Theorem* 1.1. Let $\xi_t^u = \phi(\tilde{\xi}_{t,\mathrm{BLUE}})$ be the contact process constructed from the cover tree version of the grow and explore process described in Section 3.3.1. Let $t_1 = (1-\varepsilon)\log n/c_\lambda$. To prove (1.1), it suffices to show that

$$\mathbb{P}\{\exists s \le t_1 \text{ such that } v \in \xi_s^u\} \to 0 \qquad \text{as } n \to \infty, \tag{3.1}$$

because if $\mathbb{P}\{\exists s \le t_1 s.t. v \in \xi_s^u\} \le \beta_n$ where $\beta_n \to 0$ as $n \to \infty$, then by Markov's inequality,

$$\mathbb{P}\{G : \mathbb{P}_G(\exists s \le t_1 \text{ such that } v \in \xi_s^u) \ge \sqrt{\beta_n}\} \le \sqrt{\beta_n}.$$

Now the event that $v \in \xi_s^u$ for some $s \le t_1$ coincides with the event that at least one vertex in $\bigcup_{s \le t_1} \xi_s^u$ is assigned label v. Consequently, to bound the probability of this event, it suffices to show that:

(1) as $n \to \infty$, $\mathbb{P}\{|\bigcup_{s \le t_1} \tilde{\xi}_s^u| \le n^{1-\varepsilon/2}\} \to 1$; and

(2) given $|\bigcup_{s \le t_1} \tilde{\xi}_s^u| \le n^{1-\varepsilon/2}$, the conditional probability that label v is not used in the labeling process before time t_1 approaches 1.

Assertion (1) follows directly from Corollary 2.2. To prove assertion (2), observe that, because all labels other than u are equally likely to be used in the labeling process, and because these probabilities add up to at most $n^{1-\varepsilon/2}$, the chance that label v appears in $\bigcup_{s \le t_1} \tilde{\xi}_s^u$ is at most of order

$$n^{1-\varepsilon/2}/(n-1) \to 0. \qquad \square$$

4. A second moment argument.

4.1. *Heuristics and strategy.* In this section, we shall prove assertion (1.2) of Theorem 1.1. This states that for any two vertices $u, v \in [n]$ the conditional probability, given the graph G, that $v \in \xi_{t_+}^u$ converges to p_λ^2 as $n \to \infty$, where

$$t_+ := (1+\varepsilon)c_\lambda^{-1} \log n.$$

Following is a heuristic explanation. Recall that the contact process is self-dual: in particular, the Poisson processes used in the standard graphical construction can be reversed without change of distribution. Thus, the event that $v \in \xi_t^u$ has the same $\mathbb{P}_G$-probability as the event that $u \in \xi_t^v$, and these events have the same $\mathbb{P}_G$-probability that two *independent* contact processes ξ_s^v and ξ_s^u started at u and v will intersect at time $t/2$. This will only happen if both contact processes survive for time $t/2$; and since the contact processes on G are stochastically dominated by contact processes on covering trees $\mathbb{T}$, the probability that both will survive for a large time t (call this event *quasi-survival*) will be bounded by, and approximately equal to, p_λ^2. Hence, for large t, with high probability,

$$\mathbb{P}_G\{v \in \xi_t^u\} = \mathbb{P}_G\{\xi_{t/2}^u \cap \xi_{t/2}^v \ne \varnothing\} \le p_\lambda^2(1+o(1)). \tag{4.1}$$

This argument shows that p_λ^2 is the largest possible asymptotic value for the probability in relation (1.2). To show that this value is actually attained, we will argue that conditional on the event of simultaneous quasi-survival for two independent contact processes ξ_s^u and ξ_s^v, the random sets $\xi_{t_+/2}^u$ and $\xi_{t_+/2}^v$ will almost certainly overlap. To see this, observe that on the event of quasi-survival, the cardinality of $\xi_{t_+/2}^u$ will be at least $n^{1/2+\alpha}$ for some $\alpha > 0$ depending on ε, and that $\xi_{t_+/2}^v$ is approximately distributed as a random subset of $[n]$ of cardinality $n^{1/2+\alpha}$. Since two such independent random subsets will intersect with high probability, this suggests that

$$\mathbb{P}_G\{\xi_{t+/2}^u \cap \xi_{t+/2}^v \neq \varnothing\} \approx p_\lambda^2.$$

Unfortunately, because the labeling processes used in constructing the two contact processes ξ_s^v and ξ_s^u will interfere, the random sets $\xi_{t_+/2}^u$ and $\xi_{t_+/2}^v$ are not independent. To circumvent this difficulty, we will use the two-stage construction described in Section 3.3.3, in which synchronous labeling and coloring are done up to an initial time t_1, and then sequential coloring, with sector priority rules, is adopted for the second stage. Set

$$\begin{aligned}
t_1 &= (1-\varepsilon)\log n/2c_\lambda,\\
t_2 &= (1+3\varepsilon)\log n/2c_\lambda \quad \text{and}\\
\Delta t &= t_2 - t_1 = 2\varepsilon \log n/c_\lambda.
\end{aligned}$$

We will show, using the estimates of Section 2, that with probability near 1, the labeling processes induced by the contact processes $\tilde{\xi}^u$ and $\tilde{\xi}^v$ up to time t_1 do not interfere. We will then follow the labeling processes induced by the descendant (post-t_1) contact processes of the *pioneer points* at time t_1 in *pairs*. The number of such pairs will be large (on the event of quasi-survival), but the contact process engendered by any particular pioneer point will infect only a (relatively) small number of vertices, so for any particular pair it will be unlikely that their induced labeling processes interfere.

For notational ease, we will let the descendant processes evolve for times 1 and $\Delta t + 1$ (since time t_1) rather than for equal times, and we will use as our target time $t_+ + 2$ instead of t_+. Thus, our objective will be to show that

$$\mathbb{P}\{\xi_{t_2+1}^u \cap \xi_{t_1+1}^v \neq \varnothing\} \geq \big(1-o(1)\big)p_\lambda^2. \tag{4.2}$$

STRATEGY OF THE PROOF. The event of interest in (4.2) involves the two processes ξ_t^u and ξ_t^v, which, conditional on the realization of the graph G, are assumed to evolve as independent contact processes on G. We shall assume throughout that these processes are constructed using the two-stage grow-and-explore process described in Section 3.3, with sequential updating and sector-priority rules, so that ξ_t^u and ξ_t^v are partial projections of contact processes $(\tilde{\xi}_t^u)_{t\leq t_2+1}$ and $(\tilde{\xi}_t^v)_{t\leq t_1+1}$ on distinct covering trees $\mathbb{T}^u$ and $\mathbb{T}^v$. Up to time t_1, labeling will be synchronous; then sequential updating will be used, first with the labels for $\mathbb{T}^u$ assigned using $\tilde{\xi}_t^u$ in the time interval $t \in [t_1, t_2+1]$ and then with labels assigned on $\mathbb{T}^v$ using

$\tilde{\xi}^v_t$ in the time interval $t \in [t_1, t_1 + 1]$. We will refer to the time interval $[0, t_1]$ as *stage 1*, and the two subsequent updating stages as *stage 2a* and *stage 2b*.

Fix $0 < \delta \ll \varepsilon < 1/32$. By the results of Section 2, conditional on the event that both contact processes $\tilde{\xi}^u_t$ and ξ^v_t survive for time t_1 the sets of pioneer points, $\tilde{\zeta}^u_{t_1}$ and $\tilde{\zeta}^v_{t_1}$ will, with high probability, each have cardinality at least $n^{(1-\varepsilon)(1-\delta)/2}$, while $\bigcup_{s\le t_1} \tilde{\xi}^u_s$ and $\bigcup_{s\le t_1} \tilde{\xi}^v_s$ will have fewer than $n^{(1-\varepsilon)(1+\delta)/2}$ elements. Consequently (cf. Proposition 4.1 below), with probability near 1, *no vertex label* $v \in [n]$ *is sampled more than once* in stage 1, and consequently no vertex in either $\bigcup_{s\le t_1} \tilde{\xi}^u_s$ or $\bigcup_{s\le t_1} \tilde{\xi}^v_s$ is colored red. Henceforth, we shall refer to this as a *favorable stage 1*.

Our goal is to show that conditional on a favorable stage 1, if we run sequential updating on $\tilde{\xi}^u_s$ for another time $\Delta t + 1$, and then on $\tilde{\xi}^v_s$ for another time 1, then with high probability, there will be at least one common label between them at the end. To accomplish this, we will follow the labeling processes along the infection trails descendant from the pioneer points $i \in \tilde{\zeta}^u_{t_1}$ and $j \in \tilde{\zeta}^v_{t_1}$; and furthermore, we shall only keep track of the labels assigned by the descendant *severed* contact processes $\tilde{\eta}^i_s$ and $\tilde{\eta}^j_s$ in free branches attached to these pioneer points. Since these free branches are nonoverlapping, the descendant severed contact processes are all (conditionally, given stage 1) independent. Observe, however, that the induced labeling processes are not independent, because they use a common pool of unused half-edges (see Section 3.3.3).

For each pair (i, j), let I_{ij} be the indicator of the event that at the end of stage 2 there is a label used by both $\tilde{\eta}^i_{\Delta t+1}$ and $\tilde{\eta}^j_1$ (the event I_{ij} will be modified slightly below). We will use a second moment argument to show that $\sum_i \sum_j I_{ij} \to \infty$ with high probability, which will imply that $\xi^u_{t_2+1} \cap \xi^v_{t_1+1} \neq \varnothing$ with high probability. $\square$

We now formulate this argument in detail. Fix $0 < \delta \ll \varepsilon$, and define F (the event that there is a favorable stage 1) to be the intersection of the following 3 events:

$$F_1 = \{\min(|\tilde{\zeta}^u_{t_1}|, |\tilde{\zeta}^v_{t_1}|) \ge n^{(1-\varepsilon)(1-\delta)/2}\};$$

$$F_2 = \left\{\left|\bigcup_{s\le t_1} \tilde{\xi}^u_s\right| + \left|\bigcup_{s\le t_1} \tilde{\xi}^v_s\right| \le n^{(1-\varepsilon)(1+\delta)/2}\right\};$$

$$F_3 = \{\text{no label is used more than once in stage 1}\}.$$

Clearly, on the event F, all vertices in $\tilde{\xi}^u_{t_1}$ and $\tilde{\xi}^v_{t_1}$ are colored BLUE.

PROPOSITION 4.1.

$$\mathbb{P}(F) \ge (1 - o(1)) p_\lambda^2 \qquad \textit{as } n \to \infty.$$

PROOF. For the two contact processes on the cover tree, with probability at least p_λ^2, both survive up to time t_1. Thus, it follows from Corollary 2.2 and 2.6 that F_1 and F_2 hold simultaneously with probability at least $(1 - o(1)) p_\lambda^2$.

To show that F_3 occurs, it is enough to show that with conditional probability approaching 1, given $F_1 \cap F_2$, no label $i \in [n]$ is drawn more than once in the combined exploration processes induced by $\bigcup_{s \le t_1} \tilde{\xi}_s^u$ and $\bigcup_{s \le t_1} \tilde{\xi}_s^v$. Recall that in the construction of the labeling functions (Section 3.3.3), whenever a vertex of $\mathbb{T}^u$ and $\mathbb{T}^v$ is assigned a label not forced by the existing neighborhood structure, two half-edges are drawn from the pool U_t of unused half-edges and glued together. Each label $i \in [n]$ appears on at most d half-edges in U_t, and on the event F_2, at most $2n^{(1-\varepsilon)(1+\delta)/2}$ draws are made in stage 1. Therefore, the probability that there will be no repeated labels is at least

$$\prod_{m=1}^{A} \left(1 - \frac{(d-2)m + d + 1}{dn - 2m + 1}\right),$$

where $A = 2n^{(1-\varepsilon)(1+\delta)/2}$. It is a routine exercise in elementary analysis to show that this product approaches 1 as $n \to \infty$. $\square$

In stage 2, unlike stage 1, some labels will, with high probability, be drawn more than once, and so some of the vertices infected by the contact process $\tilde{\xi}_t^u$ in the time period $t \in [t_1, t_2 + 1]$ will be colored red. This complicates the task of proving (4.2), because only *blue* vertices in the covering contact processes $\tilde{\xi}_t^u$ and $\tilde{\xi}_t^v$ project to vertices in ξ_t^u and ξ_t^v. It is for this reason that we must examine the descendant severed contact processes $\tilde{\eta}_s^i$ (where the vertices i range over the set of pioneer points $\tilde{\zeta}_{t_1}^u$) *individually*. We will show that for any fixed $i \in \tilde{\zeta}_{t_1}^u$, with conditional probability $\to$ 1, the descendant severed contact process $(\tilde{\eta}_s^i)_{s \le \Delta t + 1}$ will have no vertices colored red. The key to this is the following estimate on the total number of vertices infected in stages 1 and 2.

PROPOSITION 4.2. *Denote by $\mathcal{S}$ the σ-algebra generated by the grow and explore processes induced by the contact processes $\tilde{\xi}^u$ and $\tilde{\xi}^v$ in stage* 1. *Define G_α to be the event that the total number of vertices infected by $\tilde{\xi}_t^u$ at any time $t \le t_2 + 1$ is less than $n^{((1+3\varepsilon)(1+\delta)+\alpha)/2}$. Then there exists $B < \infty$ such that*

$$\mathbb{P}(G_\alpha^c \mid \mathcal{S})\mathbf{1}_F \le Bn^{-\alpha/2} \qquad \textit{for all large } n. \tag{4.3}$$

PROOF. This is a direct consequence of Corollary 2.2 and the Markov property of the contact process. On F, the number of vertices infected by time t_1 is no larger than $n^{(1-\varepsilon)(1+\delta)/2}$. Each of the vertices infected at time t_1 engenders its own descendant contact process, to which the bound (2.2) applies. This, together with the Markov inequality, gives (4.3). $\square$

Henceforth, we will set $G = G_{\varepsilon(1+\delta)}$.

By construction, on the event F all vertices infected during stage 1 are colored blue, so there is a one-to-one correspondence between infections on the cover

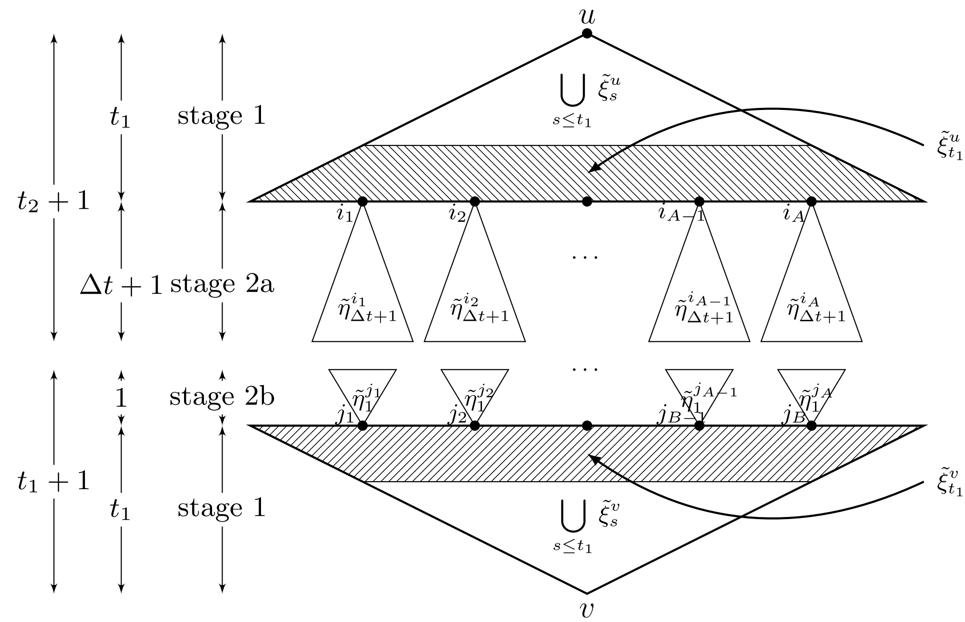

FIG. 4. $i_1,\dots,i_A$ *are all pioneer points of* $\tilde\xi^u_{t_1}$; $j_1,\dots,j_B$ *are all pioneer points of* $\tilde\xi^v_{t_1}$. *We will run independent severed contact processes inside these branches associated with the pioneer points.*

trees and infections on the finite graph up to time t_1. Thus, in particular, each pioneer point $i\in\zeta^u_{t_1}$ will correspond to an infected vertex in the projected contact process $\xi^u_{t_1}$. Denote by $\tilde\eta^i$ the severed contact process in the free branch (if there is more than one free branch, choose the "leftmost") attached to i initiated by i at time t. For notational convenience, we shall write the dependence on time as $(\tilde\eta^i_s)_{s\ge0}$ instead of $(\tilde\eta^i_{s+t_1})_{s\ge0}$; thus, for any $s\ge0$ the infected set $\tilde\eta^i_s$ consists of all vertices in $\tilde\xi^u_{t_1+s}$ connected to i by infection trails of duration s that lie entirely in the free branch of the tree connected to i. Similarly, for each pioneer point $j\in\zeta^v_{t_1}$ let $(\tilde\eta^j_s)_{s\ge0}$ denote the corresponding severed contact process. See Figure 4 for a graphical illustration. The processes $\tilde\eta^i_s$ will be followed up to time $s=\Delta t+1$. When we refer to a *pioneer point* of $\tilde\eta^i_{\Delta t}$, we will mean a vertex z in $\tilde\eta^i_{\Delta t}$ such that for some branch $\mathcal{B}_z$ of the tree adjacent to z, $\bigcup_{s\le\Delta t}\tilde\eta^i_s$ contains no vertices in $\mathcal{B}_z$. (Note, however, that $\mathcal{B}_z$ can contain vertices of $\bigcup_{s\le t_2}\tilde\xi^u_s$.)

Next, we will define the "intersection event" I_{ij} for $i\in\tilde\zeta^u_{t_1}$ and $j\in\tilde\zeta^v_{t_1}$. (We will use the same notation I_{ij} for both the event and its indicator.) First, define Z_{ij} to be the set of all vertices infected by either $\tilde\xi^u$ before time t_2+1 or $\tilde\xi^v$ before time t_1+1 *minus* the set of vertices infected by $\tilde\eta^i$ before time $\Delta t+1$ or by $\tilde\eta^j$ in the time interval $[0,1]$. We will say that I_{ij} occurs if in stage 2 all of the following events happen:

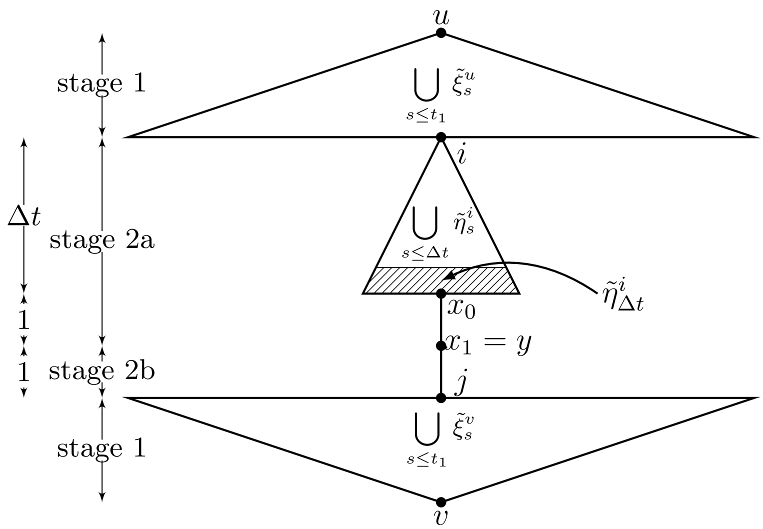

FIG. 5. *The event I_{ij}. x_0 is a pioneer point of $\tilde{\eta}^i_{\Delta t}$.*

(I0) $|\bigcup_{s\le\Delta t+1}\tilde{\eta}^i_s| \le n^{2(1+\delta)\varepsilon}$;

(I1) vertices in $\bigcup_{s\le\Delta t}\tilde{\eta}^i_s$ are assigned distinct labels;

(I2) no labels assigned to vertices in $\bigcup_{0\le s\le\Delta t}\tilde{\eta}^i_s$ occur in the set of labels assigned to vertices in Z_{ij};

(I3) in the severed contact process $(\tilde{\eta}^j_s)_{0\le s\le 1}$, vertex j infects a neighboring vertex y in its free branch before time 1, which remains infected until time 1, and $\tilde{\eta}^j$ produces no other infections by time 1;

(I4) some pioneer point x_0 of $\tilde{\eta}^i_{\Delta t}$ infects the neighbor in its free branch (call this neighbor x_1) in the time interval $[\Delta t, \Delta t+1]$, and the infection at x_1 stays alive until $\Delta t+1$ without infecting other vertices; and

(I5) x_1 is assigned the same label l_* as y.

See Figure 5 for a graphical illustration. The conditions (I1), (I2), (I3) ensure that the vertices of $\tilde{\eta}^i$ and $\tilde{\eta}^j$ will be assigned the color *blue* and, therefore, will project to infected vertices in the contact processes ξ^u, ξ^v, respectively. Thus, if (I4) and (I5) also occur, then the intersection $\xi^u_{t_2+1}\cap\xi^v_{t_1+1}$ will contain the vertex l_*.

Recall that $\mathcal{S}$ is the σ-algebra generated by the grow and explore processes induced by the contact processes $\tilde{\xi}^u$ and $\tilde{\xi}^v$ in stage 1. (Note: $\mathcal{S}$ includes the information about the choices of free branches for the pioneer points in $\tilde{\xi}^u_{t_1}$ and $\tilde{\xi}^v_{t_1}$; since these pioneer points might each have several free branches, the choices will in general involve auxiliary randomization.) We will prove that with probability approaching 1 as $n\to\infty$, there exists some pair (i, j) such that the intersection event I_{ij} occurs. To accomplish this, it will suffice to show that with probability $\to 1$ as $n \to \infty$,

$$\text{(FM)} \qquad \sum_{i \in \tilde{\zeta}^u_{t_1}} \sum_{j \in \tilde{\zeta}^v_{t_1}} \mathbb{P}\{I_{ij} \cap G | \mathcal{S}\} \to \infty \qquad \text{on } F,$$

and

$$\begin{aligned} &\sum_{i \in \tilde{\zeta}^u_{t_1}} \sum_{j \in \tilde{\zeta}^v_{t_1}} \sum_{i' \in \tilde{\zeta}^u_{t_1}} \sum_{j' \in \tilde{\zeta}^v_{t_1}} \mathbb{P}\{I_{ij} \cap I_{i'j'} \cap G | \mathcal{S}\} \\ \text{(SM)} \qquad &= (1 + o(1)) \biggl(\sum_{i \in \tilde{\zeta}^u_{t_1}} \sum_{j \in \tilde{\zeta}^v_{t_1}} \mathbb{P}\{I_{ij} \cap G | \mathcal{S}\} \biggr)^2 \qquad \text{on } F. \end{aligned}$$

The remainder of Section 4 is devoted to proving (FM) and (SM). Since we are only interested in the behavior of the quantities in (FM) and (SM) on the event F, we will henceforth, when convenient, omit the qualifying phrase "on the event F."

4.2. *1st moment calculation*. *Proof of* (FM). Since $\mathbb{P}\{I_{ij} | \mathcal{S}\}$ is the same for all pairs (i, j), it suffices to estimate a single term.

By Corollary 2.2, the (conditional) probability that the number of vertices infected by the severed contact process $\tilde{\eta}^i_s$ in the time interval $s \in [0, \Delta t + 1]$ exceeds $n^{2(1+\delta)\varepsilon}$ is less than $Bn^{-2\delta}$. Hence, the conditional probability, given $\mathcal{S}$, that the event (I0) in the definition of I_{ij} occurs converges to 1 in probability as $n \to \infty$.

Next, we show that, with conditional probability near 1 all vertices infected by the severed contact process $\tilde{\eta}^i$ in the time interval $[0, \Delta t + 1]$ will be colored blue, and no label assigned to one of these vertices will be used more than once in stages 1 and 2. Recall that by Proposition 4.2, $\mathbb{P}(G^c) \le Bn^{-\varepsilon/2}$, where G is the event that no more than $n^{(1+4\varepsilon)(1+\delta)/2}$ vertices will be infected by $\tilde{\xi}^u$ before time $t_2 + 1$. Furthermore, on F the number of vertices infected by $\tilde{\xi}^v$ before time t_1 is no larger than $n^{(1-\varepsilon)(1+\delta)/2}$. Consequently, on the event $F \cap G$, at every step of stage 2 the number of unused half-edges satisfies

$$|U_t| \ge dn - n^{(1+4\varepsilon)(1+2\delta)/2}.$$

Consequently, by an argument like that used in the proof of Proposition 4.1, the probability that G and (I0) both occur, that there will be no repeated labels drawn in the labeling of $\bigcup_{s \le \Delta t+1} \tilde{\eta}^i_s$, and that these labels will not overlap with those assigned to vertices in Z_{ij} is at least

$$\bigl(1 - d^{-1} n^{-1+(1+4\varepsilon)(1+2\delta)/2}\bigr)^{n^{2(1+\delta)\varepsilon}} \longrightarrow 1.$$

This proves that, on the event F of a favorable stage 1, the conditional probability, given $\mathcal{S}$, that the events (I0), (I1) and (I2) in the definition of I_{ij} both occur converges to 1 in probability as $n \to \infty$. Now we consider the events (I3), (I4) and (I5). The event (I3) involves only the graph structure at the vertex j and the neighboring vertex y in its free branch; clearly, this event has a positive (conditional) probability q_λ independent of n. Furthermore, the event (I3) is conditionally independent of both the contact process $\tilde{\xi}^u$ and the exploration (labeling) processes. Now consider the severed contact process $\tilde{\eta}^i$: there is a positive probability q_λ^* that this survives (see [19]), and conditional on the event that it survives, there is high probability that the cardinality of its set of pioneer points at time Δt is at least $n^{2(1-\delta)\varepsilon}$, by Proposition 2.7. Hence, there is high conditional probability (given that $\tilde{\eta}^i$ survives) that the number of these pioneer points which satisfy the requirement of (I4) is at least $\frac{1}{2}q_\lambda n^{2(1-\delta)\varepsilon}$. Finally, consider the event that for one of these pioneer points x_0 the infected nearest neighbor x_1 is assigned the same label l_* as the vertex y in (I3): since for any x_0 the label choices for x_0 and y involve sampling from the pool of unused half-edges, the chance that both x_0 and y choose half-edges attached to vertex label l_* is at least $(d-1)/(dn)$. Because the number of these pioneer points which satisfy the requirement of (I4) is (with high probability) at least $\frac{1}{2}q_\lambda n^{2(1-\delta)\varepsilon}$, it follows that

$$\mathbb{P}\{I_{ij} \cap G \mid \mathcal{S}\} \geq \bigl(1-o(1)\bigr)\frac{1}{2}q_\lambda^2 n^{2(1-\delta)\varepsilon}\frac{(d-1)}{dn} \geq C_1 n^{2((1-\delta)\varepsilon)-1}.$$

Therefore, on the event F,

$$\begin{aligned}\sum_{i\in\tilde{\zeta}_{t_1}^u}\sum_{j\in\tilde{\zeta}_{t_1}^v}\mathbb{P}\{I_{ij}\cap G|\mathcal{S}\} &\geq n^{(1-\varepsilon)(1-\delta)/2}n^{(1-\varepsilon)(1-\delta)/2}C_1 n^{2(1-\delta)\varepsilon-1}\\ &= C_2 n^{\varepsilon-\delta(1+\varepsilon)} \to \infty.\end{aligned}$$

4.3. *2nd moment calculation.* The second moment estimate (SM) is proved by similar arguments, but is complicated by the fact that we must keep track of 3 or 4 severed contact processes rather than just 2. Expand the second moment as follows:

$$\sum_{i\in\tilde{\zeta}_{t_1}^u}\sum_{j\in\tilde{\zeta}_{t_1}^v}\sum_{i'\in\tilde{\zeta}_{t_1}^u}\sum_{j'\in\tilde{\zeta}_{t_1}^v}\mathbb{P}\{I_{ij}\cap I_{i'j'}\cap G|\mathcal{S}\} = I + II + III + IV,$$

where

$$\begin{aligned} I &= \sum_{i\in\tilde{\zeta}_{t_1}^u}\sum_{j\in\tilde{\zeta}_{t_1}^v}\mathbb{P}\{I_{ij}\cap G|\mathcal{S}\},\\ II &= \sum_{i\in\tilde{\zeta}_{t_1}^u}\sum_{j\neq j'\in\tilde{\zeta}_{t_1}^v}\mathbb{P}\{I_{ij}\cap I_{ij'}\cap G|\mathcal{S}\},\\ III &= \sum_{i\neq i'\in\tilde{\zeta}_{t_1}^u}\sum_{j\in\tilde{\zeta}_{t_1}^v}\mathbb{P}\{I_{ij}\cap I_{i'j}\cap G|\mathcal{S}\},\\ IV &= \sum_{i\neq i'\in\tilde{\zeta}_{t_1}^u}\sum_{j\neq j'\in\tilde{\zeta}_{t_1}^v}\mathbb{P}\{I_{ij}\cap I_{i'j'}\cap G|\mathcal{S}\}.\end{aligned}$$

Now (SM) becomes

$$I + II + III + IV = \big(1 + o(1)\big)I^2 \qquad \text{as } n \to \infty.$$

We have already shown that on the event F we have $I \to \infty$ as $n \to \infty$, so

$$I = o\big(I^2\big).$$

Therefore, to complete the proof, we must establish the following assertions:

$$II = o\big(I^2\big), \tag{4.4}$$

$$III = o\big(I^2\big) \quad \text{and} \tag{4.5}$$

$$IV = \big(1 + o(1)\big)I^2. \tag{4.6}$$

4.3.1. *Proof of* (4.4). Let y be the neighbor of j in j's free branch, and y' the neighbor of j' in j''s free branch. In order that $I_{ij} \cap I_{ij'}$ occur, there must be pioneer points $x_0, x_0' \in \tilde{\eta}^i_{\Delta t}$ (possibly the same), with neighboring vertices x, x' in their free branches, such that y and x are assigned the same label l and y' and x' are assigned the same label l'. On the event that (I1)–(I2) hold for both Z_{ij} and $Z_{ij'}$, the labels l, l' will not have been used by any other vertex other than x, x', y or y'. Moreover, on the event $F \cap G$, the total number of vertices infected by either $\tilde{\xi}^u$ or $\tilde{\xi}^v$ during stages 1 and 2 is not more than $n^{(1+4\varepsilon)/2}$, and so the set of unused labels will always be at least $n - n^{(1+4\varepsilon)/2}$. Hence, for any choice of pioneer points $x, x' \in \tilde{\eta}^i_{\Delta t}$, regardless of the order in which x, x', y, y' are assigned labels, the (conditional) probability that x, y are assigned a common label l and x', y' a common label l' is no larger than

$$\left(\frac{1}{n - n^{(1+4\varepsilon)/2}}\right)^2 \le \frac{4}{n^2} \qquad \text{for large } n.$$

The proof of (FM) shows that with conditional probability $\to 1$, the number of pioneer points in $\tilde{\eta}^i_{\Delta t}$ is bounded above by $n^{2(1+\delta)\varepsilon}$, and hence the number of possible choices of the pair x, x' is no larger than $n^{4(1+\delta)\varepsilon}$. Consequently, on the event F,

$$P\big\{(I_{ij} \cap I_{ij'}) \cap G \mid \mathcal{S}\big\} \le \frac{8n^{4(1+\delta)\varepsilon}}{n^2}$$

with probability approaching 1 as $n \to \infty$. But on F [see (F2)], the number of possible triples i, j, j' does not exceed $(n^{(1-\varepsilon)(1+\delta)/2})^3$, and so (using the fact that $\delta \ll \varepsilon < 1/32$)

$$\begin{aligned} II = \sum_i \sum_j \sum_{j'} P\big\{(I_{ij} \cap I_{ij'}) \cap G \mid \mathcal{S}\big\} &\le \frac{16n^{4(1+\delta)\varepsilon + 3(1-\varepsilon)(1+\delta)/2}}{n^2} \\ &= O\big(n^{-\frac{1}{2} + \frac{3\delta}{2} + 8\varepsilon}\big) = o(1), \end{aligned}$$

which proves (4.4), since by (FM) the quantity I in (4.4) becomes large.

4.3.2. *Proof of* (4.5). This is similar to the proof of (4.4). For a given triple i, i', j, the number of possible triples x, x', y (where , x, x' are the nearest neighbors of pioneer points $x_0, x_0' \in \tilde{\eta}^i_{\Delta t}$, and y is the nearest neighbor of j in its free branch) is bounded above by $n^{4(1+\delta)\varepsilon}$ on the event $F \cap G$. For any such possible triple x, x', y, the conditional probability that x, x', and y will all be assigned the same label is no larger than $4/n^2$, by an argument like that in the proof of (4.4). Therefore, by (4.3), for sufficiently large n,

$$\mathbb{P}\{(I_{ij} \cap I_{i'j}) \cap G \mid \mathcal{S}\}\mathbf{1}_F \le \frac{8n^{4(1+\delta)\varepsilon}}{n^2}.$$

On the event F, the number of possible triples i, i', j is once again bounded above by $(n^{(1-\varepsilon)(1+\delta)/2})^3$, and so

$$III = \sum_i \sum_{i'} \sum_j P\{(I_{ij} \cap I_{i'j}) \cap G \mid \mathcal{S}\} \le O(n^{-\frac{1}{2}+\frac{3\delta}{2}+8\varepsilon}) = o(1).$$

4.3.3. *Proof of* (4.6). The conditional probabilities $P\{I_{ij} \cap I_{i'j'}|\mathcal{S}\}$ are the same for all quadruples i, j, i', j' such that $i \ne i'$ and $j \ne j'$, so it will suffice to show that there are constants $\beta_n \to 0$ such that with probability approaching 1,

$$\begin{aligned} &|P\{(I_{ij} \cap I_{i'j'}) \cap G \mid \mathcal{S}\} - P\{I_{ij} \cap G|\mathcal{S}\}P\{I_{i'j'} \cap G \mid \mathcal{S}\}| \\ &\quad \le \beta_n P\{I_{ij} \cap G|\mathcal{S}\}P\{I_{i'j'} \cap G|\mathcal{S}\} \end{aligned} \tag{4.7}$$

on F.

As in the proof of (4.4), let y be the neighbor of j in j's free branch, and y' the neighbor of j' in j''s free branch. In order that the event I_{ij} occur, there must be a pioneer point $x_0 \in \tilde{\eta}^i_{\Delta t}$ with neighboring vertex x in its free branch such that y and x are assigned the same label l. Similarly, for $I_{i'j'}$, there must be a pioneer point $x_0' \in \tilde{\eta}^{i'}_{\Delta t}$ with neighboring vertex x' in its free branch such that y' and x' are assigned the same label l'. These events are not conditionally independent given $\mathcal{S}$, because although the numbers of pioneer points $x_0 \in \tilde{\eta}^i_{\Delta t}$ and $x_0' \in \tilde{\eta}^{i'}_{\Delta t}$ are independent, the *coloring* of vertices in different sectors might interfere (cf. Section 3.3.2), and so the occurrence of (say) I_{ij}, which is correlated with the number of *blue* pioneer points in $\tilde{\eta}^i$, will also correlate (negatively) with the number of blue pioneer points in $\tilde{\eta}^{i'}$. However, we will show that these correlations are small as $n \to \infty$.

Define

$$N_i = \#\text{pioneer points } x \text{in } \tilde{\eta}^i_{\Delta t};$$

$$N_{i'} = \#\text{pioneer points } x \text{in } \tilde{\eta}^{i'}_{\Delta t};$$

$$M_i = \left|\bigcup_{s \le \Delta t} \tilde{\eta}^i_s\right|;$$

$$M_{i'} = \left|\bigcup_{s\le\Delta t} \tilde{\eta}_s^{i'}\right|; \qquad \text{and}$$

$$M_* = \left|\bigcup_{t\le t_1} \tilde{\xi}_t^u\right| + \left|\bigcup_{t\le t_1} \tilde{\xi}_t^v\right|,$$

and let $\mathcal{H}_{ii'}$ be the σ-algebra generated by the severed contact processes $(\tilde{\eta}_s^i)_{s\le\Delta t}$ and $(\tilde{\eta}_s^{i'})_{s\le\Delta t}$. (Note: $\mathcal{H}_{ii'}$ does *not* contain information generated by the labeling or coloring processes.) Clearly, N_i, $N_{i'}$, M_i and $M_{i'}$ are all measurable with respect to $\mathcal{H}_{ii'}$, and M_* is measurable with respect to $\mathcal{S}$. Recall that the events (I0) in the definition of I_{ij} and $I_{i'j'}$, which we now denote by I_{ij}^0 and $I_{i'j'}^0$, are defined by

$$I_{ij}^0 = \{M_i \le n^{2(1+\delta)\varepsilon}\} \quad \text{and}$$
$$I_{i'j'}^0 = \{M_{i'} \le n^{2(1+\delta)\varepsilon}\};$$

since these depend only on the values of M_i and $M_{i'}$, they are measurable relative to $\mathcal{H}_{ii'}$.

Recall that on the event $F \cap G$, the total number of vertices infected by either $\tilde{\xi}^u$ or $\tilde{\xi}^v$ during stages 1 and 2 is not more than $n^{(1+4\varepsilon)(1+\delta)/2}$, so at any time during stages 1 and 2 the pool of unused half-edges will have cardinality at least $dn - n^{(1+4\varepsilon)(1+\delta)/2} \ge dn - dn^{(1+5\varepsilon)/2}$, regardless of whether or not I_{ij} and/or $I_{i'j'}$ occur. Consequently, since the events I_{ij} and $I_{i'j'}$ *require* that the labels assigned to y and y', respectively, are also given to the vertices x adjacent to pioneer points x' in $\tilde{\eta}_{\Delta t}^i$ and $\tilde{\eta}_{\Delta t}^{i'}$ in their free branches, respectively, we must have

$$\begin{aligned}
&\mathbb{P}(I_{ij} \cap G \mid \mathcal{S} \vee \mathcal{H}_{ii'}) \le I_{ij}^0 N_i / \bigl(n - n^{(1+5\varepsilon)/2}\bigr),\\
&\mathbb{P}(I_{i'j'} \cap G \mid \mathcal{S} \vee \mathcal{H}_{ii'}) \le I_{i'j'}^0 N_{i'} / \bigl(n - n^{(1+5\varepsilon)/2}\bigr) \quad \text{and}\\
&\mathbb{P}(I_{ij} \cap I_{i'j'} \cap G \mid \mathcal{S} \vee \mathcal{H}_{ii'}) \le I_{ij}^0 I_{i'j'}^0 N_i N_{i'} / \bigl(n - n^{(1+5\varepsilon)/2}\bigr)^2.
\end{aligned} \tag{4.8}$$

Next, we derive *lower* bounds for the conditional probabilities in (4.8). The event I_{ij} will occur if (i) all of the labels assigned to vertices in $\bigcup_{s\le\Delta t} \tilde{\eta}_s^i$ are distinct and are not assigned to any of the vertices in the set Z_{ij}, and (ii) for some pioneer point $x' \in \tilde{\eta}_{\Delta t}^i$, the label assigned to the free-branch neighbor x is also assigned to y. Regardless of the order in which vertices are assigned labels in stage 2, if G occurs then for each of the M_i vertices to be labeled there is (conditional) probability at least $1 - n^{5\varepsilon/2-1/2}$ that it will get a label not yet used. Therefore,

$$\begin{aligned}
&\mathbb{P}(I_{ij} \mid \mathcal{S} \vee \mathcal{H}_{ii'} \vee G)\mathbf{1}_{F\cap G}\\
&\qquad \ge \bigl(1 - n^{5\varepsilon/2-1/2}\bigr)^{M_i}\bigl(1 - (1 - n^{-1})^{N_i}\bigr)\mathbf{1}_{F\cap G} I_{ij}^0 \quad \text{and}\\
&\mathbb{P}(I_{i'j'} \mid \mathcal{S} \vee \mathcal{H}_{ii'} \vee G)\mathbf{1}_{F\cap G}\\
&\qquad \ge \bigl(1 - n^{5\varepsilon/2-1/2}\bigr)^{M_{i'}}\bigl(1 - (1 - n^{-1})^{N_{i'}}\bigr)\mathbf{1}_{F\cap G} I_{i'j'}^0.
\end{aligned} \tag{4.9}$$

By similar reasoning,

$$\begin{aligned}
&\mathbb{P}(I_{ij} \cap I_{i'j'} \mid \mathcal{S} \vee \mathcal{H}_{ii'} \vee G)\mathbf{1}_{F\cap G} \\
&\quad \geq (1 - n^{5\varepsilon/2-1/2})^{M_i+M_{i'}}(1-(1-n^{-1})^{N_i})(1-(1-n^{-1})^{N_{i'}}) \\
&\qquad \times \mathbf{1}_{F\cap G} I_{ij}^0 I_{i'j'}^0.
\end{aligned} \tag{4.10}$$

Clearly,

$$N_i \leq M_i \leq n^{2(1+\delta)\varepsilon} \quad \text{on } I_{ij}^0 \quad \text{and}$$
$$N_{i'} \leq M_{i'} \leq n^{2(1+\delta)\varepsilon} \quad \text{on } I_{i'j'}^0;$$

since $0 < \varepsilon < 1/32$, the ratios of the lower bounds in (4.9) and (4.10) to the corresponding upper bounds in (4.8) converge to 1 as $n \to \infty$. Finally, since

$$P(G \cap I_{ij}^0 \cap I_{i'j'}^0 \mid \mathcal{S}) \longrightarrow 1$$

as $n \to \infty$, inequality (4.7) follows.

5. Exponential extinction time. The goal of this section is to prove Theorem 1.3.

Recall that for a graph $G = (\mathcal{V}_G, \mathcal{V}_E)$, the edge expansion parameter is defined as

$$\Psi_E(G, k) = \min_{S \subset \mathcal{V}_G, |S| \leq k} \frac{|E(S, S^c)|}{|S|},$$

where $E(S, S^c) \subset \mathcal{V}_E$ is the set of edges with one vertex in S and the other vertex in S^c. Theorem 4.16 [10] implies the following fact.

THEOREM 5.1. *Let $d \geq 3$. Then for every $\delta > 0$ there exists $\varepsilon > 0$ such that for asymptotically almost every $G \sim \mathcal{G}(n, d)$, $\Psi_E(G, \varepsilon n) \geq d - 2 - \delta$.*

Fix an integer $M > 0$. Suppose $U \subset \mathcal{V}_G$ is of size αn, where $\alpha > 0$. We remove every vertex in U whose M-neighborhood is not a tree, and denote the remaining vertex set by U'. We claim $|U'| = \alpha n - o(n)$. Here, we are using the following fact proved in [14] (Lemma 3.2).

PROPOSITION 5.2. *Asymptotically almost every $G \sim \mathcal{G}(n, d)$ has at most $o(n)$ vertices whose $\lfloor \log_{d-1} \log n \rfloor$-neighborhoods in G are not tree-like.*

Since the cardinality of U' and U are on the same order of magnitude, without loss of generality, let us assume that all vertices in U have tree-like M-neighborhoods in G.

We classify vertices in U into 2 categories by looking at their M-neighborhoods in G in the following way. For $v \in U$, let $B(v, M)$ be the induced subgraph containing all vertices in v's M-neighborhood in G. $B(v, M) \setminus \{v\}$ has d connected components, call them $C_1(v), C_2(v), \dots, C_d(v)$. If $C_i(v)$ contains no other vertices in U, call it a *free branch of depth M* of v.

- Color v *black* if at least one of $C_1(v), C_2(v), \dots, C_d(v)$ is a free branch of depth M of v.
- Color v *white* if none of $C_1(v), C_2(v), \dots, C_d(v)$ is a free branch of depth M of v.

PROPOSITION 5.3. *Fix $M \in \mathbb{N}$. There exists $\varepsilon = \varepsilon(M) > 0$, such that for asymptotically almost every $G \sim \mathcal{G}(n, d)$ the following statement holds: for any set $U \subset \mathcal{V}_G$ satisfying $|U| \le \varepsilon n$ and that every vertex in U has its M-neighborhood in G being a tree, then U has at least $|U|/4$ black vertices.*

PROOF. We will construct a subset of vertices $W \subset \mathcal{V}_G$. First of all, W contains all vertices in U. Moreover, we are going to add some vertices into W based on the white vertices of U. Let $v \in U$ be a white vertex. In each of $C_1(v), C_2(v), \dots, C_d(v)$, there must be at least another vertex in U. Suppose $x \in U \cap C_i(v)$, then for the pair (v, x), we add into W every vertex along the (unique) geodesic between v and x. We repeat this operation for every possible pair (v, x) to obtain W.

Such constructed W contains 3 types of vertices: black vertices of U, white vertices of U, and the vertices which are added by the above procedure (color them *grey*). Now let us count their contributions to $E(W, W^c)$:

- A white vertex will contribute 0 edge to $E(W, W^c)$. This is because all of its d neighboring vertices are already in W by our construction.
- A black vertex can contribute at most d edges to $E(W, W^c)$, possibly fewer.
- A grey vertex can contribute at most $d-2$ edges to $E(W, W^c)$, possibly fewer. This is because by our construction a grey vertex must be sitting on the geodesic between two other vertices in U and, therefore, at least 2 out of its d neighboring vertices are already in W.

Suppose in U there are w white vertices, b black vertices. Then g, the number of grey vertices in W, satisfies $g \le (d + d(d-1) + \cdots + d(d-1)^{M-1})w := N_M w$.

By Theorem 5.3, there exists ε_M such that on a typical random regular graph G, $\Psi_E(G, \varepsilon_M n) \ge d - 2 - (3d-8)/(3N_M + 4)$. This forces the following inequality (provided $b + w + g \le \varepsilon_M n$),

$$0w + db + (d-2)g \ge E(W, W^c) \ge \left(d - 2 - \frac{3d-8}{3N_M + 4}\right)(w + b + g).$$

Together with $g \le N_M w$, this implies that

$$\frac{b}{b+w} \ge \frac{1}{4}.$$

Let $\varepsilon'_M = \varepsilon_M/(N_M+1)$ (this guarantees that if $b+w \le \varepsilon'_M n$ then $b+w+g \le \varepsilon_M n$). As long as $|U| \le \varepsilon'_M n$ and every vertex in U has its M-neighborhood in G being a tree, then U has at least $|U|/4$ black vertices. $\square$

Denote by Δ_M the finite rooted tree of depth M in which the root O (at depth 0) has one neighbor at depth 1, and each vertex at depth $1 \le j < M$ has $d-1$ neighbors at depth $j+1$. Let $\{\eta_t^{O,\Delta_M}\}_{t\ge 0}$ be a contact process on Δ_M with initial configuration $\{O\}$. The following statement is an easy corollary of Proposition 2.3.

COROLLARY 5.4. *For every $N > 0$, there exist $T > 0$ and $M \in \mathbb{N}$ such that $\mathbb{E}|\eta_T^{O,\Delta_M}| \ge N$.*

PROOF OF THEOREM 1.3. Let G be a typical graph as in Proposition 5.1, 5.2 and 5.3. Choose $M \ge 1$ and T such that $\mathbb{E}|\eta_T^{O,\Delta_M}| \ge 10$; Corollary 5.4 guarantees that this is possible. Without loss of generality, assume $T \ge 1$. Furthermore, choose $L > 0$ large enough such that $\mathbb{E}\min(|\eta_T^{O,\Delta_M}|, L) \ge 9$.

Let $\varepsilon = \varepsilon(2M)$ be the constant in Proposition 5.3. As long as $U \subset \mathcal{V}_G$ is of size εn, then there will be at least $\varepsilon n/2$ vertices in U such that each vertex has its $2M$-neighborhood being a tree, and by Proposition 5.3 there will be at least $\varepsilon n/8$ black vertices. We enumerate the black vertices to be $v_1, v_2, \dots, v_k$ where $k \ge \varepsilon n/8$.

Each vertex v_i has one free branch of depth $2M$ (if there is more than one, choose the "leftmost"). Add v_i (and the edge connected to v_i) to its free branch of depth M (as a subgraph of the branch of depth $2M$) to obtain a subgraph isomorphic to Δ_M. Since the free branches are of depth $2M$, the copies of Δ_M attached to different vertices v_i are disjoint. For each v_i, run an independent contact process on its copy of Δ_M, and denote this by $\{\eta_t^{v_i}\}_{t\ge 0}$. By a standard construction, we can couple $\bigcup_{i=1}^k \eta_T^{v_i}$ and $\bigcup_{i=1}^k \xi_T^{v_i}$ so that $\bigcup_{i=1}^k \eta_T^{v_i}$ is dominated by $\bigcup_{i=1}^k \xi_T^{v_i}$.

Let $X_i = \min(|\eta_T^{v_i}|, L)$; by construction, $\mathbb{E}X_i \ge 9$ and $0 \le X_i \le L$. Furthermore, the random variables $(X_i)_{1\le i\le k}$ are i.i.d. variables, so, by Hoeffding's inequality,

$$\mathbb{P}\left\{\sum_{i=1}^k (X_i - 9) \le -k\right\} \le \exp\left(-\frac{2k}{L^2}\right) \le \exp\left(-\frac{\varepsilon n}{4L^2}\right);$$

thus, after time T, with probability at least $1-\exp(-\varepsilon n/(4L^2))$, we will observe at least $8k \ge \varepsilon n$ infections in $\bigcup_{i=1}^k \eta_T^{v_i}$.

Therefore, as long as the contact process has initial configuration with cardinality at least εn, then with probability at least $1-\exp(-\varepsilon n/(4L^2))$, after time T, the infected set will have cardinality at least εn. Consequently, if $\beta = \varepsilon/(8L^2)$ then

$$\mathbb{P}_G\{\xi^G_{\exp(\beta n)T} \ne \varnothing\} \ge 1 - \exp(\beta n)\exp\left(-\frac{\varepsilon n}{4L^2}\right) = 1 - o(1). \qquad \square$$

REMARK 5.5. One can slightly change the above proof to show the following statement. There exists a constant $\varepsilon_0 > 0$. For every $0 < \varepsilon \le \varepsilon_0$, there exists $\beta_\varepsilon > 0$, such that for asymptotically almost every $G \sim \mathcal{G}(n, d)$, for any $U \subset \mathcal{V}_G$ with $|U| = \varepsilon n$, we have

$$\mathbb{P}_G\{\xi^U_{\exp(\beta_\varepsilon n)} \neq \varnothing\} = 1 - o(1).$$

6. Asymptotic infection density. Recall that $0 < \varepsilon < 1/32$. Throughout this section, let $t_+ = (1+\varepsilon)\log n/c_\lambda$. Let $g_n(\varepsilon)$ be as in Theorem 1.1. Say that a pair of vertices $(u, v) \in [n] \times [n]$ is *good* if

$$\mathbb{P}_G\{v \in \xi^u_{t_+}\} \ge (1 - g_n(\varepsilon)) p^2_\lambda.$$

Say that a vertex $u \in [n]$ is *good* if the set

$$\{v \in [n] : v \neq u, (u, v) \text{ is a good pair}\}$$

has cardinality at least $(1 - \sqrt[4]{g_n(\varepsilon)})(n-1)$. A simple application of the Markov inequality, together with Theorem 1.1, yields the following proposition.

PROPOSITION 6.1. *For asymptotically almost every G, the number of good pairs is at least $(1 - \sqrt{g_n(\varepsilon)})n(n-1)$, and the number of good vertices is at least $(1 - \sqrt[4]{g_n(\varepsilon)})n$.*

The choice of $\sqrt{g_n}$ and $\sqrt[4]{g_n}$ in the definition of good pair/vertex and in the above proposition is not crucial; it is enough that they are $o(1)$. Proposition 6.1 shows that the good pairs and vertices are indeed typical.

PROPOSITION 6.2. *Suppose $u \in [n]$ is a good vertex, and ξ^u_t is a contact process with initial state $\{u\}$ on G. Then there exist constants $h_n(\varepsilon) \to 0$ as $n \to \infty$ such that for asymptotically almost every $G \sim \mathcal{G}(n, d)$,*

$$(1 + h_n(\varepsilon)) p_\lambda \ge \mathbb{P}_G\{\xi^u_{t_+} \neq \varnothing\} \ge (1 - h_n(\varepsilon)) p_\lambda.$$

PROOF. First, $\mathbb{P}_G\{\xi^u_{t_+} \neq \varnothing\} \le (1 + o(1)) p_\lambda$, because

$$\mathbb{P}_G\{\xi^u_{t_+} \neq \varnothing\} \le \mathbb{P}\{\tilde{\xi}_{t_+} \neq \varnothing\} = (1 + o(1)) p_\lambda, \tag{6.1}$$

where $(\tilde{\xi}_t)_{t \ge 0}$ is a contact process on $\mathbb{T}$ with the root as the initial configuration.

It remains to show the lower bound. Set

$$S_u = \sum_{v \in V(G), v \neq u} \mathbf{1}_{\{v \in \xi^u_{t_+}\}}.$$

Since u is a good vertex,

$$\mathbb{E}_G S_u \ge (1 - o(1)) p^2_\lambda n,$$

where here and for the remainder of this section, $o(1)$ terms only depend on n but not u or G. On the other hand,

$$S_u^2 = \sum_{v\in[n]\setminus\{u\}} \sum_{w\in[n]\setminus\{u\}} \mathbf{1}_{\{v\in\xi_{t_+}^u, w\in\xi_{t_+}^u\}}$$

and

$$\mathbb{E}_G S_u^2 = \sum_{v\in[n]\setminus\{u\}} \sum_{w\in[n]\setminus\{u\}} \mathbb{P}_G\{v \in \xi_{t_+}^u, w \in \xi_{t_+}^u\}.$$

Let $t_+ = t_{+,1} + t_{+,2}$, where $t_{+,1} = t_{+,2} = t_+/2$, The event $\{v \in \xi_{t_+}^u, u \in \xi_{t_+}^u\}$ will occur if and only if there exits open paths (in the percolation structure used to construct the contact process) starting from u that reach both v and w in time t_+. This requires that both of the following events happen:

(a) u infects some (random) subset $Z \subset [n]$ at time $t_{+,1}$; and
(b) Z infects both v and w in time interval $[t_{+,1}, t_+]$.

Let ξ_t^u, ξ_t^v, ξ_t^w be 3 mutually independent contact processes, with initial configurations $\{u\}$, $\{v\}$ and $\{w\}$, respectively. By duality, we can reverse the time axis in the second event; this shows that events (a)–(b) are no easier than observing the following two events:

(c) ξ_t^u survives to time $t_{+,1}$;
(d) ξ_t^v and ξ_t^w both survive to time $t_{+,2}$.

It is easy to see that the $\mathbb{P}_G$-probability of observing both the events above, is no larger than $(1 + o(1))p_\lambda^3$, because of (6.1).

Therefore, for each pair (v, w),

$$\mathbb{P}_G\{v \in \xi_{t_+}^u, w \in \xi_{t_+}^u\} \le \bigl(1 + o(1)\bigr)p_\lambda^3,$$

which implies

$$\mathbb{E}_G S_u^2 \le \bigl(1 + o(1)\bigr)p_\lambda^3 n^2.$$

Now since u is a good vertex,

$$\mathbb{E}_G\{S_u \mid S_u > 0\} = \frac{\mathbb{E}_G S_u}{\mathbb{P}_G\{S_u > 0\}} \ge \frac{(1 - o(1))p_\lambda^2 n}{\mathbb{P}_G\{S_u > 0\}},$$

while

$$\mathbb{E}_G\{S_u^2 \mid S_u > 0\} = \frac{\mathbb{E}_G S_u^2}{\mathbb{P}_G\{S_u > 0\}} \le \frac{(1 + o(1))p_\lambda^3 n^2}{\mathbb{P}_G\{S_u > 0\}}.$$

However, by Jensen's inequality, $\mathbb{E}_G\{S_u \mid S_u > 0\}^2 \le \mathbb{E}_G\{S_u^2 \mid S_u > 0\}$, and so

$$\mathbb{P}_G\{S_u > 0\} \ge \bigl(1 - o(1)\bigr)p_\lambda. \qquad \square$$

The preceding proof also provides a good estimate of the size of $\xi_{t_+}^u$.

PROPOSITION 6.3. *Suppose $u \in [n]$ is a good vertex, and ξ_t^u is a contact process with initial state $\{u\}$. Fix $\varepsilon > 0$. Then there exist constants $k_n(\varepsilon) \to 0$ as $n \to \infty$ such that for asymptotically almost every G,*

$$\mathbb{P}_G\{(1-\delta)np_\lambda \le S_u \le (1+\delta)np_\lambda \mid S_u > 0\} \ge 1 - k_n(\varepsilon).$$

PROOF. From Proposition 6.2, $\mathbb{P}_G\{S_u > 0\} \ge (1-o(1))p_\lambda$. On the other hand, $\mathbb{P}_G\{S_u > 0\} \le (1+o(1))p_\lambda$. These two bounds combined with the calculation in the proof of Proposition 6.2 imply that

$$\mathbb{E}_G\{S_u^2 \mid S_u > 0\} \le (1+o(1))\mathbb{E}_G\{S_u \mid S_u > 0\}^2.$$

The result now follows by Chebyshev's inequality. □

PROOF OF ASSERTION (1.3) OF THEOREM 1.2. Let

$$S = \sum_{v \in [n]} \mathbf{1}_{\{v \in \xi_{t_+}^G\}}.$$

By duality of the contact process, Proposition 6.1 and Proposition 6.2,

$$\mathbb{E}_G S \ge (1-o(1))np_\lambda.$$

On the other hand, by a similar argument as in the proof of Proposition 6.2, we have

$$\mathbb{E}_G S^2 \le (1+o(1))n^2 p_\lambda^2.$$

Therefore,

$$\mathbb{E}_G S^2 \le (1+o(1))(\mathbb{E}_G S)^2,$$

and so (1.3) follows by Chebyshev's inequality. □

We need several lemmas before we prove assertion (1.4).

LEMMA 6.4. *There exists $\gamma > 0$, such that for asymptotically almost every $G \sim \mathcal{G}(n,d)$, for any two vertices u, v of G,*

$$\mathbb{P}_G\{v \in \xi_{2\log_{d-1} n}^u\} \ge n^{-\gamma}.$$

PROOF. By [4], the diameter of a typical random regular graph is $(1 + o(1))\log_{d-1} n$. Therefore, on such a graph, for any pair of vertices (u, v), the graph distance between u and v is no more than $(1 + o(1))\log_{d-1} n$. Assume $dist(u, v) = l$, this means we can find a sequence of vertices $(w_i)_{0\le i\le l}$, with $w_0 = u$, $w_l = v$, and that w_i is connected to w_{i+1} in the graph.

One way of observing $\{v \in \xi_{2\log_{d-1} n}^u\}$ is as follows. In time interval $[i, i + 1]$ for $0 \le i \le l - 1$, we require the infection at vertex w_i to go to w_{i+1}, and stay there alive until the end of the time interval. This will happen with probability $p \ge (1-e^{-\lambda})e^{-1}$. In time interval $[l, 2\log_{d-1} n]$, we require the infection at v to stay alive. This will have probability at least $e^{-(2\log_{d-1} n - l)}$. Therefore, the overall probability is at least $p^l e^{-(2\log_{d-1} n-l)} \ge (1-e^{-\lambda})^{2\log_{d-1} n} e^{-2\log_{d-1} n} \ge n^{-\gamma}$ where $\gamma = 2/\log(d-1) + 2\log(e^\lambda/(e^\lambda - 1))/\log(d-1)$. $\square$

LEMMA 6.5. *There exists* $\delta_0 > 0$, *such that for asymptotically almost every* $G \sim \mathcal{G}(n,d)$, *for any* $v \in \mathcal{V}_G$,

$$\mathbb{P}_G\{|\xi^v_{n^{2\gamma}}| \ge \delta_0 n \mid \xi^v_{n^{2\gamma}} \ne \varnothing\} = 1 - o(1).$$

PROOF. For asymptotically almost every $G \sim \mathcal{G}(n,d)$, there are $n - o(n)$ good vertices by Proposition 6.1. In particular, there is at least one good vertex. Fix one good vertex in G, call it w. Our idea is as follows. Subdivide the time interval $[0, n^{2\gamma}]$ into $[0,T], [T,2T], \dots, [(M-1)T, MT]$, where $M = \lfloor n^{2\gamma}/T \rfloor$ and T will be specified later. We will define an event H and ask if H occurs in each interval $[iT, (i+1)T]$. The chance that it happens in each time interval will be of order $n^{-\gamma}$. This event is constructed such that as long as in at least one of these intervals the event happens, then with probability approaching 1 at the end we will see $\delta_0 n$ infections.

Here are the details. Let $T = 2\log_{d-1} n + (1+\varepsilon)\log n / c_\lambda$, where this $0 < \varepsilon < 1/8$ is the same as in Theorem 1.1. We say that we observe a success H in time interval $[(i-1)T, iT]$ if the following two events happen:

(e) $w \in \xi^v_{(i-1)T + 2\log_{d-1} n}$;
(f) $|\xi^v_{iT}| \ge np_\lambda/2$.

Conditional on $\xi^v_{n^{2\gamma}} \ne \varnothing$, it is certain that $\xi^v_{(i-1)T} \ne \varnothing$, so by Lemma 6.4, the event $\{w \in \xi^v_{(i-1)T+2\log_{d-1} n}\}$ happens with probability at least $n^{-\gamma}$ (notice that the event $\{w \in \xi^v_{(i-1)T+2\log_{d-1} n}\}$ is positively correlated with the event $\{\xi^v_{n^{2\gamma}} \ne \varnothing\}$). Given $\{w \in \xi^v_{(i-1)T+2\log_{d-1} n}\}$, since w is a good vertex, by Proposition 6.3, $|\xi^v_{iT}| \ge np_\lambda/2$ will happen with probability at least $p_\lambda/2$.

Therefore, overall we have order $(n^{2\gamma}/\log n)$ trials, each with success probability at least $p_\lambda n^{-\gamma}/2$, so the chance of having at least 1 success is $1 - o(1)$. Given that we observe a success, which means that we observe $p_\lambda n/2$ infections at some time between $[0, n^{2\gamma}]$, from the proof of Theorem 1.3 and Remark 5.5, we know that the chance of $p_\lambda n/2$ infections not lasting exponentially long time before the size of infections shrinks to $\delta_0 n$ is exponentially small in n, where δ_0 can be taken as $\min(\varepsilon_0, p_\lambda/2)$. Therefore, the overall probability is $1 - o(1)$. $\square$

PROOF OF ASSERTION (1.4) OF THEOREM 1.2. The upper bound follows from Assertion (1.3), because

$$\xi^G_{t_n - t_+} \subset \mathcal{V}_G,$$

which implies $\xi_{t_n}^G$ is dominated by $\xi_{t_+}^{\mathcal{V}_G}$.

Now we prove the lower bound. There are 2 scenarios: $t_n > n^{2\gamma} + t_+$ or $t_n \le n^{2\gamma} + t_+$, where γ is the same as in Lemma 6.4.

CASE 1. $t_n \le n^{2\gamma} + t_+$.

Since $n^{2\gamma} < \exp(\beta n)$ for large enough n, from the proof of Theorem 1.3 we know that at time $t_n - t_+$, with chance $1 - o(1)$, for some $\varepsilon_0 > 0$,

$$|\xi_{t_n - t_+}^G| \ge \varepsilon_0 n,$$

which suggests that at time $t_n - t_+$ there are plenty of infections and, therefore, if we run the contact process for another t_+ time, a similar second moment argument as in proof of Proposition 6.2 or proof of Theorem 1.3 will prove the desired lower bound.

CASE 2. $t_n > n^{2\gamma} + t_+$.

In this case, from Lemma 6.5 we know that with probability $1 - o(1)$, at time $t_n - t_+$,

$$|\xi_{t_n - t_+}^G| \ge \delta_0 n,$$

where δ_0 is the same as in Lemma 6.5. The rest is the same as in Case 1. □

Acknowledgments. The authors would like to thank Si Tang for her help in drawing the figures, and Jean-Christophe Mourrat, Hao Can and Bruno Schapira for a number of useful comments.

REFERENCES

[1] BHAMIDI, S., VAN DER HOFSTAD, R. and HOOGHIEMSTRA, G. Universality for first passage percolation on sparse random graphs. Available at arXiv:1210.6839.

[2] BOLLOBÁS, B. (1980). A probabilistic proof of an asymptotic formula for the number of labelled regular graphs. *European J. Combin.* **1** 311–316. MR0595929

[3] BOLLOBÁS, B. (2001). *Random Graphs*, 2nd ed. *Cambridge Studies in Advanced Mathematics* **73**. Cambridge Univ. Press, Cambridge. MR1864966

[4] BOLLOBÁS, B. and FERNANDEZ DE LA VEGA, W. (1982). The diameter of random regular graphs. *Combinatorica* **2** 125–134. MR0685038

[5] CHATTERJEE, S. and DURRETT, R. (2013). A first order phase transition in the threshold $\theta \ge 2$ contact process on random r-regular graphs and r-trees. *Stochastic Process. Appl.* **123** 561–578. MR3003363

[6] CRANSTON, M., MOUNTFORD, T., MOURRAT, J.-C. and VALESIN, D. (2014). The contact process on finite homogeneous trees revisited. *ALEA Lat. Am. J. Probab. Math. Stat.* **11** 385–408. MR3249416

[7] DING, J., SLY, A. and SUN, N. Maximum independent sets on random regular graphs. Available at arXiv:1310.4787.

[8] DURRETT, R. and JUNG, P. (2007). Two phase transitions for the contact process on small worlds. *Stochastic Process. Appl.* **117** 1910–1927. MR2437735

[9] HARRIS, T. E. (1978). Additive set-valued Markov processes and graphical methods. *Ann. Probab.* **6** 355–378. MR0488377

[10] HOORY, S., LINIAL, N. and WIGDERSON, A. (2006). Expander graphs and their applications. *Bull. Amer. Math. Soc.* (*N.S.*) **43** 439–561 (electronic). MR2247919

[11] LALLEY, S. P. and SELLKE, T. (1998). Limit set of a weakly supercritical contact process on a homogeneous tree. *Ann. Probab.* **26** 644–657. MR1626499

[12] LIGGETT, T. (1996). Multiple transition points for the contact process on the binary tree. *Ann. Probab.* **24** 1675–1710. MR1415225

[13] LIGGETT, T. M. (2005). *Interacting Particle Systems*. Springer, Berlin. MR2108619

[14] LUBETZKY, E. and SLY, A. (2010). Cutoff phenomena for random walks on random regular graphs. *Duke Math. J.* **153** 475–510. MR2667423

[15] MADRAS, N. and SCHINAZI, R. (1992). Branching random walks on trees. *Stochastic Process. Appl.* **42** 255–267. MR1176500

[16] MORROW, G., SCHINAZI, R. and ZHANG, Y. (1994). The critical contact process on a homogeneous tree. *J. Appl. Probab.* **31** 250–255. MR1260587

[17] MOUNTFORD, T. S. (1993). A metastable result for the finite multidimensional contact process. *Canad. Math. Bull.* **36** 216–226. MR1222537

[18] MOURRAT, J.-C. and VALESIN, D. Phase transition of the contact process on random regular graphs. Available at arXiv:1405.0865.

[19] PEMANTLE, R. (1992). The contact process on trees. *Ann. Probab.* **20** 2089–2116. MR1188054

[20] STACEY, A. (1996). The existence of an intermediate phase for the contact process on trees. *Ann. Probab.* **24** 1711–1726. MR1415226

[21] STACEY, A. (2001). The contact process on finite homogeneous trees. *Probab. Theory Related Fields* **121** 551–576. MR1872428

[22] WORMALD, N. C. (1999). Models of random regular graphs. In *Surveys in Combinatorics*, 1999 (*Canterbury*). *London Mathematical Society Lecture Note Series* **267** 239–298. Cambridge Univ. Press, Cambridge. MR1725006

DEPARTMENT OF STATISTICS
UNIVERSITY OF CHICAGO
5734 S. UNIVERSITY AVENUE
CHICAGO, ILLINOIS 60637
USA
E-MAIL: lalley@galton.uchicago.edu
weisu@galton.uchicago.edu