Tensors in Computations IV

Lek-Heng Lim

University of Chicago

modern definition

recap: definitions

- tensors capture three great ideas
  ① equivariance
  ② multilinearity
  ③ separability
- roughly correspond to three common definitions of a tensor
  ① a multi-indexed object that satisfies tensor transformation rules
  ② a multilinear map
  ③ an element of a tensor product of vector spaces

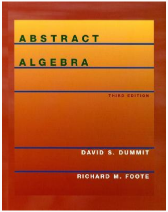
modern approach

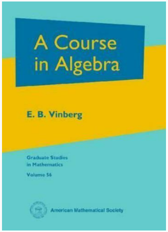
- instead of defining an object directly, define the space of all such objects
- what is vector?
- it is an element of a vector space
- what is a tensor?
- it is an element of a tensor product of vector spaces

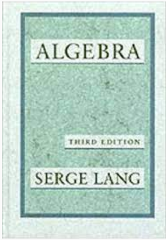
$$\mathbb{V}_1 \otimes \cdots \otimes \mathbb{V}_d$$

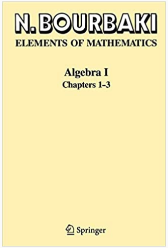
  or more generally, a tensor product of modules
- just need to define $\otimes$

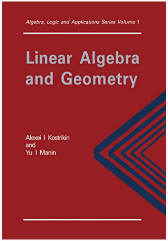
adopted in most modern books

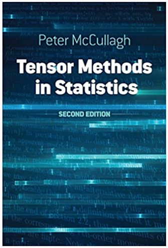


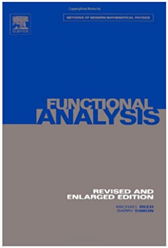


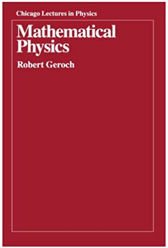












definition ③ itself has three definitions

- definition ③ itself may be defined in three different ways
  - ❶ via tensor product of function spaces
  - ❷ via tensor product of more general vector spaces
  - ❸ via the universal mapping property
- we will start with ❷ and deduce ❶
- ignore covariance and contravariance for time being

tensor product of vector spaces

1-tensors

- a vector space $\mathbb{V}$ is essentially a space in which linear combination are well-defined
- pick a basis $\mathscr{B} = \{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ of $\mathbb{V}$
- any element of $\mathbb{V}$ takes the form

$$\lambda_1 \mathbf{f}_1 + \lambda_2 \mathbf{f}_2 + \cdots + \lambda_n \mathbf{f}_n$$

  with scalar coefficients $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$
- elements of $\mathbb{R}$ = scalars = tensors of order 0
- elements of $\mathbb{V}$ = vectors = tensors of order 1

2-tensors

- suppose we now have a second vector space $\mathbb{U}$
- key to tensor product construction:

  replace scalars $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ by vectors $\mathbf{u}_1, \dots, \mathbf{u}_n \in \mathbb{U}$
- get 'linear combination'

$$\mathbf{u}_1\mathbf{f}_1 + \mathbf{u}_2\mathbf{f}_2 + \cdots + \mathbf{u}_n\mathbf{f}_n$$

  with vector coefficients $\mathbf{u}_1, \dots, \mathbf{u}_n \in \mathbb{U}$
- pick a basis $\mathscr{A} = \{\mathbf{e}_1, \dots, \mathbf{e}_m\}$ of $\mathbb{U}$, then

$$\mathbf{u}_j = \lambda_{1j}\mathbf{e}_1 + \lambda_{2j}\mathbf{e}_2 + \cdots + \lambda_{mj}\mathbf{e}_m$$

  with scalar coefficients $\lambda_{ij} \in \mathbb{R}$, $i = 1, \dots, m$, $j = 1, \dots, n$
- so linear combination becomes

$$\lambda_{11}\mathbf{e}_1\mathbf{f}_1 + \lambda_{21}\mathbf{e}_2\mathbf{f}_2 + \cdots + \lambda_{mn}\mathbf{e}_m\mathbf{f}_n = \sum_{i=1}^{m}\sum_{j=1}^{n}\lambda_{ij}\mathbf{e}_i\mathbf{f}_j$$

2-tensors

- the object

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\lambda_{ij}\mathbf{e}_i\mathbf{f}_j$$

  is called a dyadic in older literature

- modern notation:
  - write $\otimes$ for product, $\mathbf{e}_i \otimes \mathbf{f}_j$ instead of $\mathbf{e}_i\mathbf{f}_j$
  - write $\mathbb{U} \otimes \mathbb{V}$ for the set of all dyadics
- elements of $\mathbb{U} \otimes \mathbb{V}$ = dyadics = tensors of order 2

$$\begin{aligned}\mathbb{U} \otimes \mathbb{V} &= \{\text{all linear combinations of } \mathbf{u} \otimes \mathbf{v},\ \mathbf{u} \in \mathbb{U},\ \mathbf{v} \in \mathbb{V}\} \\ &= \{\text{all linear combinations of } \mathbf{e}_i \otimes \mathbf{f}_j,\ i = 1, \dots, m,\ j = 1, \dots, n\}\end{aligned}$$

3-tensors

- may recursively apply this construction to get higher-order tensors
- suppose $\mathbb{W}$ a third vector space with basis $\mathscr{C} = \{\mathbf{g}_1, \dots, \mathbf{g}_p\}$
- triadic is a 'linear combination'

$$\boldsymbol{\alpha}_1\mathbf{g}_1 + \boldsymbol{\alpha}_2\mathbf{g}_2 + \cdots + \boldsymbol{\alpha}_p\mathbf{g}_p$$

  with dyadic coefficients $\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_p \in \mathbb{U} \otimes \mathbb{V}$
- elements of $\mathbb{U} \otimes \mathbb{V} \otimes \mathbb{W}$ = triadics = tensors of order 3

$$\begin{aligned}\mathbb{U} \otimes \mathbb{V} \otimes \mathbb{W} = \{&\text{all linear combinations of } \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w},\\ &\mathbf{u} \in \mathbb{U},\ \mathbf{v} \in \mathbb{V},\ \mathbf{w} \in \mathbb{W}\}\\ = \{&\text{all linear combinations of } \mathbf{e}_i \otimes \mathbf{f}_j \otimes \mathbf{g}_k,\\ &i = 1, \dots, m,\ j = 1, \dots, n,\ k = 1, \dots, p\}\end{aligned}$$

d-tensors

- vector spaces $\mathbb{V}_1, \dots, \mathbb{V}_d$, construction gives new vector space

$$\mathbb{V}_1 \otimes \cdots \otimes \mathbb{V}_d$$

with dimension

$$\dim(\mathbb{V}_1 \otimes \cdots \otimes \mathbb{V}_d) = \dim(\mathbb{V}_1) \cdots \dim(\mathbb{V}_d)$$

- covariance and contravariance?
- replace last few vector spaces by duals $\mathbb{V}_{p+1}^*, \dots, \mathbb{V}_d^*$

$$\mathbb{V}_1 \otimes \cdots \otimes \mathbb{V}_p \otimes \mathbb{V}_{p+1}^* \otimes \cdots \otimes \mathbb{V}_d^*$$

set of d-tensors of contravariant order p covariant order $d - p$

example: stress tensor

- stress $\boldsymbol{\sigma}$ at a point has three components in directions given by unit vectors $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$

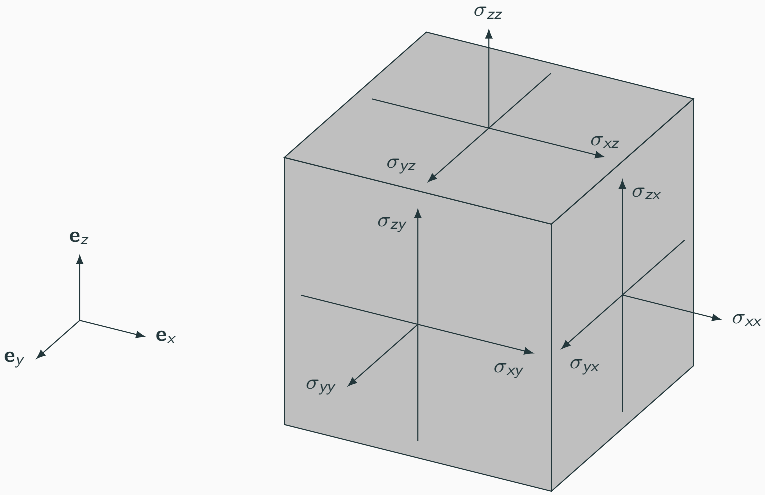
$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_x \mathbf{e}_x + \boldsymbol{\sigma}_y \mathbf{e}_y + \boldsymbol{\sigma}_z \mathbf{e}_z$$

- coefficients $\boldsymbol{\sigma}_x$, $\boldsymbol{\sigma}_y$, $\boldsymbol{\sigma}_z$ not scalars but vectors
- nature of stress: in every direction, stress in that direction has a normal component in that direction and two shear components in the plane perpendicular to it, e.g.,

$$\boldsymbol{\sigma}_x = \sigma_{xx} \mathbf{e}_x + \sigma_{yx} \mathbf{e}_y + \sigma_{zx} \mathbf{e}_z$$

- normal stress: component σ_{xx} in the direction of $\mathbf{e}_x$
- shear stress: components σ_{yx} and σ_{zx} in $\mathbf{e}_x^{\perp} = \text{span}\{\mathbf{e}_y, \mathbf{e}_z\}$
- coefficients σ_{xx}, σ_{yx}, σ_{zx} are scalars

example: stress tensor



example: stress tensor

- likewise

$$\boldsymbol{\sigma}_y = \sigma_{xy}\mathbf{e}_x + \sigma_{yy}\mathbf{e}_y + \sigma_{zy}\mathbf{e}_z,$$
$$\boldsymbol{\sigma}_z = \sigma_{xz}\mathbf{e}_x + \sigma_{yz}\mathbf{e}_y + \sigma_{zz}\mathbf{e}_z.$$

- since coefficients σ_x, σ_y, σ_z in

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_x\mathbf{e}_x + \boldsymbol{\sigma}_y\mathbf{e}_y + \boldsymbol{\sigma}_z\mathbf{e}_z$$

  are vectors, insert $\otimes$ for modern notation

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_x \otimes \mathbf{e}_x + \boldsymbol{\sigma}_y \otimes \mathbf{e}_y + \boldsymbol{\sigma}_z \otimes \mathbf{e}_z$$

- or alternatively

$$\begin{aligned}\boldsymbol{\sigma} = {} & \sigma_{xx}\mathbf{e}_x \otimes \mathbf{e}_x + \sigma_{yx}\mathbf{e}_y \otimes \mathbf{e}_x + \sigma_{zx}\mathbf{e}_z \otimes \mathbf{e}_x \\ & + \sigma_{xy}\mathbf{e}_x \otimes \mathbf{e}_y + \sigma_{yy}\mathbf{e}_y \otimes \mathbf{e}_y + \sigma_{zy}\mathbf{e}_z \otimes \mathbf{e}_y \\ & + \sigma_{xz}\mathbf{e}_x \otimes \mathbf{e}_z + \sigma_{yz}\mathbf{e}_y \otimes \mathbf{e}_z + \sigma_{zz}\mathbf{e}_z \otimes \mathbf{e}_z\end{aligned}$$

example: stress tensor

- verdict: stress is a dyadic = 2-tensor
- what kind of 2-tensor?
- set basis $\mathscr{B} = \{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$, may represent $\boldsymbol{\sigma}$ as

$$\Sigma := [\boldsymbol{\sigma}]_{\mathscr{B}} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

- normal stresses on diagonal and shear stresses off diagonal
- different basis $\mathscr{B}' = \{e'_x, e'_y, e'_z\}$ gives different representation

$$\Sigma' := [\boldsymbol{\sigma}]_{\mathscr{B}'} = \begin{bmatrix} \sigma'_{xx} & \sigma'_{xy} & \sigma'_{xz} \\ \sigma'_{yx} & \sigma'_{yy} & \sigma'_{yz} \\ \sigma'_{zx} & \sigma'_{zy} & \sigma'_{zz} \end{bmatrix}$$

- plug in change-of-basis relations

$$\begin{cases} \mathbf{e}'_x = c_{xx}\mathbf{e}_x + c_{yx}\mathbf{e}_y + c_{zx}\mathbf{e}_z \\ \mathbf{e}'_y = c_{xy}\mathbf{e}_x + c_{yy}\mathbf{e}_y + c_{zy}\mathbf{e}_z \\ \mathbf{e}'_z = c_{xz}\mathbf{e}_x + c_{yz}\mathbf{e}_y + c_{zz}\mathbf{e}_z \end{cases} \qquad C := \begin{bmatrix} c_{xx} & c_{xy} & c_{xz} \\ c_{yx} & c_{yy} & c_{yz} \\ c_{zx} & c_{zy} & c_{zz} \end{bmatrix}$$

into

$$\begin{aligned} \boldsymbol{\sigma} = {} & \sigma'_{xx}\mathbf{e}'_x \otimes \mathbf{e}'_x + \sigma'_{yx}\mathbf{e}'_y \otimes \mathbf{e}'_x + \sigma'_{zx}\mathbf{e}'_z \otimes \mathbf{e}'_x \\ & + \sigma'_{xy}\mathbf{e}'_x \otimes \mathbf{e}'_y + \sigma'_{yy}\mathbf{e}_y \otimes \mathbf{e}'_y + \sigma'_{zy}\mathbf{e}'_z \otimes \mathbf{e}'_y \\ & + \sigma'_{xz}\mathbf{e}'_x \otimes \mathbf{e}'_z + \sigma'_{yz}\mathbf{e}'_y \otimes \mathbf{e}'_z + \sigma'_{zz}\mathbf{e}'_z \otimes \mathbf{e}'_z \end{aligned}$$

- get

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} c_{xx} & c_{xy} & c_{xz} \\ c_{yx} & c_{yy} & c_{yz} \\ c_{zx} & c_{zy} & c_{zz} \end{bmatrix} \begin{bmatrix} \sigma'_{xx} & \sigma'_{xy} & \sigma'_{xz} \\ \sigma'_{yx} & \sigma'_{yy} & \sigma'_{yz} \\ \sigma'_{zx} & \sigma'_{zy} & \sigma'_{zz} \end{bmatrix} \begin{bmatrix} c_{xx} & c_{yx} & c_{zx} \\ c_{xy} & c_{yy} & c_{zy} \\ c_{xz} & c_{yz} & c_{zz} \end{bmatrix}$$

example: stress tensor

- two coordinate representations of σ satisfy transformation rule

$$\Sigma' = C^{-1} \Sigma C^{-\mathsf{T}}$$

- stress is a contravariant 2-tensor
- exact same discussion applies to any contravariant 2-tensors, e.g., inertia, polarization, strain, tidal force, viscosity
- stress important for defining piezo-electric tensor $D \in \mathbb{R}^{3\times3\times3}$, piezo-magnetic tensor $Q \in \mathbb{R}^{3\times3\times3}$, elastic tensor $S \in \mathbb{R}^{3\times3\times3\times3}$

$$d_{ijk} = -\frac{\partial^2 G}{\partial \sigma_{ij} \partial e_k}, \quad q_{ijk} = -\frac{\partial^2 G}{\partial \sigma_{ij} \partial h_k}, \quad s_{ijkl} = -\frac{\partial^2 G}{\partial \sigma_{ij} \partial \sigma_{kl}}$$

where $G = G(\boldsymbol{\sigma}, \mathbf{E}, \mathbf{H}, T)$ is Gibbs potential depending on stress $\boldsymbol{\sigma}$, electric field $\mathbf{E}$, magnetic field $\mathbf{H}$, temperature T

common pitfall

- suppose with respect to $\mathscr{B} = \{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$, stress $\boldsymbol{\sigma}$ has

$$\Sigma = [\boldsymbol{\sigma}]_{\mathscr{B}} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

- suppose with respect to $\mathscr{B}' = \{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$, stress $\boldsymbol{\sigma}'$ has

$$\Sigma' = [\boldsymbol{\sigma}']_{\mathscr{B}'} = \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{r\phi} \\ \sigma_{\theta r} & \sigma_{\theta\theta} & \sigma_{\theta\phi} \\ \sigma_{\phi r} & \sigma_{\phi\theta} & \sigma_{\phi\phi} \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

- makes perfect sense to add the 2-tensors

$$\boldsymbol{\sigma} + \boldsymbol{\sigma}'$$

- makes no sense to add the matrices

$$\Sigma + \Sigma' = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$$

formula for $\mathbf{u} \otimes \mathbf{v}$?

- is there a 'formula' to evaluate tensor product of vectors $\mathbf{u} \otimes \mathbf{v}$?
- not in general, these are abstract products of abstract vectors in abstract vector spaces
- all we may say is that $\otimes$ is associative, $+$ is associative and commutative, $\otimes$ is distributive over $+$ in the sense of

$$
\begin{aligned}
(\lambda \mathbf{u} + \lambda' \mathbf{u}') \otimes \mathbf{v} \otimes \cdots \otimes \mathbf{w} &= \lambda \mathbf{u} \otimes \mathbf{v} \otimes \cdots \otimes \mathbf{w} + \lambda' \mathbf{u}' \otimes \mathbf{v} \otimes \cdots \otimes \mathbf{w} \\
\mathbf{u} \otimes (\lambda \mathbf{v} + \lambda' \mathbf{v}') \otimes \cdots \otimes \mathbf{w} &= \lambda \mathbf{u} \otimes \mathbf{v} \otimes \cdots \otimes \mathbf{w} + \lambda' \mathbf{u} \otimes \mathbf{v}' \otimes \cdots \otimes \mathbf{w} \\
&\vdots \\
\mathbf{u} \otimes \mathbf{v} \otimes \cdots \otimes (\lambda \mathbf{w} + \lambda' \mathbf{w}') &= \lambda \mathbf{u} \otimes \mathbf{v} \otimes \cdots \otimes \mathbf{w} + \lambda' \mathbf{u} \otimes \mathbf{v} \otimes \cdots \otimes \mathbf{w}'
\end{aligned}
$$

for all $\mathbf{u}, \mathbf{u}' \in \mathbb{U}, \mathbf{v}, \mathbf{v}' \in \mathbb{V}, \ldots, \mathbf{w}, \mathbf{w}' \in \mathbb{W},\ \lambda, \lambda' \in \mathbb{R}$

interpretation

OBJECT	PROPERTY
scalar a	has magnitude $\lvert a \rvert$
vector $\mathbf{v}$	has magnitude $\lVert \mathbf{v} \rVert$ and a direction $\widehat{\mathbf{v}}$
dyad $\mathbf{v} \otimes \mathbf{w}$	has magnitude $\lVert \mathbf{v} \otimes \mathbf{w} \rVert$ and two directions $\widehat{\mathbf{v}}$, $\widehat{\mathbf{w}}$
triad $\mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w}$	has magnitude $\lVert \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w} \rVert$ and three directions $\widehat{\mathbf{u}}$, $\widehat{\mathbf{v}}$, $\widehat{\mathbf{w}}$
$\vdots$	$\vdots$
d-ad $\mathbf{u} \otimes \mathbf{v} \otimes \cdots \otimes \mathbf{w}$	has magnitude $\lVert \mathbf{u} \otimes \mathbf{v} \otimes \cdots \otimes \mathbf{w} \rVert$ and d directions $\widehat{\mathbf{u}}, \widehat{\mathbf{v}}, \dots, \widehat{\mathbf{w}}$

e.g., dyadic $\mathbf{v}_1 \otimes \mathbf{w}_1 + \mathbf{v}_2 \otimes \mathbf{w}_2 + \cdots + \mathbf{v}_r \otimes \mathbf{w}_r$ is placeholder for

$$
\begin{aligned}
&(\text{magnitude } 1, \text{ first direction } 1, \text{ second direction } 1) \\
&\quad \& \ (\text{magnitude } 2, \text{ first direction } 2, \text{ second direction } 2) \ \& \ \cdots \\
&\quad\quad \& \ (\text{magnitude } r, \text{ first direction } r, \text{ second direction } r)
\end{aligned}
$$

trivial yet important

- property of tensor product

$$(a\mathbf{v}) \otimes \mathbf{w} = \mathbf{v} \otimes (a\mathbf{w}) = a(\mathbf{v} \otimes \mathbf{w})$$

- this is why tensor products, not direct sums, are used to combine quantum state spaces
- quantum state is not described by $\mathbf{v}$ but entire one-dimensional subspace spanned by $\mathbf{v}$
- property ensures that in combining two quantum states, it matters not which vector in the subspace we pick to represent the state
- direct sum does not have this property

$$(a\mathbf{v}) \oplus \mathbf{w} \neq \mathbf{v} \oplus (a\mathbf{w}) \neq a(\mathbf{v} \oplus \mathbf{w})$$

- more sophisticated argument in [Aerts–Daubechies, 1978, 1979]

formula for $\mathbf{u} \otimes \mathbf{v}$!

as soon as we pick concrete vector spaces, we get concrete formulas for $\otimes$

- outer product of vectors $\mathbf{a} \in \mathbb{R}^m$, $\mathbf{b} \in \mathbb{R}^n$

$$\mathbf{a} \otimes \mathbf{b} := \mathbf{a}\mathbf{b}^\top = \begin{bmatrix} a_1 b_1 & \cdots & a_1 b_n \\ \vdots & \ddots & \vdots \\ a_m b_1 & \cdots & a_m b_n \end{bmatrix} \in \mathbb{R}^{m \times n}$$

- Kronecker product of matrices $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{p \times q}$

$$A \otimes B := \begin{bmatrix} a_{11} B & \cdots & a_{1n} B \\ \vdots & \ddots & \vdots \\ a_{m1} B & \cdots & a_{mn} B \end{bmatrix} \in \mathbb{R}^{mp \times nq}$$

- separable product of functions $f \colon X \to \mathbb{R}$, $g \colon Y \to \mathbb{R}$

$$f \otimes g \colon X \times Y \to \mathbb{R}, \quad f \otimes g(\mathbf{x}, \mathbf{y}) := f(\mathbf{x}) g(\mathbf{y})$$

- separable product of kernels $K \colon X \times X' \to \mathbb{R}$, $H \colon Y \times Y' \to \mathbb{R}$

$$K \otimes H\big((\mathbf{x}, \mathbf{x}'), (\mathbf{y}, \mathbf{y}')\big) := K(\mathbf{x}, \mathbf{x}') H(\mathbf{y}, \mathbf{y}')$$

tensor product via functions

separable product of functions

- separable product of functions gives another way to define $\mathbb{U} \otimes \mathbb{V}$
- vector space of real-valued functions on set X

$$\mathbb{R}^X := \{f : X \to \mathbb{R}\}$$

- separable function is

$$(\varphi \otimes \psi)(\mathbf{x}, \mathbf{y}) = \varphi(\mathbf{x})\psi(\mathbf{y}) \quad \text{for all } \mathbf{x} \in X,\ \mathbf{y} \in Y$$

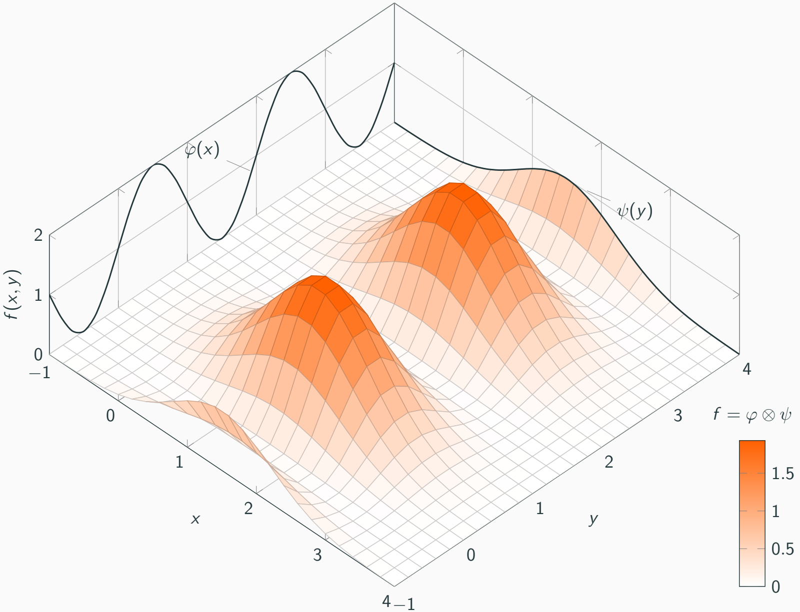
- define tensor product $\mathbb{R}^X \otimes \mathbb{R}^Y$ to be subspace of $\mathbb{R}^{X \times Y}$ comprising all finite sums of separable functions

$$\mathbb{R}^X \otimes \mathbb{R}^Y := \left\{ f \in \mathbb{R}^{X \times Y} : f = \sum_{i=1}^{r} \varphi_i \otimes \psi_i,\ \varphi_i \in \mathbb{R}^X,\ \psi_i \in \mathbb{R}^Y \right\}$$

- any $f \in \mathbb{R}^X \otimes \mathbb{R}^Y$ takes form

$$f(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{r} \varphi_i(\mathbf{x})\psi_i(\mathbf{y})$$

example: separable function



higher order

- extends to any d sets $X_1, X_2, \dots, X_d$

$$\mathbb{R}^{X_1} \otimes \mathbb{R}^{X_2} \otimes \cdots \otimes \mathbb{R}^{X_d} := \Big\{ f \in \mathbb{R}^{X_1 \times X_2 \times \cdots \times X_d} : f = \sum_{i=1}^{r} \varphi_i \otimes \psi_i \otimes \cdots \otimes \theta_i, \\ \varphi_i \in \mathbb{R}^{X_1}, \psi_i \in \mathbb{R}^{X_2}, \dots, \theta_i \in \mathbb{R}^{X_d} \Big\}$$

- each summand is a separable function

$$(\varphi \otimes \psi \otimes \cdots \otimes \theta)(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_d) := \varphi(\mathbf{x}_1)\psi(\mathbf{x}_2)\cdots\theta(\mathbf{x}_d)$$

  and

$$\mathbb{R}^{X_1 \times \cdots \times X_d} = \{ f : X_1 \times \cdots \times X_d \to \mathbb{R} \}$$

- moral: multivariate functions = tensors

another way to define $\otimes$

- if $X_1, \dots, X_d$ are finite sets, then

$$\mathbb{R}^{X_1} \otimes \cdots \otimes \mathbb{R}^{X_d} = \mathbb{R}^{X_1 \times \cdots \times X_d}$$

- any finite-dimensional vector space $\mathbb{V}$ may be regarded as real-valued functions on basis $\mathscr{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$

$$\mathbb{V} \ni \mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n \quad \longleftrightarrow \quad f : \mathscr{B} \to \mathbb{R},\ f(\mathbf{v}_i) = a_i \in \mathbb{R}^{\mathscr{B}}$$

- may define tensor product via

$$\mathbb{V}_1 \otimes \cdots \otimes \mathbb{V}_d \coloneqq \mathbb{R}^{\mathscr{B}_1} \otimes \cdots \otimes \mathbb{R}^{\mathscr{B}_d} = \mathbb{R}^{\mathscr{B}_1 \times \cdots \times \mathscr{B}_d}$$

infinite dimension?

- polynomials

$$\mathbb{R}[x_1, \ldots, x_m] \otimes \mathbb{R}[y_1, \ldots, y_n] = \mathbb{R}[x_1, \ldots, x_m, y_1, \ldots, y_n]$$

- L^2 functions

$$L^2(X) \mathbin{\widehat{\otimes}} L^2(Y) = L^2(X \times Y)$$

- Schwartz, smooth, compactly supported smooth, holomorphic

$$S(X) \mathbin{\widehat{\otimes}} S(Y) = S(X \times Y), \qquad C^\infty(X) \mathbin{\widehat{\otimes}} C^\infty(Y) = C^\infty(X \times Y)$$
$$C_c^\infty(X) \mathbin{\widehat{\otimes}} C_c^\infty(Y) = C_c^\infty(X \times Y), \qquad H(X) \mathbin{\widehat{\otimes}} H(Y) = H(X \times Y)$$

- tempered, compactly supported, distributions, analytic functionals

$$S'(X) \mathbin{\widehat{\otimes}} S'(Y) = S'(X \times Y), \qquad E'(X) \mathbin{\widehat{\otimes}} E'(Y) = E'(X \times Y)$$
$$D'(X) \mathbin{\widehat{\otimes}} D'(Y) = D'(X \times Y), \qquad H'(X) \mathbin{\widehat{\otimes}} H'(Y) = H'(X \times Y)$$

- caveat: need appropriate topological tensor product $\widehat{\otimes}$

topological tensor product

- separable Hilbert space $\mathbb{H}$ with inner product $\langle \cdot , \cdot \rangle$ and norm $\| \cdot \|$
- complete $\mathbb{H} \otimes \mathbb{H}^*$ with respect to the nuclear, Hilbert–Schmidt, spectral norms respectively

trace class $\quad \mathbb{H} \mathbin{\widehat{\otimes}_\nu} \mathbb{H}^* = \left\{ \Phi \in \mathcal{B}(\mathbb{H}) \colon \sum_{i \in I} \sum_{j \in I} |\langle \Phi(\mathbf{e}_i), \mathbf{f}_j \rangle| < \infty \right\}$

Hilbert–Schmidt $\quad \mathbb{H} \mathbin{\widehat{\otimes}_\mathsf{F}} \mathbb{H}^* = \left\{ \Phi \in \mathcal{B}(\mathbb{H}) \colon \sum_{i \in I} \|\Phi(\mathbf{e}_i)\|^2 < \infty \right\}$

compact $\quad \mathbb{H} \mathbin{\widehat{\otimes}_\sigma} \mathbb{H}^* = \left\{ \Phi \in \mathcal{B}(\mathbb{H}) \colon \begin{array}{l} X \subseteq \mathbb{H} \text{ bounded} \\ \Rightarrow \overline{\Phi(X)} \subseteq \mathbb{H} \text{ compact} \end{array} \right\}$

topological tensor product

- they have Schmidt decomposition

$$\Phi = \sum_{i=1}^{\infty} \sigma_i \mathbf{u}_i \otimes \mathbf{v}_i^*$$

- $\{\mathbf{u}_i : i \in \mathbb{N}\}$ and $\{\mathbf{v}_i : i \in \mathbb{N}\}$ orthonormal sets, $\sigma_i \geq 0$
- for trace-class, Hilbert–Schmidt, compact operators

$$\sum_{i=1}^{\infty} \sigma_i < \infty, \quad \sum_{i=1}^{\infty} \sigma_i^2 < \infty, \quad \lim_{i \to \infty} \sigma_i = 0$$

and

$$\|\Phi\|_\nu = \sum_{i=1}^{\infty} \sigma_i, \quad \|\Phi\|_{\mathsf{F}} = \left(\sum_{i=1}^{\infty} \sigma_i^2 \right)^{1/2}, \quad \|\Phi\|_\sigma = \sup_{i \in \mathbb{N}} \sigma_i$$

example: Gaussian

- quintessential example

$$f(\mathbf{x}) = \exp(\mathbf{x}^* A\mathbf{x} + \mathbf{b}^*\mathbf{x} + c)$$

  $(A + A^*)/2$ negative definite, **b** purely imaginary
- normal random variable $X \sim N(\boldsymbol{\mu}, \Sigma)$

$$\varphi_X(\mathbf{x}) = \exp\left(\mathrm{i}\boldsymbol{\mu}^\intercal\mathbf{x} - \frac{1}{2}\mathbf{x}^*\Sigma\mathbf{x}\right)$$

- Gaussian wave functions for quantum harmonic oscillator

$$\psi_{m,n,p}(x, y, z) = \left(\frac{\beta^2}{\pi}\right)^{\frac{3}{4}} \frac{H_m(\beta x)H_n(\beta y)H_p(\beta z)}{\sqrt{2^{m+n+p}\, m!\, n!\, p!}} \exp\left[-\frac{\beta^2}{2}(x^2+y^2+z^2)\right]$$

Michael Peskin

Physics is that subset of human experience which can be reduced to coupled harmonic oscillators

acknowledgment

- Simons Foundation Award no. 663281 granted to the Institute of Mathematics of the Polish Academy of Sciences for 2021–2023
- L.-H. Lim, "Tensors in computations," *Acta Numer.*, **30** (2021), pp. 555–764