



Fiber product homotopy method for multiparameter eigenvalue problems

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Abstract

We develop a new homotopy method for solving multiparameter eigenvalue problems (MEPs) called the fiber product homotopy method. For a k -parameter eigenvalue problem with matrices of sizes $n_1, \dots, n_k = O(n)$, fiber product homotopy method requires deformation of $O(1)$ linear equations, while existing homotopy methods for MEPs require $O(n)$ nonlinear equations. We show that the fiber product homotopy method theoretically finds all eigenpairs of an MEP with probability one. It is especially well-suited for a class of problems we call dimension-deficient singular problems that are generic with respect to intrinsic dimension, as the fiber product homotopy method is provably convergent with probability one for such problems as well, a fact borne out by numerical experiments. More generally, our numerical experiments indicate that the fiber product homotopy method significantly outperforms the standard Delta method in terms of accuracy, with consistent backward errors on the order of 10^{-16} without any use of extended precision. In terms of speed, it significantly outperforms

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previous homotopy-based methods on all problems and outperforms the Delta method on larger problems, and is also highly parallelizable. We show that the *fiber product MEP* that we solve in the fiber product homotopy method, although mathematically equivalent to a standard MEP, is typically a much better conditioned problem.

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1 Introduction

A multiparameter eigenvalue problem (MEP) is, in an appropriate sense, a system of linear equations

$$\begin{aligned}
 a_{11}x_1 + a_{12}x_2 + \dots + a_{1k}x_k &= b_1, \\
 a_{21}x_1 + a_{22}x_2 + \dots + a_{2k}x_k &= b_2, \\
 \vdots \quad & \\
 a_{k1}x_1 + a_{k2}x_2 + \dots + a_{kk}x_k &= b_k,
 \end{aligned}
 \tag{1}$$

where the coefficients a_{ij} 's and b_i 's are *matrices*, and where equality is interpreted to mean on a point in a product of projective spaces (this will be made precise later). These coefficients are square matrices but are of different dimensions in general, so one may not usually regard (1) as a linear system over a matrix ring. There is a rich mathematical theory behind MEP [2,3] that places it at the crossroad of linear and multilinear algebra, ordinary and partial differential equations, spectral theory and Sturm–Liouville theory, among other areas. The problem appeared as early as 1836 in the works of Sturm and Liouville on periodic heat flow in a bar, and was studied over the years by many: Klein, Lamé, Heine, Stieltjes, Pell, Carmichael, Bocher, Hilbert among them (see [2, Preface] and [3, Chapter 1]).

An MEP encompasses many known types of eigenvalue problems: Standard eigenvalue problems $Ax = \lambda x$; generalized eigenvalue problems $Ax = \lambda Bx$; quadratic eigenvalue problems $(\lambda^2 A + \lambda B + C)x = 0$; polynomial eigenvalue problems $(\lambda^m A_m + \lambda^{m-1} A_{m-1} + \dots + A_0)x = 0$; quadratic two-parameter eigenvalue problems

$$\begin{aligned}
 (A_{00} + \lambda A_{10} + \mu A_{01} + \lambda^2 A_{20} + \lambda \mu A_{11} + \mu^2 A_{02})x_1 &= 0, \\
 (B_{00} + \lambda B_{10} + \mu B_{01} + \lambda^2 B_{20} + \lambda \mu B_{11} + \mu^2 B_{02})x_2 &= 0;
 \end{aligned}$$

may all be reduced to mathematically equivalent MEPs.

Nevertheless MEP remains in the blind spot of most modern mathematicians, whether pure or applied. This is not for its lack of applications; as we pointed out, the problem in fact originated from a study of heat flow, and we will see yet other applications of MEP in Sect. 7 and that it contains eigenvalue problem and linear system, both ubiquitous in science and engineering, as special cases. We think that a main reason for the obscurity of MEPs is that there are not many effective methods for its computation and there is thus little to be gained from formulating a problem

We develop a new homotopy method to solve a multiparameter eigenvalue problem effectively, where effectiveness is measured by the following factors:

- *Speed* as measured by wall time. We record time per path, maximum time over all paths, and total track time of all paths. Our algorithm is highly parallelizable and the per-path times give good speed estimates when there are enough cores to track all paths in parallel.
- *Accuracy* as measured by the backward error. We use the normwise backward error in [14, Theorem 2] for an approximate eigenpair. Our homotopy method tracks several copies of the eigenvalue λ ; they should all converge to the same value if our method performs correctly and we include the difference between copies of λ 's as another measure of accuracy.
- *Certificates* of quadratic convergence in terms of Shub–Smale α -theory.
- *Number of divergent paths* that fail to converge to the solutions.

The last two measures only apply to methods based on homotopy continuation. We will compare our method to two existing methods:

- (i) The *Delta method* [2], which is the de facto standard method for solving MEPs by transforming them into a coupled system of generalized eigenvalue problems; we use the MULTIPAREIG package [17] in our experiments with this method. For singular MEPs, we perform Delta method after extracting the common regular parts of the Delta matrices with a staircase algorithm [17].
- (ii) The *diagonal coefficient homotopy method* recently proposed in [8] for solving MEPs, where the start system is a random choice of diagonal matrices and the homotopy is a straight-line homotopy that deforms $n_1 + \dots + n_k$ of $n_1 + \dots + n_k + k$ equations.

The numerical experiments in [8] show that the diagonal coefficient homotopy method outperforms the Delta method in terms of memory usage and is also faster for large n_1, \dots, n_k . Both methods find all eigenpairs of an MEP.

Our *fiber product homotopy method* adopts an alternative approach—we solve an MEP (2) by solving a mathematically equivalent system that we will call the *fiber product multiparameter eigenvalue problem*:¹

$$H_i(\lambda_i)\mathbf{x}_i = 0, \quad \lambda_1 = \lambda_2 = \dots = \lambda_k, \quad i = 1, \dots, k, \quad (4)$$

where $\lambda_1, \dots, \lambda_k$ are to be regarded as different copies of λ . Our corresponding homotopy has a start system that captures more structure of the MEP and allows us to deform far fewer equations. By introducing $k(k - 1)$ auxiliary unknowns, we deform at most $k(k - 1)$ of the $n_1 + \dots + n_k + k^2$ equations. For a fixed k and $n_1, \dots, n_k = O(n)$, fiber product homotopy deforms $O(1)$ equations whereas diagonal coefficient homotopy deforms $O(n)$ equations. Furthermore, fiber product homotopy deforms only *linear* equations whereas diagonal coefficient homotopy deforms *nonlinear* equations, which means that the paths of the fiber product homotopy are easier to track. As we will see later, for dimension-deficient and singular MEPs, the number of paths to track in a

¹ More precisely, we solve $H_i(\lambda_i)\mathbf{x}_i = 0, G_1 = \dots = G_k = 0, i = 1, \dots, k$, where each $G_i: \mathbb{C}^{k^2} \rightarrow \mathbb{C}^{k-1}$ is a linear function of $\lambda_1, \dots, \lambda_k$, chosen so that the resulting system is equivalent to (4). See Sect. 5.2.

fiber product homotopy can be substantially smaller than using a diagonal coefficient homotopy.

An eigenpair $(\lambda, \mathbf{x}_1, \dots, \mathbf{x}_k)$ of an MEP is said to be *regular* if the eigenvalue λ is isolated and has multiplicity one (see [14] for definitions of algebraic and geometric multiplicity). Since the expected number of regular eigenpairs to an MEP is $n_1 \cdots n_k$, one often considers only relatively small values of k to n when finding all eigenpairs. Our fiber product homotopy method is guaranteed to compute all regular eigenpairs in theory—we show in Theorem 1 that our start system in Sect. 5 is chosen correctly with probability one.

The fiber product homotopy method is motivated by several geometric insights. In Sect. 3, we define two algebraic varieties associated with a fiber product MEP: multiparameter eigenvalue variety and multiparameter eigenpair variety. *Fiber products*, a notion well-known in areas from algebraic geometry to relational database, will be reviewed in Sect. 4. The name for our method comes from the fact that a k -parameter eigenpair variety is a fiber product of k one-parameter eigenpair varieties. In Sect. 11, we rely on geometry to define a condition number for the fiber product MEP (4), which differs from the condition number for the standard MEP (3), these being two different problems, albeit having the same solutions. Our condition number arises from intersecting an algebraic variety (our multiparameter eigenpair variety) with a varying linear space (defined by our start system).

We implemented a purely numerical version (in particular, it does not use multi-precision) of fiber product homotopy in MATLAB and a mixed symbolic-numerical version in BERTINI with MACAULAY2 for comparison. We did extensive numerical experiments with both implementations: randomly generated MEPs in Sect. 8; the Mathieu two-parameter eigenvalue problem arising from a real-world application—an elliptic membrane vibration problem—in Sect. 9; and MEPs that are both dimension-deficient and singular, a challenging class of problems with a deficiency in the number of eigenpairs that breaks most other methods, in Sect. 10.

We applied a broad range of measures for speed and accuracy to our numerical experiments to stress test the robustness of our method. In Sect. 8, speed is measured via both wall time and iteration count (number of Newton steps); accuracy is measured via both relative backward error of the computed eigenpairs and the deviation in the multiple paths used to track the eigenvalues. In Sect. 9, we test the effect of reducing the number of Newton steps (by early stopping) on the accuracy of our method and certify its final quadratic convergence speed using Shub–Smale α -theory. For the dimension-deficient singular MEPs in Sect. 10, what breaks other homotopy methods is the issue of divergent paths and so we use the number of divergent paths as a measure of effectiveness. The take-away is that our method saw zero divergent paths in the cases we tested, where the MEP has exactly $\prod_{i=1}^k d_i$ eigenvalues with (d_1, \dots, d_k) the intrinsic dimension. In Sects. 8 and 10, we also provide the time it takes to track a single path, which gives a good estimate of the speed under parallel execution of our method (since each path can be tracked independently of others).

2 Homotopy methods

We recall the straight-line homotopy and describe the diagonal coefficient homotopy method used in [8] but will defer the description of our fiber product homotopy method to Sect. 5.

A homotopy deforms solutions of a *start system* $Q(z) = 0$ to solutions of a *target system* $P(z) = 0$. More precisely, a *straight-line homotopy* with path parameter t is defined as

$$H(z, t) := (1 - t)Q(z) + tP(z), \quad t \in [0, 1]. \tag{5}$$

When $t = 0$, $H(z, 0) = Q(z) = 0$ is the start system and when $t = 1$, $H(z, 1) = P(z) = 0$ is the target system.

Definition 2 A start system for the homotopy (5) is said to be *chosen correctly* [15] if the following properties hold:

- (i) the solution set of the start system $Q(z) = 0$ are known or easy to obtain;
- (ii) the solution set of $H(z, t) = 0$ for $0 \leq t < 1$ consists of a finite number of smooth paths, each parametrized by t in $[0, 1]$;
- (iii) for each isolated solution of the target system $P(z) = 0$, there is some path originating from a solution of the start system that leads to it.

Let D_{ij} denote diagonal matrices in $\mathbb{C}^{n_i \times n_i}$. The *diagonal coefficient homotopy method* for solving MEP is the straight-line homotopy given by:

$$\begin{aligned}
 H_{DC}(\lambda, x_1, \dots, x_k, t) &:= (1 - t)Q_{DC} + tP_{DC}, \\
 Q_{DC}(\lambda, x_1, \dots, x_k) &:= \begin{bmatrix} (D_{10} + \lambda_1 D_{11})x_1 \\ \vdots \\ (D_{k0} + \lambda_k D_{kk})x_k \end{bmatrix}, \\
 P_{DC}(\lambda, x_1, \dots, x_k) &:= \begin{bmatrix} H_1(\lambda)x_1 \\ \vdots \\ H_k(\lambda)x_k \end{bmatrix}
 \end{aligned} \tag{6}$$

where one regards

$$\begin{aligned}
 H_{DC}: \mathbb{C}^k \times (\mathbb{P}^{n_1-1} \times \dots \times \mathbb{P}^{n_k-1}) \times \mathbb{C} &\rightarrow \mathbb{C}^{n_1+\dots+n_k}, \\
 Q_{DC}, P_{DC}: \mathbb{C}^k \times (\mathbb{P}^{n_1-1} \times \dots \times \mathbb{P}^{n_k-1}) &\rightarrow \mathbb{C}^{n_1+\dots+n_k}
 \end{aligned}$$

as polynomial maps. The homotopy method proposed in [8] is an example of a diagonal coefficient homotopy method.

To implement the homotopy above, one has to account for the scaling of the eigenvectors by introducing k constraints. One way to do this is by scaling each x_i so that it has norm one. This is the approach undertaken in [8]. Another way to do this is to place a generic² affine constraint on each x_i , which is what we will do in our fiber product homotopy in Sect. 5.

² Here “generic” is used in its usual sense in algebraic geometry. Those unfamiliar with this notion may assume that it is synonymous with “random.”

3 Multiparameter eigenvarieties

We will define two algebraic varieties associated with an MEP.

Definition 3 In algebraic geometric terms [10], the coordinates of $H(\lambda)\mathbf{x}$ are polynomials that form a subset of $\mathbb{C}[\lambda, \mathbf{x}] = \mathbb{C}[\lambda_1, \dots, \lambda_k, x_1, \dots, x_n]$ and define an algebraic variety

$$\mathcal{EP}(H) := \{(\lambda, \mathbf{x}) \in \mathbb{C}^k \times \mathbb{P}^{n-1} : H(\lambda)\mathbf{x} = 0\}.$$

We will call this the *eigenpair variety* of H . In the context of an MEP, we will call the Cartesian product $\mathcal{EP}(H_1, \dots, H_k) := \mathcal{EP}(H_1) \times \dots \times \mathcal{EP}(H_k)$ the *multiparameter eigenpair variety* of H_1, \dots, H_k . Explicitly,

$$\begin{aligned} \mathcal{EP}(H_1, \dots, H_k) &= \{(\lambda_1, \dots, \lambda_k, \mathbf{x}_1, \dots, \mathbf{x}_k) \in \mathbb{C}^{k^2} \times \mathbb{P}^{n_1-1} \times \dots \times \mathbb{P}^{n_k-1} : \\ &H_1(\lambda_1)\mathbf{x}_1 = 0, \dots, H_k(\lambda_k)\mathbf{x}_k = 0\}. \end{aligned} \tag{7}$$

The *multiparameter eigenvalue variety* of H_1, \dots, H_k is the coordinatewise projection of the multiparameter eigenpair variety to \mathbb{C}^{k^2} and will be denoted by $\mathcal{EV}(H_1, \dots, H_k)$. Alternatively, it can be defined explicitly as

$$\begin{aligned} \mathcal{EV}(H_1, \dots, H_k) &= \{(\lambda_1, \dots, \lambda_k) \in \mathbb{C}^{k^2} : \\ &\det H_1(\lambda_1) = 0, \dots, \det H_k(\lambda_k) = 0\}. \end{aligned} \tag{8}$$

Throughout our article, \mathbb{N} will denote the set of *positive* integers.

Definition 4 An MEP is said to have *intrinsic dimension* $(d_1, \dots, d_k) \in \mathbb{N}^k$ if the total degree of the polynomial $\det H_i(\lambda_i)$ is d_i , $i = 1, \dots, k$. Such an MEP is said to be *generic with respect to intrinsic dimension* if the hypersurface defined by $\det H_i(\lambda_i)$ is generically reduced³ for $i = 1, \dots, k$.

As d_i 's are required to be positive, none of the $\det H_i(\lambda_i)$'s are the zero polynomial. By Bezout's theorem, the degree of the multiparameter eigenvalue variety $\mathcal{EV}(H_1, \dots, H_k) \subseteq \mathbb{C}^{k^2}$ is at most $\prod_{i=1}^k d_i$. So the number of isolated regular points in the intersection of $\mathcal{EV}(H_1, \dots, H_k)$ with a codimension- $k(k-1)$ affine linear space in \mathbb{C}^{k^2} is at most $\prod_{i=1}^k d_i$. We state this formally below.

Proposition 1 *An MEP with intrinsic dimension (d_1, \dots, d_k) has at most $\prod_{i=1}^k d_i$ regular eigenvalues and eigenpairs. This bound is tight if the MEP is generic with respect to intrinsic dimension.*

The next observation will be a key to our fiber product homotopy method.

Lemma 1 *Let \mathbb{D} denote the linear space defined by*

$$\mathbb{D} := \{(\lambda_1, \dots, \lambda_k, \mathbf{x}_1, \dots, \mathbf{x}_k) \in \mathbb{C}^{k^2} \times \mathbb{P}^{n_1-1} \times \dots \times \mathbb{P}^{n_k-1} : \lambda_1 = \dots = \lambda_k\}.$$

³ This is an algebraic geometry term that implies the polynomial $\det H_i(\lambda_i)$ is square-free.

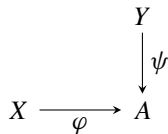
Then the intersection $\mathbb{D} \cap \mathcal{EP}(H_1, \dots, H_k)$ gives the set of eigenpairs of (2).

In fact, $\mathbb{D} \cap \mathcal{EP}(H_1, \dots, H_k)$ is the fiber product of $\mathcal{EP}(H_1), \dots, \mathcal{EP}(H_k)$ over \mathbb{C}^k , a standard notion in algebraic geometry [10]. This is the impetus for the name of our homotopy method—fiber product homotopy.

4 Fiber products

Knowledge of the fiber product’s formal properties at the level of, say, [10] is unnecessary for us. All we need is the notion of fiber product of sets—an important and well-known concept in relational database theory [16, Section 6.3].

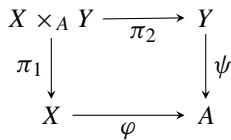
Let X, Y, A be sets and $\varphi: X \rightarrow A$ and $\psi: Y \rightarrow A$ be maps.



The fiber product $X \times_A Y$ of X and Y over A is the subset of $X \times Y$ given by

$$X \times_A Y := \{(x, y) \in X \times Y : \varphi(x) = \psi(y)\}.$$

Note that a fiber product depends on the maps φ and ψ although this is not reflected in the notation $X \times_A Y$, which is nevertheless standard. The fiber product satisfies the following commutative diagram where $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$ are projection maps:



We will illustrate fiber products with a few examples.

Example 1 (Relational database) Let $X = \{1, 2, 3, 4\}$, $Y = \{a, b, c, d, e\}$, and $A = \{-1, +1\}$. Let the maps $\varphi: X \rightarrow A$ and $\psi: Y \rightarrow A$ be given by

$$\varphi(x) = \begin{cases} +1 & x \text{ is odd,} \\ -1 & x \text{ is even,} \end{cases} \quad \psi(y) = \begin{cases} +1 & y \text{ is a vowel,} \\ -1 & y \text{ is a consonant.} \end{cases}$$

Then the fiber product of X and Y over A is

$$X \times_A Y = \{(1, a), (1, e), (2, b), (2, c), (2, d), (3, a), (3, e), (4, b), (4, c), (4, d)\}.$$

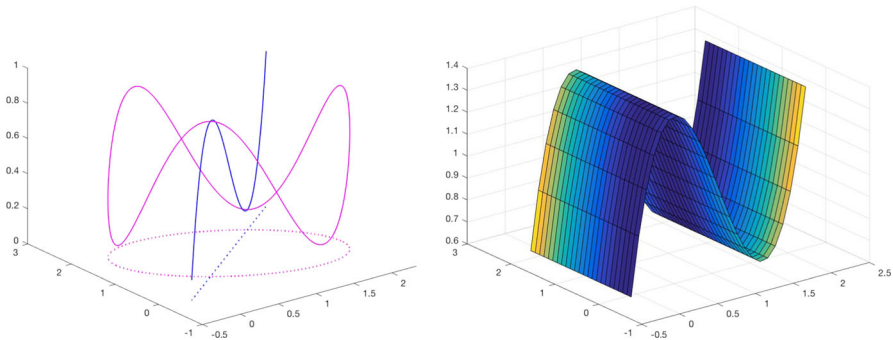


Fig. 1 $X \times_A Y$ and $X \times Y$ are subsets of \mathbb{R}^4 , but we only plot the coordinates (z_1, z_2, t_1) . *Left:* The fiber product $X \times_A Y$ is the union of the blue curve and the magenta curve. The dotted blue line and dotted magenta ellipse are the (z_1, z_2) -coordinate projections of the corresponding curve onto the $t_1 = 0$ plane. *Right:* The (z_1, z_2, t_1) -coordinate projection of the Cartesian product $X \times Y$ (color figure online)

Incidentally, this toy example underlies the JOIN operation in the structured query language (SQL) of a relational database management system (RDBMS). See [16, Section 6.3] for more information.

Example 2 (Algebraic geometry) Consider the following cubic curves in \mathbb{R}^2 :

$$X = \{(t_1, z_1) \in \mathbb{R}^2 : t_1 = z_1(z_1 - 1)(z_1 - 2) + 1\},$$

$$Y = \{(t_2, z_2) \in \mathbb{R}^2 : t_2 = z_2(z_2 - 1)(z_2 - 2) + 1\}.$$

Let $A = \mathbb{R}$ and consider the maps

$$\varphi : X \rightarrow A, \quad \varphi(t_1, z_1) = t_1 \quad \text{and} \quad \psi : Y \rightarrow A, \quad \psi(t_2, z_2) = t_2.$$

Their fiber product,

$$X \times_A Y = \{(t_1, t_2, z_1, z_2) \in \mathbb{R}^4 : (t_1, z_1) \in X, (t_2, z_2) \in Y, t_1 = t_2\},$$

is shown in Fig. 1. While the Cartesian product $X \times Y$ is an irreducible surface, the fiber product $X \times_A Y$ is a union of two curves—a point $(t_1, t_2, z_1, z_2) \in X \times_A Y$ satisfies

$$z_1 = z_2 \quad \text{or} \quad z_1^2 + z_1 z_2 + z_2^2 = 3z_1 + 3z_2 - 2.$$

One of the curves projects onto a line and the other projects onto an ellipse. Whereas the Cartesian product of two irreducible curves is always an irreducible surface, this example shows that the fiber product of two irreducible curves does not need to be irreducible.

Example 3 (*Multiparameter eigenvalue problem*) Consider a two-parameter eigenvalue problem (2) and the eigenpair varieties of H_1 and H_2 ,

$$X = \{(\lambda_1, \mathbf{x}_1) \in \mathbb{C}^2 \times \mathbb{P}^{n_1-1} : H_1(\lambda_1)\mathbf{x}_1 = 0\},$$

$$Y = \{(\lambda_2, \mathbf{x}_2) \in \mathbb{C}^2 \times \mathbb{P}^{n_2-1} : H_2(\lambda_2)\mathbf{x}_2 = 0\}.$$

Let $A = \mathbb{C}^2$ and consider the maps

$$\varphi: X \rightarrow A, \quad \varphi(\lambda_1, \mathbf{x}_1) = \lambda_1 \quad \text{and} \quad \psi: Y \rightarrow A, \quad \psi(\lambda_2, \mathbf{x}_2) = \lambda_2.$$

Their fiber product is

$$X \times_A Y = \{(\lambda_1, \lambda_2, \mathbf{x}_1, \mathbf{x}_2) \in \mathbb{C}^4 \times \mathbb{P}^{n_1-1} \times \mathbb{P}^{n_2-1} : (\lambda_1, \mathbf{x}_1) \in X, (\lambda_2, \mathbf{x}_2) \in Y, \lambda_1 = \lambda_2\} = \mathcal{EP}(H_1, H_2),$$

the two-parameter eigenpair variety of H_1, H_2 .

5 Fiber product homotopy method

Fiber products have previously appeared in numerical algebraic geometry in various contexts: study of exceptional sets [20], algorithms to intersect varieties [12], and numerical computations of Galois groups [11, Section 4]. However, the use of fiber products in a homotopy method for solving MEPs is, as far as we know, new. We will now describe this method.

5.1 Start system

We choose our start system by the following observation. Let $H(\lambda) = A_0 - \lambda_1 A_1 - \dots - \lambda_k A_k$ be a linear polynomial matrix and $L(\lambda) = A\lambda - \mathbf{b}$ be an affine linear function for some $A \in \mathbb{C}^{(k-1) \times k}$ and $\mathbf{b} \in \mathbb{C}^{k-1}$. We claim that

$$H(\lambda)\mathbf{x} = 0, \quad L(\lambda) = 0$$

is equivalent to a generalized eigenvalue problem, that we will call the *associated generalized eigenvalue problem* or associated GEP for short.

To see this, let $\lambda = \beta\mathbf{q} + \mathbf{p}$ where $\mathbf{p}, \mathbf{q} \in \mathbb{C}^k$ are such that $L(\mathbf{p}) = 0$ and $\mathbf{q} \in \ker(\nabla L)$, where ∇ denotes the total derivative (also known as total differential). Note that the previous statement says nothing more than $A\mathbf{p} = \mathbf{b}$ and $A\mathbf{q} = 0$. Eliminating \mathbf{q} and \mathbf{p} from

$$H(\beta\mathbf{q} + \mathbf{p})\mathbf{x} = 0, \quad \mathbf{q} \in \ker(\nabla L), \quad L(\mathbf{p}) = 0, \tag{9}$$

then gives us a GEP with β the generalized eigenvalue and \mathbf{x} the generalized eigenvector. We will see in Example 4 how one may obtain a GEP from (9) but expressing

the GEP in terms of general A, \mathbf{b} , and A_0, \dots, A_k is complicated and unilluminating. The *associated eigenpairs* are

$$\{(\lambda, \mathbf{x}) \in \mathbb{C}^k \times \mathbb{P}^{n-1} : \lambda = \beta \mathbf{q} + \mathbf{p}, H(\beta \mathbf{q} + \mathbf{p})\mathbf{x} = 0\}. \tag{10}$$

As each coordinate of $H(\beta \mathbf{q} + \mathbf{p})\mathbf{x}$ is homogeneous in \mathbf{x} , $H(\beta \mathbf{q} + \mathbf{p})\mathbf{x} = 0$ is well-defined for $\mathbf{x} \in \mathbb{P}^{n-1}$.

For an MEP (2), let $L_i: \mathbb{C}^k \rightarrow \mathbb{C}^{k-1}, i = 1, \dots, k$, be affine linear maps. Thus each $L_i(\lambda_i) = 0$ is an affine linear equation in $\lambda_i, i = 1, \dots, k$. We obtain k associated GEPs:

$$H_i(\lambda_i)\mathbf{x}_i = 0, \quad L_i(\lambda_i) = 0, \quad i = 1, \dots, k. \tag{11}$$

The set of solutions to (11) will be called *start solutions* or *start points* and denoted S . If the MEP has intrinsic dimension (d_1, \dots, d_k) , then the i th associated GEP has at most d_i generalized eigenvalues. Thus if S_i denotes the set of associated eigenpairs that have distinct generalized eigenvalues, then S is given by the Cartesian product

$$S = S_1 \times \dots \times S_k$$

and it has cardinality $\prod_{i=1}^k |S_i| \leq \prod_{i=1}^k d_i$.

We define $\mathcal{Q}_{\text{FP}}: \mathbb{C}^{k^2} \times \mathbb{P}^{n_1-1} \times \dots \times \mathbb{P}^{n_k-1} \rightarrow \mathbb{C}^{n_1+\dots+n_k} \times \mathbb{C}^{k(k-1)}$ by

$$\mathcal{Q}_{\text{FP}}(\lambda_1, \dots, \lambda_k, \mathbf{x}_1, \dots, \mathbf{x}_k) := \begin{bmatrix} H_1(\lambda_1)\mathbf{x}_1 \\ \vdots \\ H_k(\lambda_k)\mathbf{x}_k \\ L_1(\lambda_1) \\ \vdots \\ L_k(\lambda_k) \end{bmatrix} \tag{12}$$

and choose our *start system* to be the $\sum_{i=1}^k (k - 1 + n_i)$ equations in $\sum_{i=1}^k (k + n_i)$ variables

$$\mathcal{Q}_{\text{FP}}(\lambda_1, \dots, \lambda_k, \mathbf{x}_1, \dots, \mathbf{x}_k) = 0. \tag{13}$$

Again note that $H_i(\lambda_i)\mathbf{x}_i = 0$ is well-defined for $\mathbf{x}_i \in \mathbb{P}^{n_i-1}, i = 1, \dots, k$. We have the following easy observation.

Lemma 2 *The points in S are regular solutions to the start system (13). If the MEP is generic with respect to intrinsic dimension, then $|S| = \prod_{i=1}^k d_i$ for any generic choice of affine linear maps L_1, \dots, L_k .*

We now give an illustration of how an MEP can be transformed into a system of GEPs by imposing random affine constraints.

Example 4 Consider the two-parameter eigenvalue problem given by

$$H_1(\lambda_1) = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} + \lambda_{11} \begin{bmatrix} 11 & 13 \\ 17 & 19 \end{bmatrix} + \lambda_{12} \begin{bmatrix} 23 & 29 \\ 31 & 37 \end{bmatrix},$$

$$H_2(\lambda_2) = \begin{bmatrix} 12 & 31 \\ 15 & 71 \end{bmatrix} + \lambda_{21} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \lambda_{22} \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix},$$

where we write $\lambda_i = (\lambda_{i1}, \lambda_{i2}), i = 1, 2$. This two-parameter eigenvalue problem is singular and has only two regular eigenvalues.

We pick two random affine linear polynomials L_1 and L_2 , e.g.,

$$L_1(\lambda_1) = (0.6909 + 0.2745i)\lambda_{11} + (0.4277 - 0.1333i)\lambda_{12} - 1,$$

$$L_2(\lambda_2) = (-0.1443 + 0.5711i)\lambda_{21} + (-0.0735 + 1.8085i)\lambda_{22} - 1.$$

Then we have

$$[q_1, q_2, p_1, p_2] = \begin{bmatrix} -0.41 + 0.31i & -0.52 + 0.79i & 1.2501 - 0.4967i & 0 \\ 0.85 + 0.02i & 0.22 - 0.21i & 0 & -0.0224 - 0.5520i \end{bmatrix}.$$

The polynomial matrices for the two associated GEP are

$$H_1(\beta q_1 + p_1) = \begin{bmatrix} -11.75 + 5.46i & -13.25 + 6.45i \\ -16.25 + 8.44i & -16.75 + 9.43i \end{bmatrix} - \beta \begin{bmatrix} 15.17 + 4.02i & 19.48 + 4.79i \\ 19.55 + 6.10i & 23.87 + 6.87i \end{bmatrix},$$

$$H_2(\beta q_2 + p_2) = \begin{bmatrix} 12.04 + 1.10i & 31.04 + 1.10i \\ 15.04 + 1.10i & 71.04 + 1.10i \end{bmatrix} - \beta \begin{bmatrix} -0.08 + 0.35i & -0.08 + 0.35i \\ -0.08 + 0.35i & -0.08 + 0.35i \end{bmatrix}.$$

The first GEP has two finite eigenvalues: $-0.9978 + 1.1933i$ and $-0.5637 + 0.3035i$. The second GEP has only one finite eigenvalue at $-3.6333 - 28.4804i$. Note that the values $d_1 = 2, d_2 = 1$ are in agreement with the number of regular eigenvalues. Thus the *start solutions*, i.e., the solutions to our start system, of our homotopy is a set of two points with (λ_1, λ_2) -coordinates below.

λ_1	λ_2
$(-0.9978 + 1.1933i)q_1 + p_1$	$(-3.6333 - 28.4804i)q_2 + p_2$
$(-0.5637 + 0.3035i)q_1 + p_1$	$(-3.6333 - 28.4804i)q_2 + p_2$

5.2 Target system

For $i = 1, \dots, k$, let $R_i \in \mathbb{C}^{(k-1) \times k(k-1)}$ be generic and let $G_i : \mathbb{C}^{k^2} \rightarrow \mathbb{C}^{k-1}$ be the linear function defined by

$$G_i(\lambda_1, \dots, \lambda_k) := R_i D_k \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{bmatrix}, \quad D_k := \begin{bmatrix} I_k & -I_k & & \\ & \ddots & \ddots & \\ & & & I_k & -I_k \end{bmatrix} \in \mathbb{C}^{k(k-1) \times k^2}, \quad (14)$$

where I_k is the $k \times k$ identity matrix. If the matrices $R_1, \dots, R_k \in \mathbb{C}^{(k-1) \times k(k-1)}$ are generic, then

$$\begin{aligned} & \{(\lambda_1, \dots, \lambda_k) \in \mathbb{C}^{k^2} : G_1(\lambda_1, \dots, \lambda_k) = \dots = G_k(\lambda_1, \dots, \lambda_k) = 0\} \\ & = \{(\lambda_1, \dots, \lambda_k) \in \mathbb{C}^{k^2} : \lambda_1 = \dots = \lambda_k\}, \end{aligned} \tag{15}$$

and therefore,

$$\begin{aligned} & \{(\lambda_1, \dots, \lambda_k, \mathbf{x}_1, \dots, \mathbf{x}_k) \in \mathbb{C}^{k^2} \times \mathbb{P}^{n_1-1} \times \dots \times \mathbb{P}^{n_k-1} : \\ & \quad G_1(\lambda_1, \dots, \lambda_k) = \dots = G_k(\lambda_1, \dots, \lambda_k) = 0\} \\ & = \{(\lambda_1, \dots, \lambda_k, \mathbf{x}_1, \dots, \mathbf{x}_k) \in \mathbb{C}^{k^2} \times \mathbb{P}^{n_1-1} \times \dots \times \mathbb{P}^{n_k-1} : \lambda_1 = \dots = \lambda_k\} = \mathbb{D}, \end{aligned}$$

i.e., the linear space in Lemma 1. Hence the system

$$H_i(\lambda_i)\mathbf{x}_i = 0, \quad G_1(\lambda_1, \dots, \lambda_k) = \dots = G_k(\lambda_1, \dots, \lambda_k) = 0, \quad i = 1, \dots, k, \tag{16}$$

is equivalent to the fiber product MEP (4), which is equivalent to the original MEP in (2). We define

$$\mathbf{P}_{\text{FP}}(\lambda_1, \dots, \lambda_k, \mathbf{x}_1, \dots, \mathbf{x}_k) := \begin{bmatrix} H_1(\lambda_1)\mathbf{x}_1 \\ \vdots \\ H_k(\lambda_k)\mathbf{x}_k \\ G_1(\lambda_1, \dots, \lambda_k) \\ \vdots \\ G_k(\lambda_1, \dots, \lambda_k) \end{bmatrix}, \tag{17}$$

and choose our *target system* to be

$$\mathbf{P}_{\text{FP}}(\lambda_1, \dots, \lambda_k, \mathbf{x}_1, \dots, \mathbf{x}_k) = 0,$$

which is of course just (16). By Proposition 1, our target system (17) yields the eigenpairs of the MEP.

5.3 Fiber product homotopy

The main objective of this section is to show that our start system is chosen correctly with probability one.

Definition 5 A *fiber product homotopy* for the MEP with linear polynomial matrices H_1, \dots, H_k is a straight-line homotopy from $t = 0$ to $t = 1$ given by the polynomial map

$$\begin{aligned}
 \mathbf{H}_{\text{FP}} : \mathbb{C}^{k^2} \times (\mathbb{P}^{n_1-1} \times \dots \times \mathbb{P}^{n_k-1}) \times \mathbb{C} &\rightarrow \mathbb{C}^{n_1+\dots+n_k} \times \mathbb{C}^{(k-1)k}, \\
 \mathbf{H}_{\text{FP}}(\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_k, \mathbf{x}_1, \dots, \mathbf{x}_k, t) &:= \begin{bmatrix} H_1(\boldsymbol{\lambda}_1)\mathbf{x}_1 \\ \vdots \\ H_k(\boldsymbol{\lambda}_k)\mathbf{x}_k \\ (1-t)L_1(\boldsymbol{\lambda}_1) + tG_1(\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_k) \\ \vdots \\ (1-t)L_k(\boldsymbol{\lambda}_k) + tG_k(\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_k) \end{bmatrix}. \tag{18}
 \end{aligned}$$

Note that $\mathbf{H}_{\text{FP}} = (1-t)\mathbf{Q}_{\text{FP}} + t\mathbf{P}_{\text{FP}}$.

Throughout the article we assume that $k \geq 2$ so that we indeed have a multiparameter eigenvalue problem. If $k = 1$, then $k(k-1) = 0$, and (18) will not involve the path parameter t . If the reader is wondering whether our homotopy method applies to a standard or generalized eigenvalue problem like in [24], this shows that the answer is no.

Theorem 1 *The fiber product homotopy for an MEP (18) with intrinsic dimension (d_1, \dots, d_k) has a start system chosen correctly with probability one if the start solutions S has $|S| = \prod_{i=1}^k d_i$.*

Proof By Lemma 2, the start solutions S are known after solving k generalized eigenvalue problems of dimensions n_1, \dots, n_k respectively. Consider the variety

$$\begin{aligned}
 \mathcal{C} := \{(\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_k, \mathbf{x}_1, \dots, \mathbf{x}_k) \in \mathbb{C}^{k^2} \times \mathbb{P}^{n_1-1} \times \dots \times \mathbb{P}^{n_k-1} : \\
 L(\boldsymbol{\lambda}_1) = 0, \dots, L(\boldsymbol{\lambda}_k) = 0\}.
 \end{aligned}$$

The intersection $\mathcal{C} \cap \mathcal{EP}(H_1, \dots, H_k)$ consists of $|S| = \prod_{i=1}^k d_i$ points. As each G_i is a linear function and each $R_i \in \mathbb{C}^{(k-1) \times k(k-1)}$ is generic, $i = 1, \dots, k$, it follows from the gamma trick [21, Lemma 7.1.3] that for regular eigenpairs, the homotopy has a start system chosen correctly with probability one. \square

We will provide more extensive numerical experiments in Sect. 7 but here we illustrate our method with a small example: $k = 3$ and $n_1 = n_2 = n_3 = 2$.

Example 5 We generate an MEP by randomly choosing the 2×2 coefficient matrices A_{ij} for $i = 1, 2, 3$ and $j = 0, 1, 2, 3$. There are eight solutions to the start and target systems. Note that fiber homotopy method requires that we work with $(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \in \mathbb{C}^9 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. The end point will however be of the form $(\boldsymbol{\lambda}, \boldsymbol{\lambda}, \boldsymbol{\lambda}, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \in \mathbb{C}^9 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ where $(\boldsymbol{\lambda}, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \in \mathbb{C}^3 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is a multiparameter eigenpair.

With our homotopy (18), we deform from $t = 0$ to $t = 1$. Note that $\boldsymbol{\lambda}_i = (\lambda_{i1}, \lambda_{i2}, \lambda_{i3}) \in \mathbb{C}^3, i = 1, 2, 3$. In the left plot of Fig. 2, we track the λ_{i1} -coordinate of all eights paths for $t \in [0, 9, 1], i = 1, 2, 3$. The horizontal and vertical axes represent the real and imaginary axes. We see eight sets of three paths (colored red, blue, magenta to represent $i = 1, 2, 3$), each converging to a point that represents the first coordinate of an eigenvalue. We picked the λ_{i1} -coordinate arbitrarily and could have done the same

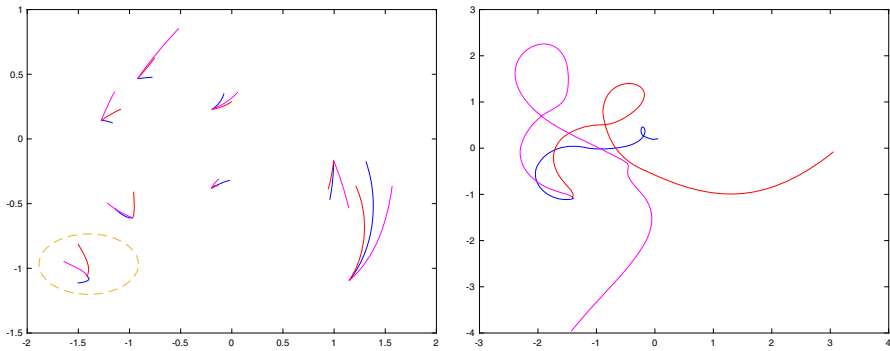


Fig. 2 Left: The λ_{i1} coordinates of all eight paths for $t \in [0.95, 1]$, $i = 1, 2, 3$. Right: The λ_{i1} coordinates of the path highlighted in the left panel for $t \in [0, 1]$, $i = 1, 2, 3$. The horizontal and vertical axes represent the real and imaginary axes

plot for any of the 15 coordinates in $(\lambda_1, \lambda_2, \lambda_3, x_1, x_2, x_3) \in \mathbb{C}^9 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. What we are witnessing is a one-dimensional projection of the homotopy path in $\mathbb{C}^9 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ converging to the eight eigenpairs of the MEP. Note that “one dimension” here means “one complex dimension” which translates to the two real dimensions we see in Fig. 2.

The left plot shows only the behavior of the homotopy path towards the end, i.e., only for $t \in [0.95, 1]$. The right plot in Fig. 2 shows the full homotopy path, i.e., for all $t \in [0, 1]$, of the λ_{i1} -coordinates for one of the eight solutions of the left plot.

We would like to emphasize that in Example 5, the homotopy path is confined to a six-dimensional subspace of $\mathbb{C}^9 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ as only $6 = k(k - 1)$ equations involve the path parameter t . The following difference between the fiber product homotopy and diagonal coefficient homotopy for MEP cannot be overstated. For the former, at most $k(k - 1)$ equations involve the path parameter, all of which are *linear* in $\lambda_1, \dots, \lambda_k$. For the latter, $n_1 + n_2 + \dots + n_k$ equations involve the path parameter, all of which are *multilinear*. Even when $k(k - 1) > n_1 + \dots + n_k$, the paths of the fiber product homotopy would still be easier to track than those of the diagonal coefficient homotopy, as we will see later.

6 Continuation procedure

Observe that there are k more variables than equations in (18) because the eigenvectors are unique only up to scaling, i.e., they are indeed points in projective spaces. To account for this arbitrary scaling, we fix a generic *affine chart* in each projective space \mathbb{P}^{n_i-1} , i.e., by introducing the affine constraints $d_i^T x_i = 1$ with $d_i \in \mathbb{C}^{n_i}$, $i = 1, \dots, k$.

Let $A_{ij} \in \mathbb{C}^{n_i \times n_i}$, $j = 0, 1, \dots, k$, $i = 1, \dots, k$ be the matrices of our MEP as in Definition 1. Recall the maps $G_i : \mathbb{C}^{k^2} \rightarrow \mathbb{C}^{k-1}$ in (14) and $L_i : \mathbb{C}^k \rightarrow \mathbb{C}^{k-1}$ in (11). For $i = 1, \dots, k$, we let $\bar{L}_i : \mathbb{C}^{k^2} \rightarrow \mathbb{C}^{k-1}$ be defined by

$$\bar{L}_i(\lambda_1, \dots, \lambda_k) := L_i(\lambda_i),$$

i.e., \bar{L}_i extends the domain of L_i from \mathbb{C}^k to \mathbb{C}^{k^2} .

In the following, we write $\mathbb{O}_{m \times n} \in \mathbb{C}^{m \times n}$ and $\mathbb{O}_n \in \mathbb{C}^n$ for the zero matrix and zero vector, and $\mathbb{1}_n \in \mathbb{C}^n$ for the all ones vector. To track the homotopy, we use an *Euler–Newton predictor-corrector* method. The Euler step gives an approximate eigenpair whereas the Newton step refines the approximation:

EULER STEP: Solve $(k^2 + n_1 + \dots + n_k) \times (k^2 + n_1 + \dots + n_k)$ linear system

$$\begin{bmatrix} \text{diag}(B_1(\mathbf{x}_1), \dots, B_k(\mathbf{x}_k)) & \text{diag}(H_1(\lambda_1), \dots, H_k(\lambda_k)) \\ \mathbb{O}_{k \times k^2} & \text{diag}(\mathbf{d}_1^T, \dots, \mathbf{d}_k^T) \\ (1-t)\nabla \bar{L}_1 + t\nabla G_1 & \mathbb{O}_{(k-1) \times (n_1 + \dots + n_k)} \\ \vdots & \vdots \\ (1-t)\nabla \bar{L}_k + t\nabla G_k & \mathbb{O}_{(k-1) \times (n_1 + \dots + n_k)} \end{bmatrix} \begin{bmatrix} \hat{\lambda}_1 \\ \vdots \\ \hat{\lambda}_k \\ \hat{\mathbf{x}}_1 \\ \vdots \\ \hat{\mathbf{x}}_k \end{bmatrix} = \begin{bmatrix} \mathbb{O}_{n_1 + \dots + n_k + k} \\ \bar{L}_1 - G_1 \\ \vdots \\ \bar{L}_k - G_k \end{bmatrix},$$

where $B_i(\mathbf{x}_i) := -[A_{i1}\mathbf{x}_i, \dots, A_{ik}\mathbf{x}_i] \in \mathbb{C}^{n_i \times k}$, $i = 1, \dots, k$. Here we regard the total derivatives as Jacobian matrices, i.e., $\nabla \bar{L}_i, \nabla G_i \in \mathbb{C}^{(k-1) \times k^2}$, $i = 1, \dots, k$. As \bar{L}_i 's and G_i 's are affine linear maps, their Jacobians are constant matrices that do not depend on λ_i 's.

PREDICTOR: This is given by

$$\mathbf{p} := \begin{bmatrix} \tilde{\lambda}_1 \\ \vdots \\ \tilde{\lambda}_k \\ \tilde{\mathbf{x}}_1 \\ \vdots \\ \tilde{\mathbf{x}}_k \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_k \\ \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_k \end{bmatrix} + h \begin{bmatrix} \hat{\lambda}_1 \\ \vdots \\ \hat{\lambda}_k \\ \hat{\mathbf{x}}_1 \\ \vdots \\ \hat{\mathbf{x}}_k \end{bmatrix} \in \mathbb{C}^{n_1 + \dots + n_k + k^2}$$

where h is the step size. The predictor is the input to the Newton step.

NEWTON STEP: Solve $(k^2 + n_1 + \dots + n_k) \times (k^2 + n_1 + \dots + n_k)$ linear system

$$\begin{bmatrix} \text{diag}(B_1(\mathbf{x}_1^{(\ell)}), \dots, B_k(\mathbf{x}_k^{(\ell)})) & \text{diag}(H_1(\boldsymbol{\lambda}_1^{(\ell)}), \dots, H_k(\boldsymbol{\lambda}_k^{(\ell)})) \\ \mathbb{O}_{k \times k^2} & \text{diag}(\mathbf{d}_1^\top, \dots, \mathbf{d}_k^\top) \\ (1-t)\nabla\bar{L}_1 + t\nabla G_1 & \mathbb{O}_{(k-1) \times (n_1 + \dots + n_k)} \\ \vdots & \vdots \\ (1-t)\nabla\bar{L}_k + t\nabla G_k & \mathbb{O}_{(k-1) \times (n_1 + \dots + n_k)} \end{bmatrix} \begin{bmatrix} \Delta\boldsymbol{\lambda}_1 \\ \vdots \\ \Delta\boldsymbol{\lambda}_k \\ \Delta\mathbf{x}_1 \\ \vdots \\ \Delta\mathbf{x}_k \end{bmatrix}$$

$$= - \begin{bmatrix} H_1(\boldsymbol{\lambda}_1^{(\ell)})\mathbf{x}_1^{(\ell)} \\ \vdots \\ H_k(\boldsymbol{\lambda}_k^{(\ell)})\mathbf{x}_k^{(\ell)} \\ \mathbf{d}_1^\top\mathbf{x}_1^{(\ell)} - 1 \\ \vdots \\ \mathbf{d}_k^\top\mathbf{x}_k^{(\ell)} - 1 \\ \mathbf{v}^{(\ell)} \end{bmatrix}$$

where B_i is as defined in the Euler step and

$$\mathbf{v}^{(\ell)} := \begin{bmatrix} (1-t)\nabla\bar{L}_1 + t\nabla G_1 \\ \vdots \\ (1-t)\nabla\bar{L}_k + t\nabla G_k \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda}_1^{(\ell)} \\ \vdots \\ \boldsymbol{\lambda}_k^{(\ell)} \end{bmatrix} + (1-t) \begin{bmatrix} \mathbb{1}_{k-1} \\ \vdots \\ \mathbb{1}_{k-1} \end{bmatrix} \in \mathbb{C}^{k(k-1)}.$$

The initial approximation $[\boldsymbol{\lambda}_1^{(0)}, \dots, \boldsymbol{\lambda}_k^{(0)}, \mathbf{x}_1^{(0)}, \dots, \mathbf{x}_k^{(0)}]^\top$ is given by \mathbf{p} .

CORRECTOR: This is given by the solution to the Newton step

$$\mathbf{c} = \begin{bmatrix} \Delta\boldsymbol{\lambda}_1 \\ \vdots \\ \Delta\boldsymbol{\lambda}_k \\ \Delta\mathbf{x}_1 \\ \vdots \\ \Delta\mathbf{x}_k \end{bmatrix} \in \mathbb{C}^{n_1 + \dots + n_k + k^2}, \tag{19}$$

which is then added to refine the approximation,

$$\begin{bmatrix} \lambda_1^{(\ell+1)} \\ \vdots \\ \lambda_k^{(\ell+1)} \\ \mathbf{x}_1^{(\ell+1)} \\ \vdots \\ \mathbf{x}_k^{(\ell+1)} \end{bmatrix} = \begin{bmatrix} \lambda_1^{(\ell)} + \Delta\lambda_1 \\ \vdots \\ \lambda_k^{(\ell)} + \Delta\lambda_k \\ \mathbf{x}_1^{(\ell)} + \Delta\mathbf{x}_1 \\ \vdots \\ \mathbf{x}_k^{(\ell)} + \Delta\mathbf{x}_k \end{bmatrix}.$$

Our Euler–Newton predictor-corrector method uses a predictor \mathbf{p} with size h , followed by Newton steps until the norm of \mathbf{c} is sufficiently small or when the maximum number of iterations is reached. If $t + h < 1$, we return to the Euler step. If $t + h \geq 1$, we update h so that $t + h = 1$, and do one final Euler step followed by Newton steps. For a rough idea of the relative costs, tracking one path, i.e., one eigenpair, typically takes a total of 50 Euler steps and a total of 400 Newton steps. We provide actual implementation details in Sect. 7.

The matrices in the Euler and Newton steps will in general look like the one depicted in Fig. 3 for a four-parameter eigenvalue problem. The k row vectors $\mathbf{d}_1^T, \dots, \mathbf{d}_k^T$ near the bottom right corner and the $(k^2 - k) \times k$ block representing $[(1-t)\nabla\bar{L}_i + t\nabla G_i]_{i=1}^k$ on the bottom left corner of the matrix are dense—recall from Lemma 2, Theorem 1, and the first paragraph of this section that we require the entries in these blocks be generic, i.e., there is zero probability that any of these entries is zero. On the other hand, if the matrices $A_{ij} \in \mathbb{C}^{n_i \times n_i}$, $j = 0, 1, \dots, k$, $i = 1, \dots, k$, defining the MEP are sufficiently sparse, then the lighter shaded blocks in the top right part of the matrix representing $H_i(\lambda_i)$ (Euler step) or $H_i(\lambda_i^{(\ell)})$ (Newton step), being the sum of $k + 1$ sparse matrices, will also be sparse. The darker shaded blocks on the top left part of the matrix representing $B_i(\mathbf{x}_i)$ (Euler step) or $B_i(\mathbf{x}_i^{(\ell)})$ (Newton step) are expected to be dense, as the column vectors in these blocks are each a product of a sparse matrix and a dense vector.

The Euler–Newton predictor-corrector method will converge to the solution of the target system if the step sizes are sufficiently small and the approximate start solution is sufficiently close to the actual start solution [1, Theorem 5.2.1]. Quantifying the “sufficiently small” and “sufficiently close” is still an active area of research, but we will see in Sects. 8–10 ample numerical evidence of stable convergence to true solutions (even for dimension-deficient singular MEPs). Indeed, the results in Sects. 8–10 are from thousands of MEPs — a single value in the tables/figures represents an aggregate over tens or hundreds of runs— and we did not encounter any instance where Euler–Newton failed to converge.

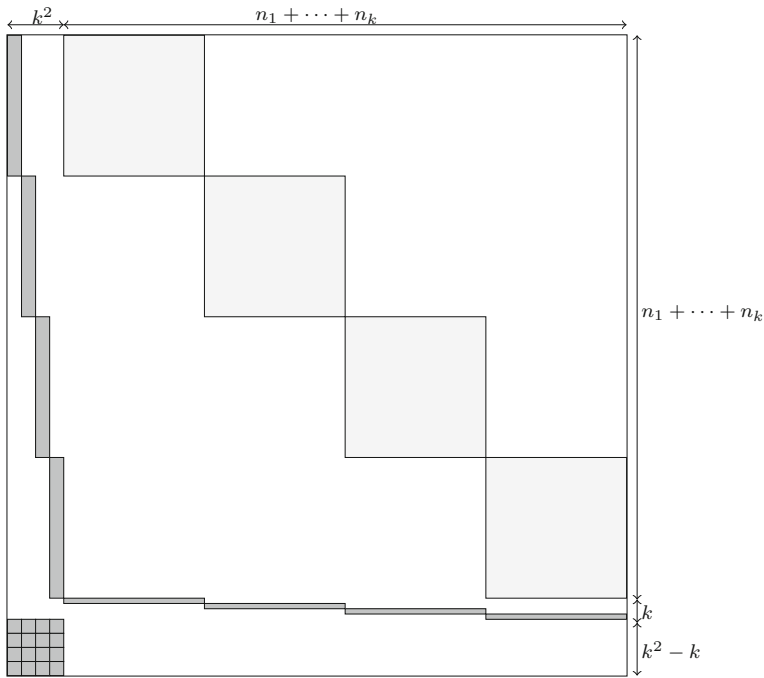


Fig. 3 Pictorial representation of a typical matrix in the Euler and Newton steps. Here $k = 4$ and $n_1 = n_2 = n_3 = n_4$. The darker blocks are almost always dense; the lighter blocks are sparse if the input matrices A_{ij} 's are sufficiently sparse; the white areas are always zero

7 Implementation

We implemented⁴ our method in both MATLAB and BERTINI/MACAULAY2, catering respectively to the numerical computing and symbolic computing communities. All experiments are carried out on single node, an Intel E5-2680v4 2.4 GHz processor with 28 threads and 56 GB of RAM, in the University of Chicago Research Computing Center. The parameters below can be readily changed by the user.

INPUTS: The inputs are the coefficients $A_{ij} \in \mathbb{C}^{n_i \times n_i}$, $j = 0, 1, \dots, k$, $i = 1, \dots, k$, of the polynomial matrices H_1, \dots, H_k , as in Definition 1.

START SOLUTIONS: Solutions to the start system (12) are obtained as follows. For each i , we set the constant terms of $L_i(\lambda_i)$ to be -1 , and generate other coefficients from the standard complex Gaussian distribution using `randn` (MATLAB) or `random CC` (MACAULAY2). In our MATLAB implementation, we determine a null vector $q_i \in \ker(\nabla L_i)$ using `null`, a particular solution p_i to $L_i(\lambda_i) = 0$ using `mldivide` (i.e., backslash), and the associated eigenpairs

⁴ https://github.com/JoseMath/MEP_Homotopy.

using `eig`. In our BERTINI/MACAULAY2 implementation, we use `bertiniZeroDimSolve` (BERTINI) with default settings to determine the solutions to $H_i(\lambda_i)x_i = 0$ and $L_i(\lambda_i) = 0$.

CONTINUATION: The bulk of our computations is in running the Euler–Newton predictor-corrector method in Sect. 6. In our MATLAB implementation, the linear systems in the Euler and Newton steps are solved using `mldivide`. We use two sets of criteria for controlling the step size and termination. In the `conservative` criterion, the step size h in the predictor p is updated as follows: if the number of Newton iterations $i_{NT} \leq 2$, we double h ; if $i_{NT} > 8$, we halve h ; else we keep the same h . The Newton iteration continues until the corrector c satisfies $\|c\|_\infty < 10^{-9}$ or when number of iterations exceeds $\max\{20, k \max(n_1, \dots, n_k) + 5\}$. In the `fast` criterion, the step size h in the predictor p is updated as follows: if the number of Newton iterations $i_{NT} \leq 2$, we double h ; if $i_{NT} > 5$, we halve h ; else we keep the same h . The Newton iteration continues until the corrector c satisfies $\|c\|_2 < 10^{-8}$ or when the number of iterations exceeds five. The maximum and minimum step sizes are 10^{-2} and 10^{-6} respectively for both two criteria. The `fast` criterion is the same one used in [8], and we will use it when comparing with the results given by the method in [8]. Otherwise, we will use the `conservative` criterion for better accuracy. These choices are heuristical but may be easily fine-tuned.

BERTINI is specifically designed with the homotopy method in mind and these tasks are built-in and automated; users need only specify the configurations they wish to change in the input file. In our BERTINI/MACAULAY2 implementation, we may simply use `runBertini` with default configurations.

STOPPING CONDITIONS: In our MATLAB implementation, we estimate the endpoint of the homotopy as follows: when $t + h > 1$, we perform the Euler step with step size $h = 1 - t$ and refine our endpoint with Newton’s method until the change in the update is sufficiently small (the default is $\|c\|_\infty < 10^{-9}$) or when the maximum number of iterations is reached (default is set to $\max\{20, k \cdot \max(n_1, \dots, n_k) + 5\}$). Again, BERTINI is designed for homotopy method and has a variety of built-in options for estimating endpoint. Among other things, BERTINI has implemented various sophisticated “endgames” to identify solutions with multiplicities larger than one and return the values of these multiplicities, a feature that is too involved to replicate in our MATLAB implementation. We use the default “fractional power series endgame.”

In the next three sections, we present our numerical results on MEPs that are (i) randomly generated (Sect. 8), (ii) from a real-world application (Sect. 9), and (iii) dimension-deficient and singular (Sect. 10). We compare speed and accuracy of our method with those of Delta and diagonal coefficient homotopy methods.

8 Numerical results I: randomly generated MEPs

Here we randomly generate our inputs $A_{ij} \in \mathbb{C}^{n_i \times n_i}$, $j = 0, 1, \dots, k$, $i = 1, \dots, k$, from the standard complex Gaussian distribution. For convenience, we assume that $n_1 = \dots = n_k = n$ so that there are just two parameters k and n to consider. We write N for the number of eigenpairs, given by Proposition 1.

We report the maximum time it takes to track one path, denoted by t_{PATH} , and the average number of Newton iterations during the path tracking, denoted by $\phi(k, n)$. The value of t_{PATH} is important as path-tracking is a task with high parallelism and t_{PATH} provides a good estimate of the time it takes to run fiber product homotopy method in parallel. Nevertheless, we are only able to report t_{PATH} for our MATLAB implementation as BERTINI deals with path-tracking in a more sophisticated and automated manner that offers users no easy way of determining the total track time of one path.

We investigate the stability of our method and the accuracy of our solutions by examining the backward error as defined in [14, Section 3]. The normwise backward error of an approximate eigenpair $(\lambda, x_1, \dots, x_k)$ of an MEP with coefficients $A_{ij} \in \mathbb{C}^{n_i \times n_i}$, $j = 0, 1, \dots, k$, $i = 1, \dots, k$, and polynomial matrices H_1, \dots, H_k , as in Definition 1 is given by

$$\begin{aligned} \eta(\lambda, x_1, \dots, x_k) &:= \min\{\varepsilon \in \mathbb{R} : (H_i(\lambda) + \Delta H_i(\lambda))x_i = 0, \\ &\quad \|\Delta A_{ij}\| \leq \varepsilon \|A_{ij}\|, \quad i = 1, \dots, k, \quad j = 0, \dots, k\}, \\ \Delta H_i(\lambda) &:= \Delta A_{i0} - \sum_{j=1}^k \lambda_j \Delta A_{ij}, \quad i = 1, \dots, k. \end{aligned}$$

To compute $\eta(\lambda, x_1, \dots, x_k)$ we take advantage of [14, Theorem 2], which says

$$\eta(\lambda, x_1, \dots, x_k) = \max_{i=1, \dots, k} \left(\frac{\|H_i(\lambda)x_i\|}{\|A_{i0}\| + \sum_{j=1}^k |\lambda_j| \|A_{ij}\|} \right).$$

8.1 Fixed k , varying n

Here we fix $k = 3$ and $n_1 = n_2 = n_3 = n$. For each value of n we generate ten three-parameter eigenvalue problems. Note that the expected number of eigenpairs in these problems is $N = n^3$. As our MATLAB implementation of fiber product homotopy method computes in parallel with 28 threads, the wall time reported in Table 1 is estimated by multiplying the real wall time by 28. In Table 1, we see that our method is faster than the timings reported for the diagonal coefficient homotopy method in [8]. The Delta method fails for larger values of n —it crashes with an out-of-memory error in every instance when $n = 30$.

Table 1 Elapsed timings (in seconds) for Delta method, fiber product homotopy method (averaged over ten runs), and diagonal coefficient homotopy method [8]

n	N n^3	Wall time			t_{PATH}	$\phi(3, n)$
		Delta Mtd.	Fiber Prod.	Diag. Coeff.		
10	1000	1.44	264.38	779.47	0.22	386
15	3375	19.14	906.00	2888.73	0.25	398
20	8000	150.59	2547.29	7857.44	0.31	402
25	15625	815.9125	5905.11	17169.53	0.37	408
30	27000	FAILED	12820.89	32786.64	0.46	418

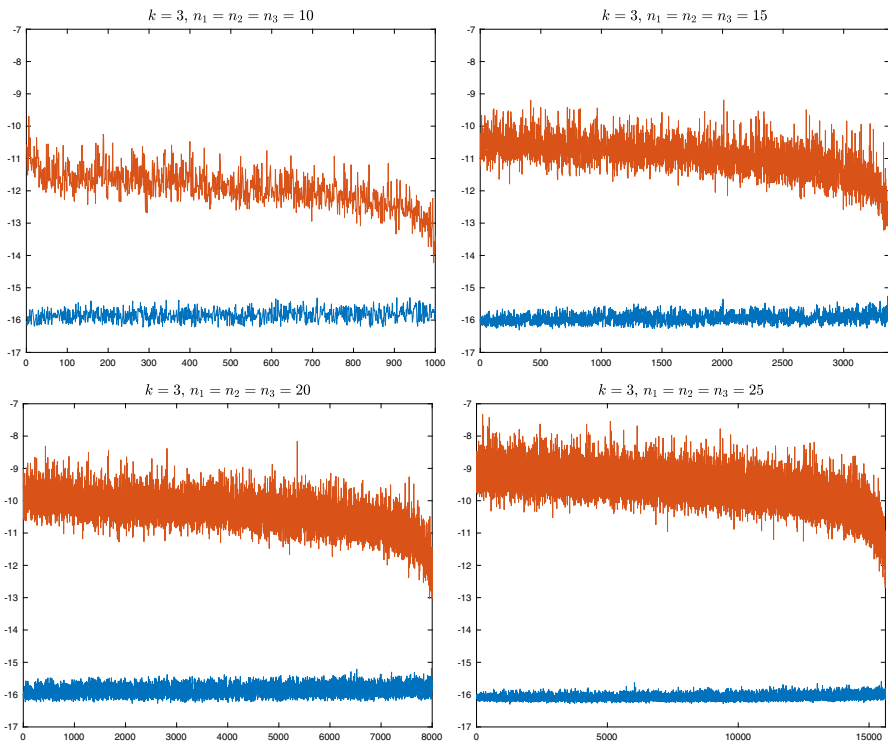


Fig. 4 The \log_{10} backward errors (vertical axis) of fiber product method (blue) and the Delta method (red) are plotted against the eigenvalues (horizontal axis) ordered by increasing norm from left to right (color figure online)

Our results for backward errors are presented in Fig. 4, where it is clear that fiber product homotopy method (blue plot) has significantly smaller backward error than the Delta method (red plot) in our numerical experiments. The experiments in Table 1 and Fig. 4 find all eigenpairs by tracking all start solutions of (12).

In our next experiment, our goal is to examine the effect of a substantial increase in n on speed and accuracy of our fiber product homotopy method. Finding all eigenpairs would take too long and serves little purpose for this next experiment. Instead,

Table 2 Timings (in seconds) and accuracy for fiber product homotopy method (100 runs)

n	Wall time			Backward error		
	Best	Avg	Worst	Best	Avg	Worst
30	0.71	1.12	4.45	7.47×10^{-17}	1.81×10^{-16}	1.42×10^{-15}
70	2.88	4.27	8.85	8.18×10^{-17}	1.66×10^{-16}	1.26×10^{-15}
150	16.08	29.07	91.45	1.05×10^{-16}	1.96×10^{-16}	1.36×10^{-15}

we randomly generate 100 MEPs for each n and track *one* random start solution. The timings in Table 2 are rough estimates of the time it would have taken to find all eigenpairs, obtained by multiplying average wall time for one randomly chosen eigenpair by n^3 . The backward errors in Table 2 are for one randomly chosen eigenpair (and not multiplied by n^3). Evidently, increasing n has negligible effect on backward errors, which are all splendidly small—on the order of 10^{-15} or less.

8.2 Fixed n , varying k

This time we fix $n_1 = \dots = n_k = 3$ and vary k from 3, \dots , 9. The expected number of eigenpairs is then $N = 3^k$. If the reader is wondering why we do not increase k in a more drastic manner, note that increasing k produces a corresponding exponential increase in the number of solutions—for each randomly generated MEP, there are 3^k eigenpairs. Also, unlike changing n , which just changes the dimension of the problem, changing k gives a *different class* of problems—for example, a two-parameter eigenvalue problem is qualitatively different from a one-parameter eigenvalue problem (i.e., a GEP)—and each k deserves a careful examination. Compared to the numbers for diagonal coefficient homotopy method in [8, Table 3], the numbers in Table 3 show that the fiber product homotopy method is significantly faster and also more stable in the sense that every path converged and did not need to be rerun.

The reader is reminded that fiber product homotopy method tracks k paths each carrying a copy of the eigenvalue; these k copies $\lambda_1, \dots, \lambda_k$ converge to the same eigenvalue $\lambda_1 = \dots = \lambda_k$ if the method runs correctly (see Example 5). This is indeed the case as the reader can see from the near zero values of $\delta(k) := \max_{2 \leq i \leq k} (\|\lambda_1 - \lambda_i\|_1)$ reported in Table 3.

8.3 Comparisons on Bertini

The results in Tables 1 and 3 compare the MATLAB implementation of fiber product homotopy method (18) to the results for diagonal coefficient homotopy method (6) reported in [8]. Here we will compare them on BERTINI.⁵ As BERTINI is designed for homotopy continuation methods, we do not include a non-homotopy method like Delta method in our comparison on this platform.

⁵ We have also used a MACAULAY2 [9] package [4] to produce the relevant input files.

Table 3 Elapsed timings (in seconds) for Delta method and the two homotopy methods, all in MATLAB. Accuracy for our method. Here $\delta(k) = \max_{2 \leq i \leq k} (\|\lambda_1 - \lambda_i\|_1)$

k	N 3^k	Wall time			$\delta(k)$	$\phi(k, 3)$ Fiber Prod.	t_{PATH}
		Delta Mtd.	Fiber Prod.	Diag. Coeff.			
4	81	0.13	50.81	77.63	2.90×10^{-15}	382	0.27
5	243	0.19	89.44	288.24	5.43×10^{-15}	403	0.26
6	729	1.32	269.92	1089.10	1.37×10^{-14}	433	0.32
7	2187	12.30	920.54	3979.94	3.18×10^{-14}	443	0.40
8	6561	172.44	3355.81	13445.96	7.35×10^{-13}	457	0.49
9	19683	FAILED	12893.37	48624.78	1.06×10^{-13}	466	0.64

Table 4 Elapsed timings (in seconds) for comparing homotopy methods and implementations

n	N n^2	BERTINI		MATLAB
		Diag. Coeff.	Fiber Prod.	Fiber Prod.
5	25	0.25	0.45	0.61
15	225	31.93	29.36	3.02
30	900	1158.61	882.68	18.99
50	2500	18661.62	14840.28	110.79

In this numerical experiment, we look at a two-parameter eigenvalue problem, i.e., $k = 2$, with $n_1 = n_2 = n$. The dimension of the problem is thus $N = n^2$. From Table 4, the fiber product homotopy method is consistently faster than the diagonal coefficient homotopy method in BERTINI. We include the timings of our MATLAB implementation of the fiber product homotopy method in the last column for comparison.

9 Numerical results II: Mathieu’s systems

An example where multiparameter eigenvalue problems surface is Mathieu’s systems, which in turn arises from studies of vibration of a fixed elliptic membrane [18,23]. We will test our MATLAB implementation of fiber product homotopy method against the Delta method on this problem.

We refer the reader to [18, Section 2] for a discussion of how a coupled system of two-point boundary value problems, representing Mathieu’s angular and radial equations, yields a two-parameter eigenvalue problem (thus $k = 2$) upon Chebyshev collocation discretization. The dimensions of the matrices n_1 and n_2 correspond to the number of points used in the discretization.

In Fig. 5, we present accuracy result for a two-parameter eigenvalue problem with $n_1 = 18$ and $n_2 = 38$ coming from a Mathieu system. The horizontal axis represents the $n_1 n_2 = 684$ eigenvalues, ordered from the smallest to largest by the norm of λ_1 .

The vertical axis measures backward errors on a log scale. The blue plot shows the backward error of our fiber product homotopy method whereas the red plot is that for

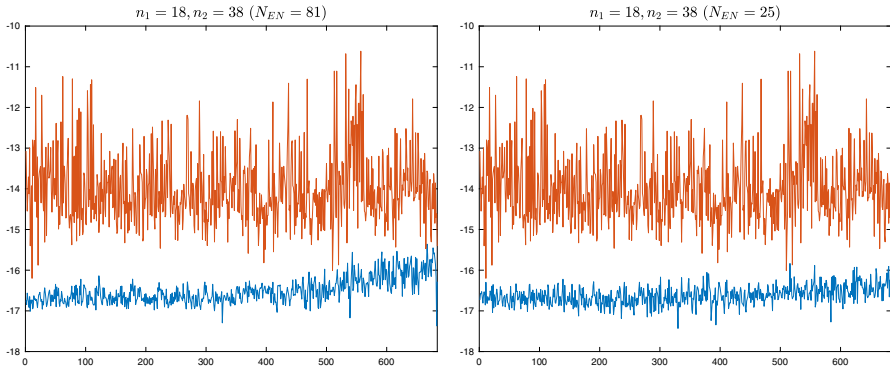


Fig. 5 The \log_{10} backward errors (vertical axis) of fiber product method (blue) and Delta method (red) plotted against eigenvalues (horizontal axis) ordered from smallest to largest. Left figure: 81 Newton iterations, taking 19 seconds. Right figure: 25 Newton iterations at the end of the path, taking 16 seconds

the Delta method. With few exceptions, fiber product homotopy produces significantly more accurate results than the Delta method by orders of magnitude. Indeed, every eigenpair computed with our method here has a backward error that is less than 10^{-15} .

The left and right figures in Fig. 5 differ by N_{EN} , the number of Newton iterations used in the Newton step at the end of the path—the left plot uses the conservative stopping condition of $81 = \max(20, k \max\{n_1, \dots, n_k\} + 5)$ iterations whereas the right plot uses an early stopping condition of 25 iterations. We see no discernible difference in the backward errors but the left plot took 19 seconds with 28 threads whereas the right plot took only 16 seconds to compute. This suggests that there is room for further fine-tuning to improve speed without sacrificing stability.

We next employ Shub–Smale α -theory [5] to certify the quadratic convergence of the Newton steps in a neighborhood of the end point. Briefly, in this theory there is a function α that takes a polynomial system $f: \mathbb{C}^n \rightarrow \mathbb{C}^n$ and an approximate solution $z \in \mathbb{C}^n$ as its input and returns a positive real number. If $\alpha(f, z)$ is less than the constant $(13 - 3\sqrt{17})/4$, then one has certified quadratic convergence of Newton’s method on the approximate solution z to the polynomial system f [5, Theorem 2, p. 160]. With our default tolerances we are able to certify the smallest 550 of the 684 eigenvalues—see Fig. 6.

10 Numerical results III: dimension-deficient singular MEPs

To each MEP with k parameters there correspond $k + 1$ Delta matrices $\Delta_0, \dots, \Delta_k$ [2]. The nonhomogeneous problem (3) is *singular* if Δ_0 is singular [13, Section 2]. Such problems have fewer than the expected $n_1 \cdots n_k$ eigenpairs. That a singular k -parameter eigenvalue problem presents computational difficulties is already evident when $k = 1$: a GEP with a singular matrix pencil $A - \lambda B$ is well-known to be challenging computationally. Among the singular MEPs, our fiber product homotopy method is well-suited for dimension-deficient MEPs that are generic with respect to intrinsic dimension, i.e., where the intrinsic dimensions $(d_1, \dots, d_k) < (n_1, \dots, n_k)$.

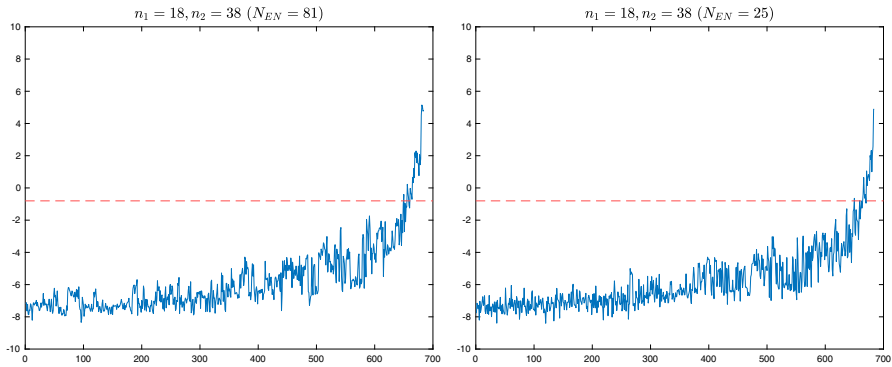


Fig. 6 Computed bound (vertical axis) on $\log_{10} \alpha$ for approximate eigenpairs of Mathieu MEP plotted against eigenvalues (horizontal axis) ordered from smallest to largest. If the computed bound is below $\log_{10}[(13 - 3\sqrt{17})/4]$ (red dashed line), the eigenpair is certified. Left figure: 81 Newton iterations. Right figure: 25 Newton iterations (color figure online)

We emphasize that these form a subclass of the singular problems — not all singular MEPs satisfy the aforementioned criteria.

Such dimension-deficient singular MEPs are inevitable when one linearizes *quadratic multiparameter eigenvalue problems* (QMEPs) [13]: Given quadratic polynomial matrices

$$\begin{aligned} Q_1(\lambda, \mu) &:= B_{00} + \lambda B_{10} + \mu B_{01} + \lambda^2 B_{20} + \lambda \mu B_{11} + \mu^2 B_{02}, \\ Q_2(\lambda, \mu) &:= C_{00} + \lambda C_{10} + \mu C_{01} + \lambda^2 C_{20} + \lambda \mu C_{11} + \mu^2 C_{02}, \end{aligned} \tag{20}$$

where $B_{ij} \in \mathbb{C}^{n_1 \times n_1}$ and $C_{ij} \in \mathbb{C}^{n_2 \times n_2}$, $i, j = 0, 1, 2$, solve

$$Q_1(\lambda, \mu) \mathbf{x}_1 = 0, \quad Q_2(\lambda, \mu) \mathbf{x}_2 = 0$$

for all possible $\lambda, \mu \in \mathbb{C}$ and nonzero $\mathbf{x}_1 \in \mathbb{C}^{n_1}$, $\mathbf{x}_2 \in \mathbb{C}^{n_2}$. It is straightforward to generalize this to a quadratic k -parameter eigenvalue problem but we will only study the case $k = 2$ here.

The quadratic two-parameter eigenvalue problem is mathematically equivalent to a two-parameter eigenvalue problem [13]:

$$\begin{aligned} H_1(\lambda, \mu) &= \begin{bmatrix} B_{00} & B_{10} & B_{01} \\ 0 & -I & 0 \\ 0 & 0 & -I \end{bmatrix} + \lambda \begin{bmatrix} 0 & B_{20} & B_{11} \\ I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & B_{02} \\ 0 & 0 & 0 \\ I & 0 & 0 \end{bmatrix}, \\ H_2(\lambda, \mu) &= \begin{bmatrix} C_{00} & C_{10} & C_{01} \\ 0 & -I & 0 \\ 0 & 0 & -I \end{bmatrix} + \lambda \begin{bmatrix} 0 & C_{20} & C_{11} \\ I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & C_{02} \\ 0 & 0 & 0 \\ I & 0 & 0 \end{bmatrix}. \end{aligned} \tag{21}$$

Whereas the original coefficient matrices in Q_i are in $\mathbb{C}^{n_i \times n_i}$, the coefficient matrices of H_i are in $\mathbb{C}^{3n_i \times 3n_i}$, $i = 1, 2$. However, the real catch is that the two-parameter

Table 5 Number of divergent paths and elapsed timings (in seconds) for the diagonal coefficient (DC), fiber product (FP) homotopy methods, and Delta method (ΔM). t_{PATH} is the time taken to track a single path, i.e., roughly how fast FP would have run in parallel

n	N $4n^2$	# of divergent paths			Wall time				t_{PATH}
		BERTINI		MATLAB	BERTINI		MATLAB		
		DC	FP	FP	DC	FP	ΔM	FP	
2	16	20	0	0	0.49	0.38	0.07	0.55	0.19
5	100	125	0	0	23.72	8.60	0.11	1.36	0.22
10	400	500	0	0	801.24	244.01	0.95	5.41	0.30
20	1600	2000	0	0	38761.58	14079.69	24.98	56.56	0.79
40	6400	8000	0	0	33 DAYS	8 DAYS	1156.38	1133.90	4.40

eigenvalue problem (21) is singular and its intrinsic dimension $(2n_1, 2n_2)$ is strictly smaller than $(3n_1, 3n_2)$.

In our experiment, we set $n_1 = n_2 = n$, randomly generate $B_{ij}, C_{ij} \in \mathbb{C}^{n \times n}$, and compare fiber product and diagonal coefficient homotopy methods in BERTINI. The difficulty of dimension-deficient MEP is conspicuously reflected in our observed results: The diagonal coefficient homotopy method has $9n^2$ start solutions and $5n^2$ of the paths tracked does not converge for all values of n we tested, which we recorded in Table 5. On the other hand, in these experiments where the MEPs are generic with respect to intrinsic dimension, our fiber product homotopy method did not produce a single divergent path. For $n = 40$, the wall time of the diagonal coefficient homotopy method in BERTINI is estimated based on the number of paths it tracked when the fiber product homotopy method finishes.

We next compare fiber product homotopy and Delta methods [17] in MATLAB. From the backward errors in Fig. 7 for dimension-deficient singular MEPs, we see a similar pattern as in the nonsingular MEPs in Fig. 4—fiber product homotopy method is more accurate than the Delta method by orders of magnitude. More importantly, Delta method loses accuracy as n increases but fiber product homotopy method maintains it at 10^{-16} even after n is doubled four times.

From the last three columns of Table 5, we see that while the Delta method beats fiber product homotopy method in terms of speed for smaller values of n , the situation reverses when $n = 40$. Furthermore, the value of t_{PATH} in the last column indicates that had we run fiber product homotopy method in parallel on a machine with sufficiently many cores, all eigenpairs can be obtained in seconds. On the other hand, even though the program `eig` in MATLAB has the ability to use multiple cores and processors for the Delta method, parallelization is not straightforward as the problem involves k large Delta matrices of sizes $n_1 \cdots n_k \times n_1 \cdots n_k$ that need to be formed and stored in advance. As such, the Delta method is infeasible for large matrices.

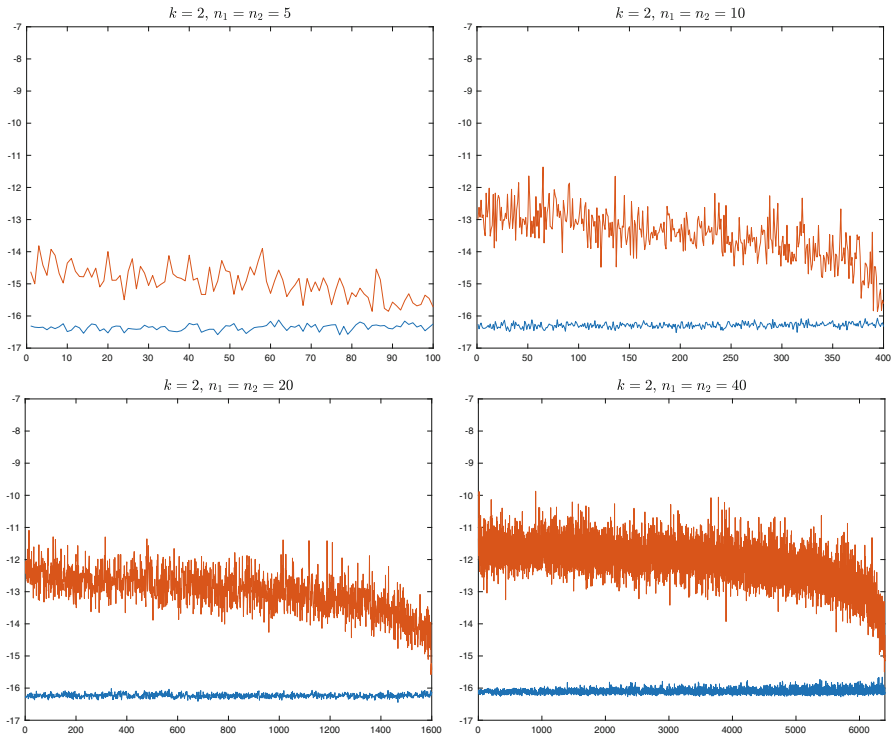


Fig. 7 The \log_{10} backward errors (vertical axis) of fiber product homotopy (blue) and Delta methods (red) for the dimension-deficient singular MEP (21), plotted against eigenvalues (horizontal axis) ordered in increasing norm from left to right (color figure online)

In this section we have limited ourselves to singular MEPs that are (i) dimension-deficient and (ii) generic with respect to intrinsic dimension. One situation when (ii) is not satisfied is an example⁶ aerospace engineering.

11 Conditioning of the fiber product MEP and standard MEP

Recall that condition number [7, Chapter 14] depends on the *problem*—two different problems that always give the same solution will in general have different condition numbers. The condition number for MEP as defined in [14] applies to the standard formulation of MEP in (2). Since we are solving a different problem (4), or more generally (16), albeit one that gives the same solutions as (2), it will have a different condition number.

⁶ The example is implemented in our GitHub codes at https://github.com/JoseMath/MEP_Homotopy/blob/master/MATLAB/examples/demo_flutter1.m for aerospace engineering [19], which remains a challenge for our method. In general, when the solutions to (3) are not the same as the regular eigenpairs of the MEP, the fiber product homotopy method may lose some regular eigenpairs. To address this issue, one has to know the connection between the $H_i(\lambda)$ and the corresponding system of generalized eigenvalue problems on Δ -matrices but this is unfortunately not well-understood.

In this section, we discuss the condition number for the fiber product MEP (4) and compare it to the condition number for the standard MEP as defined in [14]. They are expected to be quite different by the common interpretation of condition number as measuring the change in output produced by a change in input. The standard condition number for MEP in [14] measures the change in eigenvalues (output) produced by a change in the coefficient matrices (input). In fiber product homotopy method, we do not deform these coefficient matrices; instead, the relevant notion of condition number is one that measures the change in eigenvalues (output) produced by a change in the linear space defined by $G_1 = \dots = G_k = 0$ as in (16) (input). So in our setting, we require the condition number of a variety intersected with a varying linear subspace.

In the following, by a *projective linear subspace* of a projective space, we mean $\{\pi(\mathbf{x}) \in \mathbb{P}^n : M\mathbf{x} = 0, \mathbf{x} \neq 0\}$ for some $M \in \mathbb{C}^{m \times (n+1)}$ and where $\pi : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n$ is the canonical projection. Henceforth, we will write $\mathbb{M} := \pi(\ker(M) \setminus \{0\})$ for the projective linear subspace that $M \in \mathbb{C}^{m \times (n+1)}$ defines in \mathbb{P}^n . We write $[x_0 : x_1 : \dots : x_n]$ for homogeneous coordinates in \mathbb{P}^n .

11.1 Intersecting a variety with a varying linear subspace

We briefly review some relevant ideas in [6], on which our condition number in Sect. 11.2 is based. Let \mathcal{Z} be a degree- p variety in \mathbb{P}^n of codimension $n - d$. The *Hurwitz variety* of \mathcal{Z} is a subvariety of the Grassmannian $\mathbb{G}(n - d, n)$ of $(n - d)$ -dimensional projective linear subspaces in \mathbb{P}^n defined by

$$\mathcal{H}_{\mathcal{Z}} := \{\mathbb{M} \in \mathbb{G}(n - d, n) : \mathcal{Z} \cap \mathbb{M} \text{ does not consist of } p \text{ reduced points}\}.$$

The Hurwitz variety is an irreducible hypersurface defined by a polynomial called the *Hurwitz form* in the coordinate ring of the Grassmannian [22]. This variety can be regarded as the set of ill-posed instances of the problem of intersecting a variety by a varying linear space [6]. By [6, Definition 1.1 and Theorem 1.4], we have the following definition.

Definition 6 Let \mathcal{Z} be a d -dimensional irreducible projective variety in \mathbb{P}^n and $\mathbb{M} \in \mathbb{G}(n - d, n)$. Let $\mathbf{z} \in \mathcal{Z} \cap \mathbb{M}$ and α be the minimum angle between the tangent spaces $T_{\mathbf{z}}(\mathcal{Z})$ and $T_{\mathbf{z}}(\mathbb{M})$. The *intersection condition number* of \mathbb{M} at \mathbf{z} with respect to \mathcal{Z} is

$$\kappa_{\mathcal{Z}}(\mathbb{M}, \mathbf{z}) := \frac{1}{\sin \alpha}$$

if \mathbf{z} is a smooth point of \mathcal{Z} and \mathbb{M} intersects \mathcal{Z} transversally at \mathbf{z} , and $\kappa_{\mathcal{Z}}(\mathbb{M}, \mathbf{z}) := \infty$ otherwise.

The elements of $\mathcal{H}_{\mathcal{Z}}$ are precisely the projective linear subspaces where the intersection condition number is infinite [6, Theorem 1.6].

11.2 Condition number of the fiber product MEP

Consider the multiparameter eigenvalue variety $\mathcal{E}\mathcal{V}(H_1, \dots, H_k) \subseteq \mathbb{C}^{k^2}$ in (8). Throughout this section, $\mathcal{Z} \subseteq \mathbb{P}^{k^2}$ will always denote the projective variety

$$\mathcal{Z} := \overline{\{[1 : \lambda_1 : \dots : \lambda_k] \in \mathbb{P}^{k^2} : (\lambda_1, \dots, \lambda_k) \in \mathcal{E}\mathcal{V}(H_1, \dots, H_k)\}}, \tag{22}$$

where we have denoted the homogeneous coordinates of \mathbb{P}^{k^2} by

$$[\lambda_0 : \underbrace{\lambda_{11} : \dots : \lambda_{1k}}_{\lambda_1} : \dots : \underbrace{\lambda_{k1} : \dots : \lambda_{kk}}_{\lambda_k}],$$

with λ_0 as the homogenizing coordinate. The defining equations of \mathcal{Z} are homogeneous polynomials in $\lambda_0, \lambda_1, \dots, \lambda_k$ such that setting $\lambda_0 = 1$ gives the defining equations for $\mathcal{E}\mathcal{V}(H_1, \dots, H_k)$ in (8)

We will apply the notion of an intersection condition number in Sect. 11.1 to define a condition number for fiber product MEP.

Definition 7 Given a fiber product multiparameter eigenvalue problem (4), let \mathcal{Z} be the projective variety in (22), \mathbb{M}_{FP} be the projective linear space

$$\begin{aligned} \mathbb{M}_{\text{FP}} &:= \overline{\{[1 : \lambda_1 : \dots : \lambda_k] \in \mathbb{P}^{k^2} : \lambda_1 = \dots = \lambda_k\}} \\ &= \overline{\{[\lambda_0 : \lambda_1 : \dots : \lambda_k] \in \mathbb{P}^{k^2} : \lambda_1 = \dots = \lambda_k\}}, \end{aligned} \tag{23}$$

and \mathbf{z} be the point

$$\mathbf{z} := [1 : \lambda_1 : \dots : \lambda_k] \in \mathcal{Z} \cap \mathbb{M}_{\text{FP}}.$$

The condition number of the fiber product MEP (4) is given by

$$\kappa_{\text{FP}}(\lambda_1, \dots, \lambda_k, H_1, \dots, H_k) := \kappa_{\mathcal{Z}}(\mathbb{M}_{\text{FP}}, \mathbf{z}).$$

Note that $\mathcal{Z} \cap \mathbb{M}_{\text{FP}}$ consists precisely of points $[1 : \lambda_1 : \dots : \lambda_k] \in \mathbb{P}^{k^2}$ where $(\lambda_1, \dots, \lambda_k) \in \mathbb{C}^{k^2}$ are the eigenvalues of the MEP. We next show how to compute $\kappa_{\mathcal{Z}}(\mathbb{M}, \mathbf{z})$ for any projective linear subspace \mathbb{M} —as we will see later, for fiber product homotopy method, we would also be interested in $\kappa_{\mathcal{Z}}(\mathbb{M}, \mathbf{z})$ for projective linear subspaces \mathbb{M} other than \mathbb{M}_{FP} .

Theorem 2 Let $A_{ij} \in \mathbb{C}^{n_i \times n_i}$, $i = 1, \dots, k$, $j = 0, 1, \dots, k$, be the matrices of an MEP as in Definition 1. Let \mathcal{Z} be the projective variety in (22), $\mathbb{M} = \pi(\ker(M) \setminus \{0\})$ for some $M \in \mathbb{C}^{(k^2-k) \times (k^2+1)}$, and $\mathbf{z} = [1 : \lambda_1 : \dots : \lambda_k] \in \mathcal{Z} \cap \mathbb{M}$ be a smooth point of \mathcal{Z} . If $\mathbf{x}_i, \mathbf{y}_i \in \mathbb{C}^{n_i}$ are nonzero vectors⁷ such that

$$H_i(\lambda_i)\mathbf{x}_i = 0, \quad H_i(\lambda_i)^T\mathbf{y}_i = 0, \quad i = 1, \dots, k,$$

⁷ These are respectively right and left eigenvectors of the polynomial matrix.

then the intersection condition number

$$\kappa_{\mathcal{Z}}(\mathbb{M}, \mathbf{z}) = \frac{1}{\sin \alpha},$$

where α is the minimum angle between \mathbb{M} and $\mathbb{J} = \pi(\ker(J) \setminus \{0\})$,

$$J := \begin{bmatrix} \mathbf{y}_1^T A_{10} \mathbf{x}_1 & \mathbf{y}_1^T B_1(\mathbf{x}_1) & & & \\ \mathbf{y}_2^T A_{20} \mathbf{x}_1 & & \mathbf{y}_2^T B_2(\mathbf{x}_2) & & \\ \vdots & & & \ddots & \\ \mathbf{y}_k^T A_{k0} \mathbf{x}_k & & & & \mathbf{y}_k^T B_k(\mathbf{x}_k) \end{bmatrix} \in \mathbb{C}^{k \times (1+k^2)} \tag{24}$$

with $B_i(\mathbf{x}_i) := -[A_{i1}\mathbf{x}_i, \dots, A_{ik}\mathbf{x}_i] \in \mathbb{C}^{n_i \times k}$, $i = 1, \dots, k$.

Proof By Definition 6, it suffices to show that $T_{\mathbf{z}}(\mathcal{Z}) = T_{\mathbf{z}}(\mathbb{J})$. First, recall from Definition 3 that the multiparameter eigenvalue variety $\mathcal{EV}(H_1, \dots, H_k)$ is a projection of the multiparameter eigenpair variety $\mathcal{EP}(H_1, \dots, H_k)$. So the projective closure of $\mathcal{EP}(H_1, \dots, H_k)$ in $\mathbb{P}^{k^2} \times \mathbb{P}^{n_1-1} \times \dots \times \mathbb{P}^{n_k-1}$, which we will denote by $\overline{\mathcal{EP}}(H_1, \dots, H_k)$, projects onto the variety \mathcal{Z} .

For a linear polynomial matrix

$$H(\boldsymbol{\lambda}) = A_0 - \lambda_1 A_1 - \dots - \lambda_k A_k,$$

we introduce a homogenizing variable λ_0 and define a homogenized H as

$$\widehat{H}(\lambda_0, \boldsymbol{\lambda}) := \lambda_0 A_0 - \lambda_1 A_1 - \dots - \lambda_k A_k.$$

We homogenize H_1, \dots, H_k in this manner, write $\mathbf{w} := (\lambda_0, \boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_k, \mathbf{x}_1, \dots, \mathbf{x}_k)$ for brevity, and then let

$$\mathbf{H}(\mathbf{w}) := \begin{bmatrix} \widehat{H}_1(\lambda_0, \boldsymbol{\lambda}_1) \mathbf{x}_1 \\ \vdots \\ \widehat{H}_k(\lambda_0, \boldsymbol{\lambda}_k) \mathbf{x}_k \end{bmatrix}.$$

Write $\mathbf{H} = [h_1, \dots, h_{n_1+\dots+n_k}]^T$ where $h_i \in \mathbb{C}[\mathbf{w}]$, $i = 1, \dots, n_1 + \dots + n_k$, are *multihomogeneous polynomials*—homogeneous in the variables $\lambda_0, \boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_k$ and homogeneous in each of the variables $\mathbf{x}_1, \dots, \mathbf{x}_k$. Note that the h_i 's vanish on the variety $\mathcal{EP}(H_1, \dots, H_k)$; in fact, the set of polynomials that vanish on $\mathcal{EP}(H_1, \dots, H_k)$ is given by

$$\{h \in \mathbb{C}[\mathbf{w}] : \lambda_0^m h = g_1 h_1 + \dots + g_{n_1+\dots+n_k} h_{n_1+\dots+n_k}, g_i \in \mathbb{C}[\mathbf{w}], m \in \mathbb{N}\}.$$

The $(n_1 + \dots + n_k) \times (1 + k^2 + n_1 + \dots + n_k)$ Jacobian matrix,

$$\nabla \mathbf{H}(\mathbf{w}) = \left[\frac{\partial}{\partial \lambda_0} \mathbf{H}, \nabla_{\boldsymbol{\lambda}_1} \mathbf{H}, \dots, \nabla_{\boldsymbol{\lambda}_k} \mathbf{H}, \nabla_{\mathbf{x}_1} \mathbf{H}, \dots, \nabla_{\mathbf{x}_k} \mathbf{H} \right],$$

when evaluated at a point

$$w_0 := ([1 : \lambda_1 : \dots : \lambda_k], x_1, \dots, x_k) = (z, x_1, \dots, x_k) \in \overline{\mathcal{EP}}(H_1, \dots, H_k),$$

has the form

$$\nabla H(w_0) = \begin{bmatrix} A_{10}x_1 & B_1(x_1) & & & \widehat{H}_1(1, \lambda_1) & & \\ A_{20}x_1 & & B_2(x_2) & & & \widehat{H}_2(1, \lambda_2) & \\ \vdots & & & \ddots & & & \ddots \\ A_{k0}x_k & & & & B_k(x_k) & & \widehat{H}_k(1, \lambda_k) \end{bmatrix}.$$

For any $L \in \mathbb{C}^{m \times (k^2+1)}$, if each row of the matrix $[L, \mathbb{O}_{m \times (n_1+\dots+n_k)}] \in \mathbb{C}^{m \times (k^2+1+n_1+\dots+n_k)}$ is in the row space of $\nabla H(w_0)$, then $\mathbb{L} = \pi(\ker(L) \setminus \{0\})$ is such that $T_z(\mathcal{Z}) \subseteq T_z(\mathbb{L})$. This is because \mathcal{Z} is the projection of $\overline{\mathcal{EP}}(H_1, \dots, H_k)$ to \mathbb{P}^{k^2+1} in the first $k^2 + 1$ coordinates. With this in mind, we multiply $\nabla H(w_0)$ on the left by

$$\begin{bmatrix} y_1^\top & & & \\ & y_2^\top & & \\ & & \ddots & \\ & & & y_k^\top \end{bmatrix} \in \mathbb{C}^{k \times (n_1+\dots+n_k)}$$

to obtain the $k \times (1 + k^2 + n_1 + \dots + n_k)$ matrix

$$\begin{bmatrix} y_1^\top A_{10}x_1 & y_1^\top B_1(x_1) & & & \mathbb{O}_{n_1+\dots+n_k}^\top \\ y_2^\top A_{20}x_1 & & y_2^\top B_2(x_2) & & \mathbb{O}_{n_1+\dots+n_k}^\top \\ \vdots & & & \ddots & \vdots \\ y_k^\top A_{k0}x_k & & & & y_k^\top B_k(x_k) \mathbb{O}_{n_1+\dots+n_k}^\top \end{bmatrix} = [J, \mathbb{O}_{k \times (n_1+\dots+n_k)}],$$

showing that $T_z(\mathcal{Z}) \subset T_z(\mathbb{J})$. When J is full rank, we have that

$$\text{codim } T_z(\mathbb{J}) = k = \text{codim } T_z(\mathcal{Z})$$

when z is a smooth point. It follows that $T_z(\mathcal{Z}) = T_z(\mathbb{J})$. □

We show how we obtain the condition number $\kappa_{\text{FP}}(\lambda_1, \dots, \lambda_k, H_1, \dots, H_k)$. Consider the following one-parameter family of projective linear spaces induced by a one-parameter family of matrices: For a fixed $t \in [0, 1]$, let

$$M_t := (1-t) \begin{bmatrix} -\mathbb{1}_{k-1} & \nabla L_1 & & & \\ -\mathbb{1}_{k-1} & & \nabla L_2 & & \\ \vdots & & & \ddots & \\ -\mathbb{1}_{k-1} & & & & \nabla L_k \end{bmatrix} + t \begin{bmatrix} \mathbb{O}_{k-1} & \nabla G_1 \\ \mathbb{O}_{k-1} & \nabla G_2 \\ \vdots & \vdots \\ \mathbb{O}_{k-1} & \nabla G_k \end{bmatrix} \in \mathbb{C}^{k(k-1) \times (k^2+1)}, \tag{25}$$

where L_1, \dots, L_k are as in (11) and G_1, \dots, G_k are as in (14). Let

$$\mathbb{M}_t := \pi(\ker(M_t) \setminus \{0\}) \subseteq \mathbb{P}^{k^2}$$

be the corresponding projective linear subspace. Note that \mathbb{M}_t dehomogenizes (by setting $\lambda_0 = 1$) to the affine linear space in \mathbb{C}^{k^2} defined by

$$\begin{aligned} 0 &= (1 - t)L_1(\lambda_1) + tG_1(\lambda_1, \dots, \lambda_k), \\ &\vdots \\ 0 &= (1 - t)L_k(\lambda_k) + tG_k(\lambda_1, \dots, \lambda_k), \end{aligned}$$

where the right-hand sides are the last $k(k - 1)$ linear polynomials in (18). Therefore, for any $t \in [0, 1)$, the projective closure of the set of solutions to $\mathbf{H}_{\text{FP}}(\lambda_1, \dots, \lambda_k, \mathbf{x}_1, \dots, \mathbf{x}_k, t) = 0$ defined in (18) projects onto $\mathcal{Z} \cap \mathbb{M}_t$.

When $t = 1$, by (15) and (23), we have

$$\mathbb{M}_1 = \{[\lambda_0 : \lambda_1 : \dots : \lambda_k] \in \mathbb{P}^{k^2} : \lambda_1 = \dots = \lambda_k\} = \mathbb{M}_{\text{FP}},$$

and for $\mathbf{z} = [1 : \lambda_1 : \dots : \lambda_k] \in \mathcal{Z} \cap \mathbb{M}_1$, we obtain, by Definition 7,

$$\kappa_{\mathcal{Z}}(\mathbb{M}_1, \mathbf{z}) = \kappa_{\text{FP}}(\lambda_1, \dots, \lambda_k, H_1, \dots, H_k).$$

The intersection condition number $\kappa_{\mathcal{Z}}(\mathbb{M}_t, \mathbf{z})$ for $t \in [0, 1)$ is also useful as it measures conditioning of the subproblems encountered during path tracking in the fiber product homotopy method. The next theorem shows that $\kappa_{\mathcal{Z}}(\mathbb{M}_t, \mathbf{z})$ is almost always finite and Example 6 indicates that it is typically small.

Theorem 3 *For any $t \in [0, 1)$, $\kappa_{\mathcal{Z}}(\mathbb{M}_t, \mathbf{z})$ is finite with probability one.*

Proof Since the Hurwitz variety $\mathcal{H}_{\mathcal{Z}}$ comprises projective linear subspaces \mathbb{M} with $\kappa_{\mathcal{Z}}(\mathbb{M}, \mathbf{z}) = \infty$, it suffices to show that for any $t \in [0, 1)$, $\mathbb{M}_t \notin \mathcal{H}_{\mathcal{Z}}$ with probability one. By Theorem 1, the fiber product homotopy $\mathbf{H}_{\text{FP}}: \mathbb{C}^{k^2} \times (\mathbb{P}^{n_1-1} \times \dots \times \mathbb{P}^{n_k-1}) \times \mathbb{C} \rightarrow \mathbb{C}^{n_1+\dots+n_k} \times \mathbb{C}^{k(k-1)}$ has a start system chosen correctly with probability one. Thus $\mathbf{H}_{\text{FP}}(\lambda_1, \dots, \lambda_k, \mathbf{x}_1, \dots, \mathbf{x}_k, t) = 0, t \in [0, 1)$, has smooth solution paths with probability one.

If $\mathbb{M}_t \in \mathcal{H}_{\mathcal{Z}}$, then these solution paths would not be smooth as the projective closure of the set of solutions to $\mathbf{H}_{\text{FP}}(\lambda_1, \dots, \lambda_k, \mathbf{x}_1, \dots, \mathbf{x}_k, t) = 0$ projects onto $\mathcal{Z} \cap \mathbb{M}_t$. Thus, with probability one, $\kappa_{\mathcal{Z}}(\mathbb{M}_t, \mathbf{z})$ is finite for $t \in [0, 1)$. \square

For comparison, $\kappa(\lambda, H_1, \dots, H_k)$, the condition number of a standard MEP (2) as defined in [14] captures how small perturbations in the inputs $A_{ij}, i = 1, \dots, k, j = 0, \dots, k$, affect the eigenvalue $\lambda = (\lambda_1, \dots, \lambda_k)$, i.e.,

$$\limsup_{\varepsilon \rightarrow 0} \left\{ \|\Delta\lambda\|/\varepsilon : (A_{i0} + \Delta A_{i0} - \sum_{j=1}^k (\lambda_j + \Delta\lambda_j)(A_{ij} + \Delta A_{ij}))(x_i + \Delta x_i) = 0, \|\Delta A_{ij}\| \leq \varepsilon \|A_{ij}\|, i = 1, \dots, k, j = 0, \dots, k \right\}$$

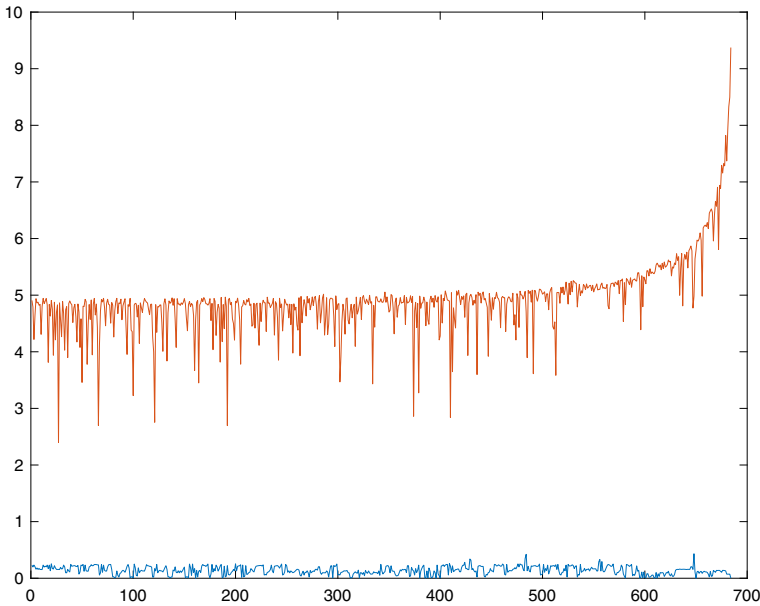


Fig. 8 The condition numbers (vertical axis) $\kappa_{\text{FP}}(\lambda_1, \lambda_2, H_1, H_2)$ for fiber product homotopy (blue) and $\kappa(\lambda, H_1, H_2)$ for Delta method (red) plotted respectively against eigenvalues λ_1 and λ (horizontal axis), ordered in increasing norm. The vertical axis is in \log_{10} -scale (color figure online)

where $(\mathbf{x}_1, \dots, \mathbf{x}_k)$ is the eigenvector, i.e., $H_i(\lambda)\mathbf{x}_i = 0, i = 1, \dots, k$. We will also let $(\mathbf{y}_1, \dots, \mathbf{y}_k)$ be the left eigenvector, i.e., $H_i(\lambda)^T \mathbf{y}_i = 0, i = 1, \dots, k$.

Let $\theta_i := \|A_{i0}\| + \sum_{j=1}^k |\lambda_j| \|A_{ij}\|, i = 1, \dots, k$. The θ -weighted norm,

$$\|M\|_{\theta} := \max\{\|Mz\|_2 : z \in \mathbb{C}^k, |z_i| = \theta_i, i = 1, \dots, k\},$$

for any $M \in \mathbb{C}^{k \times k}$. By [14, Theorem 6], $\kappa(\lambda, H_1, \dots, H_k) = \|M^{-1}\|_{\theta}$ where

$$M := \begin{bmatrix} \mathbf{y}_1^* A_{11} \mathbf{x}_1 & \mathbf{y}_1^* A_{12} \mathbf{x}_1 & \cdots & \mathbf{y}_1^* A_{1k} \mathbf{x}_1 \\ \mathbf{y}_2^* A_{21} \mathbf{x}_2 & \mathbf{y}_2^* A_{22} \mathbf{x}_2 & \cdots & \mathbf{y}_2^* A_{2k} \mathbf{x}_2 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{y}_k^* A_{k1} \mathbf{x}_k & \mathbf{y}_k^* A_{k2} \mathbf{x}_k & \cdots & \mathbf{y}_k^* A_{kk} \mathbf{x}_k \end{bmatrix}.$$

We next show that the same Mathieu problem formulated as a fiber product MEP (4) and a standard MEP (2) can have vastly different condition numbers.

Example 6 [Conditioning of Mathieu problem] We generate an instance of the Mathieu problem in Sect. 9 with $n_1 = 18$ and $n_2 = 38$. In Fig. 8, we plot the fiber product multiparameter eigenvalue problem condition number $\kappa_{\text{FP}}(\lambda_1, \lambda_2, H_1, H_2) = \kappa_{\mathcal{Z}}(\mathbb{M}_1, z)$ and the standard multiparameter eigenvalue problem condition number $\kappa(\lambda, H_1, H_2)$, against λ_1 and λ respectively in increasing norms.

The difference is striking—we emphasize that the vertical axis of the figure is in \log_{10} -scale. The values of $\kappa_{\text{FP}}(\lambda_1, \lambda_2, H_1, H_2)$ are all close to 1 and certainly less than 10. So small perturbations of the projective linear space \mathbb{M}_1 yield small changes in the points of intersection of \mathcal{Z} with \mathbb{M}_1 , which correspond to the eigenvalues we seek. On the other hand, the values of $\kappa(\lambda, H_1, H_2)$ are vastly larger, with a majority exceeding 10^5 and those near the right end of the plot corresponding to the largest eigenvalues are as large as 10^9 .

Apart from $\kappa_{\mathcal{Z}}(\mathbb{M}_1, \mathbf{z})$, we have also computed the intersection condition number $\kappa_{\mathcal{Z}}(\mathbb{M}_t, \mathbf{z})$ as t varies from 0 to 1 in every path we tracked. We ran our implementation five times and found that $\kappa_{\mathcal{Z}}(\mathbb{M}_t, \mathbf{z})$ does not exceed 21.85 for every t we encountered in every path and in every run.

12 Conclusions

The fiber product homotopy method solves an MEP by solving a mathematically equivalent problem, the fiber product MEP, via a homotopy algorithm designed to exploit its structure. Our numerical experiments show that the fiber product homotopy method: (i) is much faster than diagonal coefficient homotopy method on all instances we tested and is faster than the Delta method on large instances; (ii) is extremely accurate, producing relative backward errors on the order of 10^{-16} , especially in comparison with the 10^{-11} errors in the Delta method; (iii) is unique in that it maintains the same high degree of accuracy across a robust range of parameters — irrespective of the dimensions of the matrices or the magnitudes of the eigenvalues.

We proffer two insights to explain its strength: (a) it deforms only linear equations whereas the diagonal coefficient homotopy method deforms nonlinear equations; (b) the problem that it solves, the fiber product MEP, is much better conditioned than the equivalent standard MEP with the same solutions.

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